ANOTHER PROOF OF T. TRACZYK'S THEOREM

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1. Introduction. This paper is concerned with the properties of a minimal extention of a Boolean algebra. T. Traczyk [2] had proved that a minimal extension of a weakly α-distributive Boolean algebra satisfying the α-chain condition is also weakly α-distributive. The tools with which he had proved this theorem was the family of α-partitions of a Boolean algebra $\boldsymbol{B}$ which has a covering of $\boldsymbol{B}$ which refines every member of the family. In this way his proof was so algebraic but so elegant. Motiviated by this theorem, I thought that whether this theorem can be proved by another method i. e. topological one or not. However, an answer for this problem is affirmative by showing the theorem which is as follows: A minimal extension of an α-representative Boolean algebra satisfying the α-chain condition is also α-representative. In the first place I prove this theorem by using topological properties of the Stone space of a Boolean algebra and in the second place give another proof of the previous Traczyk's theorem. All sorts of theorems, concepts and terminologies which are used in this paper due to the work of Prof. Sikorski [1].

2. Terminology and notation. Throughout this paper, α denotes an infinite cardinal number. The symbol $\cup$ will be used both for the Boolean join and for set-theoretical union. The symbol $\cap$ will be used both for the Boolean meet and for the set-theoretical intersection. The zero element of a Boolean algebra will be denoted by 0 and the unit by 1. The empty set will be denoted by ϕ and the complement of a set A will be denoted by A'. The difference of two sets A and C is written by $A-C$. The symbol $|T|$ stands for the cardinal number of the set T.

If A is a subalgebra of a Boolean algebra $\boldsymbol{B}$ and $a_t \in A$ for every $t \in T$, then the set $\{a_t : t \in T\}$ may have two joins, one taken in A and the other in $\boldsymbol{B}$; we denote these joins whenever they exist, by $\cup^{A}_{t \in T} a_t$ and $\cup^{B}_{t \in T} a_t$ respectively. A subalgebra A of a Boolean algebra $\boldsymbol{B}$ is said to be α-regular subalgebra of $\boldsymbol{B}$ provided, for $\{a_t : a_t \in A, t \in T, |T| \leqq \alpha\}$, if the join $\cup^{A}_{t \in T} a_t$ exists, it is also the join of all a_t in $\boldsymbol{B}$, i. e. the equality

$$\cup^{B}_{t \in T} a_t = \cup^{A}_{t \in T} a_t$$

holds.

A set $\boldsymbol{D}$ of elements of a Boolean algebra $\boldsymbol{B}$ is said to be dense in $\boldsymbol{B}$ provided, for every element $a \in \boldsymbol{B}, a \neq 0$, then there exists an element $b \in \boldsymbol{D}$ such that $0 \neq b \subset a$.

H. M. Stone had proved that every Boolean algebra $\boldsymbol{B}$ is isomorphic to a field of all both open and closed subsets of a compact totally disconnected space. Such a space is said to be the Stone space of $\boldsymbol{B}$ and such a isomorphism is called the Stone isomorphism. For every Boolean algebra and every cardinal number α, $S(\boldsymbol{B})$ denotes the Stone space of $\boldsymbol{B}$, $F_0(\boldsymbol{B})$ the field of all both open and closed subsets of $S(\boldsymbol{B})$, and $F_\alpha(\boldsymbol{B})$ the smallest α-field of subsets of $S(\boldsymbol{B})$ containing $F_0(\boldsymbol{B})$.

3. Topological properties. A subset A of a topological space X is said to be α-closed provided it is the intersection of at most α both open and closed subsets. A subset A of topological space X is said to be α-nowhere dense provided it is a subset of a nowhere dense α-closed set. Any union of at most α sets which are all α-nowhere dense in X is called set of the α-category. Obviously, the class of all sets of the α-category in X forms α-ideal in the class of all subsets of X. We define that $A \subset X$ is said to be have the Baire α-property if there exists an open set G such that $A-G$ and $G-A$ are sets of the α-category. In other words, A has Baire α-property if

$$A=(G-N_1) \cup N_2$$

where G is open, and N_1, N_2 are of the α-category.

Lemma 1. If every nowhere dense set in a topological space X is α-nowhere dense, then the class of sets which have all Baire α-property is a α-field of subsets of X.

Proof. The symbol B denotes the closure of a set B. If A has the Baire α-property, that is, there exists an open set G such that $A-G$ and $G-A$ are sets of the α-category, so has its complement. In fact, $(\bar{G})'=G_0$ be the complement of the closure of G. Then G_0 is open, and

$$A'-G_0=\bar{G}-A \subset (\bar{G}-G) \cup (G-A).$$

Since $\bar{G}-G$ is the boundary set of open set G, it is a nowhere dense. According to the hypothesis, $\bar{G}-G$ is an α-nowhere dense. On the other hand,

$$G_0-A'=A-\bar{G} \subset A-G.$$

Hence, it follows that $A'-G_0$ and G_0-A' are of the α-category.

If all sete A_t $(t \in T,\ T \leqq \alpha)$ have the Baire α-property, so their union A. In fact, let G_t be open sets such that A_t-G_t and G_t-A_t are sets of the α-category $t \in T$. Let G be the union of all sets G_t. Since

$$A-G \subset \cup_{t \in T}(A_t-G_t),\quad G-A \subset \cup_{t \in T}(G_t-A_t),$$

the sets $A-G$ and $G-A$ are of the α-category.

Lemma 2. If every nowhere dense set in a topological space X is α-nowhere dense, then every set of the smallest α-field containing the field of all both open and closed subsets of X has the Baire α-property.

Proof. It is clear that every both open and closed set has the Baire α-property. By Lemma 1, the class of sets which have all the Baire α-property forms the α-field, consequently, the smallest α-field containing the field of all both open and closed sets is contained in the class of sets which have all the Baire α-property.

4. The α-representativity. A Boolean algebra is said to be α-representative provided it is isomorphic to an α-regular subalgebra of a factor algebra $\boldsymbol{F}/\boldsymbol{I}$ where $\boldsymbol{F}$ is an α-field of sets and $\boldsymbol{I}$ is an α-ideal of $\boldsymbol{F}$. If α-complete Boolean algebra $\boldsymbol{B}$ is α-representative, then its isomorphic image is α-subalgebra $\boldsymbol{F}'/\boldsymbol{I}$ of $\boldsymbol{F}/\boldsymbol{I}$ where $\boldsymbol{F}'$ is an α-subfield of $\boldsymbol{F}$.

Thus an α-complete Boolean algebra is called α-representative if and only if it is isomorphic to a factor algebra $\boldsymbol{F}/\boldsymbol{I}$ where $\boldsymbol{F}$ is an α-field of sets, and $\boldsymbol{I}$ is α-field of $\boldsymbol{F}$.

A Boolean algebra $\boldsymbol{B}^{\infty}$ is said to be an minimal extension of $\boldsymbol{B}$ provided that it has following two properties:

(i) $\boldsymbol{B}^{\infty}$ is complete,

(ii) $\boldsymbol{B}^{\infty}$ contains a dense subalgebra isomorphic to $\boldsymbol{B}$.

The existence of such $\boldsymbol{B}^{\infty}$ is guaranteed [see [1] 35.1].

All minimal extensions of $\boldsymbol{B}$ are isomorphic to each other [see [1] 33.4].

For a Boolean algebra $\boldsymbol{B}$, let $\boldsymbol{i}$ be the Stone isomorphism of $\boldsymbol{B}$ onto $F_0(\boldsymbol{B})$ and let $\Delta_\alpha(\boldsymbol{B})$ be the α-ideal of all sets $A \epsilon F_\alpha(\boldsymbol{B})$ which are of the α-category in $S(\boldsymbol{B})$.

Theorem 1. A minimal extension of an α-representative Boolean algebra $\boldsymbol{B}$ satisfying the α-chain condition is also α-representative.

Proof. The symbol $[A]$ denotes the element of the factor algebra of $F_\alpha(\boldsymbol{B})/\Delta_\alpha(\boldsymbol{B})$. The formula

$$\boldsymbol{i}^*(a) = [\boldsymbol{i}(a)] \quad (a \epsilon \boldsymbol{B})$$

defines an isomorphism from $\boldsymbol{B}$ into $F_\alpha(\boldsymbol{B})/\Delta_\alpha(\boldsymbol{B})$. It is clear that the $\boldsymbol{i}^*$ is an homomorphism from $\boldsymbol{B}$ into $F_\alpha(\boldsymbol{B})/\Delta_\alpha(\boldsymbol{B})$. If $\boldsymbol{i}^*(a)=0$, then $[\boldsymbol{i}(a)]=0$. Hence, $\boldsymbol{i}(a)$ is contained in $\Delta_\alpha(\boldsymbol{B})$. Since $\boldsymbol{B}$ is α-representative, it follows that $\boldsymbol{i}(a)=\Phi$ [see [1] 29.3 r_1]. This means that $\boldsymbol{i}^*$ is the isomorphism.

Now we shall show that the isomorphic image $\boldsymbol{i}^*(\boldsymbol{B})$ of $\boldsymbol{B}$ is a dense subalgebra of $F_\alpha(\boldsymbol{B})/\Delta_\alpha(\boldsymbol{B})$. Let us recall that $\boldsymbol{B}$ satisfies the α-chain condition, consequently, every nowhere dense set in $S(\boldsymbol{B})$ is α-nowhere dense [see [1] § 22, example (G)].

By lemma 2, every set in $F_\alpha(\boldsymbol{B})$ has the Baire α-property. Thus, every set A in $F_\alpha(\boldsymbol{B})$ can be represented as follws:

$$A=(G-A_1)\cup A_2$$

where G is an open set and A_1, A_2 are of the α-category. If $[A]\neq 0$, then G is not empty, i. e. there exists an element $a\neq 0\,(a\in\boldsymbol{B})$ such that $i(a)\subset G$. Consequently $i^*(a)=[i(a)]\neq 0$. We have $[A]\supset[i(a)]$. In fact,

$$\begin{aligned} i(a)-A=i(a)\cap A'&=i(a)\cap(G'\cup A_1)\cap A'_2 \\ &=i(a)\cap A_1\cap A'_2\subset A_1. \end{aligned}$$

Since $i(a)-A$ is contained in $F_\alpha(\boldsymbol{B})$, it is a set of the α-category. This means that $i(a)-A\in\Delta_\alpha(\boldsymbol{B})$ i. e. $[A]\supset[i(a)]$.

Finally, we shall show that $F_\alpha(\boldsymbol{B})/\Delta_\alpha(\boldsymbol{B})$ is a complete Boolean algebra. $\boldsymbol{B}$ satisfies the α-chain condition, therefore the subalgebra $i^*(\boldsymbol{B})$ of $F_\alpha(\boldsymbol{B})/\Delta_\alpha(\boldsymbol{B})$ satisfies the α-chain condition. Since the subalgebra $i^*(\boldsymbol{B})$ is dense, $F_\alpha(\boldsymbol{B})/\Delta_\alpha(\boldsymbol{B})$ satisfies the α-chain condition. Consequently, α-complete Boolean algebra $F_\alpha(\boldsymbol{B})/\Delta_\alpha(\boldsymbol{B})$ is complete Boolean algebra. [see [1] 20.2].

This proves that $F_\alpha(\boldsymbol{B})/\Delta_\alpha(\boldsymbol{B})$ is a minimal extension of $\boldsymbol{B}$. Hence $\boldsymbol{B}^\infty$ is α-representative.

Theorem 2. Let $\boldsymbol{B}$ be an α-representative Boolean algebra which satisfies the α-chain condition. Let F_α be the α-field of all subsets of $S(\boldsymbol{B})$ having Baire α-property and the let Δ_α be the α-ideal of all sets $A\in F_\alpha$ of the α-category in $S(B)$. Then F_α/Δ_α is a minimal extension of the Boolean algebra $\boldsymbol{B}$.

Proof. This proof is formally quite similar to the proof of theorem 1.

5. The weakly α-distributivity. A Boolean algebra $\boldsymbol{B}$ is called weakly α-distributive if satisfies the identity

$$\bigcap_{t\in T}\bigcup_{s\in S} a_{ts}=\bigcup_{\varphi\in F(S)^T}\bigcap_{t\in T} a_{t\varphi(t)}$$

where $F(S)$ is the set of finite subsets of S, $a_{t\varphi(t)}=\bigcup_{s\in\varphi(t)} a_{ts}$, $|T|\leqq\alpha$, $|S|\leqq\alpha$ subject to hypothesis that the join $\bigcup_{s\in S} a_{ts}$ as well as the meet $\bigcap_{t\in T}\bigcup_{s\in S} a_{ts}$ and $\bigcap_{t\in T} a_{t\varphi(t)}$ exist in $\boldsymbol{B}$.

Theorem 3. (T. Traczyk) A minimal extention of weakly α-distributive Boolean algebra $\boldsymbol{B}$ satisfying the α-chain condition is also weakly α-distributive.

Proof. Since every weakly α-distrbutive Boolean algebra is α-representative, by Theorem 2, F_α/Δ_α is a minimal extension of $\boldsymbol{B}$. We shall show that F_α/Δ_α is weakly α-distributive.

Let $\{a_{ts}: t\in T, s\in S, |T|\leqq\alpha, |S|\leqq\alpha\}$ be a subset of F_α/Δ_α. By the completeness of F_α/Δ_α, $\bigcup_{s\in S} a_{ts}$ for every $t\in T$, $\bigcap_{t\in T}\bigcup_{s\in S} a_{ts}$ and $\bigcap_{t\in T} a_{t\varphi(t)}$ for every $\varphi\in F(S)^T$

exist.

For every a_{ts}, there is an element $A_{ts} \in F_\alpha$ such that $a_{ts}=[A_{ts}]$. Moreover, every set $A \in F_\alpha$ is of the form

$$A=(G-A_1)\cup A_2$$

where G is open and $A_1, A_2 \in \Delta_\alpha$. Consequently $[A]=[G]$, that is, every element a of F_α/Δ_α is of the form $[G]$ where G is open. Hence, for every a_{ts}, there exists an open set G_{ts} in $S(B)$ such that $a_{ts}=[G_{ts}]$. It follows easily that

$$a_{t\varphi(t)}=\cup_{s\in\varphi(t)} a_{ts}=\cup_{s\in\varphi(t)}[G_{ts}]$$
$$=[\cup_{s\in\varphi(t)} G_{ts}]=[G_{t\varphi(t)}].$$

Suppose now that $\cap_{t\in T}\cup_{s\in S} a_{ts}\neq 0$. Then, by the α-completeness of F_α, we obtain

$$\cap_{t\in T}\cup_{s\in S} a_{ts}=\cap_{t\in T}\cup_{s\in S}[G_{ts}]$$
$$=\cap_{t\in T}[\cup_{s\in S} G_{ts}]=[\cap_{t\in T}\cup_{s\in S} G_{ts}]\neq 0$$

That is,

$$\cap_{t\in T}\cup_{s\in S} G_{ts}$$

is not contained in Δ_α. Besides it is of the form

$$\cap_{t\in T}\cup_{s\in S} G_{ts}=(G-A_1)\cup A_2$$

where G is open and A_1, A_2 belong to Δ_α. Thus G is not empty. Since the Boolean algebra B is weakly α-distributive, the sets A_1, A_2 are nowhere dense sets [see [1] 30.1 (w_4)].

Consequently, $\cap_{t\in T}\cup_{s\in S} G_{ts}$ contains a non-empty both open and closed subset C. $S(B)$ being the compact space, the closed subset C is compact. Since $C\subset\cup_{s\in S} G_{ts}$ for every $t\in T$, that is, $\{G_{ts}: s\in S\}$ is a open covering of C, there is a finite subcovering of C. Consequently, there exists a $\varphi\in F(S)^T$ such that $C\subset \cap_{t\in T} G_{t\varphi(t)}$.

C being a non-empty open set, C is not contained in Δ_α. Hence, we have that

$$\cap_{t\in T} a_{t\varphi(t)}=\cap_{t\in T}[G_{t\varphi(t)}]$$
$$=[\cap_{t\in T} G_{t\varphi(t)}]\supset[C]\neq 0$$

Thus, F_α/Δ_α is weakly α-distributive Boolean algebra [see [1] 30.1 (w_1)].

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