SOME NOTES ON THE QUEUES WITH MULTIPLE INPUTS

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Introduction

Some studies of queues with different types of customers having independent Poisson inputs have been published. Previous works on the subject have been confined to a special case when the type of service distribution is exponential. In the case each server has its own special service distribution, the analysis has been developed [1]. But if each type of customers has its own speciality, the analysis of many server queuing system seems to be difficult. Recently the case of a single server queuing system has studied analytically by Ancker and Gafarian [2]. In this paper we shall examine the problem of a single-server queuing system with multiple Poisson inputs and general service distributions by applying the technique devised by Keilson and Kooharian. [3] Next we shall investigate the equilibrium behaviors of a two-server queuing system for two different types of customers having independent Poisson arrivals with rates λ_1, λ_2 and exponential service times with rates μ_1, μ_2.

1. Statement of the first Problem

The first and secand types of customers arrive at a service mechanism in independent Poisson streams with mean rates λ_1 and λ_2 respectively. The service-time distributions of both types are general with probability densities $D_1(x)$ and $D_2(x)$ respectively and the customers are serviced under the first-come, first-served queue discipline. Let $\eta_1(x)\triangle$ be the first-order probability that a customer of type 1 completes the service in the interval $(x, x+\triangle)$, if the customer has already been in the service for a time x, and similarly $\eta_2(x)\triangle$ for a customer of type 2.

The relation between $\eta_i(x)$ and the probability density $D_i(x)$ is given by

$$D_i(x)=\eta_i(x)\, exp\left[-\int_0^x \eta_i(y)\,dy\right].$$

We now define the following probabilities:

$P_i^{(n)}(x, t)\,dx\ (n\geqq 0, i=1, 2)$ is the probability that at time t there are n customes in the queue, excluding service, and a customer of type i is being served with elapsed service time lying between x and $x+dx$.

$P_0(t)$ is the probabilty that at time t there is neither a customer of type 1 nor one of type 2 in the system.

$w(t)$ is the waiting time of a customer arriving at time t.

2. Formulation of Equations and Their Solutions

In order to derive the difference-differential equations for the process, we follow Keilson and kooharian [3] and relate as usual the probabilities at time $t+\triangle$ to those at time t. These arguments lead to the equation

$$P_1^{(n)}(x+\triangle, t+\triangle)=P_1^{(n)}(x,t)(1-\lambda\triangle)(1-\eta_1(x)\triangle)+P_1^{(n-1)}(x,t)\lambda\triangle \qquad (n\geqq 1)$$

which, as $\triangle\to 0$ becomes

$$\frac{\partial P_1^{(n)}(x,t)}{\partial t}+\frac{\partial P_1^{(n)}(x,t)}{\partial x}+\{\lambda+\eta_1(x)\}P_1(x,t)=\lambda P_1^{(n-1)}(x,t) \qquad (n\geqq 1). \tag{2.1}$$

Similarly, we have

$$\frac{\partial P_2^{(n)}(x,t)}{\partial t}+\frac{\partial P_2^{(n)}(x,t)}{\partial x}+\{\lambda+\eta_2(x)\}P_2^{(n)}(x,t)=\lambda P_2^{(n-1)}(x,t) \qquad (n\geqq 1). \tag{2.2}$$

For $n=0$ we similarly find

$$\frac{\partial P_1^{(0)}(x,t)}{\partial t}+\frac{\partial P_1^{(0)}(x,t)}{\partial x}+\{\lambda+\eta_1(x)\}\, P_1^{(0)}(x,t)=0 \tag{2.3}$$

and

$$\frac{\partial P_2^{(0)}(x,t)}{\partial t}+\frac{\partial P_2^{(0)}(x,t)}{\partial x}+\{\lambda+\eta_2(x)\}\, P_2^{(0)}(x,t)=0. \tag{2.4}$$

Finally for $P_0(t)$ we have

$$\frac{dP_0(t)}{dt}+\lambda P_0(t)=\int_0^\infty P_1^{(0)}(x,t)\,\eta_1(x)\,dx+\int_0^\infty P_2^{(0)}(x,t)\,\eta_2(x)\,dx. \tag{2.5}$$

These equations are to be solved under the following boundary conditions:

$$P_1^{(n)}(0,t)=\alpha\int_0^\infty P_1^{(n+1)}(x,t)\,\eta_1(x)\,dx+\alpha\int_0^\infty P_2^{(n+1)}(x,t)\,\eta_2(x)\,dx, \tag{2.6}$$

$$P_1^{(0)}(0,t)=\alpha\int_0^\infty P_1^{(1)}(x,t)\,\eta_1(x)\,dx+\alpha\int_0^\infty P_2^{(1)}(x,t)\,\eta_2(x)\,dx+\lambda_1 P_0(t) \tag{2.7}$$

$$P_2^{(n)}(0,t)=\beta\int_0^\infty P_2^{(n+1)}(x,t)\,\eta_2(x)\,dx+\beta\int_0^\infty P_1^{(n+1)}(x,t)\,\eta_1(x)\,dx, \tag{2.8}$$

$$P_2^{(0)}(0,t)=\beta\int_0^\infty P_2^{(1)}(x,t)\,\eta_2(x)\,dx+\beta\int_0^\infty P_1^{(1)}(x,t)\,\eta_1(x)\,dx+\lambda_2 P_0(t) \tag{2.9}$$

and the intial condition $P_0(0)=1$ (i. e. the system starts with no customers),

where $\qquad \lambda=\lambda_1+\lambda_2, \quad \alpha=\frac{\lambda_1}{\lambda} \quad$ and $\quad \beta=\frac{\lambda_2}{\lambda}$.

We define the following generating functions:

$$G_i(z,x,t)=\sum_{n=0}^{\infty} z^n P_i^{(n)}(x,t) \qquad i=1,2. \tag{2.10}$$

Multiplying equations (2.1)—(2.4) with appropriate powers of z, adding and using (2.10) we have

$$\frac{\partial G_1(z,x,t)}{\partial t}+\frac{\partial G_1(z,x,t)}{\partial x}+\{\lambda(1-z)+\eta_1(x)\}G_1(z,x,t)=0, \tag{2.11}$$

$$\frac{\partial G_2(z,x,t)}{\partial t}+\frac{\partial G_2(z,x,t)}{\partial x}+\{\lambda(1-z)+\eta_2(x)\}G_2(z,x,t)=0 \tag{2.12}$$

and the boundary conditions become

$$\begin{aligned}G_1(z,o,t)=&\frac{\alpha}{z}\int_0^\infty G_1(z,x,t)\,\eta_1(x)\,dx+\frac{\alpha}{z}\int_0^\infty G_2(z,x,t)\,\eta_2(x)\,dx\\&-\frac{\alpha}{z}\int_0^\infty G_1(o,x,t)\,\eta_1(x)\,dx-\frac{\alpha}{z}\int_0^\infty G_2(o,x,t)\,\eta_2(x)\,dx+\lambda_1P_0(t),\end{aligned} \tag{2.13}$$

$$\begin{aligned}G_2(z,o,t)=&\frac{\beta}{z}\int_0^\infty G_2(z,x,t)\,\eta_2(x)\,dx+\frac{\beta}{z}\int_0^\infty G_1(z,x,t)\,\eta_1(x)\,dx\\&-\frac{\beta}{z}\int_0^\infty G_2(o,x,t)\,\eta_2(x)\,dx-\frac{\beta}{z}\int_0^\infty G_1(o,x,t)\,\eta_1(x)\,dx+\lambda_2P_0(t).\end{aligned} \tag{2.14}$$

Equation (2.5) becomes

$$\frac{dP_0(t)}{dt}+\lambda P_0(t)=\int_0^\infty G_1(o,x,t)\,\eta_1(x)\,dx+\int_0^\infty G_2(o,x,t)\,\eta_2(x)\,dx. \tag{2.15}$$

Setting

$$G_i(z,x,t)=H_i(z,x,t)\,e^{-\int_0^x \eta_i(y)\,dy}\qquad i=1,2,$$

in (2.11) and (2.12), we find that

$$\frac{\partial H_1(z,x,t)}{\partial t}+\frac{\partial H_1(z,x,t)}{\partial x}+\lambda(1-z)\,H_1(z,x,t)=0 \tag{2.16}$$

and

$$\frac{\partial H_2(z,x,t)}{\partial t}+\frac{\partial H_2(z,x,t)}{\partial x}+\lambda(1-z)\,H_2(z,x,t)=0. \tag{2.17}$$

The solution of (2.16) and (2.17) is given by

$$H_i(z,x,t)=H_i^*(z,t-x)\,e^{-\lambda(1-z)x},\qquad i=1,2. \tag{2.18}$$

Substituting (2.13) and (2.14) in (2.15) and using (2.18), we have

$$\begin{aligned}&\frac{dP_0(t)}{dt}+\lambda(1-z)\,P_0(t)+\frac{z}{\alpha}\,H_1^*(z,t)\\&=\int_0^\infty H_1^*(z,t-x)\,D_1(x)\,e^{-\lambda(1-z)x}dx+\int_0^\infty H_2^*(z,t-x)\,D_2(x)\,e^{-\lambda(1-z)x}dx\end{aligned} \tag{2.19}$$

and

(2.20) $$\frac{dP_0(t)}{dt}+\lambda(1-z)P_0(t)+\frac{z}{\beta}H_2^*(z,t)$$
$$=\int_0^\infty H_2^*(z,t-x)D_2(x)e^{-\lambda(1-z)x}dx+\int_0^\infty H_1^*(z,t-x)D_1(x)e^{-\lambda(1-z)x}dx.$$

Applying the similar argument in Keilson and Kooharian [3] to (2.19) and (2.20) we have

(2.21) $$\frac{dP_0(t)}{dt}+\lambda(1-z)P_0(t)+\frac{z}{\alpha}H_1^*(z,t)$$
$$=\int_0^t H_1^*(z,t-x)D_1(x)e^{-\lambda(1-z)x}dx+\int_0^t H_2^*(z,t-x)D_2(x)e^{-\lambda(1-z)x}dx$$

and

(2.22) $$\frac{dP_0(t)}{dt}+\lambda(1-z)P_0(t)+\frac{z}{\beta}H_2^*(z,t)$$
$$=\int_0^t H_2^*(z,t-x)D_2(x)e^{-\lambda(1-z)x}dx+\int_0^t H_1^*(z,t-x)D_1(x)e^{-\lambda(1-z)}dx.$$

Let the Laplace transform of the function $F(t)$ be denoted by $f(s)$, i.e. let

$$f(s)=\int_0^\infty e^{-st}F(t)\,dt.$$

Applying the Laplace transform to the equations (2.22), and employing the initial condition mentioned above we get

(2.23) $$\{s+\lambda(1-z)\}p_0(s)=\left[d_1\{s+\lambda(1-z)\}-\frac{z}{\alpha}\right]h_1^*(z,s)+d_2\{s+\lambda(1-z)\}h_2^*(z,s)+1$$

and

(2.24) $$\{s+\lambda(1-z)\}p_0(s)=\left[d_2\{s+\lambda(1-z)\}-\frac{z}{\beta}\right]h_2^*(z,s)+d_1\{s+\lambda(1-z)\}h_1^*(z,s)+1$$

or equivalently,

(2.25) $$h_1^*(z,s)=\frac{\alpha[\{s+\lambda(1-z)\}p_0(s)-1]}{\alpha d_1\{s+\lambda(1-z)\}+\beta d_2\{s+\lambda(1-z)\}-z}$$

and

(2.26) $$h_2^*(z,s)=\frac{\beta[\{s+\lambda(1-z)\}p_0(s)-1]}{\alpha d_1\{s+\lambda(1-z)\}+\beta d_2\{s+\lambda(1-z)\}-z}.$$

Thus (2.25) and (2.26) were completely determined except for $p_0(s)$, which we will determine by the usual argument.

Now it is easy to show that the equation

(2.27) $$z-\alpha d_1\{s+\lambda(1-z)\}-\beta d_2\{s+\lambda(1-z)\}=0$$

has one and only one root inside the unit circle $|z|=1$ for $R(s)>0$, $\lambda_1/\mu_1+\lambda_2/\mu_2<1$, where

$$\mu_i = 1 \Big/ \int_0^\infty x D_i(x)\, dx$$

is the mean rate of service of type i. The proof is similar to that of Takács [4] and will not be given. Let z_s be that root.

Accordingly, under the familiar stability condition $\lambda_1/\mu_1 + \lambda_2/\mu_2 < 1$ $p_0(s)$ can be explicitly given as

$$p_0(s) = \frac{1}{s+\lambda[1-\alpha d_1\{s+\lambda(1-z_s)\} - \beta d_2\{s+\lambda(1-z_s)\}]}. \tag{2.28}$$

3. Distribution of the Length of the Waiting Line

Now if $\Phi(z, t)$ represent the generating function of the distribution of the queue length at time t, it is given by

$$\Phi(z, t) = P_0(t) + \sum_{n=1}^\infty z^n \{P_1^{(n)}(t) + P_2^{(n)}(t)\} \tag{3.1}$$

where $P_i^{(n)}(t) = \int_0^t P_i^{(n-1)}(x, t)\, dx$ is the probability that at time t there are n customers in the system and a customer of type i is being served. In terms of $H_i^*(z, t)$ $(i=1, 2)$, (3.1) reduces to

$$\Phi(z, t) = P_0(t) + z\int_0^t e^{-\lambda(1-z)x}\{H_1^*(z, t-x)\, e^{-\int_0^x \eta_1(y)dy} + H_2^*(z, t-x) e^{-\int_0^x \eta_2(y)dy}\} dx. \tag{3.2}$$

Using the relation

$$exp\{-\int_0^x \eta_i(y)\, dy\} = 1 - \int_0^x D_i(y)\, dy$$

we find the Laplace transform of $\Phi(z, t)$ as follows:

$$\phi(z, s) = \int_0^\infty e^{-st}\Phi(z, t)\, dt$$

$$= \frac{(1-z)(s+\lambda-\lambda z)p_0(s) + z\{1-[\alpha d_1(s+\lambda-\lambda z) + \beta d_2(s+\lambda-\lambda z)]^{-1}\}}{(s+\lambda-\lambda z)\{1-z[\alpha d_1(s+\lambda-\lambda z)+\beta d_2(s+\lambda-\lambda z)]^{-1}\}}. \tag{3.3}$$

The generating function of the steady-state distribution, which exists for

$$\frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} < 1,$$

is obtained from (3.3) by the property of Laplace transform, viz.

$$\lim_{s\to 0} sf(s) = \lim_{t\to\infty} F(t).$$

From (3.3) we have

$$\lim_{t\to\infty} \Phi(z,t)=\frac{(1-z)P_0}{1-z\{\alpha d_1(\lambda-\lambda z)+\beta d_2(\lambda-\lambda z)\}^{-1}},$$

where P_0 is the steady-state null probability

$$P_0=\lim_{t\to\infty} P_0(t)=1-\frac{\lambda_1}{\mu_1}-\frac{\lambda_2}{\mu_2}.$$

4. Distribution of Waiting Times

The Laplace transform of the waiting-time distribution is readily derived by a usual method. If the system is empty at t, then $w(t)=0$. Otherwise $w(t)$ is the sum of the service times of those customers already waiting and the remaining service time of the current customer.

We introduce the following functions

$$F(x,t)=P\{w(t)\leqq x\}$$

and

$$\Psi(u,t)=\int_0^\infty e^{-ux}dF(x,t).$$

Since the service times are distributed independently, we have

$$\Psi(u,t)=P_0(t)+\int_0^t dx\sum_{n=0}^\infty \frac{P_1^{(n)}(x,t)[\alpha d_1(u)+\beta d_2(u)]^n}{1-\int_0^x D_1(y)\,dy}\cdot\int_0^\infty e^{-uv}D_1(x+v)\,dv$$

$$+\int_0^t dx\sum_{n=0}^\infty \frac{P_2^{(n)}(x,t)[\alpha d_1(u)+\beta d_2(u)]^n}{1-\int_0^x D_2(y)\,dy}\int_0^\infty e^{-uv}D_2(x+v)\,dv$$

$$=P_0(t)+\int_0^t dx H_1^*(\alpha d_1(u)+\beta d_2(u),t-x)\,e^{-\lambda[1-\alpha d_1(u)-\beta d_2(u)]x}\int_0^\infty e^{-uv}D_1(x+v)dv$$

$$+\int_0^t dx H_2^*(\alpha d_1(u)+\beta d_2(u),t-x)\,e^{-\lambda[1-\alpha d_1(u)-\beta d_2(u)]x}\int_0^\infty e^{-uv}D_2(x+v)dv.$$

Let $\Gamma(u,s)=\int_0^\infty e^{-st}\Psi(u,t)\,dt.$ Using the property of the Laplace transform we find that

$$\Gamma(u,s)=\frac{up_0(s)-1}{u-s-\lambda\{1-\alpha d_1(u)-\beta d_2(u)\}}$$

and hence that

$$\Psi(u,t)=e^{-ut-\lambda[1-\alpha d_1(u)-\beta d_2(u)]}\{1-u\int_0^t e^{-u\tau+\lambda\tau[1-\alpha d_1(u)-\beta d_2(u)]}P_0(\tau)d\tau\}.$$

We note that these results may be derived by substituting $D(x)=\alpha D_1(x)+\beta D_2(x)$ into the results derived by Keilson and Kooharian [3] and Heathcote [5].

5. Statement of the second Problem

Two types of customers, namely type 1 and type 2 customers, arrive randomly and independently at a service station with two channels.

Arrival times are distributed within classes according to exponential functions with arrival rates λ_1, λ_2. The service-time distributions are also exponential with service rates μ_1, μ_2 and queue discipline is first-come, first-served.

We now define the following steady-state probabilities:

$p_{20}^{(n)}$ $(n\geqq 0)$ is the probability that there are n customers in the queue and two of type 1 customers are being served.

$p_{11}^{(n)}$ $(n\geqq 0)$ is the probability that there are n customers in the queue and two of distinct types are being seveed.

$p_{02}^{(n)}$ $(n\geqq 0)$ is the probability that there are n customers in the queue and two of type 2 customers are being served.

p_{10} is the probability that one of type 1 customers is being served in one of two channels.

p_{01} is the probability that one of type 2 customers is being served in one of two channels.

p_0 is the probability that there is neither a type 1 customer nor a type 2 customer in the system.

6. Distribution of the Length of the Waiting Line

The steady-state equations for this process are

(6.1) $\mu_1 p_{10}+\mu_2 p_{01}=(\lambda_1+\lambda_2) p_0,$

(6.2) $(\lambda_1+\lambda_2+\mu_1) p_{10}=2\mu_1 p_{20}+\mu_2 p_{11}+\lambda_1 p_0,$

(6.3) $(\lambda_1+\lambda_2+\mu_2) p_{01}=\mu_1 p_{11}+2\mu_2 p_{02}+\lambda_2 p_0,$

(6.4) $(\lambda_1+\lambda_2+2\mu_1) p_{20}^{(0)}=2\mu_1 \alpha p_{20}^{(1)}+\mu_2 \alpha p_{11}^{(1)}+\lambda_1 p_{.0},$

(6.5) $(\lambda_1+\lambda_2+\mu_1+\mu_2) p_{11}^{(0)}=2\mu_1 \beta p_{20}^{(1)}+\mu_1 \alpha p_{11}^{(1)}+\mu_2 \beta p_{11}^{(1)}+2\mu_2 \alpha p_{02}^{(1)}+\lambda_2 p_{10}+\lambda_1 p_{01},$

(6.6) $(\lambda_1+\lambda_2+2\mu_2) p_{02}^{(0)}=\mu_1 \beta p_{11}^{(1)}+2\mu_2\beta p_{02}^{(1)}+\lambda_2 p_{01},$

(6.7) $(\lambda_1+\lambda_2+2\mu_1) p_{20}^{(n)}=2\mu_1 \alpha p_{20}^{(n+1)}+\mu_2 \alpha p_{11}^{(n+1)}+(\lambda_1+\lambda_2) p_{20}^{(n-1)},$ $(n\geqq 1),$

(6.8) $(\lambda_1+\lambda_2+\mu_1+\mu_2)p_{11}^{(n)}=2\mu_1\beta p_{20}^{(n+1)}+\mu_1\alpha p_{11}^{(n+1)}+\mu_2\beta p_{11}^{(n+1)}+2\mu_2\alpha p_{02}^{(n+1)}+(\lambda_1+\lambda_2)p_{11}^{(n-1)},$ $(n\geqq 1),$

(6.9) $(\lambda_1+\lambda_2+2\mu_2) p_{02}^{(n)}=\mu_1 \beta p_{11}^{(n+1)}+2\mu_2\beta p_{02}^{(n+1)}+(\lambda_1+\lambda_2) p_{02}^{(n+1)},$ $(n\geqq 1).$

We introduce the following generating functions:

(6.10) $$F_{20}(z)=\sum_{n=0}^{\infty} p_{20}^{(n)} z^n,\ F_{11}(z)=\sum_{n=0}^{\infty} p_{11}^{(n)} z^n,\ F_{02}(z)=\sum_{n=0}^{\infty} p_{02}^{(n)} z^n.$$

We multiply equations (6.4)—(6.9) by appropriate powers of z and add:

$$\{\lambda z^2-(\lambda+2\mu_1)z+2\mu_1\alpha\}F_{20}(z)+\mu_2\alpha F_{11}(z) = \lambda_1 p_{10}(1-z)+\alpha(\mu_1 p_{10}-\lambda_1 p_0)\equiv A(z), \tag{6.11}$$

$$2\mu_1\beta F_{20}(z)+\{\lambda z^2-(\lambda+\mu_1+\mu_2)z+(\mu_1\alpha+\mu_2\beta)\}F_{11}(z)+2\mu_2\alpha F_{02}(z) = (\lambda_1 p_{01}+\lambda_2 p_{10})(1-z)+\alpha(\mu_2 p_{01}-\lambda_2 p_0)+\beta(\mu_1 p_{10}-\lambda_1 p_0)\equiv B(z), \tag{6.12}$$

$$\{\lambda z^2-(\lambda+2\mu_2)z+2\mu_2\beta\}F_{02}(z)+\mu_1\beta F_{11}(z) = \lambda_2 p_{01}(1-z)+\beta(\mu_2 p_{01}-\lambda_2 p_0)\equiv C(z) \tag{6.13}$$

where
$$\lambda=\lambda_1+\lambda_2,\quad \alpha=\frac{\lambda_1}{\lambda}\quad \text{and}\quad \beta=\frac{\lambda_2}{\lambda}.$$

From (6.11), (6.12) and (6.13) we have

$$(\lambda z-2\mu_1)F_{20}(z)+(\lambda z-\mu_1-\mu_2)F_{11}(z)+(\lambda z-\mu_2)F_{02}(z)=-\lambda(p_{10}+p_{01}) \tag{6.14}$$

From (6.11), (6.12) and (6.13) we obtain

$$F_{20}(z)=H_1(z)/\triangle(z),\quad F_{11}(z)=H_2(z)/\triangle(z),\quad F_{02}(z)=H_3(z)/\triangle(z)$$

where

$$\triangle(z)=\begin{vmatrix} \lambda z^2-(\lambda+2\mu_1)z+2\mu_1\alpha & \mu_2\alpha & 0 \\ 2\mu_1\beta & \lambda z^2-(\lambda+\mu_1+\mu_2)z+\mu_1\alpha+\mu_2\beta & 2\mu_2\alpha \\ 0 & \mu_1\beta & \lambda z^2-(\lambda+2\mu_2)z+2\mu_2\beta \end{vmatrix} \tag{6.15}$$

$$=z(z-1)\{\lambda z^2-(\lambda+\mu_1+\mu_2)z+\mu_1\alpha+\mu_2\beta\}\{\lambda^2z^2-\lambda(\lambda+2\mu_1+2\mu_2)z+2\lambda(\mu_1\alpha+\mu_2\beta)+4\mu_1\mu_2\}$$

and $H_i(z)$ is the determinant obtained from $\triangle(z)$ by replacing the i-th column by a column vector

$$\begin{pmatrix} A(z) \\ B(z) \\ C(z) \end{pmatrix}.$$

Now it is easy to show that the equation

$$\lambda z^2-(\lambda+\mu_1+\mu_2)z+\mu_1\alpha+\mu_2\beta=0$$

has two real roots z_1, z_2 such that $0<z_1<1, z_2>1$. Furthermore it is seen that the equation $\lambda^2z^2-\lambda(\lambda+2\mu_1+2\mu_2)z+2\lambda(\mu_1\alpha+\mu_2\beta)+4\mu_1\mu_2=0$ has no root whose absolute value is less than unity under the condition $\lambda_1/\mu_1+\lambda_2/\mu_2<2$.

Hence it turns out that $\triangle(z)=0$ has one and only one real root in $0<|z|<1$ under the condition $\lambda_1/\mu_1+\lambda_2/\mu_2<2$.

Let the value of the root be ξ then using the relation

$$\lambda\xi^2-(\lambda+\mu_1+\mu_2)\xi+\mu_1\alpha+\mu_2\beta=0 \tag{6.16}$$

we get

$$H_i(\xi)=0,\qquad i=1,2,3.$$

When any one of these equations holds, another two equations follow from it. This shows a relation among p_0, p_{10} and p_{01}.

Using a relation $H_1(\xi)-H_2(\xi)=0$, we have

$$\{(1-\xi)\lambda_2+\beta\mu_1\}\{(\mu_1-\mu_2)\xi+\mu_1\alpha+\mu_2\beta\}p_{10}+\{(1-\xi)\lambda_1+\alpha\mu_2\}\{(\mu_1-\mu_2)\xi-\mu_1\alpha-\mu_2\beta\}p_{01}=2\lambda_1\beta(\mu_1-\mu_2)\xi p_0. \tag{6.17}$$

Taking account of the total probability, it holds that

(6.18) $$p_0+p_{10}+p_{01}+F_{20}(1)+F_{11}(1)+F_{02}(1)=1.$$

From the equations (6.11), (6.13) and (6.14) it is evident that quantities $F_{20}(1)$, $F_{11}(1)$ and $F_{02}(1)$ satisfy the following relations:

(6.19) $$-2\mu_1\beta F_{20}(1)+\mu_2\alpha F_{11}(1)=\alpha(\mu_1 p_{10}-\lambda_1 p_0),$$

(6.20) $$\mu_1\beta F_{11}(1)-2\mu_2\alpha F_{02}(1)=\beta(\mu_2 p_{01}-\lambda_2 p_0),$$

(6.21) $$(\lambda-2\mu_1)F_{20}(1)+(\lambda-\mu_1-\mu_2)F_{11}(1)+(\lambda-2\mu_2)F_{02}(1)=-\lambda(p_{10}+p_{01}).$$

Solving these equations for $F_{20}(1)$, $F_{11}(1)$ and $F_{02}(1)$, we have

(6.22) $$F_{20}(1)=H_1/\triangle,\ F_{11}(1)=H_2/\triangle,\ F_{02}(1)=H_3/\triangle,$$

where

$$\triangle=\begin{vmatrix} -2\mu_1\beta & \mu_2\alpha & 0 \\ 0 & \mu_1\beta & -2\mu_2\alpha \\ \lambda-2\mu_1 & \lambda-\mu_1-\mu_2 & \lambda-2\mu_2 \end{vmatrix}$$

and H_i is the determinant obtained from $\triangle$ by replacing the i-th column by a column vector

$$\begin{pmatrix} \alpha(\mu_1 p_{10}-\lambda_1 p_0) \\ \beta(\mu_2 p_{01}-\lambda_2 p_0) \\ -\lambda(p_{10}+p_{01}) \end{pmatrix}.$$

Hence unknowns p_0, p_{10} and p_{01} can be evaluated from (6.1), (6.17) and (6.18). However, this method is complicate, so we use the following results

(6.23) $$\begin{aligned} p_{10}+F_{11}(1)+2F_{20}(1)&=\rho_1, \\ p_{01}+F_{11}(1)+2F_{02}(1)&=\rho_2, \end{aligned}$$

where $$\rho_1=\frac{\lambda_1}{\mu_1} \quad\text{and}\quad \rho_2=\frac{\lambda_2}{\mu_2}.$$

The above results was shown in the case of the many server queuing system by T. Kawamura [6], but in the case of two servers it is easily proved as follows:

From (6.19), (6.20) and (6.1), we have

(6.24) $$\mu_1\beta(p_{10}+F_{11}(1)+2F_{20}(1))-\mu_2\alpha(p_{01}+F_{11}(1)+2F_{02}(1))=0.$$

Also, from (6.18), (6.21) and (6.1), we have

(6.25) $$\mu_1(p_{10}+F_{11}(1)+2F_{20}(1))+\mu_2(p_{01}+F_{11}(1)+2F_{02}(1))=\lambda_1+\lambda_2.$$

From (6.24) and (6.25) equation (6.23) can be derived.

From (6.18) and (6.23), we have

(6.26) $$p_{10}+p_{01}=2(1-p_0)-\rho_1-\rho_2.$$

Thus, by (6.1) and (6.26), we find that

(6.27) $$p_{10}=\frac{(\lambda+2\mu_2)p_0-\mu_1(2-\rho_1-\rho_2)}{\mu_1-\mu_2}$$

(6.28) $$p_{01}=\frac{(\lambda+2\mu_1)\,p_0-\mu_2\,(2-\rho_1-\rho_2)}{\mu_2-\mu_1}$$

for $\mu_1\neq\mu_2$.

Note that for equal service rate $\mu_1=\mu_2\equiv\mu$

$$(\lambda+2\mu)\,p_0-\mu\,(2-\rho_1-\rho_2)=0$$

and therefore

$$p_0=\frac{1-\dfrac{\lambda}{2\mu}}{1+\dfrac{\lambda}{2\mu}}=\frac{1-\rho}{1+\rho},$$

where $\rho=\frac{\lambda}{2\mu}$. In this case the above system reduces to that of the two-server case with Poisson arrival (arrival rate $\lambda=\lambda_1+\lambda_2$) and identical exponential service. From (6.17), (6.27) and (6.28), we have

(6.29)

$$p_0=\left[\frac{\{(\mu_1-\mu_2)(\alpha\mu_1^2-\beta\mu_2^2)-\lambda(\mu_1\alpha+\mu_2\beta)^2\}\xi+(\mu_1\alpha+\mu_2\beta)\{-\alpha\mu_1^2+2\mu_1\mu_2-\beta\mu_2^2+\lambda_1\mu_1+\lambda_2\mu_2\}}{\{2(\mu_1-\mu_2)(\alpha\mu_1^2-\beta\mu_2^2)-\lambda\mu_1\mu_2-\lambda_1\mu_1^2-\lambda_2\mu_2^2-(\alpha\mu_1+\beta\mu_2)\lambda^2\}\xi+(\mu_1\alpha+\mu_2\beta)\{-2\alpha\mu_1^2+4\mu_1\mu_2-2\beta\mu_2^2+(2\lambda_2+\lambda_1)\,\mu_1+(2\lambda_1+\lambda_2)\,\mu_2+\lambda^2\}}\right](2-\rho_1-\rho_2),\qquad(\mu_1\neq\mu_2),$$

Now we shall derive the expected number of customers in the queue. From (6.11), (6.13) and (6.14), we have

(6.30) $$-2\mu_1\beta F'_{20}(1)+\mu_2\alpha F'_{11}(1)=-(\lambda-2\mu_1)\,F_{20}(1)-\lambda_1 p_{10},$$

(6.31) $$\mu_1\beta F'_{11}(1)-2\mu_2\alpha F'_{02}(1)=-(\lambda-2\mu_2)\,F_{02}(1)-\lambda_2 p_{01},$$

(6.32) $$\begin{aligned}(\lambda-2\mu_1)\,F'_{20}(1)+(\lambda-\mu_1-\mu_2)\,F'_{11}(1)+(\lambda-2\mu_2)\,F'_{02}(1)&\\ =-\lambda\,\{F_{20}(1)+F_{11}(1)+F_{02}(1)\}&\\ =-\lambda\,(1-p_0-p_{10}-p_{01}).&\end{aligned}$$

After some algebra the mean queue length is given by

$$L_q=F'_{20}(1)+F'_{11}(1)+F'_{02}(1)$$

(6.33) $$=\frac{1}{\mu_1\,\mu_2\,(2-\rho_1-\rho_2)}\{(\mu_2-\mu_1)\,(\lambda-2\mu_1)\,F_{20}\,(1)+(\mu_1-\mu_2)(\,\lambda-2\mu_2)\,F_{02}(1)+(\mu_2\,\lambda_1+\mu_1\,\lambda_2)-(\lambda^2+\mu_2\,\lambda_1+\mu_1\,\lambda_2)\,p_0\}.$$

The result (6.33) holds, of course, for $\mu=\mu_2\,(\equiv\mu)$. The result (6.33) is identical, as it should be if $\mu_1=\mu_2=\mu$, with the mean queue length

$$L_q=\frac{2\rho^3}{1-\rho^2},\qquad \rho\equiv\frac{\lambda}{2\mu}$$

in the classical queuing process $M/M/2$.

7. Distribution of Waiting Times

We now define the following density functions:

$f_n(t)\,(n\geqq 0)$ is the probability density function of the waiting time conditioned on there being n customers in the queue and two of type 1 customers being served.

$g_n(t)\,(n\geqq 0)$ is the probability density function of the waiting time conditioned on there being n customers in the queue and two of distinct types being served.

$h_n(t)\,(n\geqq 0)$ is the probability density function of the waiting time conditioned on there being n customers in the queue and two of type 2 customers being served.

The following relations among these probalility density functions can easily be seen to hold:

$$(7.1)\quad f_n(t)=\int_0^t 2\mu_1 e^{-2\mu_1 x}\alpha f_{n-1}(t-x)\,dx+\int_0^t 2\mu_1 e^{-2\mu_1 x}\beta g_{n-1}(t-x)\,dx,\qquad (n\geqq 1),$$

$$(7.2)\quad g_n(t)=\int_0^t \{\mu_1 e^{-(\mu_1+\mu_2)x}\alpha g_{n-1}(t-x)+\mu_1 e^{-(\mu_1+\mu_2)x}\beta h_{n-1}(t-x)+\mu_2 e^{-(\mu_1+\mu_2)x}\alpha f_{n-1}(t-x)$$
$$+\mu_2 e^{-(\mu_1+\mu_2)x}\beta g_{n-1}(t-x)\}\,dx,\qquad (n\geqq 1)$$

and

$$(7.3)\quad h_n(t)=\int_0^t 2\mu_2 e^{-2\mu_2 x}\beta h_{n-1}(t-x)\,dx+\int_0^t 2\mu_2 e^{-2\mu_2 x}\alpha g_{n-1}(t-x)\,dx,\qquad (n\geqq 1).$$

Let the Laplace transform of $f(t)$ be denoted by $f^*(s)$, i.e. let

$$f^*(s)=\int_0^\infty e^{-st}f(t)\,dt.$$

Then we have the following relations:

$$(7.4)\quad f_n^*(s)=\frac{2\mu_1\alpha}{2\mu_1+s}f_{n-1}^*(s)+\frac{2\mu_1\beta}{2\mu_1+s}g_{n-1}^*(s),\qquad (n\geqq 1),$$

$$(7.5)\quad g_n^*(s)=\frac{\mu_1\alpha}{\mu_1+\mu_2+s}g_{n-1}^*(s)+\frac{\mu_1\beta}{\mu_1+\mu_2+s}h_{n-1}^*(s)+\frac{\mu_2\alpha}{\mu_1+\mu_2+s}f_{n-1}^*(s)$$
$$+\frac{\mu_2\beta}{\mu_1+\mu_2+s}g_{n-1}^*(s),\qquad (n\geqq 1)$$

and

$$(7.6)\quad h_n^*(s)=\frac{2\mu_2\beta}{2\mu_2+s}h_{n-1}^*(s)+\frac{2\mu_2\alpha}{2\mu_2+s}g_{n-1}^*(s),\qquad (n\geqq 1).$$

From (7.5) we obtain

$$g_n^*(s)=\frac{\mu_1}{\mu_1+\mu_2+s}(\alpha g_{n-1}^*(s)+\beta h_{n-1}^*(s))+\frac{\mu_2}{\mu_1+\mu_2+s}(\alpha f_{n-1}^*(s)+\beta g_{n-1}^*(s)).$$

The use of (7.4) and (7.6) in the above expression leads to

$$g^*_n(s)=\frac{\mu_1}{\mu_1+\mu_2+s}\cdot\frac{2\mu_2+s}{2\mu_2}h^*_n(s)+\frac{\mu_2}{\mu_1+\mu_2+s}\cdot\frac{2\mu_1+s}{2\mu_1}f^*_n(s). \tag{7.7}$$

From (7.4), (7.6) and (7.7) we obtain

$$f^*_{n+1}(s)-\left\{\frac{2\alpha\mu_1}{2\mu_1+s}+\frac{2\beta\mu_2}{2\mu_2+s}+\frac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2+s}\right\}f^*_n(s)+\left\{\frac{4\alpha\beta\mu_1\mu_2}{(2\mu_1+s)(2\mu_2+s)}\right.$$
$$\left.+\frac{2\alpha^2\mu_1^2}{(\mu_1+\mu_2+s)(2\mu_1+s)}+\frac{2\beta^2\mu_2^2}{(\mu_1+\mu_2+s)(2\mu_2+s)}\right\}f^*_{n-1}(s)=0. \tag{7.8}$$

Here it can be shown that the equation

$$X^2-\left\{\frac{2\alpha\mu_1}{2\mu_1+s}+\frac{2\beta\mu_2}{2\mu_2+s}+\frac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2+s}\right\}X+\left\{\frac{4\alpha\beta\mu_1\mu_2}{(2\mu_1+s)(2\mu_2+s)}\right.$$
$$\left.+\frac{2\alpha^2\mu_1^2}{(\mu_1+\mu_2+s)(2\mu_1+s)}+\frac{2\beta^2\mu_2^2}{(\mu_1+\mu_2+s)(2\mu_2+s)}\right\}=0$$

has the roots

$$\frac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2+s}\quad\text{and}\quad\frac{2\alpha\mu_1}{2\mu_1+s}+\frac{2\beta\mu_2}{2\mu_2+s}.$$

From (7.8) the expression for $f^*_n(s)$ is obtained in the form

$$f^*_n(s)=A_1\left(\frac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2+s}\right)^n+B_1\left(\frac{2\alpha\mu_1}{2\mu_1+s}+\frac{2\beta\mu_2}{2\mu_2+s}\right)^n. \tag{7.9}$$

Employing the obvious relations we obtain

$$f^*_0(s)=\frac{2\mu_1}{2\mu_1+s}$$

and

$$f^*_1(s)=\frac{2\mu_1\alpha}{2\mu_1+s}\cdot\frac{2\mu_1}{2\mu_1+s}+\frac{2\mu_1\beta}{2\mu_1+s}\cdot\frac{\mu_1+\mu_2}{\mu_1+\mu_2+s}.$$

Using these relations we have

$$A_1=\frac{2\mu_1}{2\mu_1+s}\cdot\frac{\dfrac{2\beta\mu_2}{2\mu_2+s}-\dfrac{(\mu_1+\mu_2)\beta}{\mu_1+\mu_2+s}}{\dfrac{2\alpha\mu_1}{2\mu_1+s}+\dfrac{2\beta\mu_2}{2\mu_2+s}-\dfrac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2+s}} \tag{7.10}$$

and

$$B_1=\frac{2\mu_1}{2\mu_1+s}\cdot\frac{\dfrac{2\alpha\mu_1}{2\mu_1+s}+\dfrac{(\mu_1+\mu_2)\beta}{\mu_1+\mu_2+s}-\dfrac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2+s}}{\dfrac{2\alpha\mu_1}{2\mu_1+s}+\dfrac{2\beta\mu_2}{2\mu_2+s}-\dfrac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2+s}}. \tag{7.11}$$

Similarly, using above relations, we have

$$g^*_n(s)=A_2\left(\frac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2+s}\right)^n+B_2\left(\frac{2\alpha\mu_1}{2\mu_1+s}+\frac{2\beta\mu_2}{2\mu_2+s}\right)^n \tag{7.12}$$

and

$$h_n^*(s) = A_3\left(\frac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2+s}\right)^n + B_3\left(\frac{2\alpha\mu_1}{2\mu_1+s}+\frac{2\beta\mu_2}{2\mu_2+s}\right)^n \tag{7,13}$$

where

$$A_2 = \frac{\dfrac{2\alpha\mu_1^2}{2\mu_1+s}+\dfrac{2\beta\mu_2^2}{2\mu_2+s}-\dfrac{(\alpha\mu_1+\beta\mu_2)(\mu_1+\mu_2)}{\mu_1+\mu_2+s}}{(\mu_1+\mu_2+s)\left(\dfrac{2\alpha\mu_1}{2\mu_1+s}+\dfrac{2\beta\mu_2}{2\mu_2+s}-\dfrac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2+s}\right)}, \tag{7.14}$$

$$B_2 = \frac{\dfrac{2\alpha\mu_1\mu_2}{2\mu_1+s}+\dfrac{2\beta\mu_1\mu_2}{2\mu_2+s}}{(\mu_1+\mu_2+s)\left(\dfrac{2\alpha\mu_1}{2\mu_1+s}+\dfrac{2\beta\mu_2}{2\mu_2+s}-\dfrac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2+s}\right)}. \tag{7.15}$$

In addition, A_3 and B_3 are given by substituting β, α, μ_2 and μ_1 for α, β, μ_1 and μ_2 in A_1 and B_1 respectively.

Using these relations, the Laplace transform for the waiting-time density function is

$$\sum_{n=0}^{\infty}\{p_{20}^{(n)}f_n^*(s)+p_{11}^{(n)}g_n^*(s)+p_{02}^{(n)}h_n^*(s)\} \tag{7.16}$$

$$= A_1F_{20}\left(\frac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2+s}\right)+A_2F_{11}\left(\frac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2+s}\right)+A_3F_{02}\left(\frac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2+s}\right)$$

$$+B_1F_{20}\left\{\frac{2\alpha\mu_1}{2\mu_1+s}+\frac{2\beta\mu_2}{2\mu_2+s}\right\}+B_2F_{11}\left\{\frac{2\alpha\mu_1}{2\mu_1+s}+\frac{2\beta\mu_2}{2\mu_2+s}\right\}+B_3F_{02}\left\{\frac{2\alpha\mu_1}{2\mu_1+s}+\frac{2\beta\mu_2}{2\mu_2+s}\right\}.$$

On differentiating (7.16) and taking $s=0$, the mean waiting-time, W, is given by

$$W=\left(\frac{\alpha}{2\mu_1}+\frac{\beta}{2\mu_2}\right)L_q+\left\{\frac{1}{2\mu_1}+\frac{\beta(\mu_2-\mu_1)}{2\mu_2(\beta\mu_1+\alpha\mu_2)}\right\}F_{20}(1) \tag{7.17}$$

$$+\left\{\frac{1}{2\mu_2}+\frac{\alpha(\mu_1-\mu_2)}{2\mu_1(\beta\mu_1+\alpha\mu_2)}\right\}F_{02}(1)+\frac{1}{2(\beta\mu_1+\alpha\mu_2)}F_{11}(1)$$

$$+\frac{\mu_1-\mu_2}{2(\beta\mu_1+\alpha\mu_2)}\left[B\left\{\frac{1}{\mu_2}F_{20}\left(\frac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2}\right)+\frac{1}{\mu_1+\mu_2}F_{11}\left(\frac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2}\right)\right\}\right.$$

$$\left.-\alpha\left\{\frac{1}{\mu_1}F_{02}\left(\frac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2}\right)+\frac{1}{\mu_1+\mu_2}F_{11}\left(\frac{\alpha\mu_1+\beta\mu_2}{\mu_1+\mu_2}\right)\right\}\right].$$

For $\mu_1=\mu_2\equiv\mu$ the mean waiting time is given by

$$W=\frac{1}{\lambda}\cdot\frac{2\rho^3}{1-\rho^2}=\frac{1}{\lambda}L_q$$

which tallies with the result of the classical queuing process $M/M/2$.

However the mean waiting time in the queuing system for two different types of customer's having independent Poisson arrivals with rates λ_1 and λ_2 can easily be evaluated by using the following formula

$$W=\frac{1}{\lambda_1+\lambda_2}L_q.$$

The outline of the proof of this formula is as follows (see[7]).
We define the following notations (in the steady state):

(i) $G(x)$ is the distribution of the waiting time,

(ii) $X_j\,(j\geqq 1)$ is the interarrival time,

(iii) Q is the queue size observed at the epoch just before the service of a customer begins,

(iv) $F_m(x)=P(\sum_{i=1}^{m}X_i<x)$.

Then we have

$$P(Q\geqq n)=\int_0^\infty P(X_1+X_2+\cdots\cdots+X_n<x)\,dG(x).$$

From the above expression we obtain

$$L_q=\int_0^\infty (\sum_{m=1}^{\infty}F_m(x))\,dG(x).$$

In our queuing system with two distinct types of customers, if the input is Poissonian with arrival rate $\lambda=\lambda_1+\lambda_2$ then

$$\sum_{m=1}^{\infty}F_m(x)=\begin{cases}\lambda x & \text{for } x>0,\\ 0 & \text{for } x\leqq 0.\end{cases}$$

Thus we have

$$L_q=\lambda\int_0^\infty x\,dG(x)=\lambda W.$$

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