ON HAUPTVERMUTUNG AND TRIANGULATION OF n-MANIFOLDS

Dedicated to Professor Hidetaka Terasaka on his sixtieth birthday.

By

TATSUO HOMMA

1. Introduction

In this paper we shall show that **Hauptvermutung for combinatorial n−manifolds** and the **triangulation theorem for topological n−manifolds** can be reduced to **Lemma 1** which is a piecewise linear approximation theorem of homeomorphisms of an n−space R^n into R^n. Moreover we shall prove that any topological n−manifold has one and only one combinatorial stucture under the assumption of Lemma 1. Throughout this paper an *n−manifold* means a compact metric space any point of which has a neighborhood homeomorphic to an n−space R^n. Then any n−manifold is a closed (i. e. compact without boundary) topological n−manifold. A *polyhedral n−manifold M* is an n−manifold which has a triangulation μ. A *combinatorial n−manifold* is a polyhedral n−manifold M such that the star $ST(v)$* of any vertex v of M is a *combinatorial n−cell* (i. e. a polyhedral n−cell piecewise linearly homeomorphic to an n−simlex). Throughout this paper we shall assume the following lemma:

Lemma 1 *Let $\triangle$ and $\tilde{\triangle}$ be n−simplexes such that $\tilde{\triangle}$ is linearly imbedded in Int$(\triangle)$ and let K be a polyhedron which is piecewise linearly imbedded in $\triangle$. Then if ε is any positive number and f is a homeomorphism of $\triangle$ into R^n such that $f|K$ is piecewise linear, there is a homeomorphism $g|\triangle\rightarrow R$ such that*

$$g|K\cup\partial\triangle=f|K\cup\partial\triangle$$

$$g|\tilde{\triangle} \text{ is piecewise linear}$$

and $$d(f,g)^{**}<\varepsilon$$

---

* A star $ST(v)$ is the sum of all simlexes of M having v as a vertex.

** $d(f,g)=Sup\{d(f(x),g(x))\mid x\in\triangle\}$.

2. Hauptvermutung of n-manifolds

In this section we shall show that **Lemma 1** implies the following theorem:

Theorem 1 *Let f be a homeomorphism of a combinatorial n-manifold M_1 onto a combinatorial n-manifold M_2. Then for any positive numer ε there is a piecewise linear homeomorphism g of M_1 onto M_2 satisfying*

$$d(f,g)<\varepsilon.$$

Proof of Theorem 1 Since M_2 is a combinatorial n-manifold. We can take a finite number of combinatorial n-cells $\{C_1, C_2, \cdots, C_k\}$ which are piecewise linearly imbedded in M_2 and satisfies

$$\bigcup_{i=1}^{k} Int(C_i)=M_2$$

Since M_1 is a combinatorial n-manifold and f is a homeomorphism of M_1 onto M_2. We can find a positive number ε' and two sets of combinatorial n-cells $\{B_1, B_2, \cdots, B_h\}$ $\{\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_h\}$, piecewise linearly imbedded in M_1, such that

$$\tilde{B}_i \subset Int(B_i), \quad \bigcup_{i=1}^{h} \tilde{B}_i=M_1$$

and for any B_i there is a C_j satisfying

$$U_{\varepsilon'}(f(B_i))\subset Int(C_j), \qquad \cdots(1)$$

where $U_{\varepsilon'}(f(B_i))$ is the ε'-neighborhood of $f(B_i)$.

We put $$\delta=\frac{\varepsilon'}{h} \qquad \cdots(2)$$

We shall inductively construct a sequence of homeomorphisms of M_1 onto M_2 $f_0=f, f_1, f_2, \cdots, f_h$ such that

$$d(f_{i-1}, f_i)<\delta \qquad \cdots(3)$$

and $f_i \cdot \tilde{B}_1\cup\tilde{B}_2\cup\cdots\cup\tilde{B}_h$ is piecewise linear. $\cdots(4)$

we assume that $f_0, f_1, \cdots, f_{i-1}$ have been constructed and we put

$$K=(\tilde{B}_1\cup\tilde{B}_2\cup\cdots\cdots\cup\tilde{B}_{i-1})\cap B_i=K_i \qquad \cdots(5)$$

From (3) we have $d(f_{i-1}, f)<i\delta$ and then from (1), (2) we have

$$f_{i-1}(B_i)\subset U_{i\delta}(f(B_i))\subset U_{\varepsilon'}(B_i)\subset Int(C_j).$$

By **Lemma 1** there is a homeomorphism f_i' of B_i into C_j such that

$$f_i' | K_i \cup \partial B_i = f_{i-1} | K_i \cup \partial B_i \qquad \cdots(6)$$

$$d(f_i', f_{i-1} | B) < \delta$$

and $f_i' | \tilde{B}_i$ is piecewise linear.

According to (6) we can extend f_i' to a homeomorphism of M_1 onto M_2 by the formula

$$f_i | B_i = f_i' | B_i$$

$$f_i | \overline{M_1 - B_i} = f_{i-1} | \overline{M_1 - B_i}.$$

Then it is clear that f_i is the required homeomorphism and consequently we get f_h. It is easy to see that f_h is the required piecewise linear homeomorphism g of M_1 onto M_2.

3. Triangulation of n-manifolds

At first under the assumption of Lemma 1 we shall prove the following lemma :

Lemma 2 *Let f_1 and f_2 be homeomorphisms of combinatorial n-cells C_1 and C_2 into an n-manifold M and let $\tilde{C}_1$, $\tilde{C}_2$ and K be two combinatorial n-cells and a polyhedron such that $\tilde{C}_1$ and $\tilde{C}_2$ are piecewise linearly imbedded in $Int(C_1)$ and $Int(C_2)$ respectively, K is piecewise linearly imbedded in C_1, $f_1(K) \subset f_2(C_2)$ and $f_2^{-1} f_1 | K$ is piecewise linear. Then for any positive number ε there is a homeomorphism $\tilde{f}_1$ of C_1 into M such that*

$$d(f_1, \tilde{f}_1) < \varepsilon$$

$$\tilde{f}_1 | K = f_1 | K$$

and $f_2^{-1} \tilde{f}_1 \, \tilde{f}_1^{-1}(\tilde{f}_1(\tilde{C}_1) \cap f_2(\tilde{C}_2))$ *is piecewise linear.*

Proof of Lemma 2 Let $B_1, B_2, \cdots, B_k$ be combinatorial n-cells piecewise linearly imbedded in $Int(C_1)$ such that

$$Int(B_1) \cup Int(B_2) \cdots \cup Int(B_k) \supset f_1^{-1}(f_1(\tilde{C}_1) \cap f_2(\tilde{C}_2))$$

and $\qquad f_1(B_i) \subset f_2(Int(C_2)) \qquad i = 1, 2, \cdots, k.$

Put $\varepsilon' = d(f_1(B_1 \cup B_2 \cup \cdots \cup B_k), M - f_2(C_2))$ and $\varepsilon'' = d(f_1(\tilde{C}_1 - Int(B_1) \cup \cdots \cup Int(B_k))$,

$f_2(\tilde{C}_2))$, where $d(X, Y) = Inf\{d(x, y) \mid x \in X, y \in Y\}$.

We take a positive number δ satisfying

$$k\delta < \varepsilon, \varepsilon', \varepsilon''.$$

We shall inductively construct a sequence of homeomorphisms of C_1 into M, $\varphi_0 = f_1, \varphi_1, \cdots, \varphi_k$, such that

$$d(\varphi_{i-1}, \varphi_i) < \delta \qquad \cdots (7)$$

$$\varphi_i \mid K = f_1 \mid K$$

and $f_2^{-1} \varphi_i \mid B_1 \cup B_2 \cup \cdots \cup B_i$ is piecewise linear.

We assume that $\varphi_0, \varphi_1, \cdots \varphi_{i-1}$ have been constructed. Since $\varphi_{i-1}(B_i) \subset U_{(i-1)\delta}(f_1(B_i)) \subset U_{\varepsilon'}(f_1(B_i)) \subset f_2(Int(C_2))$. We have a combinatorial n-cell D_i, piecewise linearly imbedded in $\mathrm{Int}(C_1)$, such that

$$\varphi_{i-1}(D_i) \subset f_2(Int(C_2)) \text{ and } Int(D_i) \supset B_i$$

By Lemma 1 we can take a homeomorphisms $\tilde{\varphi}_i$ of D_i into $f_2(Int\, C_2)$ such that

$$d(\varphi_{i-1}, \tilde{\varphi}_i) < \delta$$

$$\tilde{\varphi}_i \mid ((B_1 \cup \cdots \cup B_{i-1} \cup K) \cap D_i) \cup \partial D_i = \varphi_{i-1} \mid ((B_1 \cup \cdots \cup B_{i-1} \cup K) \cap D_i) \cup \partial D_i$$

and $f_2^{-1} \tilde{\varphi}_i \mid B_i$ is piecewise linear.

We extend $\tilde{\varphi}_i$ to a homeomorphism φ_i of C_1 into M by the formula

$$\varphi_i \mid D_i = \tilde{\varphi}_i \mid D_i$$

$$\varphi_i \mid \overline{C_1 - D_i} = \varphi_{i-1} \mid \overline{C_1 - D_i}$$

Consequently we get the sequence $\varphi_0, \varphi_1, \cdots, \varphi_k$ of homeomorphisms and we put

$$\varphi_k = \tilde{f}_1.$$

It is clear that

$$d(f_1, \tilde{f}_1) < k\delta < \varepsilon$$

$$\tilde{f}_1 \mid K = f_1 \mid K$$

and $f_2^{-1} \tilde{f}_1 \mid B_1 \cup B_2 \cup \cdots \cup B_k$ is piecewise linear.

Furthermore since $d(\tilde{f}_1, f_1) < k\delta < \varepsilon''$

and $\varepsilon'' = d(f_1(\tilde{C}_1 - Int(B_1) \cup \cdots \cup Int(B_k)), f_2(\tilde{C}_2))$.

We have $\tilde{f}_1(\tilde{C}_1 - B_1 \cup \cdots \cup B_k)) \cap f_2(\tilde{C}_2)$

$$\subset U_{\varepsilon''}(f_1(\tilde{C}_1 - B_1 \cup \cdots \cup B_k)) \cap f_2(\tilde{C}_2) = \phi.$$

Then we have $\tilde{f}_1(\tilde{C}_1) \cap f_2(\tilde{C}_2) \subset \tilde{f}_1(B_1 \cup \cdots \cup B_k)$.

Hence $f_2^{-1}\tilde{f}_1 | \tilde{f}_1^{-1}(\tilde{f}_1(\tilde{C}_1) \cap f_2(\tilde{C}_2))$ is piecewise linear and we have proved Lemma 2.

From Lemma 2 we shall prove the main theorem of this section as follows:

Theorem 2 *Any n-manifold M is combinatorial.*

Proof of Theorem 2 We shall use a double induction. Let $f_1, f_2, \cdots, f_k$ be homeomorphisms of combinatorial n-cells $C_1, C_2, \cdots, C_k$, into M respectively such that

$$f_1(Int(C_1)) \cup \cdots \cup f_k(Int(C_k)) = M.$$

Then we can take combinatorial n-cells $\tilde{C}_1, \cdots, \tilde{C}_k$, which are piecewise linearly imbedded in $Int(C_1), \cdots, Int(C_k)$ respectively such that

$$f_1(Int(\tilde{C}_1)) \cup \cdots \cup f_k(Int(\tilde{C}_k)) = M.$$

It is sufficicnt to prove that for any ε there is a sequenee $\varphi_1, \cdots, \varphi_k$ of homeomorphisms of $\tilde{C}_1, \cdots, \tilde{C}_k$ into M respectively such that

$$d(\varphi_i, f_i) < \varepsilon$$

and $\varphi_j^{-1}\varphi_i | \varphi_i^{-1}(\varphi_i(\tilde{C}_i) \cap \varphi_j(\tilde{C}_j))$ is piecewise linear for $i \neq j$. Put $\varphi_1 = f_1$ and assume that $\varphi_1, \varphi_2, \cdots, \varphi_{i-1}$ have been constructed. Using Lemma 2 we get a homeomorphism ψ_1 of C_i into M such that

$$d(\psi_1, f_i) < \frac{\varepsilon}{i}$$

and $\varphi_1^{-1}\psi_1 | \psi_1^{-1}(\psi_1(\tilde{C}_i) \cap \varphi_1(\tilde{C}_1))$ is piecewise linear. Put $K_1 = \psi_1^{-1}(\psi_1(\tilde{C}_i) \cup \varphi_1(\tilde{C}_1))$.

Similarly as ψ_1 we get a homeomorphism ψ_2 of C_i into M such that

$$d(\psi_2, \psi_1) < \frac{\varepsilon}{i}$$

$$\psi_2 | K_1 = \psi_1 | K_1$$

and $\varphi_2^{-1}\psi_2 | \psi_2^{-1}(\psi_2(\tilde{C}_i) \cap \varphi_2(\tilde{C}_2))$ is piecewise linear.

Put $K_2 = K_1 \cup \psi_2^{-1}(\psi_2(\tilde{C}_i) \cap \varphi_2(\tilde{C}_2))$. Similarly as ψ_1, ψ_2 we get a homeomorphism ψ_3 of C_i into M such that

$$d(\psi_3, \psi_2) < \frac{\varepsilon}{i}$$

$$\psi_3 | K_2 = \psi_2 | K_2$$

and $\varphi_2^{-1}\psi_3\ \psi_3^{-1}(\psi_3(\tilde{C}_i)\cap\varphi_3(\tilde{C}_3))$ is piecewise linear.

By induction we consequently get a homeomorphism ψ_{i-1} of C into M denoted by φ_i such that

$$d(\varphi_i,f_i)=d(\psi_{i-1},f_i)\leqq d(\psi_{i-1}\psi_{i-2})+\cdots+d(\psi_1,f_i)<\varepsilon$$

and $\varphi_j^{-1}\varphi_i\,|\,\varphi_i^{-1}(\varphi_i(\tilde{C}_i)\cap\varphi_j(\tilde{C}_j))$ is piecewise linear for $i>j$. Then it is clear that φ_i is the required homeomorphism of C_i into M. Hence we have proved Theorem 2.

It is clear that Theorem 1 and Theorem 2 imply the following;

Corollary 1 *Any n-manifold has one and only one combinatorial structure.*

Furthermore Theorem 2 obviously implies the following;

Corollary 2 *Any n-manifold has a triangulation.*

Reference

T. Homma On imbedding of polyhedra into manifold. Yokohama Math. Jour. Vol. 10, No. 1 (1962).

(Received September 10, 1963)