ON EXISTENCE OF SOME SUBALGEBRA OF A GIVEN BOOLEAN ALGEBRA WHICH IS COUNTABLY INFINITE.

By

TÔRU MORI

§1. **Preface.** Our main purpose in this paper is to show that every countably infinite Boolean algebra has a countably infinite atomic subalgebra. To this end, we use here the result that every countable Boolean algebra is absolute quotient retract (see §2). This is a consequence of a result in *P. R. Halmos* [1]. To prove this result we need various theorems which are more complicated. In this paper we use only the existence of a monomorphism. For this reason, we omit this proof. Preliminary arguments described in §2 and various theorems which are proved in detail in §2 and §4 are due to *P. R. Halmos* [1] and *R. Sikorski* [3]. In §3, I show the theorems which seem to be new facts to me, and prove them in detail. I wish to thank very much to these mathematicians. Finally, in §4 I solve a problem (Theorem 4) presented by *Halmos's* book [1], as an application of Theorem 3.

§2. **Basic definitions and theorems.** A compact Hausdorff space X said to be a Boolean space if every open set is the union of these simultaneously closed and open (abbr. *clopen*) sets that it happens to include. It is true that the class of all clopen sets is a field. The field of all clopen sets in a Boolean space is called the *dual algebra* of X.

By symbol 2 we understand the Boolean algebra whose members are only 0 and 1. It will be convenient to construct it as a topological space as well, endowed with the diserete topology. A product space 2^I (that is, all functions on a set I to the discrete space 2, with the product topology) is called a ***Cantor space.*** The discrete space 2 is compact and Hausdorff. Therefore it is well known that such a space 2^I is compact and Hausdorff (*Tychonoff's* theorem). We shall denote the value of a funetion x in 2^I at an element i of I by $x(i)$. The sets of the form $\{x\in 2^I\colon x(i)=\delta\}$, where $i\in I$ and $\delta\in 2$, constitute a subbase for the product topology of 2^I; finite intersections of them constitute a base. Since the complement of each set of the indicated form is another set of the same form, so that each such a set is clopen, it follows that 2^I is a Boolean space.

(2.1) *If X is a compact Hausdorff space and if* **B** *is a separating field of clopen subsets of X, then X is a Boolean space and* **B** *is the field of all clopen subsets of X* (**B** *is said to be a separating field provided that for any two distinct points x, y in X, there exists a set p in* **B** *such that $x\in p$ and $y\notin p$*).

The fact that **B** separates points implies the **B** separate points and closed sets. This fact follows from a standard compactness argument. Suppose, in fact, that F is the closed set and x is a point not in F. Compactness yields a finite cover of F by sets in **B** none of which contains x; the finite union is a set in **B** that separates F and x.

This means that there exists a clopen set U such that $x \in U$ and $U \cap F = \phi$. Hence **B** is a base for X; this already implies that X is Boolean. Every clopen set in X is finite union of sets of **B**, because it is closed, accordingly, compact. Since **B** is closed under formation of finite unions, the proof is complete.

(2.2) ***Every closed subset Y of a Boolean space X is a Boolean space with respect to the relative topology. Every clopen set of Y is the intersection of Y with some clopen subset of X.***

Since the clopen sets form a base in X, their intersections with Y are the same for Y. If Q is a clopen set in Y, then it is open in Y, and, therefore, there exists an open set U in X such $Q = Y \cap U$. The clopen subsets of U in X cover the closed set Q, therefore, by compactness, there exists a finite class of clopen subsets of U whose union, say P, covers Q. Since $Q \subset P \subset U$ and $Y \cup U = Q$, it follows that $Q = Y \cap P$.

(2.3) ***For every non-zero element p of every Boolean algebra A there is a 2-valued homomorphism f on A such that $f(p)=1$.***

First we shall prove that there exists a maximal ideal **M** in **A** such that $p \notin \mathbf{M}$. In fact, there is an ideal in **A** which does not contain p. For example, the set **B** of all subelement of p (this means the elements q with $q \subset p$) can be constructed as a Boolean algebra, as follows: 0, meet, and join in **B** are the same as in **A**, but 1 and q' in **B** are defined to be the element p and $p \cap q'$ of **A**. The mapping $h(s) = p \cap s$ is an **B**-valued homomorphism on **A**. Then $h^{-1}(0)$ is the desired ideal. Consider the partially ordered set (inclusion) of all ideals not containing p, and apply Zorn's lemma. Then there is an ideal $\mathbf{M}_0$ not containing p that is maximal among all the ideals not containing p. It remains to prove that $\mathbf{M}_0$ is maximal among all proper ideals. We note first that if $p' \notin \mathbf{M}_0$, then the ideal generated by $\mathbf{M}_0 \cup \{p'\}$ is strictly greater than $\mathbf{M}_0$. This generated ideal consists of all elements of the form $q \cup r$ where $q \in \mathbf{M}_0$ and $r \subset p'$, and, consequently, it does not contain p. The known maximality property of $\mathbf{M}_0$ implies therefore that $p' \in \mathbf{M}_0$. The same property implies also that if **M** is an ideal strictly greater than $\mathbf{M}_0$ then $p \in \mathbf{M}$. It follows that every ideal strictly greater than $\mathbf{M}_0$ contains both p and p', therefore equals **A**. This implies that $\mathbf{M}_0$ is a maximal.

If we write $f(q)=0$ or 1 according as the element $q \in \mathbf{M}_0$ or $q \in \mathbf{M}$, then f is a 2-valued homomorphism on **A**. (Here we use the fact that an ideal $\mathbf{M}_0$ in a Boolean algebra **A** is maximal if and only if either $p \epsilon \mathbf{M}_0$ or $p' \epsilon \mathbf{M}_0$, but not both, for each p in **A**).

(2.4) ***The set X of all 2-valued homomorphism on a Boolean algebra* A *is a closed subset of the Cantor space* $2^{\mathbf{A}}$ *of all 2-valued functions on* A.**

The definition of topology in $2^{\mathbf{A}}$ implies for each fixed p in **A** the value $x(p)$ depends continuously on the point x of $2^{\mathbf{A}}$ Since the set of points where two continuous functions are equals is always a closed set, for two continuous functions $x(p'), (x(p))'$, it follows that $\{x: x(p')=(x(p))'\}$ is closed in $2^{\mathbf{A}}$ for each p in **A**. Forming the intersection of all these sets, we conclued that those 2-valued functions that preserve complementation form a closed subset of $2^{\mathbf{A}}$. A similar argument, involving sets such as $\{x: x(p \cup q)=x(p) \cup x(q)\}$, justifies the same conclusion for the join-preserving functions.

(2.2) implies that the set X of all 2-valued homomorphism on Boolean algebra becomes a Boolean space with respect to relative topology. We shall call that Boolean space the ***dual space*** of **A**.

The following assertion, known as the Stone representation theorem, is the most fundamental result about the relation between Boolean algebras and Boolean spaces.

(2.5) ***If* B *is the dual algebra of the dual space X of* A, *and if* $f(p)=\{x \epsilon X: x(p)=1\}$ *for each p in* A, *then f is an isomorphism from* A *onto* B.**

Since $\{x \epsilon X: x(p)=1\}=\{x \epsilon 2^{\mathbf{A}}: x(p)=1\} \cap X$, it follows that $f(p)$ is clopen for each p in **A**, and hence that f maps **A** into **B**. The proof that f is a homomorphism is purely mechanical. Thus,

$$f(p \cup q)=\{x: x(p \cup q)=1\}=\{x: x(p) \cup x(q)=1\}$$
$$=\{x: x(p)=1\} \cup \{x: x(q)=1\}=f(p) \cup f(q)$$
$$f(p')=\{x: x(p')=1\}=\{x: x(p)=0\}=\{x: x(p)=1\}'=(f(p))'$$

If $f(p)=\phi$, that is $\{x: x(p)=1\}=\phi$, then (2.3) implies that $p=0$; this means that f is one-to-one. Since the range of every Boolean homomorphism is a Boolean algebra, the clopen sets of the form $\{x: x(p)=1\}$ constitute a field. Since two distinct 2-valued homomorphisms on **A** must disagree on some element of **A**, the field is separating, and consequently, (2.1) implies that f maps **A** onto **B**.

(2.6) ***Let X be the dual space of a Boolean algebra* A. A *is atomic if and only if the set of all isolated points is dense in X.* A *is non-atomic if and only if X is dense in itself, i. e. if X has no isolated points.***

Let f be an isomorphism of **A** onto the dual algebra **B** of the dual space X of **A**. Then an element $p \epsilon \mathbf{A}$ is an atom if and only if $f(p)$ is a singleton of X. Of course, a singleton $\{x\}$ belongs to **B** if and only if x is an isolated point of X.

(2.7) ***If Y is the dual space of the dual algebra $\mathbf{A}$ of a Boolean space X, and φ_x for each $x \in X$ is the 2-valued homomorphism that sends each element p of $\mathbf{A}$ onto 1 or 0 according as $x \in p$ or $x \notin p$, then $\psi(\psi(x)=\varphi_x)$ is homeomorphism from X onto Y.***

To prove that ψ is continuous, it is sufficient to prove that the inverse image of every clopen subset of Y is clopen in X. The proof follows from the fact that every clopen subset of Y is of the form $\{y: y(p)=1\}$, where $p \in \mathbf{A}$. This fact follows from (2.5). Moreover we obtain $\psi^{-1}(\{y: y(p)=1\})=p$. Therefore we conclude also that inverse image of non-empty clopen set in Y is never empty; since clopen sets forms a base for Y, this implies that the range of the function ψ is dense in Y. The continuity of ψ i. e. the compactness of $\psi(X)$ and the density of its range together imply that ψ maps X onto Y. Since the clopen set separates points in X, distinct points of X determine distinct 2-valued homomorphism on $\mathbf{A}$, so that ψ is one-to-one.

A set $\mathbf{G}$ of generators of a Boolean algebra $\mathbf{A}$ is said to be a set of ***free generators*** of $\mathbf{A}$ if every mapping f of $\mathbf{G}$ into an arbitrary Boolean algebra $\mathbf{B}$ can be extended to a homomorphism of $\mathbf{A}$ into $\mathbf{B}$. A Boolean algebra $\mathbf{A}$ is said to be ***free*** provided it contains a set $\mathbf{G}$ of free generators of $\mathbf{A}$. It is well known that an infinite Boolean algebra with $\mathfrak{m}$ generators has $\mathfrak{m}$ elements.

(2.8) ***A countably infinite Boolean algebra $\mathbf{A}$ is free if and only if it is non-atomic.***

First, suppose that $\mathbf{A}$ is free, then $\mathbf{A}$ is isomorphie with the dual algebra $\mathbf{B}$ of a Cantor space $2^{\mathbf{I}}$ (the power of I is $\aleph_0$) which is a free Boolean algebra with a set of free generators of the same power as I. Because two free Boolean algebra with same power of free generators are isomorphic. In addition, the Cantor space $2^{\mathbf{I}}$ is dense in itself. Therefore, by (2.7) the dual space of $\mathbf{B}$ is dense in itself. i. e. by (2.6) $\mathbf{B}$ is non-atomic. Hence, $\mathbf{A}$ is non-atomic.

We note that since $\mathbf{A}$ is countable, the dual space X of $\mathbf{A}$ is metrizable.[(1)] Conversely, suppose $\mathbf{A}$ is non-atomic, then X is totally disconnected, perfect, compact metric space[(2)] and also $2^{\mathbf{I}}$ (the power of I is $\aleph_0$) is similar. Since any two totally disconnected, perfect, compact metric space are homeomorphic,[(3)] by

(1) The Boolean space of a Boolean algebra $\mathbf{A}$ is metrizable if and only if $\mathbf{A}$ is countable. (see Sikorski [3])

(2) A Boolean space is equivalent to a totally disconnected compact Hausdorff space. (see Halmos [1])

(3) Any two totally disconnected, perfect, compact metric spaces are homeomorphic. (see Hocking and Young [6])

(2.5) **A** is isomorphic with the dual algebra **B** of $2^{\mathbf{I}}$. By the fact that **B** is free, **A** is free also.

(2.9) *The closure of a set which is dense in itself is dense in itself.* Let E be a set which is dense in itself, then $E \subset E'$. Since $E \subset \bar{E}$ we get $E' \subset (\bar{E})'$ and therefore $E \subset (\bar{E})'$. But $\bar{E}$ is closed, hence $(\bar{E})'$ is closed and, since $(\bar{E})'$ contains E, it must contain $\bar{E}$.

(2.10) *A finite Boolean algebra is atomic.*

It follows from the finiteness and the atomicity of the algebra.

A homomorphism f from a Boolean algebra **B** to a Boolean algebra **A** is said to be ***monomorphism*** provided that it is one-to-one into, i.e. if $f(p)=f(q)$, then $p=q$. A homomorphism f is said to be ***epimorphism*** provided that it is onto, i.e. every element of **A** is equal to $f(p)$ for some p in **B**.

We say that a Boolean algebra **B** is ***absolute quotient retract*** if to every epimorphism f from a Boolean algebra **A** to **B** there corresponds a monomorphism g from **B** to **A** such that, again, the composite $f \circ g$ is the identity i on **B**.

(2.11) *Every countable Boolean algebra is absolute quotient retract.* (see *Halmos* [1])

§ 3. Existence theorems.

THEOREM 1. *Every countably infinite Boolean algebra* **A** *contains an countably infinite free subalgebra provided that its dual space X contains a non-empty perfect subset D.*

Proof. If $D=X$, then by (2.6) **A** is non-atomic and, consequently, by (2.8) **A** is free itself. In the case $D \neq X$, let **B** be the dual algebra of X and let h be an isomorphism from **A** onto **B**. If we write $\mathbf{J}=\{p \epsilon \mathbf{A} : h(p) \cap D=\phi\}$, then **J** is an ideal which is not empty. Because if $p_1 \epsilon \mathbf{J}$, $p_2 \mathbf{J} \epsilon$, then $h(p_1) \cap D=\phi$ and $h(p_2) \cap D=\phi$. This implies $(h(p_1) \cup h(p_2)) \cap D=h(p_1 \cup p_2) \cap D=\phi$, i.e. $p_1 \cup p_2 \epsilon \mathbf{J}$. And if $p_2 \subset p_1 \epsilon \mathbf{J}$, then $\phi=h(p_1) \cap D \supset h(p_2) \cap D$ i.e. $h(p_2) \cap D=\phi$. Hence $p_2 \epsilon \mathbf{J}$. Since $X-D$ is non-empty open set, **J** is non-empty.

D is a closed set in X. Therefore, by (2.2) D is a Boolean space with respect to relative topology and $\{h(p) \cap D : p \epsilon \mathbf{A}\}=\mathbf{B}_D$ is the dual algebra of the space D. If we define the mapping h_0 by the formula

$$h_0=(h(p) \cap D)=[p]_J,$$

then the mapping h_0 is an isomorphism from $\mathbf{B}_D$ onto $\mathbf{A}/\mathbf{J}$, where $[p]_J$ is the class containing an element $p \epsilon \mathbf{A}$, i.e. the element of the factor algebra $\mathbf{A}/\mathbf{J}$. First, to show this definition of h_0 is unambigous and h_0 is one-to-one, we shall prove that $h(p_1) \cap D=h(p_2) \cap D$ is equivalent to $[p_1]_J=[p_2]_J$.

If $h(p_1)\cap D=h(p_2)\cap D$, then $h(p_1)\cap h(p'_2)\cap D=\phi$. This impliea $h(p_1-p_2)\cap D=\phi$, i. e. $p_1-p_2\in\mathbf{J}$. Similarly, we have $p_2-p_1\in\mathbf{J}$. Therefore we have $[p_1]_J=[p_2]_J$. Conversely, i $[p_1]_J=[p_2]_J$, that is, $p_1-p_2\in\mathbf{J}$. and $p_2-p_1\in\mathbf{J}$, then $h(p_1-p_2)\cap D=\phi$ and $h(p_2-p_1)\cap D=\phi$ respectively. For example, from $h(p_1-p_2)\cap D=\phi$, we have $h(p_1)\cap h(p_2')\cap D=\phi$. It follows from this that $h(p_1)\cap D\subset h(p_2)$, accordingly, $h(p_1)\cap D\subset h(p_2)\cap D$. Similarly we have $h(p_2)\cap D\subset h(p_1)\cap D$. Hence we have $h(p_1)\cap D=h(p_2)\cap D$. By the definition of h_0, it is clear that h_0 is onto.

Second, we shall prove that the mapping h_0 is an isomorphism. By the fact that h is an isomorphism, we obtain

$$h_0((h(p_1)\cap D)\cup(h(p_2)\cap D)\cup(h(p_2)\cap D))=h_0(h(p_1)\cup p_2)\cap D=[p_1\cup p_2]_J$$
$$=[p_1]_J\cup[p_2]_J=h_0(h(p_1)\cap D)\cup h_0(h(p_2)\cap D)$$

Since $\overline{h(p)\cap D}=D-h(p)\cap D=X\cap D-(p)\cap D=(X-h(p))\cap D=h(p')\cap D$, where—— is complement in space D, we have

$$h_0(\overline{h(p)\cap D})=h_0(h(p')\cap D)=[p']_J=[p]_J'=(h_0(h(p)\cap D))'.$$

Let Y be the dual space of the dual algebra $\mathbf{B}_D$ of D, then by (2.7), the space D is homeomorphic to the space Y. Since the subset D of X is dense in itself, D is dense in itself as a Boolean space. Hence the space Y is dense in itself. Therefore, by (2.6) $\mathbf{B}_D$ is non-atomic. It follows from this that $\mathbf{A}/\mathbf{J}$ is also non-atomic. Since $\mathbf{A}$ is countably infinite, $\mathbf{A}/\mathbf{J}$ is countable. If $\mathbf{A}/\mathbf{J}$ is finite, then by (2.10), $\mathbf{A}/\mathbf{J}$ is atomic. This contradicts the fact that $\mathbf{A}/\mathbf{J}$ is non-atomic. Therefore $\mathbf{A}/\mathbf{J}$ is certainly countably infinite. Hence $\mathbf{A}/\mathbf{J}$ is countably infinite non-atomic Boolean algebra.

Finally, from the fact that $\mathbf{A}/\mathbf{J}$ is countably infinite, by (2.11) $\mathbf{A}/\mathbf{J}$ is absolute quotient retract. Since there is a natural homomorphism of $\mathbf{A}$ onto $\mathbf{A}/\mathbf{J}$, i. e. epimorphism f, there corresponds a monomorphism g from $\mathbf{A}/\mathbf{J}$ to $\mathbf{A}$ such that $f\circ g=i$. By the result of the preceding paragraph, $\mathbf{A}/\mathbf{J}$ is countably infinite non-atomic, therefore $g(\mathbf{A}/\mathbf{J})$ is also countably infinite non-atomic subalgebra of $\mathbf{A}$. And yet, by (2.8), $g(\mathbf{A}/\mathbf{J})$ is free subalgebra of $\mathbf{A}$.

THEOREM 2. *Every countably infinite free Boolean algebra* **A** *contains a countably infinite atomic subalgebra.*

Proof. Let **B** be the field of all subsets of a countably infinite set X. Then the subfield of B which generated by the set of all singletons of **B** is a countably infinite atomic subfield i. e. countably infinite atomic subalgebra of the Boolean algebra **B**. It follows from this that there exists surely a countably infinite atomic Boolean algebra. Let **A**′ is one of such algebras.

Since **A** is the countably infinite free Boolean algebra, by means of the definition of a free Boolean algebra, **A** contains a set **G** of countably infinite free generators of **A**. Since **A**$'$ and **G** have the same power $\aleph_0$, there is a one-to-one mapping f from **G** onto **A**$'$. Accordingly, f can be extended to a homomorphism h of **A** onto **A**$'$. By (2.11), **A**$'$ is absolute quotient retract. Therefore, there corresponds a monomorphism g of **A**$'$ to **A** such that $h\circ g=i$. As the result of the fact that **A**$'$ is a countably infinite atomic Boolean algebra, $g(\mathbf{A}')$ is also same.

THEOREM 3. *Every countably infinite Boolean algebra,* **A** *contains a countably infinite atomic subalgebra.*

Proof. Let C be the set of all isolated points of the dual space X of **A**.

First suppose that $\bar{C}=X$. Then C is dense in X. Therefore, by (2.6) **A** is atomic itself.

Second, suppose that $\bar{C}=\phi$. Then $C=\phi$. Therefore, X has no isolated points. By (2.6), **A** is non-atomic, hence by (2.8), **A** is a countably infinite atomic subalgebra. It follows from Therem 2 that **A** contains a countably infinite atomic Subalgebra.

Finally, suppose that $\bar{C}\neq\phi$ anc $\bar{C}\neq X$. Then $X-\bar{C}$ is non-empty. If we write $G=X-\bar{C}$, then the open set G is dense in itself. This follows: if x is an arbitrary point of G, then there exists a neighborhood N of x since G is open. While x is not an isolated point of X. Hence x is a limit of X. Accordingly, there exists a point $y\in X$ in N which is distinct from x. Since y belongs to G, there exists a point $y\in G$ in every neighborhood of x which is distinct from x. Therefore, $G(\neq\phi)$ is dense in itself. If we write $\bar{G}=D$, then by (2.9), $D\neq\phi$ is a perfect subset in X. By means of Theorem 1, **A** contains a countably infinite free subalgebra. Hence, by Theorem 2, **A** contains a countably infinite atomic subalgebra. The proof is complete.

§ 4. **Application.** A set **D** of elements of a Boolean algebra **A** is said to be dense (in **A**) provided, for every element $p\in\mathbf{A}$, $p\neq 0$, there exists an element $q\in\mathbf{D}$ such that $0\neq q\subset p$.

(4.1) *Let* **A** *be an atomic Boolean algebra and, for every* $p\in$**A**, *let* $h(p)$ *be the set of all atoms contained in* p. *Then* h *is an isomorphism of* **A** *into the field* **B** *of all subsets of the set* X *of all atoms of* **A**. *And that* h(**A**) *is dense subalgebra of* **B**.

First we shall prove that h is an isomorphism of **A** into **B**. To prove this we use the fact that an element $a(\neq 0)$ of **A** is an atom if and only if for every element $p\in\mathbf{A}$, either $a\subset p$ or $a\cap p=0$.

If $p\neq q$, then at least one of $p'\cap q$ and $p'\cap q$ is not 0. For example, if $p\cap q'\neq 0$ then an atom b such that $b\subset p\cap q'$ is not contained in q and simultaneously b is contained in p. Hence $f(p)\neq f(q)$. This implies that h is one-to-one.

For every $a \epsilon h(p \cup q)$ i. e. $a \subset p \cup q$, at least one of $a \cap p$ and $a \cap q$ is not 0. If $a \cap p \neq 0$, then $a \subset p$. Hence $h(p \cup q) \subset h(p) \cup h(q)$. Conversely, inverse inclusion is clear. Therefore we obtain $h(p \cup q) = h(p) \cup h(q)$. Next, for every $a \epsilon (h(p))'$, we have $a \cap p = 0$ i. e. $a \subset p'$. Hence $(h(p))' \subset h(p')$. Inverse inclusion is evident. Therefore we obtain $(h(p))' = h(p')$. These imply that h is a homomorphism, accordingly an isomorphim of **A** into **B**.

Finally we shall prove that $h(\mathbf{A} = \mathbf{B}'$ is dense subalgebra of **B**. Since $\mathbf{B}'$ is an isomorphic image of a Boolean algebra **A**, it is clear that $\mathbf{B}'$ is a subalgebra of **B**. Let $\{a_t\}$ be an arbitrary non-empty element of **B**. And we select a finite number of elements $a_1, a_2, a_3, \cdots\cdots, a_n$ from $\{a_t\}$. Then $\bigcup_{i=1}^{n} a_i$ belongs to **A** and we obtain $h(\bigcup_{i=1}^{n} a_i) \subset \{a_t\}$. This implies that $\mathbf{B}'$ is a dense subalgebra of **B**.

(4.2) *If h_0 is a homomorphism of a proper subalgebra $\mathbf{A}_0$ of a Boolean algebra $\mathbf{A}$ into a complete Boolean algebra $\mathbf{A}'$ and if $p_0 \epsilon \mathbf{A}\text{-}\mathbf{A}_0$, then h_0 can be extended to a homomorphism h^* of the subalgebra $\mathbf{A}_1$ generated by $\mathbf{A}_0$ and p_0 into $\mathbf{A}'$.*

To prove this, let us recall $\mathbf{A}_1$ is the set of all elements $p \epsilon \mathbf{A}$ which can be represented in the form

(1) $$p = (p_1 \cup p_0) \cap (p_2 - p_0)$$

where $p_1, p_2 \epsilon \mathbf{A}_0$. Let b_1 be the join (in the complet algebra $\mathbf{A}'$) of all elements $h_0(p)$ where $p \epsilon \mathbf{A}_0$ and $p_0 \subset p$. Similarly, let b_2 be the meet (in the complete algebra $\mathbf{A}'$) of all element $h_0(p)$ where $p \epsilon \mathbf{A}_0$ and $p_0 \subset p$. By definition, $b_1 \subset b_2$. Choose an element $b \epsilon \mathbf{A}'$ such that $b_1 \subset b \subset b_2$. By definition.

(2) if $r, s \epsilon \mathbf{A}_0$ and $r \subset p_0 \subset s$, then $h_0(r) \subset b \subset h_0(s)$.

If $p \epsilon \mathbf{A}_1$ is an element of the form (1), we define

(3) $$h^*(p) = (h_0(p_1) \cap b) \cup (h_0(p_2) - b)$$

To verify the unambiguity of this definition, we have to show that the element on the right side of (3) does not depend on the representation of $p \epsilon \mathbf{A}_1$ in the form (1). Suppose that (1) holds and that simultaneously

(1′) $$p = (q_1 \cap p_0) \cup (q_2 - p_0)$$

where $q_1, q_2 \epsilon \mathbf{A}_0$. It follows from (1) and (1′) that

$$p_2 - q_2 \subset p_0, \qquad q_2 - p_2 \subset p_0$$
$$p_0 \subset p_1' \cup q_1, \qquad p_0 \subset p_1 \cup q_1'$$

Bacause, multiplying $p_0 \cap q_1'$ the both sides of

$$(p_1 \cap p_0) \cup (p_2 \cap p_0') = (q_1 \cap p_0) \cup (q_2 \cap p_0')$$

we have $p_1 \cap q_1' \cap p_0 = 0$, hence $p_0 \subset p_1' \cup q_1$, and the others are similar. Consequently, by (2)

$$h_0(p_2) - h_0(q_2) \subset b, \qquad h_0(q_2) - h_0(p_2) \subset b,$$
$$b \subset (h_0(p_1))' \cup h_0(q_1), \qquad b \subset h_0(p_1) \cup (h_0(q_1))',$$

which implies

$$(h_0(p_1)\cap b)\cup(h_0(p_2)-b)=(h_0(q_1)\cap b)\cup(h_0(q_2)-b)$$

Thus (3) defines uniquely a mapping h^* of $\mathbf{A}_1$ into $\mathbf{A}'$. It is easy to verify that for every $p=(p_1\cap p_0)\cup(p_2\cap p_0')\in\mathbf{A}_1$, we have $p'=(p_1'\cap p_0)\cup(p_2'\cap p_0')$. It follows from this that h is a homomorphism. If $p\in\mathbf{A}_0$, then $p=(p\cap p_0)\cup(p-p_0)$ and by (3), $h^*(p)=(h_0(p)\cap b)\cup(h_0(p)-b)=h_0(p)$,
i. e. h is an extension of h_0

(4.3) *Let* $\mathbf{A}_0$ *be a subalgebra of a Boolean algebra* $\mathbf{A}$. *Every homomorphism* h_0 *of* $\mathbf{A}_0$ *into a complete Boolean algebra* $\mathbf{A}'$ *can be extended to a homomorphism of* $\mathbf{A}$ *into* $\mathbf{A}'$.

Let f be an homomorphism of a subalgebra $\mathbf{A}_f$ of $\mathbf{A}$ into $\mathbf{A}'$; suppose $\mathbf{A}_0$ is a proper subalgebra of $\mathbf{A}_f$ and the f is an extension of h_0. Let F be the class of all such homomorphisms f. If $f, g\in F$, let us define the relation $f\leqq g$ to mean that $\mathbf{A}_f\subset\mathbf{A}_g$ and that g is an extension of f. This relation defines a partial ordering of F. Moreover, F is non-empty, for certainly h^* in (4.2) belongs to F.

Now suppose that S is a completely ordered subset of F. We shall define an element $k\in F$ which is an upper bound of S. Let $\mathbf{A}_k$ be the union of all the set $\mathbf{A}_f$ corresponding to elements $f\in S$. This set $\mathbf{A}_k$ is a subalgebra of $\mathbf{A}$. For suppose $p_1, p_2\in\mathbf{A}_k$. Then there exist elements $f_1, f_2\in S$ such that $p_i\in\mathbf{A}_{fi}$ $(i=1, 2)$. We may suppose $f_1\leqq f_2$, since S is completely ordered. Then $\mathbf{A}_{f_1}\subset\mathbf{A}_{f_2}$, and so $p_1\cup p_2\in\mathbf{A}_{f_2}\subset\mathbf{A}_k$. Next, suppose $p\in\mathbf{A}_k$. Then there exists a element $f\in S$ such that $p\in\mathbf{A}_f$. Therefore $p'\in\mathbf{A}_f\subset\mathbf{A}_n$, since $\mathbf{A}_f$ is a subalgebra of $\mathbf{A}$. Now, suppose $p\in\mathbf{A}_k$. Then $p\in\mathbf{A}_f$ for some $f\in S$. We shall define $k(p)=f(p)$. This definition is unambigous, for, if $p\in\mathbf{A}_{f_1}$ and $p\in\mathbf{A}_{f_2}$, where $f_1, f_2\in S$, we have $f_1(p)=f_2(p)$ by the fact that S is completely ordered. The proof that k is a homomorphism is like the proof that $\mathbf{A}_k$ is a subalgebra. It is clear that $k\in F$ and that $f\leqq k$ for every $f\in S$.

We now know that the F satisfies the conditions of Zorn's lemma and must, therefore, contain a maximal element, say h. The $\mathbf{A}_h$ must equal to $\mathbf{A}$, for otherwise, we could regard $\mathbf{A}_h$ as the $\mathbf{A}_0$ in the first part of the proof and thus obtain an element $g\in F$ with $g\neq h, h<g$, contrary to the maximality of h. The proof of the theorem is now complete, for h has the properties required in the theorem.

(4.4) *Let* $\mathbf{A}_0$ *be a dense subalgebra of a Boolean algebra* $\mathbf{A}$ *and let* h_0 *be an isomorphism of* $\mathbf{A}_0$ *into a complete Boolean algebra* $\mathbf{A}'$. *The isomorphism* h_0 *can be extended to an isomorphism* h *of* $\mathbf{A}$ *into* $\mathbf{A}'$.

By (4.3), the isomorphism h_0 can be extended to an homomorphism h of $\mathbf{A}$ into $\mathbf{A}'$. If $p\in\mathbf{A}, p\neq 0$, then there exists an $p_0\in\mathbf{A}_0$ such that $0\neq p_0\subset p$. Consequently $0\neq h_0(p_0)=h(p_0)\subset h(p)$. Thus $h(p)\neq 0$ which proves that h is an isomorphism.

THEOREM 4. *Every infinite complete Boolean algebra has a subalgebra that is isomophic to the field of all subsets of a countably infinite set.*

Proof. Let **A** be an infinite complete Boolean algebra. Then **A** has a countably infinite subset **D**. The subalgebra **A**′ generated by the set **D** is countably infinite. Therefore, by Theorem 3, it has countably infinite atomic subalgebra $\mathbf{A}_0$ of **A**′. This shows that **A** contain a countably infinite atomic subalgebra $\mathbf{A}_0$. Let X be the set of all atoms of $\mathbf{A}_0$ and let **B** be the field of all subsets of the set X. Then, by (4.1), there exists an isomorphism h of $\mathbf{A}_0$ into **B**. And that $h(\mathbf{A}_0)$ is a dense subalgebra of **B**. The set X is clearly conntable, but X must be countably infinite, for $h(\mathbf{A}_0)$ is countably infinite. Since the inverse mapping h^{-1} of h is an isomorphism, we can apply (4.4) to h^{-1}. Thus this shows that the property required in the theorem is reasonable.

REFERENCES

[1] Paul. R. Halmos, *Boolean algebras.* (1959)
[2] Paul. R. Halmos, *Injective and projective Boolean algebras,* (Proceedings of Symposia in Pure Mathematics Vol. II *Lattice Thcory*) Amer. Math. Soc. (1961)
[3] Roman Sikorki, *Boolean algebras,* Berlin: Springer-Verlage (1960)
[4] Roman Sikorski, *Cartesian products of Boolean algebras,* Fund. Math. 37 (1950)
[5] W. Hurewicz and H. Wallman, *Demension Theory,* Princeton U. P. (1941)
[6] J. G. Hocking and G. S. Young, *Topology,* Reading Mass.: Addison Wesley (1961)

DEPARTMENT OF MATHEMATICS, SAGA UNIVERSITY, SAGA.

(Received July, 28, 1963)