SINGULARITIES OF n-SPHERES IN $(n+2)$-SPACE

by

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Introduction

R. H. Fox and J. W. Milnor have bestowed some consideration on the singularities of a 2-sphere S^2 in euclidean 4-space R^4 from any imbedding S^2 in R^4 [1 and 2: p. 1655].* Below is given a partial explanation of the consideration mentioned above :——

An oriented polygonal simple closed curve S^1 either in (oriented) euclidean 3-space R^3 or in the (oriented) 3-sphere S^3 will be called a ***knot***, and their combinatorial equivalence class will be called the same ***knot type***.

Given an oriented polyhedral surface M^2 in 4-space R^4, one can measure the local singularity of M^2 at a point x as follows: Choose a small sphere S^3 in R^4 with center at x. Then S^3 intersects M^2 in an oriented closed curve S^1. We say that x is a ***singular*** or ***non-singular*** point according as S^1 is knotted or unknotted. In either case, if S^1 belongs to the knot type k, we will say that the singularity at the point x is of ***type k***.

Let k be a knot type with representative S^1 in R^3. Let H^4 be the half space $R^3\times[0, \infty)$ in $R^4=R^3\times(-\infty, \infty)$. We define that k is of ***trivial kuot cobordism class***, if there exists a non-singular, polyhedral 2-cell E^2 in H^4 with S^1 as its boundary.

By a non-singular 2-cell, we mean a 2-cell such that each interior point is non-singular in the above sense, and such that each boundary point x is non-singular in the following sense. A small sphere S^3 with center x intersects the half space H^4 in a 3-cell E^3, and intersects the cell E^2 in an arc E^1 spanning E^3. We require that this arc E^1 will be unknotted in E^3.

The product of two knot types can be defined in the usual way. (See 3: Definition 4).

One of Fox and Milnor's results is as follows:——

A collection $\{k_1, k, \cdots\cdots, k_n\}$ *of knot types can occur as the collection of singularities of a 2-sphere in 4-space if and only if the product* $k_1 k_2 \cdots\cdots k_n$ *is of the trivial knot cobordism class.*

By Argument [5: p. 119] —— the argument that the only euclidean space

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* The bracketed uumber indicates the number of reference given at the end of this paper.

in which we can knot S^n is $(n+2)$-dimensions, although the Schoenflies-Mazur-Brown theorem gives an unknotting of S^n in S^{n+1}, —— we will generalize the result into an n-sphere in euclidean $(n+2)$ -space from a combinatorial point of view, but in the simple case which is 1-flat (see p. 7).

It is hoped, therefore, that such terms as spheres, elements, spaces and manifolds —— all used in this paper —— will be deemed oriented and combinatorial, so long as a notice is not given.

V. K. A. M. Gugenheim [3, 4] states that an n-sphere $(n \geqq 2)$ in an $(n+2)$ -sphere may be regarded as a generalization of the knot —— the generalization called ***n-knot*** —— and that the same can be said of several other concepts of the knot which is not yet generalized; for example, a congruence class of n-knots is called an ***n-knot type***.

In this paper, the knot cobordism class —— the term created by Fox, Milnor and Kervaire [Math. review. 8511 (1961) and 2] —— will be adopted into n- knots.

The main theorems are Theorems 2 and 3 of 4 which lead to the conclusion of this paper, one of the essential parts of which, however, Theorem 1 makes.

1. Definitions and Notations*

1. 1.

Let R^q be a q-dimensional metric euclidean space. By ***simplex*** we shall mean closed euclidean simplex, by ***complex***, rectilinear closed locally finite simplicial complex of some euclidean space. Let K be a complex; we denote by $|K|$ the point set covered by the simplexes of K; such a point set will be called a ***polyhedron***, and K a ***partition*** (or simplicial subdivision) of the polyhedron. Polyhedra having isomorphic partitions will be said to be ***equivalent***. By a ***q-element*** we shall mean a polyhedron equivalent to a q-simplex, by a ***q-sphere***, one equivalent to the boundary of a $(q+1)$-simplex.

Let K be a complex and A one of its simplexes. The set of all simplexes of K having A as a face is denoted by $St(A, K)$ and referred to as the ***star of A in K***. The set of simplexes of K which are faces opposite A in some simplex of $St(A, K)$ is defined by $Lk(A, K)$ and referred to as the ***link of A in K***.

A combinatorial q-manifold M is defined as a complex M such that $|St$

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* The author of this paper is indebted to Prof. Gugenheim for his exact and exhaustive definitions and natations.

$(A, M)|$ is a q-element for every simplex $A \subset M$. A polyhedron is called a ***q-manifold***, if it has a partition which is a combinatorial q-manifold.

Let M^n $(n>2)$ be an n-manifold in the euclidean half m-space $H_+^m = R^{m-1} \times [(0, \infty)$ $(m>n)$ and let K, L be the given partitions of M^n, H_+^m such that K is a subcomplex of L. Let P be any vertex of K and let E^n and E^m be $|St(P, K)|$ and $|St(P, L)|$, respectively.

If K is a combinatorial n-manifold (homogeneous complex), ∂K will denote its mod 2 boundary; and if $P = |K|$, we shall write $\partial P = \partial |K|$ and int $|K| = |K| - |\partial K|$.

Let K_1, K_2 be isomorphic partitions of polyhedra P, Q and let $\varphi: P \longrightarrow Q$ be the homeomorphism obtained by mapping each simplex of K_1 linearly onto its correlate simplex of K_2. We call φ a ***piecewise linear homeomorphism onto*** or *PLO*. Let M^q be an orientable q-manifold. An orientation preserving *PLO* $\varphi: M^q \longrightarrow M^q$ is said to be ***positive*** in M^q, and we call it a $+PLO$.

Let $P, Q \subset R^n$ be polyhedra and let $\varphi: R^n \longrightarrow R_n$ be the $+PLO$ such that $\varphi P = Q$. Then we say that P, Q are ***congruent*** in R^n and write $P \equiv Q$ in R^n.

A q-manifold will be called ***strongly connected*** if the following condition is satisfied in the manifold: Let A^q, B^q be q-simplexes of the manifold; then there is a sequence

$$A^q = \triangle_0^q, \triangle_1^{q-1}, \triangle_1^q, \cdots, \triangle_k^{q-1}, \triangle_k^q = B^q$$

of q-and $(q-1)$-simplexes of the manifold such that successive simplexes of the sequence are incident.

Let $X \subset M^q$ be a point set of a q-manifold M^q; it will be said to disconnect M^q if the condition of strong connectivity is no longer satisfied in $M^q - X$.

A polyhedron P is said to be ***locally imbedded*** in an n-manifold M^n if

(i) there is an n-element $E^n \subset M^n$ such that $P \subset$ int E^n;

(ii) P dose not disconnect M^n.

By $[P, M]$ we shall denote the pair consisting of

(i) a polyhedron P locally imbedded in the orientable n-manifold M^n (where M^n shows that it is not yet oriented).

(ii) The oriented manifold M^n (,,)

We call $[P, M^n]$ a ***pair***.

Let $[P, M^n]$ be a given pair and N^n an n-manifold. Since P is locally imbedded, there is an n-lement $E^n \subset M^n$ such that $P \subset$ int E^n. We can clearly find a piecewise linear homeomorphism into $\varphi: E^n \longrightarrow N^n$, where E^n has the orientation induced by M^n. Then $\varphi P = Q$ is locally imbedded in N^n with $Q \subset$ int F^n, where $F^n = \varphi E^n$. In this case we say that the pairs $[P, M^n]$ and $[Q, N^n]$ are ***congruent***, and we write $[P, M^n] \equiv [Q, N^n]$.

For $n<m$, an n-sphere S^n either in m-sphere S^m or in m-space R^m will be called (n, m)-*knot* and each their congruence class will be called (n, m)-*knot type* and we denote a set of congruence classes of n-spheres of m-sphere or m-space, by the use of $[n, m]$.

1. 2

Punctured knot.* Let M^n be an n-manifold in the upper half m-space $H^m_+ (=R^{m-1}\times[0,\infty])$, and let K, L be the partitions of M^n, H^m_+, respectively, such that K is the subcomplex of $L, \partial M^n \subset R^{m-1} (=R^{n-1}\times 0)$ and int $M^n \subset R^{m-1}\times(0,\infty)$. Let P be a vertex of K. By $[P, K, L]$ we will denote $[|Lk(P,K)|, |LK(P,L)|]$ in brief.

Case I: $P\in$ int M^n

It is clear that the pair $[P, K, L]$ represents an element if $[n-1, m-1]$ which is a set of congruence classes of $(n-1, m-1)-$knot type.

Case II: $P\in \partial M^n$

In this case is given a generalized way of interpreting of $[P, K, L]$ as a knot as follows: —

$Lk[P, L]$ is an $(m-1)-$element E^{m-1}_1 such that $\partial Lk(P, L)\subset R^{m-1}\times 0$ and int $Lk(P, L)\subset R^{n-1}\times(0,\infty)$. $|\partial Lk(P, L)|$ is an $(m-2)-$sphere S^{m-2}. In R^{m-1}, there is an $(m-1)-$element E^{m-1}_2 with S^{m-2} as its boundary. Then $E^{m-1}_1\cup E^{m-1}_2$ is an $(m-1)-$sphere S^{m-1} by the Alexander's theorem [6: p. 314].

"For the same reason $|Lk\ P, K)|$ is an $(n-1)-$element E^{n-1}_1 such that ∂Lk $(P, \dot{K})\subset R^{m-1}\times 0$ and int $(P, K)\subset R^{m-1}\times(0, \infty)$. In this case, $\partial|Lk(P, K)|$ is an $(n-2\cdot$ $-$sphere S^{n-2}, and $\partial M^n\cap R^{m-1}$ is an $(n-1)-$element E^{n-1}_2 with S^{n-2} as its boundary. Then $E^{n-1}_1\cup E^{n-1}_2$ is an $(n-1)-$sphere S^{n-1} and $S^{n-1}\subset S^{m-1}$.

$[S^{n-1}, S^{m-1}]$ which has been constructed now, is an $(n-1, m-1)-$knot."

Definition 1. *Let $P\in\partial M^n$ be the vertex of K. The pair $[P, K, L]$ is called the punctured $(n-1, m-1)-$knot, and its knot type is called k, if the knot $[S^{n-1}$, $S^{n+1}]$, which was constructed in the above, has the knot type*** k.

2. Trivial Knot Cobordism Class

Proposition 1 given below is on the basis of Gugenheim's argument [4: p. 135.]

Proposition 1. *The $(n-1, m-1)-$knot types for the common vertex of any two partitions of an n-manifold M^n in an m-space R^m are exactly the same as*

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* The author h?s adopted the term "punctured knot", as he called, for convenience, after the name of a "punctured" oriented n-sphere which was described by Prof. Solomon Lefshetz on page 170 of his book entitled "Introduction To Topolo8y", 1949, and [4: p. 132; 7. 14].

** The $(n-1.\ m-1)$knot is represented by the $(n-1)-$knot or simply knot if $(m-1)$ or $(n-1.\ m-1)$ is clear. The same ean be said of the knot type.

each other.

Proof. Let x be a vertex of K and let L_1 and L_2 be any two partitions of R^m with x as their common vertex. $B_1=Lk(x, L_1)$ and $B_2=Lk(x, L_2)$ are composed of all the opposite faces of x in the x-containg simplexes of L_1 and L_2, respectively. Then ray $\overrightarrow{xy}(y\in B_1)$ and B_1 have only one point y in common. Since $B_2=LK(x, L_2)$, $\overrightarrow{xy}\cap B_2$ has only one point z. Then there is homeomorphism $\varphi: B_1 \longrightarrow B_2$ by $\varphi(y)=z$. It is also known that φ is $+PLO$ by [4: p. 135: 7. 32].

By $p*A$ we denote the complex composed of all the simplexes that have both p and the vertices of each simplex of A—as vertices also of the complex. We call the complex the ***join*** of p and A.

Proposition 2. *Let p be a point of a manifold M. Then we can find the partition of M with p as its vertex.*

Proof. If K be any partition of M, p is either a vertex of K or an inteoior point of a simlex A of K. If p is a vertex of K, K is one of the required partitions. If p is in int A, the join of ∂A and p which is denoted $\partial A*p$, is a partition of A. Hence the partitions of K with $p*\{\partial St(A, K)\}$ as its subcomplex are the required partitions of M.

Proposition 2 holds good with either an m-space R^m or a half m-space H^m. Hence the knot type of M^n in R^m or H^m at p, which is defined in the both cases of p, is in int M^n and is in ∂M^n, is determined uniquely (proposition 1).

For $n<m$, we call a pair $[S_1^n, S_2^n]$ a trival (n, m)-knot, so long as S_1^n is congruent to the boundary of an $(n+1)$-simplex in S_1^m.

From propositions 1 and 2, for any point p of M^n, the $(n-1, m-1)$-knot type at p is determined uniquely.

We can say, therefore, that any point p of M^n in R^m is ***nonsingular*** or ***singular*** according as the $(n-1)$-knot type k at p is tuivial or not, and that the singularity of M^n at the point p is of ***type*** k, if the knot type belengs to k.

Proposition 3. *Let $\triangle^r$ $(r=1, 2, \cdots, n)$ be an r-simplex of an n-manifold M^n in euclidean $(n+2)$-space R^{n+2} and let p_1, p_2 be two points of int $\triangle^r$. Then their singularities will be one and the same.*

Proof. Let K, L be the same means as that of Proposition 2. Each of the two elements $|St(p_1, K)|$ and $|St(p_2, K)|$ of p_1 and p_2 in K, respectively, is $|St(\triangle^r, K)|$, if we subdivide K by p_1 and p_2, each partition independent of the other. That is:

(1) $|St(p_1, K)| = |St(p_2, K)| = |St(\triangle^r, K)|$

By the same process, it is also foumd that

(2) $|St(p_1, L)| = |St(p_2, L)| = |St(\triangle^r, L)|$

Combining each of the three terms of (1) with its correlate term of (2), we find:

(3) $[p_1, K, L]=[p_2, K, L]$.

From propositions 1 and 2 and the above mentioned (3) we will lead to the conclusion that the proof of proposition 3 is complete.

Note: Proposition 3 hold good with p_i ($i=1$ or 2) which, is in $\partial\triangle^r$ if (1) and (2) are hold.

Definition 2. *For any r-simplex $\triangle^r$ ($r=0.1, \cdots, n$) of K in L, we call the knot type of $[\triangle^r, K, L]$ the singularity of $\triangle^r$, which may contain the trivial knot type.*

We say that M^n is 0-*flat* if any r-simplex of M^n is *non*-singular concerning each r and that, M^n is 1-flat* if any r-simplex of M^n is non-singular concerning each $r>0$.

In the sense of 0-flat, 1-flat, we require that any above r-simlex in both the interior and the boundary of M^n, will be nonsingular.

Definition 3. *We say that the $(n-1)$-knot type k belongs to the trivial knot cobordism class, if a 0-flat combinatorial oriented n-element E^n lies in H^{n+2} ($H_-^{n+2}=R^{n+1}\times(-\infty, 0]$ or $H_+^{n+2}=R^{n+1}\times[0, \infty)$) with S^{n-1} in R^{n+1} as its boundary, such that int E^n lies in $H^{n+2}-R^{n+1}$ and $[S^{n-1}, R^{n+1}]$ has the knot type k.*

3. Product of Some Two $(n-1)$-Knots

3.1.

Let M^q be a q-manifold and E^q a q-element such that

$$M^q\cap E^q=\partial M^q\cap\partial E^q=E^{q-1}$$

a $(q-1)$-element. We say that M^q and E^q *have regular contact* in E^{q-1}.

Let $M^q\subset R^n$ be a q-munifold and $E_1^q\subset M^q$ a q-element such that

$$\partial M^q\cap\partial E_1^q\supset E^{q-1},$$

a $(q-1)$-element. Keep $E_2^q\subset R^n$ in regular contact with M^q in E^{q-1} and let q-element $E_1^q\cup E_2^q$ be flat in R^n. Then we call E_2^q a *flat attachment* to M^q.

Let E_1^q, E_2^q be disjointed oriented q-elements of R^n and let g^q be a flat oriented q-element which is a positive flat attachment both to E_1^q and to E_2^q; we say that g^q is a *positive flat connection* between E_1^q and E_2^q. In this case, $G^q=E_1^q\cup g^q\cup E_2^q$ is an oriented q-element and we call that G^q the *connected sum* of E_1^q and E_2^q and denote $G^q=E_1^q\# E_2^q$, or $G^q=[E_1^q, g^q, E_2^q]$.

Let E^q and S^q be a q-element and a q-sphere of R^n, respectively. We

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* It will be notice that the definition of p-flat in this paper is different from one of the p-flat in [4: p. 135; 7. 33].

say that $E^q(S^q)$ is *flat* in R^n, if it is congruent to a q-simplex (the boundary of a $(q+1)$-simplex). And if $E^q(S^q)$ is flat in R^n, then it is of course 0-flat, but the converse is not true.

For $i=1,2$,

(i) Let S_1^q, S_2^q be oriented and disjointed q-spheres of R^n.

(ii) Let $E_1^n, E_2^n \subset R^n$ be disjointed n-elements such that $S_i^q \subset E_i^n$ and $S_i^q \cap E_i^n \supset e_i^q$ q-element.

(iii) Let g^{q+1} be a flat $(q+1)$-element such that

$$g^{q+1} \cap E_i^n = \partial g^{q+1} \cap \partial E_i^n = e_i^q$$

and such that g^{q+1}, S_i^q induce opposite orientation in e_i^q. Then

(iv) $U^q = \mathrm{cl}\ [S_1^q \cup S_2^q \cup g^{q+1} - e_1^q - e_2^q]$

(where cl indicates 'closure') is a q-sphere of R^n, which can be oriented so that $S_i^q \subset U^q$; in this case, we say that g^{q+1} is a ***positive flat connection*** between S_1^q and S_2^q and we write

$$U^q = (S_1^q, S_2^q, g^{q+1}) \text{ or } U^q = S_1^q \# S_2^q$$

and call that U^q is the ***connected sum*** of S_1^q and S_2^q.

Let $S_i^r (r<q; i=1,2)$ be each sphere in S^q and let $g^{r+1} \subset S^q$ be a positive flat connection between S_1^r and S_2^r, and let $e_i^r \subset S_i^r$ be r-elements as e_i^q in the process of making (S_1^q, S_2^q, g^{q+1}). Suppose that we can make $U^r = (S_1^r, S_2^r, g^{r+1})$ in U^q.

Definition 4. *Under the above preparations, we call the* knot $[U^r, U^q]\,(r<q)$ the product of the knots $[S_1^r, S_1^q]$ and $[S_2^r, S_2^q]$.

3.2.

Let S^n be an n-sphere in R^{n+2} and let K, L be partitions of S^n, R^{n+2} respectively, such that K is a subcomplex of L. By L', L'' we will understand the first, second barycentric partitions of L, respectively, and let K', K'' be the first, second barycentric partitions of K. Let p_1, p_2 be the only two singular points of S^n, each belonging to types k_1, k_2, respectively.

Remember : ——

$E_i^{n+2} = |St(p_i, L'')|$, $\quad S_i^{n+1} = \partial E_i^{n+2}$ $(i=1,2)$,

$E_i^n \ = |St(p_i, K'')|$, $\quad S_i^{n-1} = \partial E_i^n$ $\quad (i=1,2)$.

p_1 and p_2 are connected with each other by a simple polygonal arc E^1 of K. Let

$$p_1 = x_0, x_1, \cdots\cdots\cdots, x_{l-1}, x_l = p_2$$

be the vertices of E^1 in L' in this order and suppose

$F_i^{n+2} = |St(x_i, L'')|$, $\quad (i=0,1,\cdots\cdots,l)$,

$F_i^{n+2} \cap F_{i+1}^{n+2} = E_i^{n+1}$, $\quad i=0,1,\cdots\cdots,l-1)$.

This is possible since F_i^{n+2} and F_{i+1}^{n+2} retain regular contact with each other, judging from the construction of F_i^{n+2}.

Specially put

$F_0^{n+2}=E_1^{n+2}, \qquad F_l^{n+2}=E_2^{n+2}.$

Put

$g^{n+2}=F_1^{n+2}\cup F_2^{n+2}\cup\cdots\cdots\cup F_{l-1}^{n+2},$

and put

$E_0^{n+1}=e_1^{n+1}, \qquad E_{l-1}^{n+1}=e_2^{n+1}.$

Theorem 1. *We can make the product of* $[p_1, K'', L'']$ *and* $[p_2, K'', L'']$, *under the above preparations.*

Let K be a subcomplex of a q-manifold M^q. By a ***regular neighborhood*** of K in M^q, we shall mean a subcomplex $U(K, M^q)$ of M^q such that:

(1) $U(K, M^q)$ is a q-manifold.

(2) $U(K, M^q)$ contracts geometrically into K.

[7: p. 293].

We shall prove Theorem 1 by the following lemmas.

Lemma 1. *Let* N^m *be an* m*-manifold in an* m*-manifold* W^m *with no boundary, and let* N^m *and* $E^m\subset W^m$ *have the positive regular contact in an* $(m-1)$*-element* E^{m-1}. *Then*

$$N^m\equiv N^m\cup E^m.$$

Lemma 2. *Let.* $M^q\subset R^n$ $(q<n)$ q*-manifold with no boundary and let* E_i^q $(i=1,2)$ *be disjointed flat* q*-elements in* M^q. *Let* $F_i=U(E_i^q, M^q)$ *be any reqular neighborhood of* E_i^q *in* M^q. *Then there are*

(i) $+PLO\ \varphi_1:\ E_1^q\longrightarrow E_2^q,$

(ii) $+PLO\ \varphi_2:\ M^q\longrightarrow M^q,$

such that

$$\varphi_2\,|\,E_1^q=\varphi_1.$$

(iii) $+PLO\ \varphi:\ R^n\longrightarrow R^n,$

such that

$$\varphi M^q=M^q,$$
$$\varphi E_1^q=\varphi_1,$$
$$\varphi F_1=F_2.$$

Lemma 3. *We can make a connected sum of* S_1^{n+1} *and* S_2^{n+1} *in* R^{n+2}.

Lemma 4. *We can make a connected sum of* S_1^{n-1} *and* S_2^{n-1} *in* $S^{n+1}(=S_1^{n+1}\#S_2^{n+1})$.

Proof of Lemma 1. From the hypothesis of this argument, E^m is a flat attachment to N^m by [8 p. 101; Lemma 2]. Consequently,

$$N^m\equiv N^m\cup E^m$$

by [4: p. 129; Theorem 6*].

Proof of Lemma 2.

(i) Since E_i^q $(i=1,2)$ are flat $q-$elements in M^q, there are

$$+PLO\ \theta_i:\ E_i^q \longrightarrow \triangle^q$$

in M^q, where $\triangle^q$ is a $q-$simlex. Then

$$\varphi_1=\theta_2^{-1}\theta_1$$

is a $+PLO$ and $\varphi_1 E_1^q=E_2^q$.

(ii) We can choose a polyhedron $P_1\subset M^q$ such that $P_1\subset M^q-E_1^q-E_2^b$ and P_1 does not disconnect M^q.

Then there is a$+PLO\ \varphi_2: M^q\longrightarrow M^q$ such that

$$\varphi_2\,|\,E_1^q=\varphi_1,$$
$$\varphi_2\,|\ P=1,$$
$$\varphi_2\approx 1,$$

by the theorem on homogeneity of manifolds [3: p. 32; Theorem 3].

(iii)

(a) $F_i=U\ E_i^q, M^q)$ $(i=1,2)$ are flat $q-$elements which is clear from proposition 1 and the definition of the regular neighborhood. In passing any star neigoborhood is a regular neighborhood. By (ii), there is $+PLO\ \varphi_2: M^q\longrightarrow M^q$, such that $\varphi_2\ |E_1^q=\varphi_1$. Then $\varphi_2 F_1 (\subset M^q)$ is $\bar{U}(E_2^q, M^q)$ a regular neighborhod of E_2^q in M^q.

(b) There is $a+PLO\ \theta_3: M^q\longrightarrow M^q$ such that

$$\theta_3\varphi_2 F_1=F_2,$$
$$\theta_3\varphi_2\,|\,E_1^q=\varphi_1$$

by [7: p. 293; Theorem 23_q].

(c) Let $U(E_i^q, M^q)=F_i$. $U(F_i, M^q)=\bar{F}_i$ $(i=1,2)$, then $\bar{F}_i$ are flat $q-$elements such that $F_i\subset\mathrm{int}\ \bar{F}_i$. We can choose a polyhedron P_2 such that

$$P_2\subset R^n-\bar{F}_1^q-\bar{F}_2^q,$$

and that P_2 dose not disconnect M^q. Then there is a $+PLO\ \varphi_3: R^n\longrightarrow R^n$ such that

$$\varphi_3 M^q=M^q,$$
$$\varphi_3\,|\,P_2=1,$$
$$\varphi_3\,|\,F_1=\theta_3\varphi_2$$

by [3: p. 33; Theorem 9] which is the generalized theorem of [3: p. 32; Theorem 3].

That is : There is a $+PLO\ \varphi=\varphi_3: R^n\longrightarrow R^n$ such that

$$\varphi M^q=M^q,$$
$$\varphi\ U(E_1^q, M^q)=U(E_2^q, M^q),$$
$$\varphi E_1^q=E_2^q,$$
$$\varphi\,|\,P_2=1,$$

which proves Lemma 2.

Proof of Lemma 3. The following conditions are satisfied: —

(i) $S_1^{n+1} \cap S_2^{n+1} = \phi$ since $E_1^{n+2} \cap E_2^{n+2} = \phi$, where $E_i^{n+2} = St(p_i, L'')$, $(i=1, 2)$ in R^{n+2}.

(ii) $S_i^{n+1} \subset E_i^{n+2} (i=1, 2)$ since $S_i^{n+1} = \partial E_i^{n+2}$,
$S_i^{n+1} \cap \partial E_i^{n+2} \supset e_i^{n+1} (i=1, 2)$,

where

$$E_1^{n+2} \cap F_1^{n+2} = e_1^{n+1}, \quad F_{l-1}^{n+2} \cap E_2^{n+2} = e_2^{n+1}.$$

(iii) F_1^{n+2} and $F_{i+1}^{n+2} (i=0, 1, \cdots l-1)$ have the positive regular contact in E_i^{n+1} which is clear from the construction of F_i^{n+2}. Then $F_i^{n+2} \cup F_{i+1}^{n+2} (i=1, 2, \cdots, l-2)$ are flat $(n+2)$-elements in R^{n+2} by Lemma 1.

Besides —

For $i \neq j$,

$$F_i^{n+2} \cap F_j^{n+2} = \phi \text{ if } i \neq j \pm 1,$$
$$F_j^{n+2} \cap E_i^{n+2} = \phi \text{ if } i=1, 2; j=2, 3, \cdots\cdots, l-2,$$

judging from $F_i^{n+2} = St(x_i, L'')$. Then g^{n+2} is a flat $(n+2)$-element and

$$E_i^{n+1} = g^{n+2} \cap E_i^{n+2} = \partial g^{n+2} \cap \partial E_i^{n+2} (i=1, 2).$$

And g^{n+2} and $S_i^{n+1} (i=1, 2)$ induce opposite orientations in E_i^{n+1}.

(iv) put

$$S^{n+1} = cl\ [S_1^{n+1} \cup S_2^{n+1} \cup \partial g^{n+2} - e_1^{n+1} - e_2^{n+1}].$$

Then

$$S_i^{n+1} \subset S^{n+1} (i=1, 2).$$

Thus, the conditions of the connected sum of the two spheres S_1^{n+1} and S_2^{n+1} are satisfied, which proves Lemma 3 complerely.

Proof of Lemma 4. In order to connect S_1^{n-1} and S_2^{n-1} in S^{n+1}, it is sufficient to prove the existence of the positive flat connection between S_1^{n-1} and S_2^{n-1} in S^{n+1} since $S_i^{n-1} (i=1, 2)$ are in S^{n+1}.

(a) Remember: —

(i) E^1 is a simple polygonal arc of K and

$$p_1 = x_0, x_1, \cdots\cdots x_{l-1}, x_l = p_2$$

are the vertices of E^1 in L' in this order. Let A_1 be the middle point of $\overline{p_1 x_1}$ and let A_2 be that of $x_{l-1} p_2$.

(ii) $F_i^{n+2} = |St(x_i, L'')|$, $(i=0, \cdots\cdots, l)$,

specially $E_i^{n+2} = |St(p_i, L'')|$, $(i=1; 2)$,

$F_i^{n+2} \cap F_{i+1}^{n+2} = E_i^{n+1}$, $(i=0, \cdots\cdots l-1)$,

specially $E_1^{n+2} \cap F_1^{n+2} = e_1^{n+1}$, $E_2^{n+2} \cap F_{l-i}^{n+2} = e_2^{n+1}$.

$g^{n+2} = F_1^{n+2} \cup F_2^{n+2} \cup \cdots\cdots \cup F_{l-1}^{n+2}$.

(iii) $F_i^n = |St(x_i, K'')|$, $(i=0, 1, 2, \cdots\cdots, l)$,

specially $E_i^n = St(P_i, K'')$, $(i=1, 2)$.

$F_i^n \cap F_{i+1}^n = E_i^{n-1}$ $(i=0,1, \cdots l-1)$,

specially $E_1^n \cap F_2^n = e_1^{n-1}$, $E_1^n \cap F_{l-1}^n = e_2^{n-1}$.

$g^n = F_1^n \cup F_2^n \cup \cdots \cup F_{l-1}^n$.

Then each one of (iii) is the intersection of S^n and that of (ii) which is correlate.

(b) F_i^n $(i=1, 2, \cdots, l-1)$ are flat $n-$elements in S^n, and F_i^n and F_{i+1}^n $(i=1, 2, \cdots\cdots, l-1)$ have the regular contact in E_i^{n-1}. Then $E_i^n \cap E_{i+1}^n$ are flat $n-$elements by Lemma 1. Thus g^n is a flat $n-$element. Since g^n and $E_i^n (i=1.2)$ have the positive regular contact in e_i^{n-1}, g^n is a positive flat connection between S_1^{n-1} and S_2^{n-1}.

(c) Put

$I^{n-1} = I_1 \times I_2 \times \cdots\cdots \times I_{n-1}$, $I^{n+2} = I^{n-1} \times I_n \times I_{n+1} \times I_{n+2}$,

where I_j is the interval $[-1, 1]$ of the $j-$axis of rectrngular coordinates in R^{n+2}.

$$I_1^{n+1} = I^{n-1} \times I_n \times I_{n+1} \times (-1),$$
$$I_2^{n+1} = I^{n-1} \times I_n \times I_{n+1} \times 1.$$
$$I^n \ = I^{n-1} \times 0 \times 0 \times I_{n+2}.$$
$$I_1^{n-1} = I^{n-1} \times 0 \times 0 \times (-1),$$
$$I_2^{n-1} = I^{n-1} \times 0 \times 1.$$

And

$$h^n = I^{n-1} \times 0 \times \{([0, 1] \times (-1)) \cup (1 \times [-1, 1]) \cup ([1, 0] \times 1)\},$$
$$B_1 = (0, 0, \cdots\cdots\cdots, 0, -1),$$
$$B_2 = (0, 0, \cdots\cdots\cdots, 0,1).$$

(d) $[\partial g^n, \partial g^{n+2}]$, $[\partial I^n, \partial I^{n+2}]$ are each a trivial $(n-1, n+1)-$knot. Then there is $a+PLO\ \varphi_1$ such that

$\varphi_1 \partial g^{n+2} = \partial I^{n+2}$, $\varphi_1 \partial g^n = \partial I^n$.

Since e_i^{n-1} $i=1,2)$ are $U(A_i, e_i^{n-1})$,

$\varphi_1 (A_i) \epsilon \varphi_1 e_i^{n-1} \subset \partial I^n$.

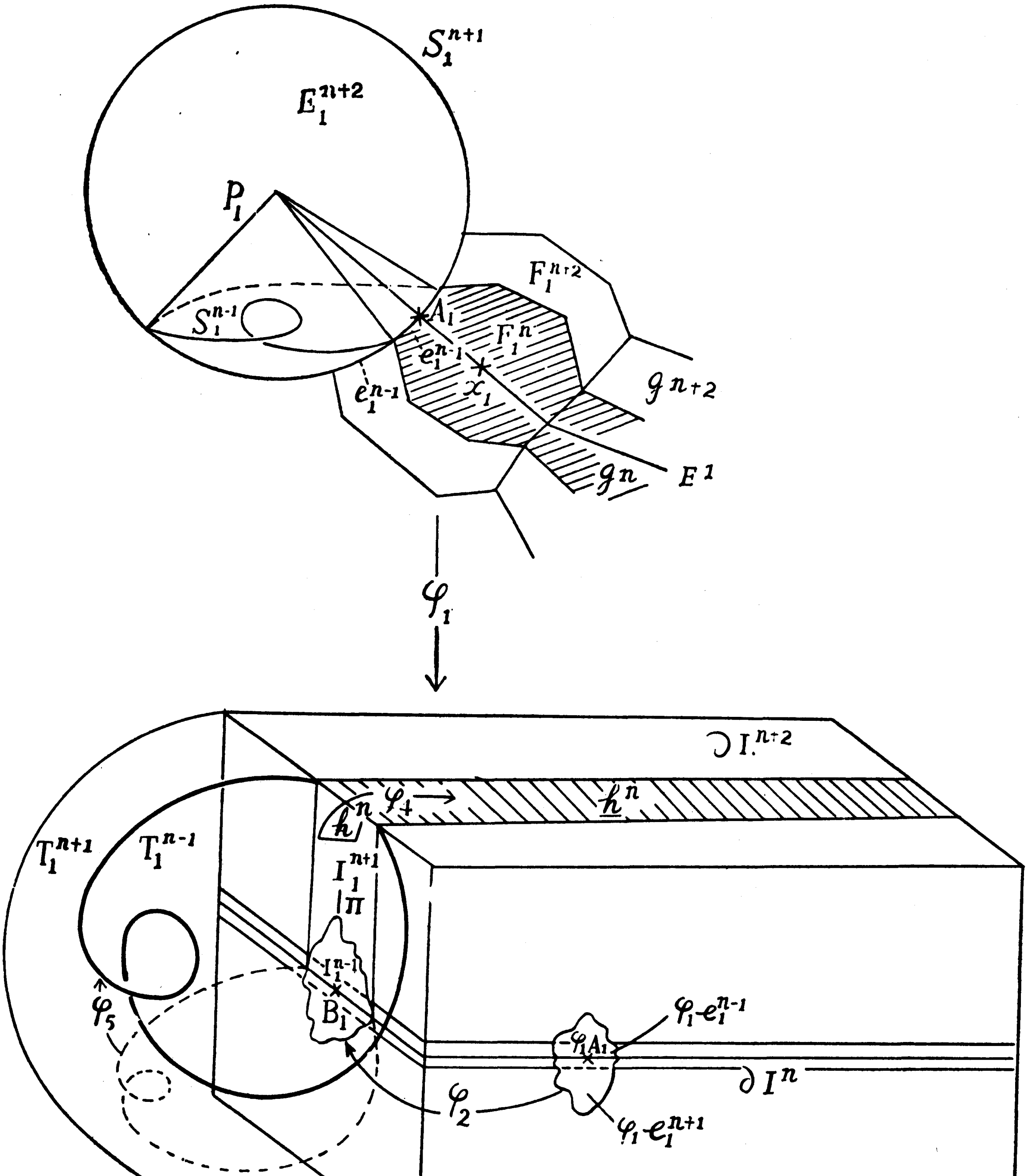
(e_1) Since e_1^{n-1} is $U(A_1, e_1^{n-1})$ and I_1^{n-1} is $U(B_1, I_1^{n-1})$, there is a $+PLO\ \varphi_{21}$ $: R^{n+2} \longrightarrow R^{n+2}$ by Lemma 2 such that

$$\varphi_{21} \partial I^n = \partial I^n,$$
$$\varphi_{21} \varphi_1 A_1 = B_1,$$
$$\varphi_{21} \varphi_1 e_1^{n-1} = I_1^{n-1}.$$

(e_2) We can choose an $(n-1)-$element P_2 in ∂I^n Lemma 2 such that

$$P_2 \supset \{(\varphi_{21} \varphi_1 e_1^{n-1}) \cup I_1^{n-1}\}.$$

Then by Lemma 2 again, there is a $+PLO\ \varphi_{22} : R^{n+2} \longrightarrow R^{n+2}$ such that



A bird's-eye view of the proof of Lemma 4

$$\varphi_{22}\,\partial I^n=\partial I^n,$$
$$\varphi_{22}\,\varphi_{21}\,\varphi_1\,A_2=B_2,$$
$$\varphi_{22}\,\varphi_{21}\,\varphi_1\,e_2^{n-1}=I_2^{n-1},$$
$$\varphi_{22}\,|\,P_2=1.$$

Put

$$\varphi_2=\varphi_{22}\,\varphi_{21}.$$

(f) Since e_i^{n+1} are $U(e_i^{n-1},e_i^{n+1})$ and I_i^{n+1} are $U(I_i^{n-1},I_i^{n+1})$, by Lemma 2 there are $+PLO\ \pi_i\ (i=1.2)$:

$$+PLO\ \pi_i\,\varphi_2\,\varphi_1\,e_i^{n+1}=I_i^{n+1},$$

under the same notice in the way of (e_1) and (e_2).

Put $\pi=\pi_2\pi_1$, and the result is:

There is a $+PLO\ \varphi_3=\pi\varphi_2\varphi_1: R^{n+2}\longrightarrow R^{n+2}$

such that

$$\varphi_3\,\partial g^{n+2}=\partial I^{n+2},$$
$$\varphi_3\,e_i^{n+1}=I_i^{n+1},$$
$$\varphi_3\,|\,e_i^{n-1}=\pi_i\,\varphi_2\,\varphi_1.$$

Then

$$\varphi_3\{\mathrm{cl}\,[\partial g^{n+1}-e_1^{n+1}-e_2^{n+1}]\}=\mathrm{cl}\,[\partial I^{n+2}-(I_1^{n+1}\cup I_2^{n+1})].$$

(g) In ∂I^{n+2}, there is $+PLO\ \varphi_4:\partial I^{n+2}\longrightarrow \partial I^{n+2}$

such that

$$\varphi_4\,h^n=I^{n-1}\times 0\times\{1\times[-1,1]\}=\underline{h}^n$$
$$\varphi_4\,I_1^{n-1}=I^{n-1}\times 0\times 1\times(-1)$$
$$\varphi_4\,I_2^{n-1}=I^{n-1}\times 0\times 1\times 1.$$

If we choose $U(h^n,\partial I^{n+2})$, $U(\underline{h}^n,\partial I^{n+2})$, there is a $+PLO\ \varphi_5: R^{n+2}\longrightarrow R^{n+2}$, by Lemma 2 such that

$$\varphi_5\,|\,h^n=\varphi_4.$$
$$\varphi_5\,U(h^n\partial I^{n+2})=U(\underline{h}^n\partial I^{n+2})$$

(h) Let $T_i^{n-1}, T_i^{n+1} (i=1,2)$ be $\varphi_4\,\varphi_3\,S_i^{n-1}, \varphi_4\,\varphi_3\,S_i^{n+1}$, respectively. From the above process, there is a $+PLO\ \psi=\varphi_5\,\varphi_3: R^{n+2}\longrightarrow R^{n+2}$, such that

$$\psi\,\partial g^{n+2}=\partial I^{n+2},$$
$$\psi\,S_i^{n+1}=T_i^{n+1},$$
$$\psi\,\partial g^n=\partial\underline{h}^n,$$
$$\psi\,S_i^{n-1}=T_i^{n-1}.$$

Since $\underline{h}^n$ is in the $(n-1)$-sphere $T_1^{n+1}\# T_2^{n+1}, \psi^{-1}\,\underline{h}^n$ is in S^{n+1} and it is a pos'tive flat connection between S_1^{n-1} and S_2^{n-1}. This is the very n-element which we have long been seeking for.

All these prove Lemma 4 completely.

Proof of Theorem 1. This proof is clear from Lemmas 3 and 4 and

Definition 4.

4. Main Theorems

Under the above preliminaries, the main theorems are as follows: ——

Theorem 2. *If a collection $\{k_1, k_2, \cdots\cdots, k_m\}$ of $(n-1)$-knot types can occur as a collection of all singularities of a 1- flat n-sphere $S^n (n>2)$ in euclidean $(n+2)$-space R^{n+2}, then the product $k_1 k_2 \cdots\cdots k_m$ is one of the trivial knot cobordism classes.*

Theorem 3. (A converse of Theorem 2).

If the poduct $k_1 k_2 \cdots\cdots k_m$ of $(n-1)$-knot types is a trivial knot cobordism class, then a collection $\{k_1, k_2, \cdots\cdots k_m\}$ of 1-flat $(n-1)$-knot types can occur as a collection of singularities of S^n in R^{n+2}.

The notations used in Chapter 3, will also be used in Chapter 4.

Proof of Theorem 2. Remembering that K is a subcomplex of L, let K, L be partitions of S^n, R^{n+2}, and K'', L'' be the second barycentric partitions of K, L, respectively. And suppose that $p_i\ (i=1, 2, \cdots\cdots, m)$ are all singular points in M^n. From this it is clear that we can choose a polyhedron E^{n+2} in R^{n+2} such that

$$E_3^{n+2} \cup E_4^{n+2} \cup \cdots\cdots \cup E_m^{n+2} \subset E^{n+2},$$
$$(E_1^{n+2} \cup E_2^{n+2}) \cap E^{n+2} = \phi$$

and E^{n+2} does not disconnect R^{n+2}.

Then, from [3: p. 37; 3. 22 Lemma], there is an $(n+3)$-element $F^{n+2} \subset R^{n+2}$ disjointed from E^{n+2} such that $E_1^{n+2} \cup E_2^{n+2} \subset \text{int } F^{n+2}$.

Theorem 1, therefore, lead to making a product of $[p_1, K'', L'']$ and $[p_2, K'', L'']$ in int F^{n+2}. In the analogous way, we can make another product of this product and $[p_3, K'', L'']$.

Repeat this method, and we can make a product of all knots, which are $[p_1, K'', L'']$, $[p_2, K'', L'']$, $\cdots\cdots\cdots\cdots\cdots$, $[p_m, K'', L'']$.

Suppose the following:

$$E^{n+2} = E_1^{n+2} \# E_2^{n+2} \# \cdots\cdots \# E_m^{n+2}$$
$$S^{n+1} = S_1^{n+1} \# S_2^{n+1} \# \cdots\cdots \# S_m^{n+1}$$
$$e^n = E_1^n \# E_2^n \# \cdots\cdots \# E_m^n$$
$$S^{n-1} = S_1^{n-1} \# S_2^{n-1} \# \cdots\cdots \# S_m^{n-1}$$
$$f^n = cl[S^n - e^n].$$

Then

$$S^{n-1} \subset S^{n+1},$$
$$e^n \subset S^n, \text{ but } e^n \not\subset S^{n+1},$$
$$\partial e^n = S^{n-1},$$

and int e^n has all singularities of S^n.

Reference [6:p.316; 2δ] illustrates that f^n is an $n-$element with S^{n-1} as its boundary. f^n has no singularity of S^n from the property of e^n.

E^{n+2} is an $(n+2)-$element and it is flat in R^{n+2} from Newman's theorem on homogeneity of manifolds [3;p;32]. Then E^{n+2} is congruent to a euclidean $(n+2)-$ball V^{n+2} which is not a topological ball. By means of Stereographic projection,

$$\partial V^{n+2} \equiv R^{n+1} \cup \{\infty\}$$

where R^{n+1} is a euclidean $(n+1)-$space. As the result, there is

$$+PLO\ \varphi:\ [R^{n+2} \cup \{\infty\}] \longrightarrow [R^{n+2} \cup \{\infty\}]$$

such that

$$\varphi(\partial E^{n+2}) = R^{n+1} \cup \{\infty\},$$
$$\varphi f^n \subset H_-^{n+2},$$
$$\varphi f^n \text{ is } 0-\text{flat in } R^{n+2},$$
$$\varphi\, \mathrm{int} f^n \subset H_-^{n+2} - R^{n+1},$$
$$\varphi\, \partial f^n = \varphi\, S^{n-1}. \subset R^{n+1}.$$

All this has led to the complete proof of Theorem 2.

To prove Theorem 3.

Proposition 4. *Let* S^{n-1} *be the* $0-$*flat* $(n-1)-$*sphere in a euclidevn* $(n+1)-$*space* R^{n+1}. *If* p *is a point of* $R^{n+1} \times (0.\infty)$, *then* $E^n - p$ *is* $0-$*flat in* $R^{n+1} \times [0.\infty)$, *where* $E^n = p * S^{n-1}$.

Proof. Let K_1, L_1 be the partitions of S^{n-1}, R^{n+1}, respectively, such that K_1 is the subcomplex of L_1 and let the partition K_2 of E^n is a partition constructed from K_1 by the standard cone and let L_2 be a partition of $R^{n+1} \times [0, \infty)$ containing K_2 as its subcomlex. Then all the simlexes of $E^n - p$ are either $\triangle^r$ $(r = 0, 1, \cdots, n-1)$ in K_1 or $p * \triangle^r$.

Case I. The Simplex $\triangle^r$ in K_1

By the hypothesis, S^{n-1} is $0-$flat. All simplexes of S^{n-1}, therefore, are non$-$singular.

Case II. The occasion of the Simplex being $\triangle^r * p$

Let β be any point of $E^n - p$ and $\alpha = \overrightarrow{p\beta} \cap S^{n-1}$, where α is a vertex of the subcomplex of K_1. (We denote this subcomplex of K_1 by same K_1). Since K_1 is a subcomplex of L_1,

$$\begin{aligned} Lk(\beta, L_2) &= \partial\{p * St(\alpha, L_1)\} \\ &\supset \partial\{p * St(\alpha, K_1)\} \\ &= St(\alpha, K_1) \cup \{p * Lk(\alpha, K_1)\} \\ &= Lk(\beta, K_2) \end{aligned}$$

Then there is $[\beta, K_2, L_2]$.

Since the knot $[\alpha, K_1, L_1]$ is trivial, on account of the hypothesis, there is a flat $(n-1)$-element e^{n-1} in $Lk(\alpha, L_1)$ with $Lk(\alpha, K_1)$ as its boundary. That is:

$$e^{n-1} \equiv \triangle^{n-1} \text{ in } Lk(\alpha, L_1),$$

where $\triangle^{n-1}$ is an $(n-1)$-simplex. Then

$$p * e^{n-1} \equiv p * \triangle^{n-1} = \triangle^n \text{ in } L_2.$$

where $\triangle^n$ is an n-simplex. That is:
There is an $(n-1)$-element $(\partial\triangle^n - \triangle^{n-1})$ in L_2 with $\partial\triangle^{n-1}$ as its boundary. Therefore, $[\alpha, K_2. L_2]$ is trivial.

On the other hand, from Definition 1 and proposition 3 and its note.

$$[\alpha, K_2, L_2] = [\beta, K_2, L_2].$$

Consequently, $[\beta, K_2, L_2]$ is trivial.

Since β is any point of $p * \triangle^r - p$, the proof of case II is complete, which naturally will lead to the complete proof of proposition 4.

Proof of Theorem 3. From the hypothesis of Theorem 3 will follow the conclusion that there is an $(n-1)$-sphere S^{n-1} in a euclidean $(n+1)$-space $R^{n+1} \subset R^{n+2} = R^{n+1} \times (-\infty, \infty)$, which has the knot type $k_1 k_2 \cdots\cdots k_m$. Since the knot type $k_1 k_2 \cdots k_m$ belongs to the trivial knot cobordism class, there exists a 0-flat combinatorial n-element e^n in H_-^{n+2} with S^{n-1} as its boundary, such that int e^n is in $H_-^{n+2} - R^{n+1}$.

And the same hypothesis will lead to the conclusion S^{n-1} is the connected sum of $S_1^{n-1}, S_2^{n-1}, \cdots\cdots, S_m^{n-1}$, each having knot types $k_1, k_2, \cdots, k_m$ in R^{n+1}, respectively. We can choose all the points $p_i\ (i=1, 2, \cdots, m)$ in $R^{n+1} \times (0, \infty)$, such that $p_i * S_i^{n-1}$ and $p_j * S_j^{n-1}$ are pairwise disjointed for $i \neq j$ as is clear from proposition 4. And $e_i^n = p_i * S_i^{n-1}$ will have the only one point p_i as its own singularity, which is also concluded from the same prorosinion.

On the o her hand, we can choose each positive flat connection $g_i^n\ (i=1, 2, \cdots, m-1)$ in H_+^{n+2}, such that g_i^n connects e_i^n with e_{i+1}^n in R^{n+1} and that they might be pairwise disjointed. Hence the connected sum is given below:—

$$f^n = e_1^n \# e_2^n \# \cdots\cdots \# e_m^n.$$

f^n whose boundary is an $(n-1)$-sphere S^{n-1}, has points as its own singularity, each $p_i\ (i=1, \cdots, m)$.

e^n and f^n are n-elements with their boundary called S^{n-1} and the interior parts of the two have no point in common with each other. $e^n \cup f^n$, therefore, is an n-sphere, according to J. W. Alexander's theorem [6: p. 314]. In this way, we have found our way to the conclusion that $e^n \cup f^n$ is the very sphere which we have long been reaching for.

And all this will give the complete proof of Theorem 3.

P. S.: At first, ***the trivial knot cobordism class*** was called ***nullequivalent*** by Fox and Milnor, as shown in (I), who, however, have been dis-satisfied with this terminology. In result, Fox, in his work entitled "A Quich Trip Through Knot Theary", adopted the name ***slice knot*** proposed by Edwin E. Moise.

To my regret, however, it was after this paper was written when I met with this terminology. And this is the reason why I used the former terminology in this paper.

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