LINEAR FUNCTIONALS ON A BANACH SPACE WITH SEMI-NORMS

Dedicated to Professor K. Kunugui on his sixtieth birthday.

By

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We have given the definitions and sketched simle properties of many norms space in the previous paper [7], as the generalization of the Saks space or the two norm space (see [1], [2], [3], [4], [8]). In the paper, we explained that $<\Xi^*, \| \|^*>$ is closed in $<X, \| \|>$ and was given the conditions of the reflexivity of $<X, \| \|>$. But, the proof of Proposition 2 is not exact, so is not complete also the condition of the reflexivity of $<X, \| \|>$ using the proposition (last theorem in [7]). In the present paper, we shall give the complete proof by the new topology (bounded mixed topology) and the condition of the reflexivity of $<X, \| \|>$. Moreover, we shall prove that the conjugate space $<\Xi^*, \| \|^*, \| \|>$ of the many norms space is a dual many norms space.

In § 1 of the present paper, $b\gamma$-topology will be defined in the many norms space $<X, \| \|, \| \|^*>$ and be shown the space $\Xi_{b\gamma}$ of the $b\gamma$-continuous linear functionals is coincident with the closure of the space Ξ^* of τ^*-continuous linear functionals. This fact was the property of the space Ξ_γ in the two-norms space. It becomes clear in § 2 the conjugate space $<\Xi^*, \| \|^*, \| \|>$ is a dual many norms space. It will be discussed in § 3 make generalized in what form in our case many properties of the biconjugate space in the two-norms space. In § 4, the γ-reflexivity of the many norms space corresponds with Theorem 3.7 in § 3, [3] only will be considered. A deeper study on it must be made in future.

§ 1. Topologies of a many norms space. Let X be a Banach space with the norm $\| \|$ and complete by norm topology, that is, a Banach space. Moreover, we define that each element x of X has some semi-norms $\| x \|_\alpha^*$ $(\alpha \in \Lambda)$ satisfying the conditions $\| x+y \|_\alpha^* \leqq \| x \|_\alpha^* + \| y \|_\alpha^*$, $\| ax \|_\alpha^* = |a| \ \| x \|_\alpha^*$. ($\| x \|_\alpha^* = 0$ does not imply $x=0$)

The norm $\| x \|$ and semi-norms $\| x \|_\alpha^*$ $(\alpha \in \Lambda)$ have the relation

$$(M) \qquad \| x \| = sup_{\alpha \in \Lambda} \| x \|_\alpha^*$$

By the norm $\| x \|$ of x in X, it is a locally convex, linear topological space; denote the topology by τ.

The neighbourhoods of zero by semi-norms $\| x \|_\alpha^*$ $(\alpha \in \Lambda)$ are $U(x ; \| x \|_{\alpha_i}^* < \varepsilon,\ i=1, 2, \cdots, n :\ \alpha_i \in \Lambda)$, its topology is denoted by τ^*, the system of the neighbourhoods of zero by $U(\tau^*)$. The space X with topology τ^* is a linear topological space:

(1) if $U \in U(\tau^*)$ and $\lambda \in R$, $\lambda \neq 0$, then $\lambda U \in U(\tau^*)$,

(2) if $U \epsilon U(\tau^*)$ and $\lambda \epsilon R$, $|\lambda| \leqq 1$, then $\lambda U \subset U$,

(3) if $U \epsilon U(\tau^*)$, then for every $x \epsilon X$, there exists $\lambda \epsilon R$, $\lambda \neq 0$ such that $\lambda x \epsilon U$,

(4) is $U, V \epsilon U(\tau^*)$, then there exists $W \epsilon U(\tau^*)$ such that $W \subset U \cap V$,

(5) if $U \epsilon U(\tau^*)$, then there therc exists $V \epsilon U(\tau^*)$ such that $V + V \subset U$.

Moreover, $U(\tau^*)$ satisfy the following condition by the assumption (M)

(6) for every $x \epsilon X, x \neq 0$, there exists $U \epsilon U(\tau^*)$ such that $x \bar{\epsilon} U$.

The space X with the topology τ^* is incomplete. Of course, $\tau^* \leqq \tau$.

The space X with the norm $\|x\|$ and semi-norms $\|x\|_\alpha^*$ $(\alpha \epsilon \Lambda)$ is denote by $<X, \| \|, \| \|^*>$ or $<X, \tau, \tau^*>$ in the same notation with the case of the two-norm space and call it "***many norms space***".

Many norms space has many important examples. (see [7])

We can introduce the mixed topology in a many norms space as in the two-norm space; For each $U \epsilon U(\tau)$ and for each sequence $U_1^*, U_2^*, \cdots \epsilon U(\tau^*)$, we shall denote by $\gamma(U_1^*, U_2^*, \cdots; U)$ the set $\bigcup_{n=1}^{\infty} (U_1^* \cap U + U_2^* \cap 2U + \cdots + U_n^* \cap nU)$.

Wiweger [8] has called this topology the ***mixed topology*** determined by the τ and τ^*.

Mixed topology is weaker than τ and stronger than τ^* as the topology τ^* is weaker than the topology τ $(\tau^* \leqq \tau)$.

Both of $U(\tau), U(\tau^*)$ are locally convex, so also the mixed topology is locally convex. We shall denote by γ-topology the mixed topology in according to Wiweger.

We define that B is a bounded set in $<X, \| \|>$ if for each $U \epsilon U(\tau)$ there exists $\lambda \epsilon R$ such that $B \subset \lambda U$.

τ^* doesn't satisfy the first countability axiom in general, so it is not also in mixed topology.

Next, if $\|x\|_\alpha^* \geqq \|x\|$ $(\alpha \epsilon \Lambda)$ and there exists the following condition instead of the condition (M) between $\|x\|_\alpha^*$ $(\alpha \epsilon \Lambda)$ and $\|x\|$:

(DM) $$\|x\| = \inf_{\alpha \epsilon \Lambda} \|x\|_\alpha^*$$

it will be named ***dual many normed*** space.

Condition (DM) will be used only in § 3.

Under the condition (M), we have

Proposition 1. *The unit sphere* $\|x\| \leqq 1$ ***is closed by the*** τ^****-topology.*** (see [8], p. 62). It satisfies moreover the conditions (o), (n), (d) in [8].

We have also in our case Proposition 1.5 in [2] (p. 124) or Theorem B in [3] (p. 268) which play an important rôle in further discussions.

Let be Ξ, Ξ_γ, Ξ^* the set of linear continuous functionals concerning τ, γ, τ^* on X, so it will be $\Xi \supset \Xi_\gamma \supset \Xi^*$.

Theorem 1. *The set* $Y=\{\xi\,;\xi\epsilon\Xi^*, \|\xi\|\leqq 1\}$ *is norming; that is,* $\|x\|=\sup\{\xi(x)\,;\xi\epsilon Y\}$ *for each* x.

Proof. We take any x_0 ($\|x_0\|=1$), $\varepsilon>0$, $S=\{x\,;\|x\|\leqq 1\}$ is convex, symmetric, τ^*-closed. Ξ^* is a linear topological space and $(1+\varepsilon)x_0$ is not in S, so there is a $\xi\epsilon\Xi^*$ such that

$$\xi(x)\begin{cases}<1 & \text{for } x\epsilon S,\\ =1 & \text{for } x=(1+\varepsilon)x_0.\end{cases}$$

by Theorem 2 in [5] (p. 21, *Mazur's* theorem in a topological space)

$|\xi(x)|<1$ for $\|x\|\leqq 1$, therefore $\|\xi\|\leqq 1$, that is, $\xi\epsilon Y$.

In the other hand,

$$\xi(x_0)=\frac{\|x_0\|}{1+\varepsilon}=\frac{1}{1+\varepsilon},$$ and ε is any positive nu nber.

So, it will be $\sup\{\xi(x)\,;\xi\epsilon Y\}=1$.

Proposition 2. If *the sequence* $\{x_n\}$ *converges to* x *by* τ^**-topology, then the set* $\{x_n\}$ *is norm bounded*; $\|x_n\|\leqq K$ (see [7]).

By the proposition, we shall introduce in X another topology: if $E\cap A$ is closed in γ-topology for each norm bounded set A, then E will be called $b\gamma$-closed. This topology will be called bounded mixed topology or bounded γ-topology ($b\gamma$-topology).

τ^*-topology is identical with γ-topology for bounded set by proposition 2.2.1 in [8]. Therefore, τ^*-topology, γ-topology or $b\gamma$-topology are coincident in bounded set.

Let $\Xi_{b\gamma}$ be the set of $b\gamma$-continuous linear functionals.

From the relation $\tau^*\leqq\tau$, it follows $\tau^*\leqq\gamma\leqq\tau$ ([7], p. 3) and moreover, $\tau^*\leqq\gamma\leqq b\gamma\leqq\tau$. Let $\Xi^*, \Xi_\gamma, \Xi_{b\gamma}, \Xi$ be the set of the linear functionals continuously concerning $\tau^*, \gamma, b\gamma, \tau$-topology respectively. And, there is the following inclusion relations between these four spaces; $\Xi^*\subset\Xi_\gamma\subset\Xi_{b\gamma}\subset\Xi$.

The closure of the space Ξ^* is Ξ_γ in the two norms space. This property has important meaning in the reflexivity in the two norm space. In our case, it has no this property in Ξ_γ, because there exists no the first countability axiom in τ^*. The space which takes the place of it in many norms space will be $\Xi_{b\gamma}$.

Lemma. Let *H be a linear closed subset of* $<X, \|\ \|>$, *and* $x_0\bar{\epsilon} H$, *then there exists a constant* A *such that* $h\epsilon H, \|\lambda x_0+h\|\leqq 1$ *imply* $\|h\|\leqq A$. (see [2], Lemma 4.1)

Theorem 2. *$b\gamma$-lınear functionals* ξ *in* $<X, \|\ \|, \|\ \|^*>$ *is represented by*

$$\xi(x)=\lim_{n\to\infty}\xi_n(x),$$

where $\xi_n(x)\in\Xi^*$ *and* $\|\xi_n-\xi\|\to 0$.

Proof. It suffices to consider only non-trivial functional ξ. Let H be the null-set of ξ, let $\xi(x_0)=1$. $Z_n=H\cap S_n$ $(S_n=(x:\|x\|\leqq n)$ is closed in $<X, \|\ \|^*>$ (see Theorem 3.2, [2]). This set is convex, symmetric and $x_0\bar{\in} Z_n$. There exists $\zeta_n(x)\in\Xi^*$ by the *Mazur*'s theorem in a topological space (see [5]) such that

$$\zeta_n(x)\begin{cases}<1/n & x\in Z_n,\\ =1 & x=x_0.\end{cases}$$

Then, $|\zeta_n(x)|<1/n$ for $x\in H\cap S_1=Z_1$.

We set $\xi_n(x)=\xi(x)-\zeta_n(x)$, so

$$|\xi_n(x)|\begin{cases}\leqq 1/n & \text{for}\quad x\in Z_n\\ =0 & \text{for}\quad x=x_0.\end{cases}$$

x is any element such that $\|x\|\leqq 1$, from lemma $\|h\|\leqq A$, $x=h+\lambda x_0, h\in H$, so $h/A\ \in Z_1$. Then $|\xi_n(x)|=|\xi_n(h)|=A|\xi_n(h/A)|\leqq A/n$. It follows $\|\xi_n\|\leqq A/n$. Thus, we have $\xi(x)=\lim_{n\to\infty}\xi_n(x)$.

Proposition 3. *If* $\{\xi_n(x)\}\in\Xi^*$, $\lim_{n\to\infty}\xi_n(x)=\xi(x)$ $(\|\xi_n-\xi\|\to 0)$ *in* $<X, \|\ \|, \|\ \|^*>$, *then* $\xi(x)$ *is a* $b\gamma-$*linear functional.*

Proof. It is clear that $\xi(x)$ is an additive, homogeneous functional. Let $\|x\|\leqq K$, zero be an accumulation point of $\{x\}$, $\xi_n(x)\in\Xi^*$ and taken n such as $\|\xi_n-\xi\|<\varepsilon/2k$, so there exist $\alpha_i\in\wedge$, $\|x\|^*_{\alpha i}<\delta(i=1,2,\cdots,m)$ and $\|x\|\leqq K, |\xi_n(x)|<\varepsilon/2$. Therefore,

$$|\xi(x)|\leqq|\xi_n(x)|+|(\xi_n-\xi)(x)|\leqq|\xi_k(x)|+K\|\xi_n-\xi\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Thus, we have the extension in our case of Theorem A in [3].

Theorem 3. $\Xi_{b\gamma}$ *is coincident with the closure of* Ξ^* *in* $<\Xi, \|\ \|>$.

Next, we shall consider the similar property concerning Ξ_γ.

Let $U_\gamma(\xi_0)$ be a neighbourhood of ξ_0 in Ξ_γ, in accordance with the set (ξ; $|\xi(x)-\xi_0(x)|<\varepsilon$, for an element x of $\gamma-$bounded set S)

Proposition 4. *If elements* $\{\xi_n\}$ *sequentially converge to* ξ *by the topology, then* ξ *belongs to* Ξ_γ.

Proof. We shall be proved at first the additivity and homogeneity of $\xi(x)=\lim_{n\to\infty}\xi_n(x)$ by the above topology. Given a $\gamma-$bounded set S containing x, y and $x+y$, there exists N such that for $n>N_1$,

$$|\xi_n(x)-\xi(x)|<\varepsilon/4,\ |\xi_n(y)-\xi(y)|<\varepsilon/4, |\xi_n(x)+\xi_n(y)-\xi(x)-\xi(y)|<\varepsilon/2.$$

But, as ξ_n is linear, $\xi_n(x+y)=\xi_n(x)+\xi_n(y)$, and

$$|\xi_n(x+y)-\xi(x+y)|<\varepsilon/2, n>N_2,\ x+y\in S.$$

Thus, it will be the relation for $n >$ max $(N_1, N_2) = N$,

$$|\xi(x+y) - \xi(x) - \xi(y)| < \varepsilon.$$

Next, taken $\varepsilon > 0$ and γ-bounded set S containing x and ax so there exists N, for $n > N$

$$|\xi_n(x) - \xi(x)| < \varepsilon/2|a|,$$
$$|\xi_n(ax) - \xi(ax)| < \varepsilon/2.$$

But, $a\xi_n(x) = \xi_n(ax)$, so

$$|a\xi(x) - \xi(ax)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For a neighbourhood by γ-topology U_γ and $x \epsilon U_\gamma$, there exists N such that for any $n > N$

$$|\xi_n(x) - \xi(x)| < \varepsilon/2,$$

$\xi_n \epsilon \Xi^*$ implies $\xi_n \epsilon \Xi_\gamma$, so we take a neighbourhood U' such that $U' \subset U_\gamma$ for $x \epsilon U'$, $|\xi_n(x)| < \varepsilon/2$,

$$|\xi(x)| \leqq |\xi_n(x)| + |\xi_n(x) - \xi(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, we have $\xi(x)$ is γ-continuous.

§ 2. **Conjugate space of many norms space.** Because the topology (τ^*-topology) defined by semi-norms $\|x\|_\alpha^*$ $(\alpha \epsilon \lambda)$ in a many norms space is weaker than the norm topology (τ-topology), the set Ξ^* of τ^*-continuous linear functionals.

We can define the norm $\|\xi\|$ to each element ξ of $\Xi^* \subset \Xi$, moreover can introduce semi-norms $\|\xi\|_\alpha^*$ $(\alpha \epsilon \Lambda)$ to $x \epsilon \Xi^*$ as follow; for $\xi \epsilon \Xi^*$ let $\|\xi\|_\alpha^*$ be the infimum of K; $|\xi(x)| \leqq K\|x\|_\alpha^*$ for x such as $\|x\|_\alpha^* \neq 0$. By this definition $\|\xi\|_\alpha^*$ $(\alpha \epsilon \Lambda)$ are semi-norms; in fact for $\|x\|_\alpha^* \neq 0$,

$$|\xi_1(x) + \xi_2(x)| \leqq |\xi_1(x)| + |\xi_2(x)|, \quad |a\xi(x)| = |a|\,|\xi(x)|.$$

From these inequality or equality, $\|\xi_1 + \xi_2\| \leqq \|\xi_1\| + \|\xi_2\|$, $\|a\xi\| = |a|\,\|\xi\|$ and $\|\xi\|_\alpha^* = 0$ does not imply $\xi = 0$.

Otherwise, in § 1 of this paper, we have defined the dual many norms space. We shall prove in the next theorem each element ξ in the dual space Ξ^* of a many norms space satisfies the condition (DM).

Theorem 4. *If X is a many norms space, its conjugate space Ξ^* is a dual many norms space.*

Proof. It is clear that $\|x\|_\alpha^* \leqq \|x\|$ implies $\|\xi\|_\alpha^* \geqq \|\xi\|$. We shall prove $\|\xi\| = \inf_{\alpha \epsilon \Lambda} \|\xi\|_\alpha^*$. From the fact that $\|x\| = \sup_{\alpha \epsilon \Lambda} \|x\|_\alpha^*$, for each element x of a many norms space, there exists an $\alpha = \alpha(x)$ such that $\|x\| + \varepsilon < \|x\|_\alpha^* < \|x\|$ where $\|x\|_\alpha^* \neq 0$. So, we have the relation

$$|\xi(x)| \leqq \|\xi\|\,\|x\| < (\|x\|_\alpha^* + \varepsilon)\,\|\xi\| \tag{1}$$

In general, for each x, $\| x \|_\alpha^* \neq 0$,

$$| \xi(x) | \leqq \| \xi \|_\alpha^* \; \| x \|_\alpha^* \tag{2}$$

The supremum of the left side of (2) for all x such as $\| x \|_\alpha^*$ is constant for fixed α is equal to the right side. Then, there exists x ($\| x \|_\alpha^* \neq 0$) such that for any $\varepsilon > 0$

$$\| \xi \|_\alpha^* \| x \|_\alpha^* < | \xi(x) | + \varepsilon \tag{3}$$

We take as x in (1) x satisfies the relation (3) and $\alpha(x)$ for new x, so

$$\| \xi \|_\alpha^* \; \| x \|_\alpha^* < | \xi(x) | + \varepsilon \;\; \leqq \;\; (\| x \|_\alpha^* + \varepsilon) \| \xi \| + \varepsilon.$$

$$\| \xi \|_\alpha^* < \| \xi \| + \varepsilon \frac{\| \xi \|}{\| x \|_\alpha^*} + \frac{\varepsilon}{\| x \|_\alpha^*} \tag{4}$$

$\| x \|_\alpha^* > 0$ and ξ are constant, so $\| \xi \|$ is also constant. We take the last two terms of (4) is easily small, then it concludes $\| \xi \| = \inf_{\alpha \in A} \| \xi \|_\alpha^*$. Thus, Ξ^* is a dual many norms space.

Given a many norms space $<X, \| \;\|, \| \;\|^*>$, the space $<\Xi^*, \| \;\|^*, \| \;\|>$ will be called γ-conjugate space to $<X, \| \;\|, \| \;\|^*>$.

Example 1. Let X be the space l^2 of sequence $x = \{t_i\}$ of real numbers such that $\sum_{i=1}^{\infty} | t_i |^2 < \infty$. The topology τ is defincd by the norm $\| x \| = (\sum_{i=1}^{\infty} | t_i |^2)^{1/2}$, and the topology τ^* is defined by the semi-norms $\| x \|_n = (\sum_{i=1}^{n} | t_i |^2)^{1/2}$ ($n = 1, 2, \cdots,$) It is clear that these norm and semi-norms $\| x \|_\alpha^*$ ($\alpha \in A$) satisfy the condition (M) The linear functionals are also in l^2 and $\xi \in l^2$ is $\xi = \{s_i\}$,

$$\frac{| \xi(x) |}{\| x \|_n^*} = \frac{\Sigma t_i s_i}{(\sum_{i=1}^{n} | t_i |^2)^{1/2}} \leqq (\sum_{i=1}^{\infty} | s_i |^2)^{1/2} \left(1 + \frac{\sum_{i=n+1}^{\infty} | t_i |^2}{\sum_{i=1}^{n} | t_i |^2}\right)^{1/2}$$

Then,

$$\sup \frac{| \xi(x) |}{\| x \|_n^*} \leqq \| \xi \| \left(1 + \frac{\sum_{n+1}^{\infty} | t_i |^2}{\sum_{i=1}^{n} | t_i |^2}\right)^{1/} = \| \xi \| (1 + \varepsilon_n).$$

Thus, we have $\| \xi \| \leqq \| \xi \|_n^* \leqq \| \xi \| (1 + \varepsilon_n)$, ($\varepsilon_n \to 0$ as $n \to \infty$)

Therfore, $\inf \| \xi \|_n^* = \| \xi \|$.

Example 2. Let T be a completely regular Hausdorff space. Let X be the space $C^*(T)$ of bounded, real-valued, continuous functions $x = x(t)$ on T. Let $\{T_\alpha\}_{\alpha \in A}$ be a family of compact subsets of T such that $\bigcup_{\alpha \in A} T_\alpha = T$. The topology τ is defined by the norm $\| x \| = \sup_{t \in T} | x(t) |$, and the topology τ^* is defined by the semi-norms $\| x \|_\alpha^* = \sup_{t \in T_\alpha} | x(t) |$. It is clear $\| x \|_\alpha^* \leqq \| x \|$, and $\sup \| x \|_\alpha^* = \| x \|$. The general form of a linear functional $\xi \in \Xi$ is $\xi(x) = \int_T x(t) \, dg(t)$

where $g(t)$ is a function of bounded variation on T, and $\|\xi\| = \text{var.}_T\, g(t)$.

$$\|\xi\|_\alpha^* \|x\|_\alpha^* \geqq \int \frac{|x(t)|}{\|x\|_\alpha^*} |dg(t)|\ \|x\|_\alpha^* \geqq \int |dg(t)|\, dt\ \|x\|_\alpha^* \geqq \|\xi\| \cdot \|x\|_\alpha^*$$

$$\|\xi\|_\alpha^* \|x\|_\alpha^* = \sup \int \frac{|x(t)|}{\|x\|_\alpha^*} \cdot |dg(t)| \cdot \|x\|_\alpha^*.$$

Therefore, there exists x such that for any $\varepsilon > 0$,

$$\|\xi\|_\alpha^* \|x\|_\alpha^* - \varepsilon \leqq \int \frac{|x(t)|}{\|x\|_\alpha^*} |dg(t)|\ \|x\|_\alpha^*$$

$$\|\xi\|_\alpha^*\ \|x\|_\alpha^* - \varepsilon \leqq \int (1+\varepsilon)\, |dg(t)| \cdot \|x\|_\alpha^*$$

$$\|\xi\|_\alpha^* - \frac{\varepsilon}{\|x\|_\alpha^*} \leqq \|\xi\|(1+\varepsilon)$$

$$\|\xi\|_\alpha^* \leqq \|\xi\| + \varepsilon\|\xi\| + \frac{\varepsilon}{\|x\|_\alpha^*}$$

Thus, $\inf_{\alpha\in A}\|\xi\|_\alpha^* = \|\xi\|$.

By similar method as Theorem 4, we can prove

Corollary. *The conjugate space of a dual many norms space is a many norms space.*

Thus, $<\Xi^*, \|\ \|^*, \|\ \|>$ is a dual many norms space and the relation of two topologies is $\tau^* \geqq \tau$, so we can define the γ-topology in it.

Proposition 5. *Ξ^* is closed concerning γ-topology of a conjugate space.*

Proof. Let ξ_0 is also an accumulation point by the topology τ and set $\xi - \xi_0 \epsilon U_\gamma$ where U_γ is a neighbourhood of zero by γ-topology. For a neighbourhood U_τ of 0 by τ-topology, there exists U_γ such that $U_\gamma \subset U_\tau$. So, for any $\delta > 0$, there exists ξ such as $\|\xi - \xi_0\| < \delta$.

$$|\xi_0(x) \leqq |\xi(x) - \xi_0(x)| + |\xi(x)| \qquad (1)$$

$$|\xi(x) - \xi_0(x)| \leqq \|\xi - \xi_0\|\ \|x\| < \delta\|x\| \qquad (2)$$

By $\|x\| = \sup_{\alpha\in A}\|x\|_\alpha^*$, there exists α_0 such that $\|x\| - \delta < \|x\|_{\alpha_0}^*$. And ξ being contained in Ξ^* is τ^*-continuous. Therefore, there exists x, $\|x\|_{\alpha i} < \delta\ (i=1, 2, \cdots, n)$ such that

$$|\xi(x)| < \varepsilon/2 \qquad (3)$$

Thus, we take x such as $\|x\|_{\alpha_i}^* < \delta\ (i=0, 1, 2, \cdots, n)$, from (1), (2) and (3)

$$|\xi_0(x)| \leqq \delta(\|x\|_{\alpha_0}^* + \delta) + \varepsilon/2 = \delta(\delta+\delta) + \varepsilon/2$$

we take $\delta \leqq \dfrac{\sqrt{\varepsilon}}{2}$, so $|\xi_0(x)| < \varepsilon$.

It is easily proved ξ_0 is an additive, homogeneous functional, so $\xi_0 \epsilon \Xi^*$. Thus Ξ^* is closed in γ-topology.

This proposition is correspond with Proposition 2.1 in [7].

Corollary, *Ξ^* is closed in τ^*-topology.*

§ 3. Biconjugate space. Given a many norms spaee $<X, \| \|, \| \|^*>$, the space $<\Xi^*, \| \|^*, \| \|>$ is a dual many norms space by Theorem 4, and it will be called the γ-conjugate to $<X, \| \|, \| \|^*>$.

Let us denote by $<\mathfrak{X}, \| \|>$ and $<\mathfrak{X}^*, \| \|^*>$ the spaces conjugate to $<\Xi, \| \|>$ and $<\Xi^*, \| \|^*>$ respectively. So,

$$\|\mathfrak{x}\| = \mathrm{Sup}\{|\mathfrak{x}(\xi)|;\ \xi\epsilon\Xi,\ \|\xi\| \leqq 1\} \text{ for } \mathfrak{x}\epsilon\mathfrak{X}.$$

$$\|\mathfrak{x}\|_\alpha^* = \mathrm{Sup}\{|\mathfrak{x}(\xi)|;\ \xi\epsilon\Xi,\ \|\xi\|_\alpha^* \leqq 1, \|\xi\|_\alpha^* \neq 0\} \quad (\alpha\epsilon\Lambda) \quad \text{for } \mathfrak{x}\epsilon\mathfrak{X}^*.$$

Next, let us denote by $<\mathfrak{x}^{(\gamma)}, \| \|>$ the space conjugate to $<\Xi^*, \| \|>$; the norm is equal to

$$\|\mathfrak{x}\| = \mathrm{Sup}\{|\mathfrak{x}(\xi)|; \xi\epsilon\Xi^*\cap\Xi,\ \|\xi\| \leqq 1\}.$$

We have corresponding to Proposition 2.2 in [3],

Proposition 6. The space $<\mathfrak{X}^{(\gamma)}, \| \|>$ ***is identical with the space conjugate to*** $<\Xi_\gamma, \| \|>$ ***or*** $<\Xi_{b\gamma}, \| \|>$.

Proof. The space $<\Xi_{b\gamma}, \| \|>$ is identical with the closure of the set $<\Xi^*, \| \|>$ by Theorem 3. Moreover,

$$\Xi_{b\gamma} \supset \Xi_\gamma \supset \Xi^*$$

the closure of $<\Xi_\gamma, \| \|>$ is also identical with it of $\Xi_{b\gamma}$. Thus, we have the proposition.

$<\mathfrak{X}^{(\gamma)}, \| \|, \| \|^*>$ is γ-conjugate to $<\Xi^*, \| \|^*, \| \|>$, whence it is the second γ-conjugate to $<X, \| \|, \| \|^*>$.

Proposition 7. *$\mathfrak{X}_\gamma$ be a set of γ-linear functionals on $<\Xi^* \| \|^*, \| \|>$. The set $<\mathfrak{X}^*, \| \|^*, \| \|>$ being a dual many norms space, the set $\mathfrak{X}_\gamma$ is identical with the closure of $\mathfrak{X}^{(\gamma)}$ in $<\mathfrak{X}^*, \| \|^*>$.*

Proof. Let $\mathfrak{x}\epsilon\mathfrak{X}^{(\gamma)}$, $\mathfrak{x}-\mathfrak{x}_0 \epsilon U(\tau^*)$, that is, $\|\mathfrak{x}-\mathfrak{x}_0\|_{\alpha_i}^* < \varepsilon$ for suitable α_i ($i=1, 2, \cdots, n$)

$$|\mathfrak{x}(\xi)| \leqq |\mathfrak{x}(\xi)-\mathfrak{x}_0(\xi)| + |\mathfrak{x}(\xi)| \tag{1}$$

$\mathfrak{x}\epsilon\mathfrak{X}^{(\gamma)}\subset\mathfrak{X}^*$, $|\mathfrak{x}(\xi)| < \varepsilon/2$ for $\|\xi\|_{\alpha_i}^* < \delta$ ($i=1, 2, \cdots, n$) and $\mathfrak{x}_0$, $\delta_2 < 1/2$ there exists α_{n+1}, $|\mathfrak{x}(\xi)-\mathfrak{x}_0(\xi)| \leqq \|\mathfrak{x}-\mathfrak{x}_0\|_{\alpha_{n+1}} \|\xi\|_{\alpha_{n+1}} < \delta_2\varepsilon$,

Thus the right side of (1) is

$|\mathfrak{x}(\xi)-\mathfrak{x}_0(\xi)| + |\mathfrak{x}(\xi)| \leqq \|\mathfrak{x}\|_\alpha^* + \varepsilon/2 < \delta_2\varepsilon + \varepsilon/2 < \varepsilon$.

Proposition 8. Let *$<X, \| \|, \| \|^*>$ be a many norms space. Then, the γ-canonical mapping embeds $<X, \| \|, \| \|^*>$ into $<\mathfrak{X}^{(\gamma)}, \| \|, \| \|^*>$ with the preservation of both $\| \|$ and $\| \|^*$, that is, $\|r_x\| = \|x\|$, $\|r_x\|^* = \|x\|^*$ for $x\epsilon X, r_x \epsilon \mathfrak{X}^{(\gamma)}$.*

Proof. By Theorem 1,

$$\|r_x\| = \sup\{|\xi(x)|; \xi\epsilon\Xi^*, \|\xi\| \leqq 1\} = \|x\|.$$

and by the definition of semi-norms $\| \|_\alpha^*$ $(\alpha\epsilon\Lambda)$ in $\mathfrak{X}^*$

$$\| r_x \|_\alpha^* = \sup \{ |\xi(x)|;\ \xi \epsilon \Xi^*,\ \|\xi\|_\alpha^* \leqq 1 \} = \|x\|_\alpha^* \ (\alpha \epsilon \Lambda).$$

§4. γ-reflexive many norms space. A many norms space $<X, \| \|, \| \|^*>$ is called γ-reflexive if the γ-canonical mapping embeds $<X, \| \|, \| \|^*>$ on $(\mathfrak{X}^{(\gamma)}, \| \|, \| \|^*>$ or equivalently, if each linear functional on $<\Xi_\gamma, \| \|>$ is of the form $\mathfrak{x}(\xi)=\xi(x)$ with $x \epsilon X$.

In our case, as in [3], the space conjugate to X with the topology $\sigma(X, \Xi_\gamma)$ is equal to Ξ_γ. This is realization of Theorem 1 (Theorem B in [3]) in our case, we have also as in [3], §3.

Theorem 5. *A many norms space* $<X, \| \|, \| \|^*>$ *is γ-reflexive if* and *only if the unit sphere S is compact in the weak topology* $\sigma(X, \Xi_\gamma)$.

We have also the the theorem corresponds to Theorem 3.7 in [3].

Theorem 6. *Let* $<X, \| \|, \| \|^*>$ *be a many norms space, then the following conditions are equivalent;*

(1) $<X, \| \|, \| \|^*>$ *is γ-reflexive and* $\Xi_{b\gamma}=\Xi$,

(2) $<X, \| \|>$ *is reflexive.*

Proof. The set $\Xi_{b\gamma}$ is closed in $<\Xi, \| \|>$ by Theorem 3 and it is also total with respect to X since Ξ^* is total by Theorem 1.

By the definition of reflexivity, any closed total subset of the space Ξ (*Dixmier* [6], p. 1061). Thus, we have $\Xi_{b\gamma}=\Xi$. Let $\mathfrak{x}$ be a linear functional on $<\Xi_{b\gamma}, \| \|>$; by the reflexivity of $<X, \| \|>$ and $\Xi_{b\gamma}=\Xi$, $\mathfrak{x}$ is of the form $\mathfrak{x}(\xi)=\xi(x)$ with an $x \epsilon X$, which means that $<X, \| \|, \| \|^*>$ is γ-reflexive and that $\Xi_{b\gamma}=\Xi$. Then the space conjugate to $<\Xi, \| \|>$ and $<\Xi_{b\gamma}, \| \|>$ are identical, whence $<X, \| \|>$ must be reflexive.

Bibliography

[1] A. Alexiewicz, *On the two-norm convergence*, Studia Math. 14 (1954) p. 49-56.

[2] A. Alexiewicz and Z. Semedeni, *Linear functionals on two-norm space*, Studia Math. 17 (1958) p. 121-140.

[3] ________________, *The two-norm spaces and their conjugate space*, Studia Math. 18 (1959) p. 275-293.

[4] ________________, *Some properties of two-norm spaces and a characterisation of reflexivity of Banach spaces*, Studia Math. 19 (1960) p. 115-132.

[5] M. M. Day, *Normed linear spaces*, (1958).

[6] J. Dixmier, *Sur un théorème de Banach,* Duke Math. 15 (1948) p. 1057-1071.

[7] M. Orihara, *Theory of a Banach space with semi-norms,* Yokohama Math. J. X (1962) p. 1-4.

[8] A. Wiweger, *Linear spaces with mixed topology,* Studia Math. 20 (1961) p. 48-68.

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