SOME FIXED POINT THEOREMS IN LOCALLY CONVEX LINEAR SPACES

By

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Let E be a locally convex topological linear space over the real number field. Throughout of this paper, we denote by G an open subset, by $\bar{G}$ the closure of G and by ∂G the boundary of it.

A continuous mapping F of $\bar{G}$ in E is said to be ***completely continuous*** if there exists a compact set K such that $F(\bar{G}) \subset K$.

Several fixed point theorems have been proved for a completely continuous mapping F of $\bar{G}$ in E. Schauder's theorem [9], which was generalized by Tychonoff [10], Hukuhara [5] and others into the case of locally convex linear spaces, asserts that the mapping F has a fixed point in $\bar{G}$ if

(S) *$\bar{G}$ is convex and $F(\bar{G}) \subset \bar{G}$.*

Rothe's theorem [8], being a generalization of the Schauder's theorem, asserts that, when the space E is a Banach space, the mapping F has a fixed point in $\bar{G}$ if

(R) *$\bar{G}$ is a closed ball and $F(\partial G) \subset \bar{G}$.*

It is easy to see that this theorem is valid even in the case of a completely continuous mapping F of a convex closed set $\bar{G}$ contained in a locally convex linear space E. Altman [1] proved a more general theorem by replacing the condition (R) by

(A) $\| F(x) - x \|^2 \geqq \| F(x) \|^2 - \| x \|^2$ *for every* $x \in \partial G$.

Therefore, in the case of the completely continuous mapping F of the closed ball $\bar{G}$ in a Banach space E, the Altman's theorem may be regarded as the most general form of the fixed point theorems. Altman proved this theorem by making use of some properties of the mapping degree, but the properties seem not to be used out. If we use up these properties, we get more general results which moreover contain the case that 0 is not necessarily an inner point so as to be a generalization of theorems of Schauder and Rothe:

Theorem 1. ***Let F be a completely continuous mapping of $\bar{G}$ in E. If there exists an element $a \in G$ such that***

(I. 1) ***if $F(x) = \alpha x + (1-\alpha) a$ for some $x \in \partial G$ then $\alpha \leqq 1$,***

then the mapping F has a fixed point in $\bar{G}$.

The proof of this theorem will be given in § 2. This is more general than the Rothe's theorem, because, when G is convex, the condition (I. 1) follows from the condition (R) for any inner point $a \in G$.

As a special ease of Theorem 1, we get

Corollary 1. *Let F be a completely continuous mapping of $\bar{G}$ in E and suppose that $0 \in G$. Then the mapping F has a fixed point in $\bar{G}$ if the following condition is satisfied*:

(I. 2) *If $F(x)=\alpha x$ for some $x \in \partial G$ then $\alpha \leqq 1$.*

It is this corollary that generalizes the theorem of Altman in which the assumption that $0 \in G$ is indispensable. It is easy to see that the condition (A) implies the condition (I. 2).

As an application of this corollary we will prove in § 3 the following theorem which is a generalization of the Birkhoff-Kellogg theorem [2]:

Theorem 2. *Let us assume that $0 \in G$ and ∂G is a retract of $\bar{G}$. If, for a completely continuous mapping F of $\bar{G}$ in E, there exists a number λ such that*

(II) $$\lambda F(\partial G) \cap \bar{G}=\phi,$$

then the mapping F has at least one proper value.

By Dugundji [4] it was proved that, in case the E was a infinite-dimensional Banach space, the surface of the unit ball was a retract of the closed unit ball. The reason why this theorem is a generalization of the Birkhoff-Kellogg theorem will be explained in § 3.

In § 4, we prove the following theorem which may be regarded as complementary with Theorem 1.

Theorem 3. *Let F be a completely continuous mapping of $\bar{G}$ in E. When the space E is finite-dimensional and G is bounded the mapping F has a fixed point in $\bar{G}$ if the following condition is satisfied*:

(III. 1) *There exists an element $a \in G$ such that if $F(x)=\alpha x+(1-\alpha)a$ for some $x \in \partial G$ then $\alpha \geqq 1$.*

When thc space E is infinnite-dimensional and G is finitely bounded, the mapping F has a fixed point in $\bar{G}$ if the following condition is satisfied:

(III. 2) *There exist an element $a \in G$ and a neighbourhood U of 0 such that if $(F(x)-(\alpha x+(1-\alpha) a) \in U$ for some $x \in \partial G$ then $\alpha \geqq 1$.*

As a special case of this theorem, we get

Corollary 2. *Let F be a completely continuous mapping of $\bar{G}$ in E and suppose that $0 \in G$. When the space E is finite-dimensional and G is bounded the mapping F has a fixed point in $\bar{G}$ if the following condition is satisfied*:

(III. 3) *If $F(x)=\alpha x$ for some $x \in \partial G$ then $\alpha \geqq 1$.*

When the space E is infinite-dimensional and G is finitely bounded the mapping F has a fixed point in $\bar{G}$ if the following condition is satisfied:

(III. 4) *There exists a neighbourhood* U *of* 0 *such that, if*

$F(x)-\alpha x \in U$ *for some* $x \in \partial G$ *then* $\alpha \geqq 1$.

As an application of Theorem 1 and Theorem 3, we prove in §5 the following theorem.

Theorem 4. *Let* E *be, moreover, a vector lattice and the linear topology of* E *be compatible with the lattice structure. Let* $\mathfrak{F}$ *be a family of completely continuous mappings* F *of* $\bar{G}$ *in* E *such that* $F(\bar{G}) \subset K (F \in \mathfrak{F})$ *for the same compact set* K.

(1) *If*

$$F(x) \geqq x \text{ for any } x \in \bar{G} \text{ and any } F \in \mathfrak{F},$$

and, for any finite number of $F_i \in \mathfrak{F}$ $(i=1,2,\cdots,n)$, *the mapping*

$$F(x)=F_1(x) \cup F_2(x) \cup \cdots \cup F_n(x)$$

of $\bar{G}$ *in* E *satisfies one of the conditions* (I. 1.), (III. 1) (*in case* E *is finite dimensional and* G *is bounded and* (III. 2) (*in case* E *is infinite-dimensional and* G *is finitely bounded*), *then the family* $\mathfrak{F}$ *has a common fixed point in* $\bar{G}$.

(2) *If*

$$F(x) \leqq x \text{ for any } x \in G \text{ and } F \in \mathfrak{F},$$

and, for any finite number of $F_i \in \mathfrak{F}$ $(i=1,2,\cdots,n)$ *the mapping*

$$F(x)=F_1(x) \cap F_2(x) \cap \cdots \cap F_n(x)$$

of $\bar{G}$ *in* E *satisfies one of the conditions stated above, then the family* $\mathfrak{F}$ *has a common fixed point in* $\bar{G}$.

Before proceeding to the proofs of these theorems, we give in §1 some remarks on the notion of the mapping degree.

§1. The Mapping Degree.

This notion, originally due to Brouwer [3], was defined by Leray [6] and Nagumo [7] for the completely continuous movements acting on locally convex linear spaces.

Let the mapping F be completely continuous on $\bar{G}$. The mapping

$$f(x)=x-F(x)$$

is called *a completely continuous movement on* $\bar{G}$. Let us assume that $a \notin f(\partial G)$, then *the mapping degree* $d(a,G,f)$ of G at a by the completely continuous movement f is defined and has the following properties:

(D. 1) *When* $f(x) \equiv x$ on $\bar{G}$, then $d(a,G,f)=1$ if $a \in G$ and $d(a,g,f)=0$ *if* $a \notin \bar{G}$.

(D. 2) *If* $d(a,G,f) \neq 0$, *then* $f(x)=a$ *for some* $x \in \bar{G}$.

(D. 3) *Let* $H_t(x)$ *be a mapping of* $\bar{G} \times [0,1]$ *into* E *such that it is continuous with respect to* $(x,t) \in \bar{G} \times [0,1]$ *and the range is contained in a compact set. Then, for the mapping* $h_t(x)=x-H_t(x)$, *the mapping degree* $d(a,G,h_t)$ *is constant if* $a \notin h_t(\partial G)$ *for any* $t \in [0,1]$.

In this paper, we need only these three properties. For the details and the other properties, we refer the reader to Nagumo [7].

§ 2. Proof of Theorem 1.

Let F be a completely continuous mapping of $\bar{G}$ in E. Suppose that there exists an element $a \in G$ such that the condition (I. 1) is satisfied. Put

$$G_a = G - a = \{x - a : x \in G\},$$

then the set G_a is open, $\bar{G}_a = \bar{G} - a$ and $\partial G_a = \partial G - a$. Define the mapping $F_a(y)$ for $y \in \bar{G}_a$ by

$$F_a(y) = F(x) - a \qquad \text{for } y = x - a,$$

then the mapping F_a is obviously continuous on $\bar{G}_a$ and $F_a(\bar{G}_a) \subset K - a$ where the set $K - a$ is compact. Consider the compact deformation:

$$h_t(y) = y - tF_a(y) \text{ for } y \in \bar{G}_a \text{ and } t \in [0, 1].$$

Since $h_0(y) \equiv y$ and 0 is an inner point of G_a, we have $d(a, G, h_0) = 1$. Therefore, if $0 \notin h_t(\partial G_a)$, we get $d(0, G_a, h_1) = 1$ from which it follows that there is an element $y \in \bar{G}_a$ such that $F_a(y) = y$.

Now, suppose that

$$h_{t_0}(y_0) = 0, \text{ namely, } y_0 = t_0 F_a(y_0)$$

for some $y_0 \in \partial G_a$ and $t_0 \in [0, 1]$. Since $y_0 \in \partial G_a$ and 0 is an inner point of G_a, the number t_0 is not zero. Putting $\alpha = 1/t_0$, we get

$$F_a(y_0) = \alpha y_0, \text{ namely, } f(x_0) - a = \alpha(x_0 - a)$$

where $y_0 = x_0 - a$. From this it follows that $F(x_0) = \alpha x_0 + (1 - \alpha) a$. The condition (I. 1) implies that $\alpha = 1$, namely, $t_0 = 1$. Therefore, in any case, there exists an element y in $\bar{G}_a$ such that $F_a(y) = y$, from which it follows that $F(x) = x$ for $x = y + a$.

§ 3. Proof of Theorem 2.

Let $r(x)$ be the retraction, namely, $r(x)$ is a continuous mapping of $\bar{G}$ into ∂G and $r(x) = x$ for any $x \in \partial G$. From the condition (II), it follows that

$$\lambda F_0(\bar{G}) \cap \bar{G} = \phi \qquad \text{where } F_0(x) = F(r(x)).$$

The mapping λF_0 is a completely continuous mapping of $\bar{G}$ in E, and has no fixed points in $\bar{G}$. Therefore, by Corollary 1, we see that there exists a number $\alpha > 1$ and an element $x_0 \in \partial G$ such that

$$\lambda F_0(x_0) = \alpha x_0.$$

Since $\lambda \neq 0$, we have $F_0(x_0) = \frac{\alpha}{\lambda} x_0$, from which it follows that

$$F(x_0) = \frac{\alpha}{\lambda} x_0,$$

because $F_0(x_0) = F(r(x_0)) = F(x_0)$.

Remark. Let E be a Banach space and $\bar{G}$ be the closed unit ball. Suppose that

$$inf\{\|F(x)\| : x\in\partial G\}=\mu>0.$$

Then, for a number λ such that $\lambda>\frac{1}{\mu}$, we have the condition (II). Therefore, combining with a result of Dugundji [4], we see. by Theorem 2, that the mapping F has a proper value. This is the Birkhoff-Kellogg theorem in the case of Banach spaces.

§ 4. Proof of Theorem 3.

Let us consider in the first place the case when E is finite-dimensional. As in the proof of Theorem 1, we consider the set G_a and the mapping F_a. Let us consider the continuous deformation:

$$h_t(y)=(2t-1)y-tF_a(y) \quad \text{for} \quad y\in\bar{G}_a \text{ and } t\in[0,1].$$

Since the space E is assumed to be finite-dimensional, we can consider the mapping degree $d(0, G_a, h_t)$ if $0\notin h_t(\partial G_a)$.

At first, since $0\in G_a$, we see easily that $0\notin h_0(\partial G_a)$ and $d(0, G_a, h_0)\neq 0$. Next, suppose that $0\in h_{t_0}(\partial G_a)$, namely,

$$(2t_0-1)y_0=t_0F_a(y_0) \text{ for } y_0\in\partial G_a \text{ and } t_0\in[0,1].$$

Since $y_0\neq 0$, t_0 is not zero. Therefore, putting $\alpha=2-\frac{1}{t_0}$, we get

$$F_a(y_0)=\alpha y_0 \text{ and } \alpha\leqq 1.$$

By the definitions of F_a and y_0, we get, for $x_0=y_0+a$,

$$F(x_0)=\alpha x_0+(1-\alpha)a.$$

Therefore, by the condition (III. 1), we have $\alpha=1$, namely, $F(x_0)=x_0$.

Finally, suppose that $0\notin h_t(\partial G_a)$ for any $t\in[0,1]$. Then, we have $d(0, G_a, h_1)\neq 0$ and $h_1(y)=y-F_a(y)$. Therefore, there exists an element $y_0\in\bar{G}_a$ such that $F_a(y_0)=y_0$. Taking $x_0=y_0+a$, we get $F(x_0)=x_0$.

Thus, in any case, the mapping F has a fixed point in $\bar{G}$.

Next, let us consider the case when E is infinite-dimensional. Let $\mathfrak{U}$ be the totality of all neighbourhoods of zero. We start by the following

Lemma. *If, for any $V\in\mathfrak{U}$, there exists an $x_V\in\bar{G}$ such that*

$$x_V-F(x_V)\in V,$$

then there exists at least one $x\in\bar{G}$ such that $F(x)=x$.

Proof. Putting $y_V=F(x_V)$, let us consider the sets:

$$A_U=\{y_V : V\subset U, V\in\mathfrak{U}\} \quad \text{for every } U\in\mathfrak{U}.$$

Since the sets $A_U\,(U\in\mathfrak{U})$ are contained in the compact set K, there exists an element $y_0\in E$ such that

$$y_0 \in \cap \{\bar{A}_U : U \in \mathfrak{U}\}.$$

At first we prove that $y_0 \in \bar{G}$. Take an arbitrary $U_0 \in \mathfrak{U}$ and take a $U_1 \in \mathfrak{U}$ such that $U_1 + U_1 \subset U_0$. Since $y_0 \in \bar{A}_{U_1}$, there is a $U_2 \subset U_1$ such that

$$y_{U_2} - y_0 \in U_1.$$

On the other hand, $x_{U_2} - y_{U_2} \in U_2$. Therefore, we have

$$x_{U_2} - y_0 = (x_{U_2} - y_{U_2}) + (y_{U_2} - y_0) \in U_2 + U_1 \subset U_1 + U_1 \subset U_0,$$

namely, for any $U_0 \in \mathfrak{U}$, there exists an $x \in \bar{G}$ such that

$$x \in y_0 + U_0,$$

which means that $y_0 \in \bar{\bar{G}} = \bar{G}$.

Now, since F is continuous at y_0, for any $U_0 \in \mathfrak{U}$, there exist a $U_1 \in \mathfrak{U}$ and a $U_2 \subset U_1$ such that

$$U_1 + U_1 \subset U_0 \text{ and } F(x) - F(y_0) \in U_1 \text{ if } x - y_0 \in U_2.$$

Take a $U_3 \in \mathfrak{U}$ such that $U_3 + U_3 \subset U_2$, then, since $y_0 \in \bar{A}_{U_3}$, there is a $U_4 \in \mathfrak{U}$ such that

$$U_4 \subset U_3 \text{ and } y_{U_4} - y_0 \in U_3,$$

hence it follows that

$$F(y_{U_4}) - F(y_0) \in U_1.$$

Therefore, we have

$$y_0 - F(y_0) = (y_0 - y_{U_4}) + (y_{U_4} - F(y_{U_4})) + (Fy_{U_4}) - F(y_0))$$
$$\in U_3 + U_4 + U_1 \subset U_2 + U_1 \subset U_1 + U_1 \subset U_0.$$

Since $U_0 \in \mathfrak{U}$ is taken arbitrarily, $y_0 = F(y_0)$.

Now, let us return to the proof of Theorem 3. Take the $U \in \mathfrak{U}$ and the element $a \in G$ in the condition (III. 2). Take a $V \in \mathfrak{U}$ such that $V \subset U$. Then, there exist a finite-dimensional linear subspace E_V of E and a continuous mapping S_V of K into E_V such that

$$S_V(x) - x \in V \quad \text{for every} \quad x \in K.$$

Considering the relative topology on E_V, the set $G_V = G \cap E_V$ is an open set which can be assumed to contain the element a and $S_V[F(x)]$ is a bounded continuous mapping of $\bar{G}_V$ in E_V. We will prove that the condition (III. 1) is satisfied by the mapping $S_V[F(x)]$. (The set G_V is bounded, because G is finitely bounded.)

Suppose that

$$S_V[F(x_0)] = \alpha x_0 + (1-\alpha) a \text{ for some } x_0 \in \partial G,$$

then, by the above mentioned property of S_V, we have

$$F(x_0) - (\alpha x_0 + (1-\alpha) a) = F(x_0) - S_V[F(x_0)] \in V \subset U,$$

and $x_0 \in \partial G_V \subset \partial G$. Therefore, by the condition (III. 2), we have $\alpha \geqq 1$. Hence, by making use of the upper half of Theorem 3, it follows the existence of a fixed point x_V of $S_V[F(x)]$ in $\bar{G}_V$:

$$S_V\,[F(x_V)] = x_V \text{ and } x_V \in \bar{G}_V \subset \bar{G}.$$

Therefore, by the lemma proved above, we get the existence of an $x_0 \in \bar{G}$ such that $F(x_0) = x_0$

Remark. From Corollary 1 and Corollary 2, we can conclude that, in the case of finite-dimensional E, if the mapping F has no fixed point in a bounded $\bar{G}$, then F has at least two proper values, one of which is greater than 1 and the other is smaller than 1. In the infinite-dimensional case, if we say that the number α is a U-almost proper value with respect to ∂G when $F(x) - \alpha x \in U$ for a point $x \in \partial G$, we can conclude that, if F has no fixed points in a finitely bounded $\bar{G}$, then F has at least one proper value which is greater than 1 and at least one U-almost proper value which is smaller than 1. In both cases, the corresponding proper vectors are on ∂G.

§ 5. **Proof of Theorem 4.**

At first we give some remarks on the lattice properties of E.

1. Since E is assumed to be a vector lattice, for any pair of elements x and y, there are defined the ***join*** $x \cup y$ and thc ***meet*** $x \cap y$, namely,

$$x \cup y \geqq x, y \qquad (x \cap y \leqq x, y)$$

and

$$z \geqq x, y\ (z \leqq x, y) \textit{ implies } z \geqq x \cup y\ (z \leqq x \cap y).$$

2. Since the topology of E is assumed to be compatible with the lattice stucture, the mapping

$$(x, y) \longrightarrow x \cup y, \qquad (x, y) \longrightarrow x \cap y$$

are continuous as the mappings of $E \times E$ into E. This implies that the sets $\{x \cup y : x, y \in K\}$ and $\{x \cap y : x, y \in K\}$ are compact, and hence it follows that, for completely continuous mappings $F_1(x)$ and $F_2(x)$ of $\bar{G}$ in E such that $F_i(\bar{G}) \subset K\ (i = 1, 2)$ for the same compact set K, the mappings $F(x) = F_1(x) \cup F_2(x)$ and $F(x) = F_1(x) \cap F_2(x)$ are completely continuous on $\bar{G}$.

Now, let us proceed to the proof of thc first half of the theorem. At first, we prove that every subfamily of $\mathfrak{F}$ has a common fixed point. Take $F_i \in \mathfrak{F}\ (i = 1, 2, \cdots, n)$ and consider the completely continuous mapping

$$F(x) = F_1(x) \cup F_2(x) \cup \cdots \cup F_n(x)$$

of $\bar{G}$ in E. Then, by the assumption, there exists at least one $x_0 \in \bar{G}$ such that $F(x_0) = x_0$. Hence it follows that

$$\begin{aligned} 0 = x_0 - F(x_0) &= x_0 - (F_1(x_0) \cup F_2(x_0) \cup \cdots \cup F_n(x_0)) \\ &= (x_0 - F_1(x_0)) \cap (x_0 - F_2(x_0)) \cap \cdots \cap (x_0 - F_n(x_0)). \end{aligned}$$

Since $x - F(x) \leqq 0$ for any $x \in \bar{G}$ and any $F \in \mathfrak{F}$, we have

$$F_i(x_0) = x_0 \quad (i = 1, 2, \cdots n).$$

Thus the intersection of finite number of closed sets

$$A_F = \{x \in \bar{G} \cap K : F(x) = x\} \qquad (F \in \mathfrak{F})$$

is always not empty. Since K is compact, we get that the set

$$\cap \{A_F : F \in \mathfrak{F}\}$$

is not empty. It is clear that each element in this set is a common fixed point of the family $\mathfrak{F}$.

The second half of this theorem can be proved similarly.

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