ON BEAURLING-LIVINGSTONE'S THEORY ON THE BANACH SPACE WITH DUALITY MAPPING

By

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As a generalizaion of the function space $L_p (p>1)$, Beaurling and Livingstone [1] introduced the notion of *the Banach space with duality mapping.* The purpose of this paper is to prove the following theorem:

Theorem. Let E be a conditionally σ-complete Banach lattice. If there is defined on E a duality mapping T with the property:

(*) $\quad x \cap y = 0$ *implies* $T(x+y) = Tx + Ty$,

then the space E has the property P of Bohnenblust [3]. *Therefore, if the dimension of E is not smaller than* 3, *the space is the abstract L_p-space.*

In § 1, the definition of the Banach lattice with duality mapping will be given. In § 2, the facts which will be used in the proof in connection with the lattice property are stated. In § 3, we prove our theorem, and in § 4 the case of the Orliez space will be discussed.

§ 1. Let E be a Banach space and E^* be its conjugate space. We put

$$S_r = \{x \epsilon E : \| x \| = r\} \quad \text{and} \quad S_r^* = \{x \epsilon E^* : \| x^* \| = r\}.$$

Two elements $x \epsilon S_1$ and $x^* \epsilon S_1^*$ are said to be mutually conjugate when $(x^*, x) = 1$, where (x^*, x) denotes the value of the linear functional x^* at x.

Throughout of this paper, we assume that, for any $x \epsilon S_1$ $(x^* \epsilon S_1^*)$ there is a uniquely defined conjugate element.

Now, the mapping T of E onto E^* was called by Beaurling and Livingstone the *duality mapping* if

1. T is one-to-one;

2. for any pair of mutually conjugate elements x and x^*, the mapping T maps the set $\{\lambda x : 0 \leqq \lambda < \infty\}$ onto the set $\{\mu x^* : 0 \leqq \mu < \infty\}$.

3. for any positive number r there is a positive number ρ such that $T(S_r) \subset S_\rho^*$ and $\rho_1 < \rho_2$ whenever $r_1 < r_2$. Beaurling-Livingstone [1] showed that, if T is the duality mapping, then, for muttally conjugate elements x and x^*, we have

(#) $$T(\lambda x) = \phi(\lambda) x^* \qquad (\lambda \geqq 0),$$

where $\phi(\lambda)$ is a strictly increasing function of λ such that $\phi(1) = 1$ and is depending only on T (not depending on the choice of x and x^*).

In the proof of our theorem, we especially make use of the fact that T is one-to-one and the equality (#).

§2. A ***Banach lattice*** E is a vector lattice on which is defined a complete norm $\|x\|$ $(x \in E)$ such that $|x| \leqq |y|$ implies $\|x\| \leqq \|y\|$. ([2], p. 246) It is said to be ***conditionally σ-complete*** if every non-void enumerable subset which has an upper bound has the least upper bound.

If E is a conditionally σ-complete vector lattice, we can define the ***projection operator*** P_x by

$$P_x y = \bigcup_{n=1}^{\infty} (y \cap n|x|) \qquad \text{for every positive } y \in E,$$

and

$$P_x y = P_x y^+ - P_x y^- \qquad \text{for arbitrary } y \in E.$$

The projection operator P_x is linear, and, in case E is a Banach lattice, we have $\|P_x\| = 1$, hence it follows that the linear functional $x^* P_x(y) = x^*(P_x y)$ is in E^*.

We will also use the fact that $x \cap y = 0$ implies $P_x y = 0$.

The notion of the conditionally σ-complete Banach lattice with the property P was introduced by Bohnenblust [3]. The space E is said to have the property P if, for any pair x and y such that

$$x = x_1 + x_2,\ x_1 \cap x_2 = 0;\ y = y_1 + y_2,\ y_1 \cap y_2 = 0;\ \|x_i\| = \|y_i\|\ (i = 1, 2),$$

we have $\|x\| = \|y\|$. Bohnenblust [3] showed that, if the dimension of a conditionally σ-complete Banach lattice with the property P is not smaller than 3, then there exists a number $p\,(1 \leqq p \leqq \infty)$ such that for any mutually orthogonal x_i $(i = 1, 2, \cdots, n)$

$$\left\| \sum_{i=1}^{n} x_i \right\| = \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}.$$

§3. We prove our theorem. It is assumed that, for any $x \in S_1$ $(x^* \in S_1^*)$ there is one and only one $x^* \in S_1^*$ $(x \in S_1)$ such that x and x^* are mutually conjugate.

We define a functional $N(x)$ on E by

$$N(x) = (Tx, x) \qquad (x \in E).$$

Then, we have

Lemma 1. $\|x\| = \|y\|$ ***if and only if*** $N(x) = N(y)$.

Proof. Assume that $\|x\| = \|y\|$, then we have by (#) that

$$N(x) = \left(T\left(\|x\| \frac{x}{\|x\|}\right), x\right) = \|x\|\,\phi(\|x\|)\left(\left(\frac{x}{\|x\|}\right)^*, \frac{x}{\|x\|}\right)$$

$$= \|x\|\,\phi(\|x\|) = \|y\|\,\phi(\|y\|) = N(y).$$

Conversely, if $N(x) = N(y)$, we have $\|x\|\,\phi(\|x\|) = \|y\|\,\phi(\|y\|)$, and, since $\phi(\lambda)$ is strictly increasing, we get $\|x\| = \|y\|$.

Lemma 2. $(Tx, y) = 0$ ***if*** $x \cap y = 0$.

Proof. We can assume that x and y are in S_1, I_n. In fact, when this lemma could be proved for $x \in S_1$ ane $y \in S_1$, then, for any $x \in E$ and $y \in E$ such that $x \cap y=0$, we get

$$(x/\|x\|) \cap (y/\|y\|)=0$$

and hence it follows that $(T(x/\|x\|), y/\|y\|)=0$, which implies that

$$(Tx, y)=\phi(\|x\|)(T(x/\|x\|), y)=0.$$

Now, let us suppose that x and y are in S_1. Then, we have

$$1=(x^*, x)=(x^*, P_x x)=(x^*P_x, x) \leqq \|x^*P_x\| \leqq \|x^*\|=1.$$

Therefore, $x^*P_x \in S_1$, and x and x^*P_x are mutually conjugate. As we assumed that the pair of mutually conjugate elements is determined uniquely, we have $x^*=x^*P_x$. Therefore, we have, if $x \cap y=0$, then

$$(x^*, y)=(x^*P_x, y)=(x^*, P_x y)=0.$$

Proof of our theorem.

Assume that

$$x=x_1+x_2,\ x_1 \cap x_2=0;\ y=y_1+y_2,\ y_1 \cap y_2=0;\ \|x_i\|=\|y_i\|\ (i=1,2),$$

then, by Lemma 2, and the relation (*), we have

$$N(x)=N(x_1+x_2)=(T(x_1+x_2), x_1+x_2)=(Tx_1+Tx_2, x_1+x_2)$$
$$=(Tx_1, x_1)+(Tx_2, x_2)=N(x_1)+N(x_2).$$

Similarly, we have $N(y)=N(y_1)+N(y_2)$.

By Lemma 1, since $\|x_i\|=\|y_i\|$ $(i=1,2)$, we have $N(x_i)=N(y_i)$ $(i=1,2)$, and so, $N(x)=N(y)$. Therefore, again by Lemma 1, we have $\|x\|=\|y\|$.

§4. In this section, we give a remark on the duality mapping acting on the Orlicz space L_Φ. Let $\varphi(u)$ be a continuous, strictly increasing function of $u \geqq 0$ such that $\varphi(0)=0$, and $\psi(u)$ be the inverse function of $\varphi(u)$. Define the function $\Phi(u)$ and $\Psi(u)$ by

$$\Phi(u)=\int_0^u \varphi(t)\,dt \qquad \text{and} \qquad \Psi(u)=\int_0^u \psi(t)\,dt,$$

then the set of all measurable functions $x(t)$ on $[a, b]$ such that

$$\int_a^b \Phi(|x(t)|)\,dt<+\infty$$

is the Orlicz space L_Φ which is a conditionally σ-complete Banach lattice when we define the order relation $x \geqq y$ in L_Φ in the sense that $x(t) \geqq y(t)$ almost everywhere in $[a, b]$. (See Orlicz [4]) The conjngate space L_Ψ of all measurable functions $x^*(t)$ on $[a, b]$ such that

$$\int_a^b \Psi(|x^*(t)|)\,dt<+\infty.$$

In the Orlicz space of this kind, the pair of mutually conjugate elements is determined uniquely, because the functions $\varphi(u$ and $\psi(u)$ are strictly increasing.

As was stated in the paper [1], the notion of duality mapping had been introduced as a generalization of the mapping:

$$Tx(t)=|x(t)|^{\frac{p}{q}}\, sgn\cdot x(t)$$

of L_p onto L_q where $q=p/(p-1)$. The natural generalization of this mapping into the case of the Orlicz space is the following:

$$Tx(t)=\varphi(|x(t)|)\; sgn\cdot x(t),$$

which is a one-to-one mapping of L_{Φ} onto L_{Ψ}. It is easy to see that this mapping T satisfies the condition (#), and hence we can conclude that, if the mapping T is the duality mapping of the Orlicz space onto its conjugate space, then the space must be a L_p-space for some $p>1$.

References.

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