ON A QUEUING PROCESS WITH QUEUE-LENGTH DEPENDENT SERVICE.

By

TAKEHISA FUJISAWA

§ 1. Introduction

In the queuing problems the investigation of "the ergodicity condition" makes us theoretical interesting. In other words, let $P_n(t)$ be the probability of there being, at time t, n customers in the system, then it is to investigate the conditions in order that $\lim_{t\to\infty} P_n(t)=p_n(p_n>0)$ exist independent of the queue-length at $t=0$. Up to this time, under the condition "first-come, first-served" the ergodicity condition has been investigated by generalizing service-time distribution, input distribution, and increasing in number of counters. In these researches there is a great difference in the point of difficulty between a single server and many servers. In the case of single server D. V. Lindley [1] gave a considerable resolution and later did Kiefer-Wolfowitz in the case of many servers.

The condition is given by

$$\frac{\lambda}{c\mu}<1$$

where μ is the mean service rate, λ the mean arrival rate and c is the number of servers.

The investigation mentioned above is the case of ordinary queue-discipline. But recently the investigation on *O. R.* and on various types of queue has come to be made earnestly. The queue-discipline from practical fields is rich in variety and investigating ergodicity in these cases is apt to be accompanied with much difficulty.

In fact, the solutions of this problem are hardly given in many papers. That is to say, in them, input-distribution and service-time distribution are limited as Poissonian, in exponential type respectively. Under these limitations differential difference equations are generally used, but in this paper we shall, under the same assumptions, treat the case where service time depends strongly on the queue-length.

This paper is concerned with the queuing system in which

1. there are $c(\geqq 1)$ counters,
2. the customers arrive "at random", *i. e.* the inter-arrival times have the negative exponential distribution $\lambda e^{-\lambda t}$,
3. arriving at the counter, the customers can not leave the queuing system without receiving service,
4. the mean service rate depends on the number of customers in the system by the following rule

$$\mu_n=\begin{cases}\mu_1 & \text{if} \quad 0<n\leqq N_0\\ \mu_2 & \text{if} \quad N_0+1\leqq n\end{cases}$$

   where N_0 is a positive constant number and $N_0>c$,
5. the queue-discipline is "first-come, first-served",
6. the service-time in each counter has the identical negative exponential distribution, and
7. infinite queue is allowed outside of counter.

In this paper we shall find the mean queue-length, the mean waiting time and ergodicity condition.

Let $P_n(t)$ denote the probability of there being, at the instant t, n customers in the system. When $n>c$, the fact of there being n customers in the system implies that c persons are actually being served and $n-c$ are waiting in the queue. Using normal methods it can be shown that the following set of equations characterize the system:

For $n=0$

$$\frac{dP_0(t)}{dt}=-\lambda P_0(t)+\mu_1 P_1(t). \tag{1}$$

For $0<n<c$

$$\frac{dP_n(t)}{dt}=-(\lambda+n\mu_1)P_n(t)+\lambda P_{n-1}(t)+(n+1)\mu_1 P_{n+1}(t) \tag{2}$$

For $c\leqq n<N_0$

$$\frac{dP_n(t)}{dt}=-(\lambda+c\mu)P_n(t)+\lambda P_{n-1}(t)+c\mu_1 P_{n+1}(t) \tag{3}$$

For $n=N_0$

$$\frac{dP_{N_0}(t)}{dt}=-(\lambda+c\mu_1)P_{N_0}(t)+\lambda P_{N_0-1}(t)+c\mu_2 P_{N_0+1}(t) \tag{4}$$

For $N_0<n$

(5) $$\frac{dP_n(t)}{dt}=-(\lambda+c\mu_2)P_n(t)+\lambda p_{n-1}(t)+c\mu_2 P_{n+1}(t).$$

We assume that $P_n(0)=\delta_{na}$ where δ_{na} is the Kronecker's delta. The steady state solutions are obtained by putting the left members of these equations equal to zero. Now we shall first find the solutions in the steady state.

§ 2. The steady state solutions

Assuming that $\lim_{t\to\infty} P_n(t)=p_n$ exists, we can then rewrite the equations (1)~(5) as follows:

For $n=0$

(6) $$-\lambda p_0+\mu_1 p_1=0$$

For $0<n<c$

(7) $$-(\lambda+n\mu_1)p_n+\lambda p_{n-1}+(n+1)\mu_1 p_{n+1}=0,$$

For $c\leqq n<N_0$

(8) $$-(\lambda+c\mu_1)p_n+\lambda p_{n-1}+(n+1)\mu_1 p_{n+1}=0,$$

For $n=N_0$

(9) $$-(\lambda+c\mu_1)p_{N_0}+\lambda p_{N_0-1}+c\mu_2 p_{N_0+1}=0,$$

For $n>N_0$

(10) $$-(\lambda+c\mu_2)p_n+\lambda p_{n-1}+c\mu_2 p_{n+1}=0.$$

From these equations the steady state solution are found to be:

(11) $$p_n=\frac{\rho_1{}^n}{n!}p_0 \qquad (0<n<c),$$

(12) $$p_n=\frac{\rho_1{}^n}{c!}\left(\frac{1}{c}\right)^{n-c}p_0 \qquad (c\leqq n\leqq N_0),$$

(13) $$p_n=\frac{1}{c!}\rho_1{}^{N_0}\rho_2{}^{n-N_0}\left(\frac{1}{c}\right)^{n-c}p_0 \qquad (N_0\leqq n)$$

where $\rho_i=\dfrac{\lambda}{\mu_i}$ $(i=1, 2)$.

The value of p_0 is determined by normalizing process which requires $\sum_{n=0}^{\infty}p_n=1$, assuming that $\rho_2<c$ (this proof is given later), and emerges as:

(14) $$p_0=\left(\sum_{n=0}^{c}\frac{\rho_1{}^n}{c!}+\frac{1}{c!}\sum_{n=c+1}^{N_0}\rho_1{}^n\left(\frac{1}{c}\right)^{n-c}+\frac{\rho_1{}^{N_0}}{c!}\left(\frac{1}{c}\right)^{N_0-c}\frac{\rho_2}{c-\rho_2}\right)^{-1}.$$

Average number of customers being served, N, is given by

$$N=\sum_{n=0}^{c} np_n=p_0\rho_1\left\{1+\rho_1+\frac{\rho_1^2}{2!}+\cdots\cdots+\frac{\rho_1^{c-1}}{(c-1)!}\right\}. \tag{15}$$

Average number of customers in the system, L, is given by

$$L=\sum_{n=0}^{\infty} np_n \tag{16}$$

$$=p_0\left\{\sum_{n=1}^{c}\frac{\rho_1^n}{(n-1)!}+\frac{c^c}{n!}\sum_{n=c+1}^{N_0} n\left(\frac{\rho_1}{c}\right)^n+\frac{c^c}{c!}\left(\frac{\mu_2}{\mu_1}\right)^{N_0}\sum_{n=N_0+1}^{\infty} n\left(\frac{\rho_2}{c}\right)^n\right\}.$$

Mean number of customers waiting for service, L_q, is given by

$$L_q=\sum_{n=c}^{\infty}(n-c)p_n \tag{17}$$

$$=p_0\left\{c\sum_{n=0}^{c}\frac{\rho_1^n}{n!}+\frac{c^c}{c!}\sum_{n=c+1}^{N_0} n\left(\frac{\rho_1}{c}\right)^n+\frac{c^c}{c!}\left(\frac{\mu_2}{\mu_1}\right)^{N_0}\sum_{n=N_0+1}^{\infty} n\left(\frac{\rho_2}{c}\right)^n\right\}-c.$$

Mean waiting time of customers is seen that

$$W=\frac{L_q}{\lambda} \tag{18}$$

$$=\frac{p_0}{\lambda}\left\{c\sum_{n=0}^{c}\frac{\rho_1^n}{n!}+\frac{c^c}{c!}\sum_{n=c+1}^{N_0} n\left(\frac{\rho_1}{c}\right)^n+\frac{c^c}{c!}\left(\frac{\mu_2}{\mu_1}\right)^{N_0}\sum_{n=N_0+1}^{\infty} n\left(\frac{\rho_2}{c}\right)^n\right\}-\frac{c}{\lambda}.$$

Multiplying equations (6), (7), (8), (9) and (10) by z^n and summing over all n, we obtain the generating function $F(z)$ in the following expression:

$$F(z)=p_0\left\{\sum_{n=0}^{c}\frac{(\rho_1 z)^n}{n!}+\frac{c^c}{c!}\sum_{n=c+1}^{N_0}\left(\frac{\rho_1 z}{c}\right)^n+\frac{c^c}{c!}\left(\frac{\rho_1}{c}\right)^{N_0}\frac{\rho_2 z^{N_0+1}}{c-\rho_2 z}\right\}. \tag{19}$$

Next the problem of queue-length distribution after a finite time has passed from the start has been of interest to many researchers and many papers on this problem have been issued, yet they are not so complete. For simplicity, the queuing system in which (1) there is only one counter, (2) the customers arrive at random and are served in the order of arrival and (3) the service-time distribution has a negative exponential type, will be discussed in this paper.

Especially, trials of finding solution by using Laplace transformation have come to be seen in some papers ([6], [7], etc.). Here by using Laplace transformation we shall treat with the model discribed before.

§ 3. The non-steady state

Using Laplace transformation

$$g_n(s)=\int_0^\infty e^{-st}P_n(t)\,dt \qquad (R(s)\geqq 0)$$

we can rewrite above-mentioned (1)∼(5) as follows:

For $n=0$

$$(1°)\qquad -(s+\lambda)g_0(s)+\mu_1 g_1(s)=-\delta_{0a},$$

For $0<n<c$

$$(2°)\qquad -(s+\lambda+n\mu_1)g_n(s)+\lambda g_{n-1}(s)+(n+1)\mu_1 g_{n+1}(s)=-\delta_{na},$$

For $c\leqq n<N_0$

$$(3°)\qquad -(s+\lambda+c\mu_1)g_n(s)+\lambda g_{n-1}(s)+c\mu_1 g_{n+1}(s)=-\delta_{na},$$

For $n=N_0$

$$(4°)\qquad -(s+\lambda+c\mu_1)g_{N_0}(s)+\lambda g_{N_0-1}(s)+c\mu_2 g_{N_0+1}(s)=-\delta_{N_0a},$$

For $n>N_0$

$$(5°)\qquad -(s+\lambda+c\mu_2)g_n(s)+\lambda g_{n-1}(s)+c\mu_2 g_{n+1}(s)=-\delta_{na}.$$

Multiplying equations (1°)∼(5°) by x^n and summing over all n, generating function, *i. e.*

$$F(x,s)=\sum_{n=0}^{\infty}g_n(s)x^n$$

is given by

$$F(x,s)=\frac{x^{a+1}+(1-x)\left\{\sum_{n=0}^{c-1}(n\mu_1-c\mu_2)g_n(s)x^n+c(\mu_1-\mu_2)\sum_{n=c}^{N_0}g_n(s)x^n\right\}}{-\lambda x^2+(s+\lambda+c\mu_2)x-c\mu_2}$$

Next we shall find the expressions of $g_n(s)$ and ergodic condition in single channel case.

I. The case a = 0

$$(1.1)\qquad -(s+\lambda)g_0(s)+\mu_1 g_1(s)=-1 \quad \text{for} \quad n=0,$$

$$(1.2)\qquad -(s+\lambda+\mu_1)g_n(s)+\lambda g_{n-1}(s)+\mu_1 g_{n+1}(s)=0 \quad \text{for} \quad 0<n<N_0,$$

$$(1.3)\qquad -(s+\lambda+\mu_1)g_{N_0}(s)+\lambda g_{N_0-1}(s)+\mu_2 g_{N_0+1}(s)=0 \quad \text{for} \quad n=N_0,$$

$$(1.4)\qquad -(s+\lambda+\mu_2)g_n(s)+\lambda g_{n-1}(s)+\mu_2 g_{n+1}(s)=0 \quad \text{for} \quad n>N_0.$$

From these equations it is seen that

$$(1.5)\qquad g_n(s)=\frac{\beta^{N_0}(\mu_1\beta-\mu_2\delta)\alpha^n}{\{(\alpha-1)\alpha^{N_0}(\mu_2\delta-\mu_1\alpha)+(\beta-1)\beta^{N_0}(\mu_1\beta-\mu_2\delta)\}\mu_1} \quad \text{for} \quad n\leqq N_0,$$

(1.6) $$g_n(s)=\frac{(\alpha\beta)^{N_0}(\beta-\alpha)\,\delta^{n-N_0}}{\{(\alpha-1)\alpha^{N_0}(\mu_2\delta-\mu_1\alpha)+(\beta-1)\beta^{N_0}(\mu_1\beta-\mu_2\delta)\}}\quad\text{for } n>N_0$$

where α and β are roots of the equation $\mu_1 x^2-(s+\lambda+\mu_1)x+\lambda=0$. On the other hand, δ is the root of the equation $\mu_2 x^2-(s+\lambda+\mu_2)x+\lambda=0$ and $|\delta|<1$.

From the equations (1.5) and (1.6), it is seen that the necessary and sufficient condition to be steady state is given by

$$\rho_2\equiv\frac{\lambda}{\mu_2}<1.$$

The steady state solution are found to be:

(1.7) $$p_n=\lim_{s\to 0} s\,g_n(s)=\frac{\rho_1^{\,n}}{1+\rho_1+\rho_1^{\,2}+\cdots+\rho_1^{\,N_0}(\rho_2+\rho_2^{\,2}+\cdots)}\quad\text{for } n\leqq N_0,$$

$$=\frac{\rho_1^{\,N_0}\rho_2^{\,n-N_0}}{1+\rho_1+\rho_1^{\,2}+\cdots+\rho_1^{\,N_0}+\rho_1^{\,N_0}(\rho_2+\rho_2^{\,2}+\cdots)}\quad\text{for } n>N_0\,.$$

Remark: The equation (1.7) coincides with (11), (12) and (13) in § 2.

II. The case $1\leqq a<N$.

(2.1) $-(s+\lambda)g_0(s)+\mu_1 g_1(s)=0$ for $n=0$,

(2.2) $-(s+\lambda+\mu_1)g_n(s)+\lambda g_{n-1}(s)+\mu_1 g_{n+1}(s)=0$ for $n\neq 0$, a and $n<N_0$,

(2.3) $-(s+\lambda+\mu_1)g_a(s)+\lambda g_{a-1}(s)+\mu_2 g_{a+1}(s)=-1$ for $n=a<N_0$,

(2.4) $-(s+\lambda+\mu_1)g_{N_0}(s)+\lambda g_{N_0-1}(s)+\mu_2 g_{N_0+1}(s)=0$ for $n=N_0$,

(2.5) $-(s+\lambda+\mu_2)g_n(s)+\lambda g_{n-1}(s)+\mu_2 g_{n+1}(s)=0$ $n>N_0$.

By solving above (2.1) $\sim$ (2.5) we get

(2.6) $$g_n(s)=\frac{(\alpha-1)\alpha^n+(1-\beta)\beta^n}{\mu_1(\alpha-\beta)}\left\{\frac{\beta^{N_0-a}(\mu_1\beta-\mu_2\delta)+\alpha^{N_0-a}(\mu_2\delta-\mu_1\alpha)}{(\alpha-1)\alpha^{N_0}(\mu_2\delta-\mu_1\alpha)+(\beta-1)\beta^{N_0}(\mu\beta-\mu_2\delta)}\right\}$$

for $0<n<a$,

(2.7) $$g_n(s)=\frac{(\alpha-1)\alpha^a+(1-\beta)\beta^a}{\mu_1(\alpha\beta)^a(\alpha-\beta)}\left\{\frac{\alpha^n\beta^{N_0}(\mu_1\beta-\mu_2\delta)+\beta^n\alpha^{N_0}(\mu_2\delta-\mu_1\alpha)}{(\alpha-1)\alpha^{N_0}(\mu_2\delta-\mu_1\alpha)+(\beta-1)\beta^{N_0}(\mu_1\beta-\mu_2\delta)}\right\}$$

for $a\leqq n\leqq N_0$,

(2.8) $$g_n(s)=\frac{\{(1-\alpha)\alpha^a+(\beta-1)\beta^a\}(\alpha\beta)^{N_0-a}\delta^{n-N_0}}{(\alpha-1)\alpha^{N_0}(\mu_2\delta-\mu_1\alpha)+(\beta-1)\beta^{N_0}(\mu_1\beta-\mu_2\delta)}\quad\text{for } n\geqq N_0$$

where α, β and δ are the same as before. We have $\rho_2<1$ as the ergodic condition in this case. In the same way as before, we obtain

(2.9) $$p_n = \frac{\rho_1^n}{1+\rho_1+\rho_1^2+\cdots+\rho_1^{N_0}+\rho_1^{N_0}(\rho_2+\rho_2^2+\cdots)} \quad \text{for } n \leqq N_0,$$

$$= \frac{\rho_1^{N_0}\rho_2^{n-N_0}}{1+\rho_1+\rho_1^2+\cdots+\rho_1^{N_0}+\rho_1^{N_0}(\rho_2+\rho_2^2+\cdots)} \quad \text{for } n > N_0.$$

III. The case $a = N_0$

(3.1) $-(s+\lambda)g_0(s)+\mu_1 g_1(s)=0$ for $n=0$,

(3.2) $-(s+\lambda+\mu_1)g_n(s)+\lambda g_{n-1}(s)+\mu_1 g_{n+1}(s)=0$ for $0<n<N_0$,

(3.3) $-(s+\lambda+\mu_1)g_{N_0}(s)+\lambda g_{N_0-1}(s)+\mu_2 g_{N_0+1}(s)=-1$ for $n=N_0$,

(3.4) $-(s+\lambda+\mu_2)g_n(s)+\lambda g_{n-1}(s)+\mu_2 g_{n+1}(s)=0$ for $n>N_0$.

From equations (3.1), (3.2), (3.3) and (3.4), we get

(3.5) $$g_n(s) = \frac{(\alpha-1)\alpha^n+(1-\beta)\beta^n}{(\alpha-1)\alpha^{N_0}(\mu_1\alpha-\mu_2\delta)+(\beta-1)\beta^{N_0}(\mu_2\delta-\mu_1\beta)} \quad \text{for } n \leqq N_0,$$

(3.6) $$g_n(s) = \frac{(\alpha-1)\alpha^{N_0}+(1-\beta)\beta^{N_0}}{(\alpha-1)\alpha^{N_0}(\mu_1\alpha-\mu_2\delta)+(\beta-1)\beta^{N_0}(\mu_2\delta-\mu_1\beta)}\delta^{n-N_0} \quad \text{for } n \geqq N_0$$

where α, β and δ are the same as before.

From the equations (3.5) and (3.6), it is seen that the necessary and sufficient condition to be steady state is given by $\rho_2 < 1$.

By Abel's theorem we can derive

(3.7) $$p_n = \lim_{s\to 0} s\,g_n(s) = \frac{\rho_1^n}{1+\rho_1+\rho_1^2+\cdots+\rho_1^{N_0}+\rho_1^{N_0}(\rho_2+\rho_2^2+\cdots)} \quad \text{for } n \leqq N_0,$$

$$= \frac{\rho_1^{N_0}\rho_2^{n-N_0}}{1+\rho_1+\rho_1^2+\cdots+\rho_1^{N_0}+\rho_1^{N_0}(\rho_2+\rho_2^2+\cdots)} \quad \text{for } n \geqq N_0.$$

IV. The case $a > N_0$

(4.1) $-(s+\lambda)g_0(s)+\mu_1 g_1(s)=0$ for $n=0$,

(4.2) $-(s+\lambda+\mu_1)g_n(s)+\lambda g_{n-1}(s)+\mu_1 g_{n+1}(s)=0$ for $0<n<N_0$,

(4.3) $-(s+\lambda+\mu_2)g_{N_0}(s)+\lambda g_{N_0-1}(s)+\mu_2 g_{N_0+1}(s)=0$ for $n=N_0$,

(4.4) $-(s+\lambda+\mu_2)g_n(s)+\lambda g_{n-1}(s)+\mu_2 g_{n+1}(s)=0$ for $n>N_0$, $n\neq a$,

(4.5) $-(s+\lambda+\mu_2)g_a(s)+\lambda g_{a-1}(s)+\mu_2 g_{a+1}(s)=0$ for $n=a>N_0$.

From these equations we can derive

(4.6) $$g_n(s) = \frac{(1-\alpha)\alpha^n+(\beta-1)\beta^n}{\{(\alpha-1)\alpha^{N_0}(\mu_2\delta-\mu_1\alpha)+(1-\beta)\beta^{N_0}(\mu_2\delta-\mu_1\beta)\}\gamma^{a-N_0}} \quad \text{for } 0 \leqq n \leqq N_0,$$

$$(4.7)\qquad g_n(s)=\frac{1}{\kappa\gamma^{a-n}}+\frac{\{(\alpha-1)\alpha^{N_0}(\mu_1\alpha-\mu_2\gamma)+(1-\beta)\beta^{N_0}(\mu_1\beta-\mu_2\gamma)\}\gamma^{N_0-a}}{\{(\alpha-1)\alpha^{N_0}(\mu_2\delta-\mu_1\alpha)+(1-\beta)\beta^{N_0}(\mu_2\delta-\mu_1\beta)\}\kappa\delta^{N_0-n}}$$

for $N_0\leqq n\leqq a$,

$$(4.8)\qquad g_n(s)=\frac{\delta^{n-a}}{\kappa}+\frac{\{(\alpha-1)\alpha^{N_0}(\mu_1\alpha-\mu_2\gamma)+(1-\beta)\beta^{N_0}(\mu_1\beta-\mu_2\gamma)\}\gamma^{N_0-a}}{\{(\alpha-1)\alpha^{N_0}(\mu_2\delta-\mu_1\alpha)+(1-\beta)\beta^{N_0}(\mu_2\delta-\mu_1\beta)\}\kappa}\delta^{n-N_0}$$

for $n\geqq a$

where α and β are the same as before, γ and δ are two roots of the equation

$$\mu_2x^2-(s+\lambda+\mu_2)x+\lambda=0$$

and $|\gamma|>1$, $|\delta|<1$. Moreover let κ be $\sqrt{(s+\lambda+\mu_2)^2-4\lambda\mu_2}$.

From the equations (4.6), (4.7) and (4.8), the ergodic condition is given by $\rho_2<1$.

Stationary probability p_n are

$$\frac{\rho_1^n}{1+\rho_1+\rho_1^2+\cdots\cdots+\rho_1^{N_0}+\rho_1^{N_0}(\rho_2+\rho_2^2+\cdots)}\quad\text{for}\quad n\leqq N_0,$$

and

$$\frac{\rho_1^{N_0}\rho_2^{n-N_0}}{1+\rho_1+\rho_1^2+\cdots\cdots+\rho_1^{N_0}+\rho_1^{N_0}(\rho_2+\rho_2^2+\cdots)}\quad\text{for}\quad n\geqq N_0.$$

The following conclusions can be derived from the above-mentioned facts. The ergodic condition in our system is given by $\rho_2<1$.

The steady distribution is independent of initial numbers of customers, a.

The total number L in the system is

$$L=\sum_{n=0}^{N_0}np_n+\sum_{n=N_0+1}^{\infty}np_n$$

$$=\frac{\sum_{n=0}^{N_0}n(\rho_2^{N_0}\rho_1^n-\rho_1^{N_0}\rho_2^n)+\frac{\rho_2\rho_1^{N_0}}{(1-\rho_2)^2}}{\rho_2^{N_0}\{1+\rho_1+\rho_1^2+\cdots+\rho_1^{N_0}+\rho_1^{N_0}(\rho_2+\rho_2+\cdots)\}}.$$

The mean number in queue, L_q is given by

$$L_q=\sum_{n=1}^{\infty}(n-1)p_n=L-1+p_0.$$

The mean waiting time, W is given by $\frac{L_q}{\lambda}$.

§ 4. The case $c=2$

In this case we get a set of following differential difference equations:

$$\text{(i)}\qquad \frac{dp_0(t)}{dt}=-\lambda P_0(t)+\mu_1P_1(t)\quad\text{for}\quad n=0,$$

(ii) $$\frac{dP_1(t)}{dt} = -(\lambda+\mu_1)P_1(t)+\lambda P_0(t)+2\mu_1 P_2(t) \quad \text{for} \quad n=1,$$

(iii) $$\frac{dP_n(t)}{dt} = -(\lambda+2\mu_1)P_n(t)+\lambda P_{n-1}(t)+2\mu_1 P_{n+1}(t) \quad \text{for} \quad 2 \leqq n < N_0,$$

(iv) $$\frac{dP_{N_0}(t)}{dt} = -(\lambda+2\mu_1)P_{N_0}(t)+\lambda P_{N_0-1}(t)+2\mu_2 P_{N_0+1}(t) \quad \text{for} \quad n=N_0,$$

(v) $$\frac{dP_n(t)}{dt} = -(\lambda+2\mu_2)P_n(t)+\lambda P_{n-1}(t)+2\mu_2 P_{n+1}(t) \quad \text{for} \quad n>N_0.$$

Letting $P_n(0)=\delta_{na}$, we can obtain the expressions of $g_n(s)$ in a similar way. Here $g_n(s)$ are the Laplace transforms of $P_n(t)$. Mereover, it can be concluded that the ergodic condition is given by

$$\frac{\lambda}{2\mu_2} < 1$$

and stationary probabilities are independent of initial number a. In fact, under the condition $\frac{\lambda}{2\mu_2} < 1$, we have

$$p_n = \frac{\rho_1{}^n}{\frac{1}{2}+\rho_1+\rho_1{}^2+\cdots\cdots+\rho_1{}^{N_0}+\rho_1{}^{N_0}(\rho_2+\rho_2{}^2+\cdots)} \quad \text{for} \quad n \leqq N_0,$$

$$= \frac{\rho_1{}^{N_0}\rho_2{}^{n-N_0}}{\frac{1}{2}+\rho_1+\rho_1{}^2+\cdots\cdots+\rho_1{}^{N_0}+\rho_1{}^{N_0}(\rho_2+\rho_2{}^2+\cdots)} \quad \text{for} \quad n \geqq N_0$$

where $\rho_i = \frac{\lambda}{2\mu_i}$ $(i=1, 2)$.

Similarly we can successively get the result in case of more counters.

§ 5. The stochastic law of the busy period

Let us denote by $G_{nk}(x)$ the probability that the busy period is at most of length x, consists of at least n services and at the end of the nth service, k customers are present in the queue. Then $G_{n0}(x)$ is the probability that the busy period consists of n services and its length is at most x. By the theorem of total probability, we can write that

(5.1) $$G_{1k}(x) = \int_0^x e^{-\lambda y}\frac{(\lambda y)^k}{k!}\mu_1 e^{-\mu_1 y}dy \quad \text{for} \quad k \leqq N_0-1,$$

(5.2) $$G_{1k}(x) = \int_0^x e^{-\lambda y}\frac{(\lambda y)^{N_0-1}}{(N_0-1)!}\,\frac{e^{-\lambda(x-y)}[\lambda(x-y)]^{k-N_0}}{(k-N_0)!}\lambda\mu_2 e^{-\mu y}e^{-\mu_2(x-y)}dy$$
for $k \geqq N_0$,

(5.3) $$G_{nk}(x) = \sum_{r=1}^{k+1}\int_0^x G_{n-1,r}(x-y)\,e^{-\lambda y}\frac{(\lambda y)^{k-r+1}}{(k-r+1)!}\mu_1 e^{-\mu_1 y}dy$$
for $k \leqq N_0-1$ and $n \geqq 2$

and

$$(5.4)\quad G_{nk}(x)=\sum_{r=1}^{N_0}\int_0^x G_{n-1,\,r}(x-y)\int_0^y e^{-(\lambda+\mu_1)u}\frac{(\lambda u)^{N_0-r}}{(N_0-r)!}\ \frac{e^{-\lambda(y-u)}[\lambda(y-u)]^{k-N_0}}{(k-N_0)!}$$

$$\times\lambda\mu_2 e^{-\mu_2(y-u)}\,du\,dy$$

$$+\sum_{r=N_0+1}^{k+1}\int_0^x G_{n-1,\,r}(x-y)\,e^{-\lambda y}\frac{(\lambda y)^{k-r+1}}{(k-r+1)!}\mu_2 e^{-\mu_2 y}\,dy$$

for $k\geqq N_0$ and $n\geqq 2$.

Introduce the Laplace-Stieltjes transform

$$\Gamma_{nk}(s)=\int_0^\infty e^{-sx}\,dG_{nk}(x)\quad\text{for}\quad R(s)\geqq 0$$

so that we get

$$(5.5)\quad \Gamma_{1k}(s)=\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^k\frac{\mu_1}{s+\lambda+\mu_1}\quad\text{for}\quad k\leqq N_0-1,$$

$$(5.6)\quad \Gamma_{1k}(s)=\left(\frac{\lambda}{s+\lambda+\mu_2}\right)^{N_0}\left(\frac{\lambda}{s+\lambda+\mu_2}\right)^{k-N_0}\frac{\mu_2}{s+\lambda+\mu_2}\quad\text{for}\quad k\geqq N_0,$$

$$(5.7)\quad \Gamma_{nk}(s)=\sum_{r=1}^{k+1}\Gamma_{n-1,\,r}(s)\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^{k-r+1}\frac{\mu_1}{s+\lambda+\mu_1}\quad\text{for}\quad k\leqq N_0-1,\ n\geqq 2$$

and

$$(5.8)\quad \Gamma_{nk}(s)=\sum_{r=1}^{N_0}\Gamma_{n-1,\,r}(s)\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^{N_0-r+1}\left(\frac{\lambda}{s+\lambda+\mu_2}\right)^{k-N_0}\frac{\mu_2}{s+\lambda+\mu_2}$$

$$+\sum_{r=N_0+1}^{k+1}\Gamma_{n-1,\,r}(s)\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^{k-r+1}\frac{\mu_2}{s+\lambda+\mu_2}\quad\text{for}\quad k\geqq N_0,\ n\geqq 2.$$

Let us introduce the generating function

$$(5.9)\quad C_n(s,z)=\sum_{k=0}^{\infty}\Gamma_{n-k}(s)z^k\quad\text{for}\quad |z|<1.$$

Then we have from (5.5) and (5.6) that

$$(5.10)\quad C_1(s,z)=\sum_{k=0}^{N_0-1}\Gamma_{1k}(s)z^k+\frac{\mu_2}{s+\mu_2+\lambda(1-z)}\left(\frac{\lambda z}{s+\lambda+\mu_1}\right)^{N_0}$$

and from (5.7), (5 8) that

$$(5.11)\quad C_n(s,z)=\sum_{k=0}^{N_0-1}\Gamma_{nk}(s)z^k+\frac{\mu_2}{s+\mu_2+\lambda(1-z)}\left\{\sum_{r=1}^{N_0}\Gamma_{n-1,\,r}(s)\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^{N_0-r+1}z^{N_0}\right.$$

$$\left.+\,C_{n-1}(s,z)-\sum_{j=0}^{N_0}\Gamma_{n-1,\,j}(s)z^j\right\}\quad\text{for}\quad n\geqq 2.$$

Hence from (5.10) and (5.11), we get

$$(5.12)\qquad \sum_{n=1}^{\infty} C_n(s,z)w^n = \frac{1}{z\{s+\mu_2+\lambda(1-z)\}-\mu_2 w}\Big\{[z\{s+\mu_2+\lambda(1-z)\}-\mu_2 w]$$

$$\times \sum_{k=0}^{N_0-1}\gamma_k(s,w)z^k + \frac{\mu_2 z^{N_0}}{\mu_1}\Big\{\lambda z\gamma_{N_0-1}(s,w)-\mu_1 w\gamma_{N_0}(s,w)\Big\}$$

where

$$\gamma_k(s,w)=\sum_{n=1}^{\infty}\Gamma_{nk}(s)w^n \quad \text{for} \quad k\geqq 0.$$

Note the identity

$$\frac{\lambda}{\mu_1 w}\gamma_{N_0-1}(s,w)=\Big(\frac{\lambda}{s+\lambda+\mu_1}\Big)^{N_0}+\sum_{r=1}^{N_0}\gamma_r(s,w)\Big(\frac{\lambda}{s+\lambda+\mu_1}\Big)^{N_0-r+1}$$

The left side of (5.12) is a regular function of z if $|z|\leqq 1$, $R(s)\geqq 0$ and $|w|<1$. In this domain, the denominator of the right side has only one root

$$a=\frac{s+\lambda+\mu_2-\sqrt{(s+\lambda+\mu_2)^2-4\lambda\mu_2 w}}{2\lambda}.$$

This must also be the root of the numerator, satisfy the relation

$$(5.13)\qquad \lambda a\gamma_{N_0-1}(s,w)-\mu_1 w\gamma_{N_0}(s,w)=0.$$

From (5.5) and (5.7), we get

$$(5.14)\qquad \begin{aligned} A\gamma_1(s,w)&=\gamma_0(s,w)-A\\ A\gamma_2(s,w)&=\gamma_1(s,w)-B\gamma_0(s,w)\\ A\gamma_3(s,w)&=\gamma_2(s,w)-B\gamma_1(s,w)\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ A\gamma_{N_0}(s,w)&=\gamma_{N_0-1}(s,w)-B\gamma_{N_0-2}(s,w) \end{aligned}$$

where

$$A=A(s,w)=\frac{\mu_1 w}{s+\lambda+\mu_1} \quad \text{and} \quad B=B(s)=\frac{\lambda}{s+\lambda+\mu_1}.$$

Using these relations among $\gamma_j(s,w)$ $(j=0,1,\cdots\cdots,N_0)$, we get

$$(5.15)\qquad \gamma_j(s,w)=\Big\{\frac{1}{A^j}-\frac{B}{A^{j-1}}\cdot{}_{j-1}C_1+\frac{B^2}{A^{j-2}}\cdot{}_{j-2}C_2-\cdots\cdots\cdots\cdots$$

$$+\frac{(-B)^{[j/2]}}{A^{j-[j/2]}}\cdot{}_{j-[j/2]}C_{[j/2]}\Big\}\gamma_0(s,w)$$

$$-\Big\{\frac{1}{A^{j-1}}-\frac{B}{A^{j-2}}\cdot{}_{j-2}C_1+\frac{B^2}{A^{j-3}}\cdot{}_{j-3}C_2-\cdots+\frac{(-B)^{[j-1/2]}}{A^{j-1-[j-1/2]}}\cdot{}_{j-1-[j-1/2]}C_{[j-1/2]}\Big\}$$

where $[j]$ denotes the greatest integer less than j and ${}_nC_k$ the combination calculation.

Using (5.13) and (5.15), we obtain the following theorem:

Theorem. *For $R(s) \geqq 0$ and $|w| \leqq 1$, we have*

$$\sum_{n=1}^{\infty} \Gamma_{n0}(s) w^n = \frac{\lambda \alpha P_{N_0-2}(A,B) - \mu_1 w P_{N_0-1}(A,B)}{\lambda \alpha P_{N_0-1}(A,B) - \mu_1 w P_{N_0}(A,B)} \tag{5.16}$$

where α is the root in z of the equation $z\{s+\mu_2+\lambda(1-z)\} - \mu_2 w = 0$ in the unit circle $|z| < 1$ and $P_j(A,B)$ is given by the identity

$$P_j(A,B) = \frac{1}{A^j} - \frac{B}{A^{j-1}} \cdot {}_{j-1}C_1 + \frac{B^2}{A^{j-2}} \cdot {}_{j-2}C_2 - \cdots\cdots + \frac{(-B)^{[j/2]}}{A^{j-[j/2]}} \cdot {}_{j-[j/2]}C_{[j/2]} .$$

Putting $w=1$ in (5.16), we get the Laplace-stieltjes transformation of busy period.

Special case: (i) $N_0 = 1$.

$$\sum_{n=1}^{\infty} \Gamma_{n0}(s) w^n = \frac{\mu_1 w}{s+\mu_1+\lambda(1-a)},$$

$$\sum_{n=1}^{\infty} \Gamma_{n0}(s) = \frac{\mu_1}{s+\mu_1+\lambda(1-a_1)} \quad \text{where } a_1 = [a]_{w=1}.$$

(ii) $N_0 = 2$.

$$\sum_{n=1}^{\infty} \Gamma_{n0}(s) w^n = \frac{\mu_1 w\{s+\mu_1+\lambda(1-a)\}}{\{s+\mu_1+\lambda(1-a)\}(s+\lambda+\mu_1) - \lambda\mu_1 w},$$

$$\sum_{n=1}^{\infty} \Gamma_{n0}(s) = \frac{\mu_1\{s+\mu_1+\lambda(1-a_1)\}}{\{s+\mu_1+\lambda(1-a_1)\}(s+\lambda+\mu_1) - \lambda\mu_1}.$$

(iii) $N_0 \to \infty$.

$$\sum_{n=1}^{\infty} \Gamma_{n0}(s) w^n = \frac{s+\lambda+\mu_1 - \sqrt{(s+\lambda+\mu_1)^2 - 4\lambda\mu_1 w}}{2\lambda},$$

$$\sum_{n=1}^{\infty} \Gamma_{n0}(s) = \frac{s+\lambda+\mu_1 - \sqrt{(s+\lambda+\mu_1)^2 - 4\lambda\mu_1}}{2\lambda}.$$

Next, we shall write the expected length of the busy period, $E_{N_0}(B)$.

For $N_0 = \textit{odd}$ number

$$E_{N_0}(B) = -\left.\frac{d\gamma_0(s,1)}{ds}\right|_{s=0}$$

$$= \frac{\left\{\frac{\rho_2}{1-\rho_2}\sum_{r=0}^{\left[\frac{N_0-2}{2}\right]}(-\rho_1)^r(1+\rho_1)^{N_0-2-2r} C_{N_0-2-r\ r} - \rho_1 \sum_{r=0}^{\left[\frac{N_0-2}{2}\right]}(N_0-2-2r)(-\rho_1)^r(1+\rho_1)^{N_0-\ -2r} C_{N_0-2r\ r}\right.}{\left\{\lambda \sum_{r=0}^{\frac{N_0-1}{2}}(-\rho)^r(1+\rho_1)^{N_0-1-2r} \cdot C_{N_0-1-r\ r} - \mu_1 \sum_{r=0}^{\left[\frac{N_0}{2}\right]}(-\rho_1)^r(1+\rho_1)^{N_0-2r} C_{N_0-r\ r}\right\}^2}$$

$$\frac{\left. + \sum_{r=0}^{\frac{N_0-1}{2}-1}(N_0-1-2r)(-\rho_1)^r(1+\rho_1)^{N_0-2-2r} C_{N_0-1-r\ r}\right\}\Big\{\{denominator\}\Big\} - \left\{\lambda \sum_{r=0}^{\left[\frac{N_0-2}{2}\right]}(-\rho_1)^r\right.}{\prime\prime}$$

$$\frac{\left.\cdot(1+\rho_1)^{N_0-2-2r} C_{N-2-r\ r} - \mu_1 \sum_{r=0}^{\frac{N_0-1}{2}}(-\rho_1)^r(1+\rho_1)^{N_0-1-2r} C_{N_0-1-r\ r}\right\}\left\{\frac{\rho_2}{1-\rho_2}\sum_{r=0}^{\frac{N_0-1}{2}}(-\rho_1)^r\right.}{\prime\prime}$$

$$\frac{(1+\rho_1)^{N_0-1-2r} C_{N_0-1-r\ r} - \rho_1 \sum_{r=0}^{\frac{N_0-1}{2}-1}(N_0-1-2r)(-\rho_1)^r(1+\rho_1)^{N_0-2-2r} C_{N_0-1-r\ r} + \sum_{r=0}^{\left[\frac{N_0}{2}\right]}(N_0-2r)(-\rho_1)^r}{\prime\prime}$$

$$\frac{\cdot(1+\rho_1)^{N_0-1-2r} C_{N_0-r\ r}}{\prime\prime}$$

For $N_0 =$ *even number*

$$E_{N_0}(B) = \frac{\left\{\frac{\rho_2}{1-\rho_2}\sum_{r=0}^{\frac{N_0-2}{2}}(-\rho_1)^r(1+\rho_1)^{N_0-2-2r} C_{N_0-2-r\ r} - \rho_1 \sum_{r=0}^{\frac{N_0}{2}-2}(N_0-2-2r)(-\rho_1)^r(1+\rho_1)^{N_0-3-2r} C_{N_0-2-r\ r}\right.}{\left\{\lambda \sum_{r=0}^{\left[\frac{N_0-1}{2}\right]}(-\rho_1)^r(1+\rho_1)^{N_0-1-2r} C_{N_0-1-r\ r} - \mu_1 \sum_{r=0}^{\frac{N_0}{2}}(-\rho_1)^r(1+\rho_1)^{N_0-2r} C_{N_0-1-r\ r}\right\}^2}$$

$$+\sum_{r=0}^{\left[\frac{N_0-2}{2}\right]}(N_0-1-2r)(-\rho_1)^r(1+\rho_1)^{N_0-2-2r}\,{}_{N_0-1-r}C_r\Big\}\Big\{\text{denominator}\Big\}-\Big\{\lambda\sum_{r=0}^{\frac{N_0-2}{2}}(-\rho_1)^r$$

$$\cdot(1+\rho_1)^{N_0-2-2r}\,{}_{N_0-2-r}C_r-\mu_1\sum_{r=0}^{\frac{N_0-1}{2}}(-\rho_1)^r(1+\rho_1)^{N_0-1-2r}\,{}_{N_0-1-r}C_r\Big\}\Big\{\frac{\rho_2}{1-\rho_2}\sum_{r=0}^{\left[\frac{N_0-1}{2}\right]}(-\rho_1)^r$$

$$(1+\rho_1)^{N_0-1-2r}\,{}_{N_0-1-r}C_r-\rho_1\sum_{r=0}^{\left[\frac{N_0-1}{2}\right]}(N_0-1-2r)(-\rho_1)^r(1+\rho_1)^{N_0-2-2r}\,{}_{N_0-1-r}C_r+\sum_{r=0}^{\frac{N_0-1}{2}}(N_0-2r)(-\rho_1)^r$$

$$(1+\rho_1)^{N_0-1-2r}\,{}_{N_0-r}C_r\Big\}$$

Putting $N_0=1$ and 2 in $E_{N_0}(B)$ and using above expressions, we get

$$E_1(B)=\frac{1}{\mu_1(1-\rho_2)},\qquad E_2(B)=\frac{1+\rho_1-\rho_2}{\mu_1(1-\rho_2)}$$

respectively. From these expressions it is seen that

$$E_1(B)<E_2(B)\qquad \text{for}\quad \rho_1>\rho_2.$$

§ 6. The modified busy period.

We shall now investigate the stochastic law of the modified busy period.

Let $\hat{G}_{nk}^{(l)}(x)=P_r$. {modified busy period consists of at least n services, is at most of length x and at the end of the nth service, k customers are present in the queue l customers are present in the system at $t=0$}

$$\hat{G}_{nk}(x)=\sum_{l=1}p_l\hat{G}_{nk}^{(l)}(x)\quad\text{where}\quad p_n=\begin{cases}p_0\rho_1^{\,n} & \text{for } n\leq N_0,\\ p_0\rho_1^{N_0}\rho_2^{\,n-N_0} & \text{for } n<N_0,\end{cases}\qquad \rho_i=\frac{\lambda}{\mu_i}\quad\text{and}$$

$$p_0=\left\{\sum_{n=0}^{N_0}\rho_1{}^n+\rho_1{}^{N_0}\frac{\rho_2}{1-\rho_2}\right\}^{-1}.$$

Let $\hat{\Gamma}_{nk}^{(l)}(s)=\int_0^\infty e^{-sx}d_x G_{nk}^{(l)}(x)$ and $\hat{\Gamma}_{nk}(s)=\int_0^\infty e^{-sx}d_x\hat{G}_{nk}(x)$ for $R(s)\geqq 0$.

Then we have

$$\hat{\Gamma}_{nk}(s)=\sum_{l=1}^{\infty}p_l\hat{\Gamma}_{nk}^{(l)}(s). \tag{6.1}$$

The expressions of $\hat{G}_{nk}{}^{(l)}(s)$ is very similar to the one given in § 5, this is,

$$\hat{G}_{1k}{}^{(l+1)}(x)=\int_0^x e^{-\lambda y}\frac{(\lambda y)^{k-l}}{(k-l)!}\mu_1 e^{-\mu_1 y}dy \quad \text{for} \quad l=0,1,2,\cdots\cdots\leqq k\leqq N_0-1, \tag{6.2}$$

$$=0 \quad \text{for} \quad k<l$$

$$\hat{G}_{1k}{}^{(l+1)}(x)=\int_0^x e^{-\lambda y}\frac{(\lambda y)^{N_0-l-1}}{(N_0-l-1)!}\frac{e^{-\lambda(x-y)}\{\lambda(x-y)\}^{k-N_0}}{(k-N_0)!}\lambda\mu_2 e^{-\mu_1 y}e^{-\mu_2(x-y)}dy \tag{6.3}$$

$$=\int_0^x e^{-\lambda y}\frac{(\lambda y)^{k-l}}{(k-l)!}\mu_2 e^{-\mu_2 y}dy \quad \text{for} \quad k\geqq N_0,\ l\geqq N_0,$$

$$\hat{G}_{nk}^{(l)}(x)=\int_0^x\sum_{j=1}^{k+1}\hat{G}_{n-1,j}^{(l)}(x-y)e^{-\lambda y}\frac{(\lambda y)^{k-j+1}}{(k-j+1)!}\mu_1 e^{-\mu_1 y}dy \ \text{for}\ k\leqq N_0-1,\ 2\leqq n, \tag{6.4}$$

$$\hat{G}_{nk}^{(l)}(x)=\int_{y=0}^x\sum_{j=1}^{N_0}\hat{G}_{n-1,j}^{(l)}(x-y)\int_{u=0}^y e^{-(\lambda+\mu_1)u}\frac{(\lambda u)^{N_0-j}}{(N_0-j)!}\frac{\{\lambda(y-u)\}^{k-N_0}}{(k-N_0)!}\lambda\mu_2 e^{-(\lambda+\mu_2)(y-u)}du\,dy \tag{6.5}$$

$$+\int_0^x\sum_{j=N_0+1}^{k+1}\hat{G}_{n-1}^{(l)}(x-y)e^{-(\lambda+\mu_2)y}\frac{(\lambda y)^{k-j+1}}{(k-j+1)!}\mu_2 dy \quad \text{for} \quad k\geqq N_0,\ n\geqq 2.$$

Hence by (6.1), we get

$$\hat{G}_{1k}(x)=\sum_{l=0}^{k}p_{l+1}\int_0^x e^{-\lambda y}\frac{(\lambda y)^{k-l}}{(k-l)!}\mu_1 e^{-\mu_1 y}dy \quad \text{for} \quad k\leqq N_0-1, \tag{6.6}$$

$$\hat{G}_{1k}(x)=\sum_{l=0}^{N_0-1}p_{l+1}\int_0^x e^{-\lambda y}\frac{(\lambda y)^{N_0-l-1}}{(N_0-l-1)!}\frac{e^{-\lambda(x-y)}\{\lambda(x-y)\}^{k-N_0}}{(k-N_0)!}\lambda\mu_2 e^{-\mu_1 y}e^{-\mu_2(x-y)}dy \tag{6.7}$$

$$+\sum_{l=N_0}^{k}p_{l+1}\int_0^x e^{-\lambda y}\frac{(\lambda y)^{k-l}}{(k-l)!}\mu_2 e^{-\mu_2 y}dy \quad \text{for} \quad k\geqq N_0,$$

$$\hat{G}_{nk}(x)=\int_0^x\sum_{j=1}^{k+1}\hat{G}_{n-1,j}(x-y)e^{-\lambda y}\frac{(\lambda y)^{k-j+1}}{(k-j+1)!}\cdot\mu_1 e^{-\mu_1 y}dy \ \text{for}\ k\leqq N_0-1,\ 2\leqq n. \tag{6.8}$$

(6.9) $$\hat{G}_{nk}(x)=\int_{y=0}^{x}\sum_{j=1}^{N_0}\hat{G}_{n-1,j}(x-y)\int_{u=0}^{y}e^{-(\lambda+\mu_1)u}\frac{(\lambda y)^{N_0-j}}{(N_0-j)!}\frac{\{\lambda(y-u)\}^{k-N_0}}{(k-N_0)!}\lambda\mu_2 e^{-(\lambda+\mu_2)(y-u)}du\,dy$$

$$+\int_0^x\sum_{j=N_0+1}^{k+1}\hat{G}_{n-1,j}(x-y)e^{-(\lambda+\mu_2)y}\frac{(\lambda y)^{k-j+1}}{(k-j+1)!}\mu_2\,dy \quad \text{for} \quad k\geqq N_0,\ n\geqq 2.$$

Forming Laplace-Stieltjes transforms of (6.6), (6.7), (6.8) and (6.9), we get

(6.10) $$\hat{\Gamma}_{1k}(s)=\sum_{l=0}^{k}p_{l+1}\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^{k-l}\frac{\mu_1}{s+\lambda+\mu_1} \quad \text{for} \quad k\leqq N_0-1,$$

(6.11) $$\hat{\Gamma}_{1k}(s)=\sum_{l=0}^{N_0-1}p_{l+1}\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^{N_0-l}\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^{k-N_0}\frac{\mu_2}{s+\lambda+\mu_2}$$

$$+\sum_{l=N_0}^{k}p_{l+1}\left(\frac{\lambda}{s+\lambda+\mu_2}\right)^{k-l}\frac{\mu_2}{s+\lambda+\mu_2} \quad \text{for} \quad k\geqq N_0,$$

(6.12) $$\hat{\Gamma}_{nk}(s)=\sum_{j=1}^{k+1}\hat{\Gamma}_{n-1,j}(s)\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^{k-j+1}\frac{\mu_1}{s+\lambda+\mu_1} \quad \text{for} \quad k\leqq N_0-1,\ n\geqq 2,$$

(6.13) $$\hat{\Gamma}_{nk}(s)=\sum_{j=1}^{N_0}\hat{\Gamma}_{n-1,j}(s)\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^{N_0-j+1}\left(\frac{\lambda}{s+\lambda+\mu_2}\right)^{k-N_0}\frac{\mu_2}{s+\lambda+\mu_2}$$

$$+\sum_{j=N_0+1}^{k+1}\hat{\Gamma}_{n-1,j}(s)\left(\frac{\lambda}{s+\lambda+\mu_2}\right)^{k-j+1}\frac{\mu_2}{s+\lambda+\mu_2} \quad \text{for} \quad k\geqq N_0,\ n\geqq 2.$$

Let us introduce the generating function

(6.14) $$\hat{C}_n(s,z)=\sum_{k=0}^{\infty}\hat{\Gamma}_{nk}(s)z^k$$

Then we get from (6.10) and (6.11),

(6.15) $$\hat{C}_1(s,z)=\sum_{k=0}^{N_0-1}\hat{\Gamma}_{1k}(s)z^k+\frac{\mu_2}{z\{s+\mu_2+\lambda(1-z)\}}\left\{\sum_{j=1}^{N_0}p_jz^j\left(\frac{\lambda z}{s+\lambda+\mu_1}\right)^{N_0-j+1}+k(z)\right.$$

$$\left.-\sum_{j=1}^{N_0}p_jz^j\right\}$$

and from (6.12), (6.13),

(6.16) $$\hat{C}_n(s,z)=\sum_{k=0}^{N_0-1}\hat{\Gamma}_{nk}(s)z^k+\frac{\mu_2}{z\{s+\mu_2+\lambda(1-z)\}}\left\{\sum_{j=1}^{N_0}\hat{\Gamma}_{n-1,j}(s)\left(\frac{\lambda z}{s+\lambda+\mu_1}\right)^{N_0-j+1}\right.$$

$$\left.+\hat{C}_{n-1}(s,x)-\sum_{j=0}^{N_0}\hat{\Gamma}_{n-1,j}(s)z^j\right\} \quad \text{for} \quad n\geqq 2$$

where $k(z)=\sum_{j=1}^{\infty}p_jz^j$

Hence from (6.15) and (6.16), we get

$$(2.17)\qquad \sum_{n=1}^{\infty} w^n \hat{C}_n(s,z)=\frac{1}{z\{s+\mu_2+\lambda(1-z)\}-\mu_2 w}\Big\{\left[z\{s+\mu_2+\lambda(1-z)\}-\mu_2 w\right]$$

$$\cdot\sum_{j=0}^{N_0-1}\hat{\gamma}_j(s,w)z^j+\frac{\mu_2}{\mu_1}\lambda z^{N_0+1}\hat{\gamma}_{N_0-1}(s,w)-\mu_2 w z^{N_0}\hat{\gamma}_{N_0}(s,w)$$

$$+\{k(z)-\sum_{j=1}^{N_0}p_j z^j\}\ \mu_2 w\Big\}$$

From (6.7) we get

$$(6.18)\qquad \lambda\alpha\hat{\gamma}_{N_0-1}(s,w)-\mu_1 w\hat{\gamma}_{N_0}(s,w)+p_0\mu_1 w\rho_1{}^{N_0}\frac{\rho_2\alpha}{1-\rho_2\alpha}=0$$

in the similar way to the one given in § 5, where $\hat{\gamma}_j(s,w)=\sum_{n=1}^{\infty}w^n\hat{f}_{nj}(s)$ and

$\alpha=\{s+\lambda+\mu_2-\sqrt{(s+\lambda+\mu_2)^2-4\lambda\mu_2 w}\}/2\lambda.$

On the other hand, using (6.10) and (6.12) we get

$$A\{\hat{\gamma}_1(s,w)+p_1\}=\hat{\gamma}_0(s,w)$$
$$A\{\hat{\gamma}_2(s,w)+p_2\}=\hat{\gamma}_1(s,w)-B\hat{\gamma}_0(s,w)$$
$$A\{\hat{\gamma}_3(s,w)+p_3\}=\hat{\gamma}_2(s,w)-B\hat{\gamma}_1(s,w)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$A\{\hat{\gamma}_{N_0}(s,w)+p_{N_0}\}=\hat{\gamma}_{N_0-1}(s,w)-B\hat{\gamma}_{N_0-2}(s,w).$$

From these relations we have

$$(6.19)\qquad \gamma_j(s,w)=\Big\{\frac{1}{A^j}-\frac{B}{A^{j-1}}\cdot{}_{j-1}C_1+\frac{B^2}{A^{j-2}}\cdot{}_{j-2}C_2-\cdots+\frac{(-B)^{\left[\frac{j}{2}\right]}}{A^{j-\left[\frac{j}{2}\right]}}\cdot{}_{j-\left[\frac{j}{2}\right]}C_{\left[\frac{j}{2}\right]}\Big\}\gamma_0(s,w)$$

$$-\Big\{p_j+\frac{1}{A}p_{j-1}+\Big(\frac{1}{A^2}-\frac{B}{A}\Big)p_{j-2}+\Big(\frac{1}{A^3}-\frac{B}{A^2}\cdot{}_2C_1\Big)p_{j-3}+\cdots\cdots\cdots$$

$$+\Big(\frac{1}{A^{j-1}}-\frac{B}{A^{j-2}}{}_{j-2}C_1+\frac{B^2}{A^{j-3}}\cdot{}_{j-3}C_2-\cdots+\frac{(-B)^{\left[\frac{j-1}{2}\right]}}{A^{j-1-\left[\frac{j-1}{2}\right]}}\cdot{}_{j-1-\left[\frac{j-1}{2}\right]}C_{\left[\frac{j-1}{2}\right]}\Big)p_1\Big\}$$

for $1\leqq j\leqq N_0$.

Let $Q_j(A,B)$ be

$$p_j+\frac{1}{A}p_{j-1}+\Big(\frac{1}{A^2}-\frac{B}{A}\Big)p_{j-2}+\Big(\frac{1}{A^3}-\frac{B}{A^2}\cdot{}_2C_1\Big)p_{j-3}+\cdots\cdots\cdots$$

$$\cdots\cdots+\Big(\frac{1}{A^{2-1}}-\frac{B}{A^{j-2}}\cdot{}_{j-2}C_1+\cdots\cdots+\frac{(-B)^{\left[\frac{j-1}{2}\right]}}{A^{j-1-\left[\frac{j-1}{2}\right]}}\cdot{}_{j-1-\left[\frac{j-1}{2}\right]}C_{\left[\frac{j-1}{2}\right]}\Big)p_1\ ,$$

then (6.19) can be written as follows:

(6.20) $$\hat{\gamma}_j(s,w)=P_j(A,B)\,\hat{\gamma}_0(s,w)-Q_j(A,B) \quad \text{for} \quad 1\leqq j\leqq N_0.$$

Hence by (6.18) and (6.20), we get

(6.21) $$\hat{\gamma}_0(s,w)=\frac{\gamma\alpha Q_{N_0-1}(A,B)-\mu_1 wP_{N_0}(A,B)-p_0\rho_1{}^{N_0}\mu_1 w\dfrac{\rho_2\alpha}{1-\rho_2\alpha}}{\lambda\alpha P_{N_0-1}(A,B)-\mu_1 wP_{N_0}(A,B)}.$$

Using (6.20) and (6.21) we can determine the quantities $\hat{\gamma}_j(s,w)$ $(1\leqq j\leqq N_0)$.

We shall now consider the special cases:

The case $N_0=1$: $\hat{\gamma}_0(s,w)=\dfrac{\mu_1 wk(\alpha)}{\alpha\{s+\mu_1+\lambda(1-\alpha)\}}=\dfrac{k(\alpha)}{\alpha}\gamma_0(s,w).$

// $N_0\to\infty\ (\mu_1=\mu_2\equiv\mu)$: $\hat{\gamma}_0(s,w)=k(\alpha).$

Note that for $N_0\to\infty$, (6.16) can be written as

$$\sum_{n=1}^{\infty}\hat{C}_n(s,z)w^n=\frac{\mu w\{k(z)-\hat{\gamma}_0(s,w)\}}{z\{s+\mu+\lambda(1-z)\}-\mu w}.$$

§ 7. The waiting time process $W(t)$.

We shall now investigate the stochastic law of the waiting time process. Let $W(t)=P_r$ {the waiting time is at most of length t}. Then we have

(7.1) $$W(t)=\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\int_0^t p_n\,d\hat{G}_{nk}(x).$$

Forming Laplace-Stieljes transform of (3.1), we get

$$\tilde{W}(s)=\int_0^{\infty}e^{-st}\,dW(t) \qquad (R(s)\geqq 0)$$

(7.2) $$=p_0\rho_1\left\{1-\left(\frac{\rho_1}{\rho_2}\right)^{N_0-1}\right\}\hat{C}_1(s,1)-p_0\rho_1{}^2\left\{1-\left(\frac{\rho_1}{\rho_2}\right)^{N_0-2}\right\}\hat{C}_2(s,1)+\cdots\cdots\cdots$$

$$\cdots+p_0\rho_1{}^{N_0-1}\left\{1-\left(\frac{\rho_1}{\rho_2}\right)\right\}\hat{C}_{N_0-1}(s,1)+p_0\left(\frac{\rho_1}{\rho_2}\right)^{N_0}\sum_{n=1}^{\infty}\rho_2{}^n\,\hat{C}_n(s,1).$$

Putting $z=1$ in (6.10) and (6.15), we have

(7.3) $$\hat{C}_1(s,1)=\frac{\mu_1}{s+\lambda+\mu_1}\left\{p_1+p_1\left(\frac{\lambda}{s+\lambda+\mu_1}\right)+p_2\right.$$

$$+p_1\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^2+p_2\left(\frac{\lambda}{s+\lambda+\mu_1}\right)+p_3$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+p_1\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^{N_0-1}+\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^{N_0-2}+\cdots\cdots\cdots\cdots+p_{N_0}\Big\}$$

$$+\frac{\mu_2}{s+\mu_2}\left\{\sum_{j=0}^{N_0}p_j\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^{N_0-j+1}+k(1)-\sum_{j=1}^{N_0}p_j\right\}.$$

Further, putting $z=1$ in (6.16), we have

$$\hat{C}_n(s,1)=\sum_{k=0}^{N_0-1}\hat{\Gamma}_{nk}(s)+\frac{\mu_1}{s+\mu_2}\left\{\sum_{j=1}^{N_0}\hat{\Gamma}_{n-1,j}(s)\left(\frac{\lambda}{s+\lambda+\mu_1}\right)^{N_0-j+1}+\hat{C}_{n-1}(s,1)\right. \tag{7.4}$$

$$\left.-\sum_{j=0}^{N_0}\hat{\Gamma}_{n-1,j}(s)\right\}\quad \text{for}\quad n\leqq 2.$$

Using (7.3) ane (7.4), we can successively determine the quantities,

$$\hat{C}_2(s,1),\ \hat{C}_3(s,1),\cdots\cdots\cdots,\hat{C}_{N_0-1}(s,1).$$

On the other hand, putting $z=1$ and $w=\rho_2$ in (6.17), (6.19) and (6.21) the quantity $\sum_{n=1}^{\infty}\rho_2{}^n\hat{C}_n(s,1)$ can be obtained. Hence from (7.2) we can get $\tilde{W}(s)$.

Special case $N_0=1$:

$$\tilde{W}(s)=p_0\frac{\rho_1}{\rho_2}\frac{1}{s+\mu_2-\lambda}\left\{\frac{\mu_1 s-\lambda\mu_1-\mu_2 s}{\mu_1}\hat{\gamma}_0(s,\rho_2)+\lambda(1-p_0)\right\}$$

where $p_0=(1-\rho_2)/(1+\rho_1-\rho_2)$.

Acknowledgement

I am indebted to Dr. T. Homma for advice and criticism during the preparation of this paper.

REFERENCES

[1] D. V. Lindley, The theory of queues with a single server, Proc. Camb. Phil. Soc., Vol. 48, 1952.

[2] W. Ledermann-G. E. H. Reuter, Spectral theory for the differential equation of simple birth and death processes, Phil. Trans. A, Vol. 246, 1954.

[3] N. T. J. Bailey, A continuous time treatment of a simple queue using generating functions, J. Roy. Stat. Soc. B, Vol. 16, 1954.

[4] ——————, Some further results in the non-equilibrium theory of a simple queue, J. Roy. Stat. Soc. B, Vol. 19, 1957.

[5] A. B. Clarke, A waiting time process of Markov type, Ann. Math. Stat. Vol. 27, 1956.

[6] B. W. Conolly, A difference equation technique applied to the simple queue, J. Roy. Stat. Soc. B, Vol. 20, 1958.

[7] ___________, A difference equation technique applied to the simple queue with arbitrary arrival interval distribution, J. Roy. Stat. Soc. B, Vol. 20, 1958.

[8] P. M. Morse, Stochastic properties of waiting lines, J. Opns. Soc. Amer. Vol. 3, 1955.

[9] G. Luchak, The continuous time solution of the equations of the single channel queue with a general class of service-time distribution by the method of generating functions, J. Roy. Stat. Soc. B, Vol. 20, 1958.

[10] Lakshmi Venkataraman, Probabilistic investigation of a single server queuing process with Poissonian input and batch service (unpublished).