A NOTE ON THE BOUNDEDNESS PROPERTY OF NON-LINEAR OPERATORS

By

SADAYUKI YAMAMURO

Let R be a real Banach space and H be a Non-linear operator defined on the whole space R.

H is said to be ***bounded*** if H transforms any bounded set to a bounded set. In the case of linear operators, this is equivalent to the existence of a positive number γ such that

(*) $$\|Hf\| \leqq \gamma \|f\| \qquad \text{for every } f \in R.$$

We shall say that a non-linear operator H defined on R is ***linearly bounded*** if H satisfies the above condition (*).

The purpose of this note is to make clear the relation between the boundedness and the linear boundedness properties of non-linear operators and to discuss related topics.

Throughout of this paper, we assume that $H0 = 0$.

§ 1. *Examples.*

1. 1. On the space L^2 on $[0, 1]$, we define

$$Hf = \sqrt{|f(t)|}.$$

This oparator H is defined for every $f \in L^2$ and $\|Hf\|^2 \leqq \|f\|$.

1. 2. On the space L^2 on $[0, 1]$, we define

$$Hf = \sqrt{|f(t)|\,/\,\|f\|}.$$

This is defined for every $f \in L^2$ and $\|Hf\|^2 \leqq \dfrac{1}{\|f\|}$.

1. 3. On the space l of summable real sequences $\{f(n)\}$, we define

$$Hf(n) = \begin{cases} |f(n)| & \text{if} \quad |f(n)| \leqq 1 \\ \dfrac{1}{f(n)-1} & \text{if} \quad |f(n)| > 1. \end{cases}$$

This operator H is defind for any $f \in l$.

1. 4. On the Banach space R, we define

$$Hf = \begin{cases} \dfrac{f}{1-\|f\|} & \text{if} \quad \|f\| < 1 \\ f & \text{if} \quad \|f\| \geqq 1. \end{cases}$$

This is defined for every $f \epsilon R$.

§ **2.** *Linear Boundedness and Boundedness.*

We shall say that a non-linear operator H defined on a Banach space R is *linearly upper bounded* if there exist numbers $\alpha, \gamma > 0$ such that

$$\|Hf\| \leqq \gamma \|f\| \qquad \text{if } \|f\| \geqq \alpha.$$

Similarly, an operator H is said to be *linearly lower bounded* if there exist numbers $\beta, \gamma > 0$ such that

$$\|Hf\| \leqq \gamma \|f\| \qquad \text{if } \|f\| \leqq \beta.$$

The notion of the linear upper boundedness is firstly due to A. Granas [1], where he called the operator with this property was quasi-bounded.

We shall also say that an operator H is *linearly upper* (or *lower*) *bounded everywhere* if, for any positive number α (or β), there exists a positive number $\gamma(\alpha)$ (or $\gamma(\beta)$) such that

$$\|Hf\| \leqq \gamma(\alpha) \cdot \|f\| \quad \text{if} \quad \|f\| \geqq \alpha.$$
$$(\text{or } \|Hf\| \leqq \gamma(\beta) \cdot \|f\| \quad \text{if} \quad \|f\| \leqq \beta).$$

It is clear that if H is linearly upper (or lower) bounded everywhere then H is linearly upper (or lower) bounded, and that, if H is linearly lower bounded everywhere, then H is bounded.

Theorem 1. (1) *If H is bounded and linearly upper (or lower) bounded, then H is linearly upper (or lower) bounded everywhere.*
(2) *A bounded operator H is linearly bounded if and only if it is linearly upper and lower bounded.*

Proof. (1). Assume that H is not linearly upper bounded everywhere, then we can find a positive number α and a sequence $f_n \epsilon R$ $(n = 1, 2, \cdots\cdots)$ such that

$$\|Hf_n\| \geqq n \cdot \|f_n\| \quad \text{and} \quad \|f_n\| > \beta.$$

The sequence f_n $(n = 1, 2, \cdots\cdots)$ is bounded, because, if $\{f_n\}$ is not bounded, the linear upper boundedness implies the existence of infinitely many f_n such that

$$\|Hf_n\| \leqq \gamma \|f_n\|$$

from which then follows an impossible inequality $n \leqq \gamma$ for infinitely many n. Therefore, since $\{f_n\}$ is bonnded, from the boundedness of H it follows that the sequence $\{Hf_n\}$ is also bounded. This contradicts the fact that $\|Hf_n\| \geqq n\beta$ $(u = 1, 2, \cdots)$.

By the same way we can prove that, if H is bounded and linealy lower bounded, then H is linearly lower bounded everywhere.

(2). If the bounded operator H is linearly upper and lower bounded, then, as was proved in (1), H is linearly upper and lower bounded everywhere, and hence it follows that H is linearly bounded.

The converse is clear.

Remark. The operator of Example 1.1 is bounded but is neither linearly upper nor lower bounded. The operator of Example 1.2 is linearly upper bounded everywhere, but is not bounded. The operator of Example 1.3 is linearly lower bounded, but is not bounded, and so it is not linearly lower bounded everywhere. The operator of Example 1.4 is linearly upper bounded, but is not linearly upper bounded, but is not linearly upper bounded everywhere.

§ **3.** *Differentiability and Linear Boundedness.*

An operator H defined on a real Banach space R is said to be ***Fréchet-differentiable at*** 0 if there exists a linear (additive and continuous) operator D such that

$$Hf = Hf - H0 = Df + r(f) \text{ and } \lim_{f\to 0} \frac{r(f)}{\|f\|} = 0.$$

We denote D by $\nabla H0$.

An operator H is said to be *asymptotically differentiable*, if there exists a linear operator H_∞ such that

$$\lim_{\|f\|\to+\infty} \frac{\|Hf - H_\infty f\|}{\|f\|} = 0.$$

Theorem 2. (1). *If H is Fréchet-differentiable at* 0, *then H is linearly lower bounded:*

$$\|Hf\| \leqq \|\nabla H0\| \cdot \|f\| \quad \text{if} \quad \|f\| \leqq \beta \quad \textit{for some } \beta.$$

(2). *If H is asymptotically differentiable, then H is linearly upper bounded:*

$$\|Hf\| \leqq \|H_\infty\| \cdot \|f\| \quad \textit{if} \quad \|f\| \geqq \alpha \quad \textit{for some} \quad \alpha > 0.$$

Proof. Let H be Fréchet-differentiable at 0. Assume that there exists a sequence $f_n \epsilon R\,(n = 1, 2, \cdots\cdots)$ such that

$$\|f_n\| < \frac{1}{n} \quad \text{and} \quad \|Hf_n\| > \|\nabla H0\| \cdot \|f_n\|.$$

Then, since we have

$$\lim_{n\to\infty} \frac{1}{\|f_n\|} \|Hf_n - (\nabla H0) f_n\| = 0,$$

we have

$$\lim_{n\to\infty} \left| \frac{\|Hf_n\|}{\|f_n\|} - \|(\nabla H0)\left(\frac{f_n}{\|f_n\|}\right)\| \right| = 0.$$

Therefore, there exists n_0 such that

$$\|\nabla H0\| < \|(\nabla H0)\left(\frac{f_n}{\|f_n\|}\right)\| \qquad (n \geqq n_0),$$

which is a contradiction. This completes the proof of (1).

By the same way we can prove (2).

Remark. The operator of Example 1.3 is linearly lower bounded but is not Fréchet-differentiable at 0.

§ **4.** *Continuity and Boundedness.*

Although these two notions (the continuity and the boundedness) are mutually independent as has been discussed in Vainberg [2], p. 29, we are going to prove that, in the case of the generalized Nemyzkii operators defined on some Banach lattice the continuity implies the boundedness. In order to prove this, we need some preliminary remarks.

A real Banach space is called a *Banach lattice* if it is a vector lattice and $|f| \leqq |g|$ implies $\|f\| \leqq \|g\|$.

A non-linear operator F defined on the Banach lattice R is called *the generalized Nemyzkii operator* if F satisfies the following condition:

$$|f| \cap |g| = 0 \quad \text{implies} \quad |f| \cap |Fg| = 0 \quad \text{and} \quad F(f+g) = Ff + Fg.$$

In the space $L^p[0, 1]$, the operator

$$Ff(t) = F(t, f(t))$$

is an emample of the generalized Nemyzkii operator when the function $F(t, \xi)$ satisfies some conditions.

Theorem 3. *Let F be the generalized Nemyzkii operator, continuous at* 0, *defined on the Banach lattice R. Then, for any real-valued function $N(f)$ ($f \in R$) with the following properties:*

1° $0 < N(f) < +\infty$ *if* $f \neq 0$;

2° $|f| \cap |g| = 0$ *implies* $N(f+g) = N(f) + N(g)$;

3° $\lim_{n\to\infty} \|f_n\| = 0$ *if and only if* $\lim_{n\to\infty} N(f_n) = 0$, and

$\lim_{n\to\infty} \|f_n\| = \infty$ *if and only if* $\lim_{n\to\infty} N(f_n) = \infty$;

4° If $N(f) > \alpha$, *there are f_1 and f_2 such that*

$f = f_1 + f_2$, $|f_1| \cap |f_2| = 0$ and $N(f_1) = \alpha$,

we have

$$N(Ff) \leqq \alpha N(f) + 1 \qquad (f \in R)$$

for some $\alpha > 0$, and hence it follows that the operator is bounded whenever there is dfined such a functional $N(f)$.

Proof. From the definiton it follows that $F0=0$. Since F is assumed to be continuous at 0, by 3°, we can find a number $\alpha>0$ such that

$$\alpha N(f)<1 \quad \text{implies} \quad N(Ff)<1.$$

For any f such that $\alpha N(f)\geqq 1$, we can find a natural number n such that

$$n\leqq \alpha N(f)<n+1.$$

By 4°, we can find $f_i(i=1, 2, \cdots\cdots, n+1)$ such that

$$f=\sum_{i=1}^{n+1} f_i, \ |f_i|\cap|f_j|=0 \ (i\neq j) \text{ and } \alpha N(f_i)<1.$$

Therefore, since the operator F is additive for orthogonal elements,

$$Ff=\sum_{i=1}^{n+1} Ff_i,$$

and, by 2°, we have

$$N(Ff)=\sum_{i=1}^{n+1} N(Ff_i)<n+1\leqq \alpha N(f)+1,$$

which is to be proved.

Remark. Example 1.1 shows that the constant 1 (or at least a non-zero constant) of the above inequality is indispensable if we do not assume that F be Fréchet-differentiable at 0.

Yokohama Municipal University.

References

[1] A. Granas: On a Class of Non-linear Mappings in Banach Spaces, Bulletin de la Academie Polonaise, 5(1957) 867–871

[2] M. M. Vainberg: Variational Methods in the Theory of Non-linear Operators (Russian), Moscow (1956)