ON THE THEORY OF SOME NON-LINEAR OPERATORS

By

SADAYUKI YAMAMURO

In this paper, we will study some problems on the spectra of the generalized Hammerstein operators defined on the Hilbert space. For this purpose, we regard the Hilbert space as a Banach lattice. In § 1, we give some remarks on the Hilbert lattice. In § 2, we will define the generalized Hammerstein operators and will give some remarks. In § 3, we will consider properties of the spectrum of the generalized Hammerstein operator in the case when its completely continuous part admits a proper elements system consisting of positive elements. Theorem 5 there is a correction of Theorem 2 of [2].

§ **1.** We assume that the space R satisfies the following three conditions:

I. *R is a real Hilbert space.* The inner product of $f \epsilon R$ and $g \epsilon R$ will be denoted by (f, g), which is a real number.

II. *R is a vector lattice.* Namely, R is a lattice with the following properties: (1) $f \geqq 0$ and $g \geqq 0$ imply $f+g \geqq 0$; (2) $f \geqq 0$ and $\alpha \geqq 0$ (α is a real number) imply $\alpha f \geqq 0$.

The following relations are well-known:

1. $|f| = f \cup (-f) = f^+ + f^-$
   where $f^+ = f \cup 0$ and $f^- = (-f)^+$.
2. $f+g = f \cup g + f \cap g$.
3. $|f-g| = f \cup g - f \cap g$.

If $|f| \leqq |g|$, *then* $\|f\| \leqq \|g\|$.

Throughout of this paper, we denote by R the space with the above three properties.

The following two lemmata are important for our subsequent discussions.

Lemma 1. *If* $0 \leqq f \leqq g$ and $h \geqq 0$, *then*

$$(f, h) \leqq (g, h).$$

Proof. We have only to prove that, for any positive element f and any positive element g, we have $(f, g) = 0$. Since we have

$$|f-g| = f \cup g - f \cap g \leqq f \cup g + f \cap g = f+g,$$

the condition 3 implies that

$$\|f-g\| \leqq \|f+g\|.$$

Therefore,

$$2(f, g)=\|f+g\|^2-\|f-g\|^2 \geqq 0.$$

Lemma 2. *For positive elements* f and g, $f \cap g=0$ *is equivalent to* $(f, g)=0$.

Proof. If $f \cap g=0$, then, by the condition III, we have

$$2(f, g)=\|f+g\|^2-\|f-g\|^2=0,$$

because, $\quad |f-g|=f+g$ and $\|f-g\|=\|(|f-g|)\|$.

Conversely, if $(f, g)=0$ for positive elements f and g, we have, by Lemma 1, that

$$0 \leqq (f \cap g, f \cap g) \leqq (f, g)=0,$$

which implies that $\quad f \cap g=0.$

§ **2.** Let K be a symmetric, completely continuous linear operator defined on R. Let F be a non-linear, continuous operator defined on R. We assume that (*) *if* $|f| \cap |g|=0$, *then* $|f| \cap |Fg|=0$ and $F(f+g)=Ff+Fg$. The operator $H=KF$ with the condition (*) is called *the generalized Hammerstein operator.*

The integral operator of Hammerstein type

$$Hf(s)=\int_0^1 K(s, t) F(t, f(t))\, dt,$$

where the kernel function $K(s, t)$ is symmetric and square-integrable on $[0, 1] \times [0, 1]$ and the function $F(t, u)$ is bonnded and continuous with respect to t in $[0, 1]$ and to u in $(-\infty, +\infty)$ and $F(t, 0)=0$ for any t in $[0, 1]$, is an example of the generalized Hammerstein operator defined on $L^2\ [0, 1]$, the space of square-integrable measurable functions on $[0, 1]$. Here, we have only to put

$$Kf(s)=\int_0^1 K(s, t) f(t) dt \text{ and } Ff(t)=F(t, f(t)).$$

We shall say that the space R is *non-atomic*, if, for any number $\alpha>0$, any element f with $\|f\|>\alpha$ can be devided in the following way:

$$f=f_1+f_2,\ |f_1| \cap |f_2|=0 \text{ and } \|f_1\|=\alpha.$$

It is evident that the space L^2 is non-atomic.

Theorem 1. *If the space* R *is non-atomic, then, for the operator* F *satisfying the condition* (*), *there exists a positive number* α *such that*

$$\|Ff\| \leqq \alpha\|f\|+1 \quad (f \epsilon R),$$

namely, the operator F *is boundcd (the image of any norm boundcd set by* F *is also norm bounded).*

Proof. Since F is continuous, we can find a positive number α such that

$$\|f\|<\alpha \text{ implies } \|Ff\|<1. \qquad \cdots\cdots\cdots\cdots (1)$$

For any $f \epsilon R$, there is an integer $n \geqq 0$ such that

$$n\alpha^2 \leqq \|f\|^2 \leqq (n+1)\alpha^2. \qquad \cdots\cdots\cdots\cdots (2)$$

Since the space is assumed to be non-atomic, there exist $f_i (i=1, 2, \cdots\cdots, n+1)$ such that

$$f = f_1 + f_2 + f_2 + \cdots\cdots + f_{n+1}, \quad |f_i| \cap |f_j| = 0 \ (i \neq j)$$
$$\text{and } \|f_i\| \leqq \alpha \ (i = 1, 2, \cdots\cdots, n+1).$$

Therfore, by (*) we have

$$Ff = Ff_1 + Ff_2 + \cdots\cdots + Ff_{n+1},$$

and, by (1) and (2), we have

$$\|Ff\|^2 = \sum_{i=1}^{n+1} \|Ff_i\|^2 < n + 1 \leqq \frac{1}{\alpha^2} \|f\|^2 + 1,$$

which means that F is bounded.

By making use of the above theorem, we have the following.

Theorem 2. *If the space R is non-atomic, the generalized* Hammerstein *operator is completely continuous, namely, it is continuous and the image of any bound set is compact.*

§ **3.** We denote by $S(H)$ or $S(K)$ the set of all proper values of the operator H or K respectively. For $\lambda \epsilon S(H)$ or $\lambda \epsilon S(K)$, we denote the set of proper elements belonging to the λ by $E_\lambda(H)$ or $E_\lambda(K)$ respectively.

In this section, *we assume that the symmetric, completely continuous linear operator K admits a prober elements system consisting of positive elements.* Namely, we assume that we can choose a positive proper element corresponding to each proper value.

For this kind of K, any element f in $K(R)$, the range of K, can be expressed in the following way:

$$f = \sum_{n=1}^{\infty} (f, \phi_n) \phi_n \qquad \cdots\cdots\cdots(3)$$

where ϕ_n is a positive, normalized, proper element belonging to the proper value $\lambda_n \epsilon S(H)$. The n-th coordinate of f is denoted by $f^{(n)}$, namely, we put

$$f^{(n)} = (f, \phi_n) \qquad (n = 1, 2, \cdots\cdots\cdots)$$

We define the real-valued functions $\Phi_n(\xi)$ of real variable ξ by

$$\Phi_n(\xi) = (F(\xi\phi_n), \phi_n) \qquad (n = 1, 2 \cdots\cdots).$$

The functions $\Phi_n(\xi)$ are obviously continuous and $\Phi_n(0) = 0$.

Theorem 3. *For any $f \epsilon K(R)$, we have*

$$Hf = \sum_{n=1}^{\infty} \lambda_n \Phi_n(f^{(n)}) \phi_n \qquad \cdots\cdots\cdots(4)$$

Proof. Since $(\phi_m, \phi_n) = 0$ $(m \neq n)$, we have, by Lemma 2, that

$$\phi_m \cap \phi_n = 0,$$

because $\phi_m > 0$ and $\phi_n > 0$. Therefore, we have

$$|f^{(m)}\phi_m| \cap |f^{(n)}| = 0,$$

which implies that

$$Ff = \sum_{n=1}^{\infty} F(f^{(n)}\phi_n),$$

because of (*), (2) and the fact that F is continuous. Applying K to the both sides of the above formula, we have by (*), that

$$\begin{aligned} Hf &= KFf \\ &= \sum_{n=1}^{\infty} \lambda_n (Ff, \phi_n)\phi_n \\ &= \sum_{m=1}^{\infty} \lambda_n (\sum_{m=1}^{\infty} F(f^{(m)}\phi_m), \phi_n)\phi_n \\ &= \sum_{n=1}^{\infty} \lambda_n (F(f^{(n)}\phi_n), \phi_n)\phi_n = \sum_{n=1}^{\infty} \lambda_n \Phi_n(f^{(n)})\phi_n, \end{aligned}$$

which is to be proved.

Theorem 4. $\lambda_n \Phi_n(1) \in S(H) \quad (n = 1, 2, \cdots\cdots\cdots).$

Proof. By (4), we have

$$H\phi_n = \sum_{m=1}^{\infty} \lambda_m \Phi_m(\phi_n{}^{(m)})\phi_m = \lambda_n \Phi_n(1)\phi_n,$$

which means that $\lambda_n \Phi_n(1) \in S(H).$

The above theorem shows that the number of elements of $S(H)$ is not less than that of $S(K)$. Although there is an example of Hammerstein operator that has finite number of proper values (See p. 201 of [1]), the spectra of non-linear orerators usually contain intervals of real line.

Now, we will show that the fact that $S(H)$ does not contain any interval is almost characteristic for the linearity of the operator H. Note that, in this section, we have assumed that the completely continuous linear operator K admits a proper elements system consisting of positive elements.

Theorem 5. *Let us assume that the functions $\Phi_n(\xi)$ $(n = 1, 2, \cdots\cdots)$ are differentiable at* the point 0. *Then, if $S(H)$ contains no intervals, the operator H is linear on the set $K(R)$, the range of K.*

Proof. For each n, since the function $\Phi_n(\xi)$ is continuous, the function $\Phi_n(\xi)/\xi$ is also continuous with respect to positive ξ or negative ξ respectively. Let us assume that there exist ξ and η such that

$$\xi > 0, \quad \eta > 0 \text{ and } \frac{\Phi_n(\xi)}{\xi} \neq \frac{\Phi_n(\eta)}{\eta}.$$

Then the continuity of $\Phi_n(\xi)/\xi$ implies that, for any number α between $\Phi_n(\xi)/\xi$ and $\Phi_n(\eta)/\eta$, there exists $\xi_\alpha > 0$ such that

$$\Phi_n(\xi_\alpha) = \alpha\xi_\alpha.$$

Then by Theorem 3, we have

$$H(\xi_\alpha \phi_n) = \sum_{m=1}^{\infty} \lambda_m \Phi_m [(\xi_\alpha \phi_n)^{(m)}] \phi_m$$

$$= \lambda_n \Phi_n(\xi_\alpha)\phi_n = \lambda_n \alpha\xi_\alpha \phi_n,$$

which means that $\lambda_n\alpha\epsilon S(H)$. Since, by the assumption, $S(H)$ contains no intervals, we conclude that there exists a number α such that

$$\Phi_n(\xi) = \alpha\xi \qquad \text{for any } \xi > 0.$$

Similarly, we can find a number β such that

$$\Phi_n(\xi) = \beta\xi \qquad \text{for any } \xi < 0.$$

The assumption that the function $\Phi_n(\xi)$ is differentiable at 0 implies that $\alpha = \beta$. Since $\Phi_n(1) = 1 \cdot \Phi_n(1)$, we have

$$\Phi_n(\xi) = \xi\Phi_n(1) \qquad \text{for every } \xi.$$

Then, since we have

$$(f+g)^{(n)} = f^{(n)} + g^{(n)}$$

for any f and g in $K(R)$, we have by Theorem 3 that

$$H(f+g) = \sum_{n=1}^{\infty} \lambda_n \Phi_n [(f+g)^{(n)}] \phi_n$$

$$= \sum_{n=1}^{\infty} \lambda_n \Phi_n(f^{(n)})\phi_n + \sum_{n=1}^{\infty} \lambda_n \Phi_n(g^{(n)})\Phi_n = Hf + Hg,$$

which is to be proved.

§ 4. In this last section, we give some remarks about the condition that the operator K admits a proper elements system consisting of positive elements.

Let K be a symmetric, positive definite, completely continuous linear operator defined on the space R. It is well known that, if K is positive, namely, if $f \geqq 0$ implies $Kf \geqq 0$, then the proper element belonging to the largest proper value can be chosen as positive. This fact is due to the following lattice property:

$$(Kf, f) \leqq (K|f|, |f|) \qquad \text{for any } f\epsilon R.$$

The n-th proper element ϕ_n is determined by calculating

$$sup\ \{(Kf, f) : \|f\| = 1 \text{ and } (f, \phi_i) = 0\ (i = 1, 2, \cdots\cdots, n\text{-}1)\}.$$

Therefore, if the subspace A_n spanned by such f that $(f, \phi_i) = 0\ (i = 1, 2, \cdots\cdots, n-1)$ is clossed by the lattice operation, the n-th proper element can be chosen as positive, because, in this case, the subspace A_n stated above contains $|f|$ whenever f is in it. In other words, the operator K with the properties stated above admits a proper elements system $\phi_n \geqq 0$ $(n=1, 2,\cdots\cdots)$ if and only if, for each n, the fact that $(f, \phi_i)=0$ $(i=1, 2,\cdots\cdots, n)$ implies the existence of such a positive element $g \in R$ that

$$(g, \phi_i)=0 \ (i=1, 2,\cdots\cdots, n) \text{ and } (Kf, f) \leqq (Kg, g).$$

In connection with this problem, it would not be of no use to discuss about the following condition:

$$(Kf, g)=0, \quad \textit{implies } (Kf, |g|)=0, \cdots\cdots\cdots\cdots(5)$$

because this condition immediately implies the fact that the subspace A_n stated above is itself a vector lattice, namely, $|f| \in A_n$ if $f \in A_n$. As is easily seen, if the operator K satisfies the condition (5), then, by the proposition stated above, K admits a proper elements system consisting of positive elements.

We shall say that a positive element $u \in R$ be a *unit* of R if $u \cap |f|=0$ implies $f=0$. Obviously, the function $f(t)=\alpha>0$ $(0 \leqq t \leqq 1)$ is a unit of the function space $L^2[0, 1]$.

In the sequel, the operator K is naturally assumed to be not identically vanishing.

Lemma 3. *If the operator K is positive, symmetric and satisfies* (5), *then* $(Ku, u) \neq 0$ *for any unit u of R.*

Proof. If $Ku=0$ for a unit u of R, then, for any element $f \in R$, we have, by (5), that

$$(u, K|f|)=(Ku, |f|)=0.$$

Therefore, by Lemma 2, we have $u \cap K|f|=0$, from which it follows that $K|f|=0$. Since $|Kf| \leqq K|f|$, because K is positive, we have

$$Kf=0 \qquad (f \in R).$$

This is a contradiction. Therefore, $Ku \neq 0$ and hence it follows that $(Ku, u) \neq 0$.

Theorem 6. *Let us assume that the operator K be positive, symmetric and satisfies the condition* (5). *Then, for any unit u of R, the element $\phi_0=Ku$ is a proper element of K and we have*

$$Kf=\lambda(f, \phi_0)\phi_0 \qquad \textit{where } \lambda=1/(Ku, u)$$

for any $f \in R$.

Proof. Since, by Lemma 3, (Ku, u) is not zero, we can put $\lambda=1/(Ku, u)$. If $(Ku, g)=0$ for an element $g \in R$, then, by (5), we have $Kg=0$ by making use of the same method as in the preceeding lemma. Therefore, for any $f \in R$, we have

$$(Kf, g)=(f, Kg)=0.$$

Namely, we have

$$(Ku, g)=0 \qquad \text{implies } (Kf, g)=0,$$

from which it follows that there exists a number α such that

$$Kf=\alpha Ku=\alpha\phi_0.$$

Then, we have

$$\alpha(\phi_0, u) = (Kf, u) = (f, \phi_0),$$

hence it follows that, since $\lambda = 1/(\phi_0, u)$,

$$\alpha = \lambda(f, \phi^0).$$

This completes th proof.

This theorem shows that, if the symmetric, positive operator satisfies the condition (5), then the set $K(R)$ is generated by a single element ϕ_0 which is positive. It is easy to see that, conversely, if the symmetric, positive operator admits the expression of Theorem 6, then K satisfies the condition (5).

Yokohama Municipal University.

References

[1] M. A. Krasnoseliski: Topological Method in the Theory of Non-linear Integral Operators (Russian), Moscow (1956).

[2] S. Yamamuro: On the Spectra of Some Non-linear Operators, II. Proc. Japan Academy. 37 (1961) 521-523