On structures of geometrically realizable triangulations on surfaces

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The author
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Preface

This thesis is written on the subject "Geometric realizations of triangulations on surfaces" and is to be submitted for the degree of Doctor of Science at Yokohama National University.

After introduction, the reader will find six chapters. General terminology of Graph Theory is found in Chapter 1.

Let G be a graph embedded into a surface F^2. A *geometric realization* of G is an embedding of F^2 into a Euclidian 3-space $\mathbb{R}^3$ with no self-intersection such that each face of G is a flat polygon. In Chapter 2, we introduce "exhibitions" which are used in some proofs of theorems about geometric realizations. We also mention the relationship between geometric realizations and exhibitions.

In Chapter 3, we introduce some known results about geometric realizations of triangulations on orientable surfaces. We also put open problems for geometric realizability of triangulations on orientable surfaces. In Chapter 4, we prove a theorem about geometric realizations of triangulations on the projective plane. (Actually, we consider a geometric realization of a triangulation on the projective plane with one face removed since every triangulation on the projective plane has no geometric realization in $\mathbb{R}^3$.) In Chapter 5, we characterize geometrically realizable triangulation on the Möbius band by using the theorem proved in Chapter 4. In Chapter 6, we put appendices.

Papers underlying the thesis

- A. Nakamoto and S. Tsuchiya, A face of a projective triangulation removed for its geometric realizability, *Discrete Computational Geometry* **47** (2012), 215–234.
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Contents

Introduction

Historically, polyhedra have been studied as objects of mathematical study. Their roots reach back to the ancient Greek mathematicians, having a first highlight in their enumeration of the famous *Platonic Solids*, i.e. convex polyhedra with equivalent faces composed of congruent convex regular polygons. The Platonic solids are called *regular solids* or *regular polyhedra*. It is known that there are exactly five Platonic solids, i.e. the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron. An account of these five Platonic solids is the final topic of Euclid's "Elements" [45]. In physics, chemistry, biology, architecture and mathematics, there are many problems which include Platonic solids (See [4, 6, 13, 29, 31, 47, 55, 63, 64, 67, 70, 79, 82, 88, 94, 95, 96, 99, 104] and so on).

If we do not require polyhedra to be convex, we can find more regular polyhedra called the *Kepler-Poinsot polyhedra* (or *regular star polyhedra*), i.e. the small stellated dodecahedron, the great stellated dodecahedron, the great dodecahedron and the great icosahedron. It is known that Kepler [66] discovered the small stellated dodecahedron and the great stellated dodecahedron, and Poinsot [84] discovered the great dodecahedron and the great icosahedron. Cauchy [28] proved that there are exactly four Kepler-Poinsot Polyhedra.

Similarly, if we allow weaker conditions, we can find more regular polyhedra. Petrie and Coxeter found three new regular polyhedra called *Petrie-Coxeter polyhedra* (or *regular skew polyhedra*), and proved the completeness of that enumeration (See [33]). In [51], Grünbaum found new regular polyhedra, and Dress proved that we can complete the list by adding just one more polyhedron [39, 40]. Therefore, Grünbaum's new regular polyhedra are called *Grünbaum-Dress polyhedra*. In [75] (or Section 7E of [76]), McMullen and Schulz gave a quick method of arriving at the full characterization for Grünbaum-Dress polyhedra. For regular polyhedra, see also [34, 35, 36, 37, 52, 73, 74].

In this thesis, we deal with convex polyhedra and non-convex polyhedra in mathematics, in particular polyhedra in computational geometry and topological graph theory.

We use the basic terminology of graph theory from [38]. A *graph* is a pair $G = (V, E)$ such that the elements of E are 2-element subsets of V. To avoid notational ambiguities, we assume that $V \cap E = \emptyset$. The elements of V are *vertices* (or *nodes*, or *points*) of the graph G, and the elements of E are its *edges* (or *lines*). We often denote the vertices (resp., edges) of G by $V(G)$ (resp., $E(G)$). A graph can be regarded as a mathematical model which expresses such structures of finite sets with some relation. An *embedding of a graph* G on a surface is a drawing of G on the surface without crossing edges. A *map* M is a fixed embedding of a simple graph on a surface F^2 such that each face of M is isomorphic to a 2-cell. In this thesis, we always consider a 2*-cell embedding*, i.e. an embedding such that each face is isomorphic to a 2-cell. A *geometric realization* of M is an embedding of F^2 into a Euclidian 3-space $\mathbb{R}^3$ with no self-intersection such that each face of M is a flat convex polygon. Similarly, we can define *geometric realizations in* $\mathbb{R}^4$. Note that there exist geometric realizations allowed self-intersection, i.e. *Kepler-Poinsot-type* realizations. For such realizations, see [20, 92, 93, 105]. In this thesis, we only deal with geometric realizations with no self-intersection. Then, which maps have geometric realizations? For the spherical case, Steinitz [97] gave a complete answers to the above question (See also [8, 50, 83, 85, 101, 106]).

Theorem 0.1 (Steinitz [97]) *A map G on the sphere has a geometric realization as a convex polyhedron if and only if G is* 3*-connected.*

By extending Theorem 0.1, Hong and Nagamochi [58] have presented a characterization of nonconvex polyhedra, which solves an open problem posed by Grünbaum [53] (see also [56, 57]). For *Klein's maps* (i.e. regular maps found by Klein [68, 69]), Schulte and Wills constructed a geometric realization of one of them [91]. For *Dyck's map* (i.e. regular maps found by Dyck [42, 43, 44]), Schulte and Wills constructed a Kepler-Poinsot-type realization of one of them [92]. Moreover, Bokowski constructed a geometric realization of Dyck's map [11]. Brehm also found a geometric realization of Dyck's map [23].

It is known that not all maps have geometric realizations. For example, a map with all vertices of degree three on a surface F^2, other than the sphere, does not have a geometric realization (See Ex. 11.1.7 and EX. 13.2.3 of [50]). Since all 6-3-*equivelar* maps on the torus (i.e. maps consisting of only 6-gons with every vertex belonging to exactly three 6-gons) are simple, they have no geometric realizations. Furthermore, in [9], Betke and Gritzmann showed a combinatorial obstruction to geometric realizability.

Theorem 0.2 (Betke and Gritzmann [9]) *Suppose that M is a map on a surface with at least two of odd degree vertices. Let W be any subset of the set of odd degree vertices of M where W is not an empty set. Let F_W denote a set of faces of M which contains at least one vertex of W. If $2|F_W| \leq |W|$, then M has no geometric realization.*

Although, the Betke-Gritzmann's obstruction rules out realizability of 6-3-equivelar maps on the torus, McMullen, Schulz, and Wills used the obstruction in order to show non-realizability for other, non-simple families of equivelar maps [77].

For general maps on orientable surfaces, Brehm [25] has proved that the realizability problem is NP-hard as a consequence of his universality theorem for realization spaces of maps (cf. [107]). So, we consider geometric realizability of "triangulations on surfaces".

A *triangulation* on a surface F^2 is a map on F^2 such that each face is triangular. In a triangulation, every triangular face contains at most three odd degree vertices. This implies that if M is a triangulation on a surface, then $|F_W| \geq |W|$ for every subset W of odd degree vertices of M (W and F_W are the same notions in Theorem 0.2). Therefore, the Betke-Gritzmann's obstruction rules out realizability of triangulations on surfaces. Moreover, it is known that, for every individual triangulation on the orientable surfaces, realizability of the triangulation in $\mathbb{R}^3$ is decidable (cf. [12] and Ch. VIII of [19]).

When does a map have a geometric realization? The problem, restricted to triangulations, was first proposed by Grünbaum [50], who conjectured that "*every triangulation on any orientable closed surface has a geometric realization*" (See also [27, 41]). Theorem 0.1 implies that every triangulation on the sphere has a geometric realization since every triangulation on a surface is 3-connected. A triangulation on a surface with n vertices called a *vertex-minimal n-vertex triangulation on the surface* if there are no triangulations on the surface with less than n vertices. Möbius found the vertex minimal 7-vertex triangulation on the torus, which is called *Möbius' torus* [80]. We obtain such a triangulation by embedding a complete graph K_7 with 7 vertices into the torus. Note that there is exactly one vertex-minimal triangulation on the torus. Jungerman and Ringel constructed the series of examples of vertex-minimal triangulations for all orientable surfaces [65]. Ringel constructed the series of examples of vertex-minimal triangulations for all non-orientable surfaces [86].

Császár constructed a geometric realization of Möbius' torus, which was called *Császár's torus* [32] (See also Ex. 13.2.3 of [50] and [72]). For a popular account and details on how to build a model of Császár's torus, see [48] and [102]. Bokowski and Eggert proved that there are exactly 72 different types of geometric realizations of Möbius' torus [17]. In reference to Császár's torus, we introduce "Szilassi's torus". Consider the *Heawood map*, which is the dual of Möbius' torus. Note that the Heawood map has 14 vertices, each of degree three, and each face is bounded by a hexagon. Moreover, the Heawood map is the smallest 6-3-equivelar map on the torus. Szilassi [98] proved that the Heawood map has a realization in $\mathbb{R}^3$ such that, for each face, all of the interior points are coplanar, and the interiors of the faces do not intersect, which is called *Szilassi's torus* or *Szilassi's polyhedron* (See also [49, 103]). Note that Szilassi's torus is not a geometric realization since it has non-convex faces.

Can we deal with geometric realizability on larger class of toroidal triangulations? Altshuler discovered triangulations on the torus with geometric realizations [2, 3]. Alaoglu and Giese constructed various geometric realizations of toroidal triangulations [1]. Grünbaum and Szilassi proved that, for any positive even integer $n \geq 14$, there exists a geometrically realizable triangulation on the torus with n faces [54]. Fendrich proved that a triangulation on the torus with up to eleven vertices has a geometric realization [46].

On realizability problem for triangulations on the torus, there exists a breakthrough brought by "exhibitions" (we give a definition of exhibitions in Chapter 2). In [5], Archdeacon, Bonnington and Ellis-Monaghan proposed exhibitions in order to prove geometric realizability problem on toroidal triangulations. By using topological graph theoretic methods and the new notion exhibitions, they have proved the following.

Theorem 0.3 (Archdeacon, Bonnington and Ellis-Monaghan [5]) *Every toroidal triangulation has a geometric realization.*

By Theorems 0.1 and 0.3, Grünbaum's conjecture is true in the spherical case and the toroidal case. In addition, Brehm and Schild [26] proved that every triangulation on the torus has a geometric realization in $\mathbb{R}^4$.

For orientable surfaces of genus from two to six, there are some results on geometric realizability for vertex-minimal triangulations. Bokowski and Brehm constructed several examples of geometric realizations of triangulations on the orientable surfaces of genus from two to four (See [15, 16, 21, 24]). Moreover, Bokowski and Lutz proved the following.

Theorem 0.4 (Bokowski and Lutz [14, 71]) *All* 865 *vertex-minimal* 10*-vertex triangulations on the orientable surface of genus two have geometric realizations.*

For the orientable surfaces of genus three and five, Hougardy, Luts and Zelke proved the following theorems.

Theorem 0.5 (Hougardy, Luts and Zelke [59]) *All* 20 *vertex-minimal* 10*-vertex triangulations on the orientable surface of genus three have a geometric realization.*

Theorem 0.6 (Hougardy, Luts and Zelke [59]) *All* 821 *vertex-minimal* 11*-vertex triangulations on the orientable surface of genus four have a geometric realization.*

Theorem 0.7 (Hougardy, Luts and Zelke [59]) *At least* 15 *of* 751.593 *vertex-minimal* 12*-vertex triangulations on the orientable surface of genus five have a geometric realization.*

In addition, Hougardy, Lutz and Zelke searched small coordinates for geometric realizations of triangulations on the orientable surfaces of genus from one to three [60, 61, 62]. We introduce their results in Chapter 3.

For orientable surfaces of genus at most five, there exist some positive results on Grünbaum's conjecture. However, Bokowski and Guedes de Oliveira proved the following.

Theorem 0.8 (Bokowski and Guedes de Oliveira [18]) *One of the* 59 *vertex-minimal* 12*-vertex triangulation on the orientable surface of genus six has no geometric realization.*

Schewe extended Theorem 0.8. Moreover, he found vertex-minimal 12-vertex triangulations on the orientable surface of genus five with no geometric realizations.

Theorem 0.9 (Schewe [90]) *All of the* 59 *vertex-minimal* 12*-vertex triangulations on the orientable surface of genus six have no geometric realizations.*

Theorem 0.10 (Schewe [89]) *At least* 3 *of* 751.593 *vertex-minimal* 12*-vertex triangulations on the orientable surface of genus five have no geometric realizations.*

Moreover, for at least one of triangulations in Theorem 0.10, it is possible to remove a triangle face from the triangulation while maintaining non-geometric realizability. Therefore, we can find triangulations on the orientable surfaces of genus at least five with no geometric realizations. Hence Grünbaum's conjecture is no longer true now. Although Theorems 0.4, 0.5 and 0.6 give partial answers for Grünbaum's conjecture, the problem deciding whether a triangulation has a geometric realization is still open for the orientable surfaces of genus from two to four.

Let us consider geometric realizability of triangulations on non-orientable surfaces, in particular, the projective plane and the Möbius band. For simple notations, we call a triangulation on the projective plane and that on the Möbius band a *projective triangulation* and a *Möbius triangulation*, respectively, throughout the thesis. Since the projective plane itself is not embeddable in $\mathbb{R}^3$, no map on the projective plane has a geometric realization in $\mathbb{R}^3$. So, Barnette [7] proved that every projective triangulation has a geometric realization in $\mathbb{R}^4$. In [26], Brehm and Schild gave a simpler proof of Barnette's result. In this thesis, we discuss geometric realizability on projective triangulations in $\mathbb{R}^3$, as follows. The surface obtained from the projective plane by removing a disk (i.e. a *Möbius band*) is embeddable in $\mathbb{R}^3$, and hence a Möbius triangulation might have a geometric realization. In [5], the following question was proposed.

Question 0.11 (Archdeacon, Bonnington and Ellis-Monaghan [5]) *Which Möbius triangulation has a geometric realization.*

In this thesis, we give a complete answer for Question 0.11. We know that every Möbius triangulation does not have a geometric realization since Brehm [22] found a Möbius

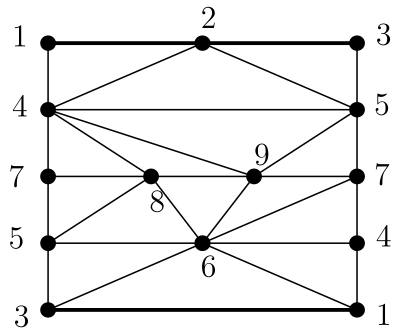


Figure 1: A Möbius triangulation with no geometric realization constructed by Brehm.

triangulation with no geometric realization, which is shown in Figure 1 (In Figure 1, identify vertices with the same label). Brehm essentially proved in [22] that for each of its spatial embeddings, the two disjoint 3-cycles 123 and 456 have a linking number at least two. See [87] for the definition of the linking number. However, two 3-cycles with straight segments in $\mathbb{R}^3$ have a linking number at most one. Hence we have the following.

Fact 0.12 *If a Möbius triangulation G has a boundary 3-cycle C and a 3-cycle C' disjoint from C which forms an annular region with C, then G has no geometric realization.*

In order to define the obstruction shown by Brehm which breaks geometric realizability of Möbius triangulations, we explain "nesting cycles". Let M be a map on a surface. For a cycle C of M bounding a 2-cell, let Int C denote the subgraph of G consisting of all vertices and edges in the interior and the boundary of C. On the other hand, let int C = Int $C - V(C)$. We say that a vertex and an edge of int C are *inner*. Let C and D are trivial cycles of M. We say that C *surrounds* D if Int C contains D. In particular, if $\mathrm{int}\, C$ contains D, then C is a *nesting cycle* of D, which is an important definition in this thesis. Let G be a projective triangulation and let f be a face of G. Let $G - f$ denote the Möbius triangulation obtained from G by removing the interior of f. Suppose that G has a face f such that the boundary 3-cycle of f has a nesting 3-cycle. In this case, we simply say that "f has a nesting 3-cycle". Then, by Fact 0.12, $G - f$ has no geometric realization. Bonnington and Nakamoto proved a positive result on geometric realizability on projective triangulations with one face removed.

Theorem 0.13 (Bonnington and Nakamoto [10]) *Every projective triangulation G has a face f such that $G - f$ has a geometric realization.*

Theorem 0.13 claims that every projective triangulation G has a face f with no nesting 3-cycle. However, for any f with no nesting 3-cycle, the theorem asserts nothing whether $G - f$ has a geometric realization. In this thesis, we would like to consider which faces f of a given projective triangulation G can be chosen so that $G - f$ is geometrically realizable.

One of the main theorems of this thesis is the following.

Theorem 0.14 *Let G be a projective triangulation and let f be a face of G. Then $G - f$ has a geometric realization if and only if f has no nesting* 3*-cycle in G.*

Theorem 0.14 claims that a nesting 3-cycle of f is the only obstruction breaking geometric realizability of $G - f$. By Theorem 0.14, geometrically realizable Möbius triangulations with the boundary cycle of length 3 are characterized. However, the theorem asserts nothing about geometric realizability of Möbius triangulations with the boundary cycle of length at least 4. We shall characterize geometrically realizable Möbius triangulations, as follows.

Theorem 0.15 *A Möbius triangulation M has a geometric realization if and only if M has no two disjoint* 3*-cycles homotopic to the boundary of M.*

Theorem 0.15 claims that the obstruction which Brehm found in [22] is the only obstruction breaking geometric realizability of Möbius triangulations.

Chapter 1

Foundations

In this chapter, we shall give the foundations of this thesis. That is, we shall present basic terminologies and notations of graph theory and topology which will be needed in the following chapters.

1.1 Graphs

A graph is a finite nonempty set of objects called *vertices* (The singular is *vertex*) together with a (possibly empty) set of unodered pairs of distinct vertices of G called *edges*. The *vertex set* of G is denoted by $V(G)$, while the *edge set* is denoted by $E(G)$. The edge $e = \{u, v\}$ is said to join the vertices u and v. If $e = \{u, v\}$ is an edge of a graph G, then u and v are *adjacent vertices* (or *neighbors*), while u and v are *incident*, as are v and e. The set of neighbors of u is denoted by $N(u)$ (or $N_G(u)$). We allow $u = v$, in which case the edge is called a *loop*. If at least two edges join u and v, then they are called *multiple edges*. A vertex is said to be *isolated* if it is incident with no edge. The degree of a vertex v is the number of edges incident with v and is denoted by $\deg(v)$ (or $\deg_G(v)$).

A graph G is said to be *simple* if G has neither loops nor multiple edges, that is, there is no edge joining a vertex and itself and there is at most one edge between each pair of vertices of G. It is clear that for each $v \in V(G)$, $\deg(v) = |N(v)|$ if G is simple.

For two graphs K and G, K is said to be a *subgraph* of G if $V(K) \subset V(G)$ and $E(K) \subset E(G)$.

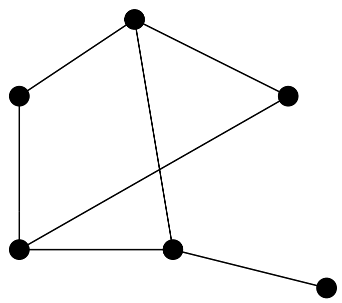


Figure 1.1: A graph

Let G be a graph and let S be a subset of $V(G)$. A subgraph K of G is said to be *induced* by S and denoted by $\langle S \rangle$ if $V(K) = S$ and $E(K)$ consists of the edges of G whose ends are both in S. Similarly, for a nonempty subset $E' \subset E(G)$, a subgraph K of G is said to be *induced* by E' if $V(K)$ consists of the ends of the edges in E' and $E(K) = E'$.

We often construct new graphs from old ones by deleting or adding some vertices and edges. For a subset W of $V(G)$, we define $G - W = \langle V(G) - V(W) \rangle$. Similarly, for a subgraph H of G, we define $G - H = \langle V(G) - V(H) \rangle$.

Let G be a graph and let

$$W := x_1 e_1 x_2 e_2 \cdots e_k x_{k+1},$$

where for $x_i \in V(G)$ and $e_i \in E(G)$, each e_i joins x_i joins x_i and x_{i+1} for $i = 1, 2, \ldots, k$. Then the sequence W is called a *walk* in G, and x_1 and x_{k+1} are called the *ends* (or the *endpoints*) of W. The number k is called the *length* of W and denoted by $|W|$. If $x_1, \ldots, x_{k+1}$ are all distinct, then W is called a *path* in G.

In a walk $W = x_1 e_1 x_2 e_2 \cdots e_k x_{k+1}$, if $x_1 = x_{k+1}$, then the walk W is said to be *closed.* A closed walk W is called a *cycle* if $x_1, \ldots, x_k$ are all distinct. For a path or cycle C in a graph G, a *chord* of C is an edge xy such that $x, y \in V(C)$ and $xy \notin E(C)$.

A graph G is said to be *connected* if for every pair of distinct vertices x and y of G, there is a path in G connecting x and y. A maximal connected subgraph of G is called a *component* of G. A graph G is called *n-connected* if for any subset $S \subset V(G)$ with $|S| < n$, the subgraph of G induced by $V(G) - S$, denoted by $G - S$, is connected.

Given an edge xy of a graph G, the graph G/xy is obtained from G by *contracting* the edge xy. To get G/xy, we identify the vertices x and y and remove all resulting

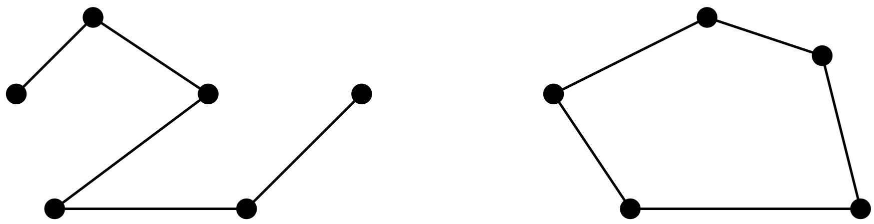


Figure 1.2: A path and a cycle

loops and multiple edges. A graph obtained by a sequence of edge-contractions is called a *contraction* of G.

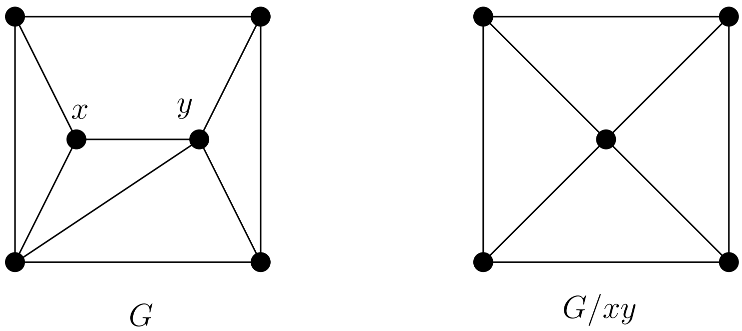


Figure 1.3: A graph G and its contraction G/xy

1.2 Embedding of graphs into surfaces

Throughout this thesis, we shall call a connected compact 2-dimensional manifold without boundaries a *closed surface*. There are two classes of closed surfaces, *orientable* ones and *non-orientable* ones. On an orientable closed surface, we can compatibly prescribe clockwise and counter clockwise orientations around all the points on it. On the other hand, we cannot do on non-orientable closed surfaces. For example, on a Möbius band, we cannot give compatible clockwise orientations to points on center line of the Möbius band (See Figure 1.4). Actually, a closed surface is orientable if and only if it does not include a Möbius band.

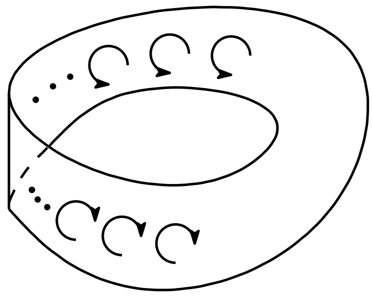


Figure 1.4: Möbius band

Let F_1^2 and F_2^2 be two closed surfaces. The closed surface obtained from F_1^2 with a disk removed and F_2^2 with a disk removed by pasting them along their boundaries is called a *connected sum* of F_1^2 and F_2^2 and denoted $F_1^2 \# F_2^2$. We can characterize orientable and non-orientable closed surfaces, as follows. A closed surface is an *orientable* surface of genus g and denoted by S_g if F^2 is homeomorphic to $\underbrace{T^2 \# \cdots \# T^2}_{g}$, where T^2 denotes the torus. On the other hand, a closed surface is a *non-orientable* surface of genus (or cross-cap number) g and denoted by $\mathbb{N}_g$ if it is homeomorphic to $\underbrace{P^2 \# \cdots \# P^2}_{g}$, where P^2 denotes the projective plane. Equivalently, $\mathbb{N}_g$ is obtained from the sphere with g pairwise disjoint disks removed by attaching g Möbius bands to each boundary of the punctured sphere.

By the classification theorem of closed surfaces, it is known that every closed surface is homeomorphic to either of an orientable surface or a non-orientable surface with some genus.

For non-orientable closed surfaces, it is also known that $\mathbb{N}_3$ and $\mathbb{N}_4$ are homeomorphic to $T^2 \# P^2$ and $T^2 \# K^2$ respectively, where T^2, P^2 and K^2 stand for the torus, the projective plane and the Klein bottle, respectively. Generally, for $\mathbb{N}_g$ and any even integer $g' < g$, N_g is homeomorphic to $\mathbb{N}_{g-g'} \# S_{\frac{g'}{2}}$.

A *closed curve* on a closed surface F^2 is a continuous function $l : S^1 \to F^2$ or its image, where S^1 is the 1-dimentional sphere, that is, $\{(x, y) \in \boldsymbol{R}^2 | x^2 + y^2 = 1\}$. A closed curve l is called *simple* if the function l is an injection. A simple closed curve l on a closed surface F^2 is called *separating* (or *non-separating*) if $F^2 - l$ is disconnected (or connected). A simple closed curve l on F^2 is said to be *trivial* if l bounds a 2-cell on F^2. Otherwise, l is said to be *essential.* Among essential simple closed curves, one with an annular neighborhood is called *2-sided* while one whose tubular neighborhood forms a Möbius band is called *1-sided.* Two closed curves l_1 and l_2 on a closed surface F^2 are said to be *homotopic* to each other on F^2 if there exists a continuous function $\Phi : [0, 1] \times S^1 \to F^2$ such that $\Phi(0, x) = l_1(x)$ and $\Phi(1, x) = l_2(x)$ for each $x \in S^1$

Let us consider a topology of several closed surfaces. See Figure 1.5, which shows developments of the projective plane and the torus, respectively. In the left of Figure 1.5, the projective place is represented as a disk D with every pair of antipodal points on the boundary of D identified. In the right of Figure 1.5, we identify the top and bottom, the left and right of the rectangle, respectively. The projective plane and the torus admit only one essential simple closed curve, up to homeomorphism.

Let us consider a topology of the Klein bottle, which admits three different types of essential simple closed curves. Figure 1.7 shows two developments of $\mathbb{N}_2$. (In Figure 1.7, we identify the top and bottom of the rectangle naturally to get an annulus, and there are two ways to get $\mathbb{N}_2$ from the annulus. One is to identify the two boundary components incoherently as in the left, and the other is to identify each pair of antipodal points of each boundary component as in the right. In particular, the expression of $\mathbb{N}_2$ in Figure 1.6 has two cross caps.) Let α, β, γ be three essential simple closed curves on $\mathbb{N}_2$ as in Figure 1.7, where each of α, β and γ in both figures stands for the same closed curve on $\mathbb{N}_2$. Observe that α is a 2-sided simple closed curve cutting $\mathbb{N}_2$ into an annulus, β is a 1-sided one cutting $\mathbb{N}_2$ into a Möbius band, and β is a 2-sided one separating $\mathbb{N}_2$ into two Möbius

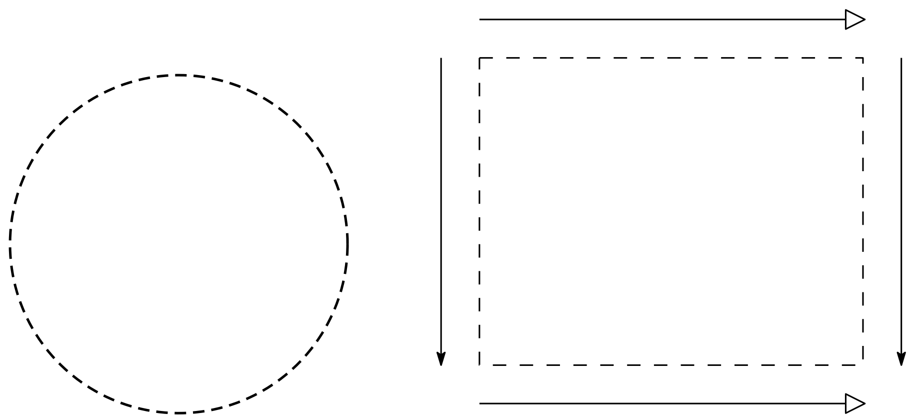


Figure 1.5: The developments of the projective plane and the torus

bands. We say that γ is an *equator*, and a cycle of a graph on $\mathbb{N}_2$ homotopic to γ is an *equator cycle*.

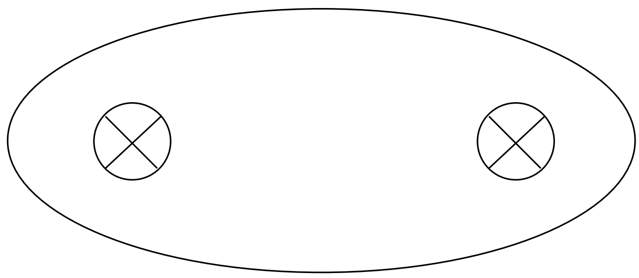


Figure 1.6: Klein bottle

When we discuss embeddings of graphs into surfaces, we regard graphs as 1-dimentional topological spaces, not only as combinatorial objects. Roughly speaking, to *embed* a graph into a surface F^2 is to draw the graph on F^2 without crossing edges. Sometimes, it is effective to regard an embedding as an injective continuous map $f : G \to F^2$. We deal with G and $f(G)$ as the same object intuitively. However, to distinguish G from the embedded one $f(G)$, we often call G an *abstract graph* while we call $f(G)$ an *embedding*. In this thesis, we often denote an embedded graph by G. When G is embedded in a closed surface F^2, then G can be regarded as a subset of F^2. Each component of $F^2 - G$ is called a *face* of G embedded in F^2. A closed walk W of G which bounds a face F of G is called the *boundary walk* of F. An embedded graph G is said to be a *2-cell embedding*, or G is

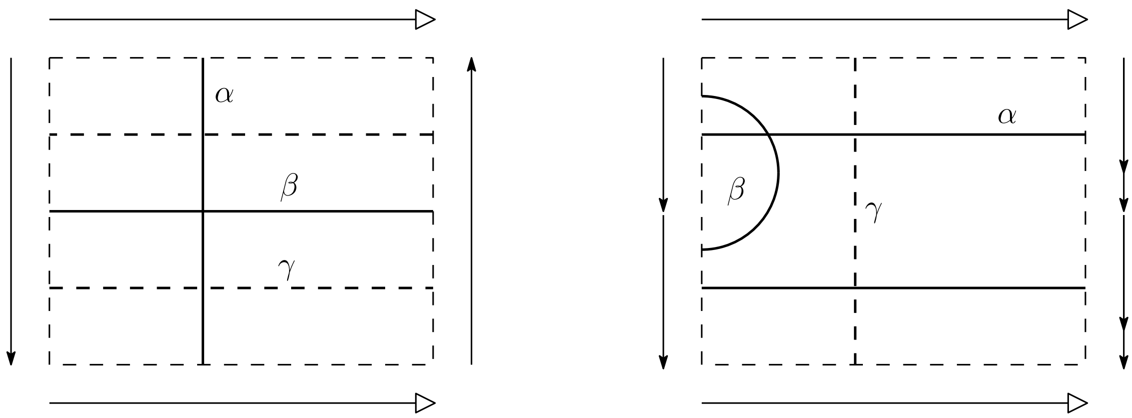


Figure 1.7: Klein bottle with a meridian α, a longitude β and an equator γ

said to be *2-cell embedded* in F^2 if each face of G is homeomorphic to an *open 2-cell*, that is, $\{(x, y) \in \boldsymbol{R}^2 | x^2 + y^2 < 1\}$. For a graph G embedded on a closed surface F^2, we denote the face set of G by $F(G)$, and denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively, as for abstract graphs. A 2-cell embedded graph on a surface F^2 is called a *map* on F^2.

Let G_1 and G_2 be two graphs embedded in closed surfacesF_1^2 and F_2^2, respectively. Two graphs G_1 and G_2 are said to be *homeomorphic* to each other if there exists a homeomorphism $h : F_1^2 \to F_2^2$ with $h(G_1) = G_2$ which induce an isomorphism from G_1 to G_2. In this case, we also say that $G_1 \subset F_1^2$ and $G_2 \subset F_2^2$ are the same ones *up to homeomorphism.*

Note that so far we have not referred to the orientability of surfaces or used *Euler's formula.* To make it explicit, the *Euler characteristic* $\epsilon(F^2)$ of a surface F^2 is defined as

$$\epsilon(F^2) = \begin{cases} 2 - 2h, & F^2 = \mathbb{S}_h, \\ 2 - k, & F^2 = \mathbb{N}_k. \end{cases}$$

The following result is well known.

Theorem 1.1 (Euler's formula.) *Let G be a multigraph which is 2-cell embedded in a surface F^2. Then, the following holds.*

$$|V(G)| - |E(G)| + |F(G)| = \epsilon(F^2)$$

A *triangulation* on a closed surface is a simple graph on the surface with each face triangular. For a graph G on a surface and a vertex v of G, the *link* of v is the boundary closed walk of the union of all faces incident to v in G. When G is a triangulation and v is a vertex of degree k in G, the link of v is a k-cycle. In a map on a surface, a closed walk is said to be *essential* if it does not bounds a 2-cell on the surface.

Chapter 2

Exhibitions

In this chapter, we consider "exhibitions" of maps, which gave a breakthrough in realizability problem of triangulations on surfaces. This notion was proposed in [5]. Exhibitions are used in [5, 10, 30], in order to prove some theorems about geometric realizations of triangulations on surfaces. After we read this chapter, we can understand why exhibitions are useful when we consider geometric realizations of triangulations on surfaces.

2.1 Geometric realizations and Exhibitions

A plane map R with a boundary walk C of length $m \geq 3$ is called a *near triangulation* if C is a cycle and if each inner face of Int C is triangular. Let R be a near triangulation and let C be the boundary cycle of R. Suppose that we are given an embedding of C in $\mathbb{R}^3$ so that each edge is a straight-line segment, and that there exists a plane $P \subset \mathbb{R}^3$ such that the image of C in $\mathbb{R}^3$ is projected to the plane P as a convex polygon. Such an embedding of C in $\mathbb{R}^3$ is called an *exhibition*.

Lemma 2.1 ([5, 10]) *Let R be a near triangulation and let C be the boundary of R. Suppose that an exhibition of C in $\mathbb{R}^3$ is given. Then the exhibition of C extends to a geometric realization of R in the convex-hull of C in $\mathbb{R}^3$.*

Let X be a map on a surface with each face bounded by a cycle. An embedding ψ of X into $\mathbb{R}^3$ is called an *exhibition* if

(i) for any face boundary C of X, $\psi(C)$ is an exhibition,

(ii) for any two distinct faces f and f' of X, the convex hulls of $\psi(f)$ and $\psi(f')$ intersect only at their common vertices and edges in the map.

An exhibition is a relaxed notion of a geometric realization since a geometric realization is an exhibition. In particular, if G is a triangulation, then an exhibition of G is equivalent to a geometric realization of G. The following is an important lemma for constructing a geometric realization of a map, and its proof can easily been obtained from the definition of exhibitions of near triangulations and maps.

Lemma 2.2 *Let G be a triangulation on a surface F^2 and let X be a sub-map of G which is a 2-cell embedding of F^2. If X has an exhibition, then G has a geometric realization.* ■

By Lemma 2.2, we can solve some realizability problems using graph theoretic methods.

2.2 Computer program for exhibitions

The following two computer programs, Programs 1 and 2, check that the coordinates of the vertices given really satisfy the above two conditions (i) and (ii), respectively.

Program 1. Let $A = a_1a_2 \dots a_n$ be a cycle embedded in $\mathbb{R}^3$ so that each edge a_ia_{i+1} is a straight segment, where $a_i \in \mathbb{R}^3$ for each i.

Input: $\mathbb{R}^3$-coordinates of point sets $a_1, a_2, \dots, a_n$, and a plane F.

Output: *True* if the projection of A to F is a convex polygon, and *false* otherwise.

The following is the procedures in Program 1.

Step 1. Let $A' = a'_1a'_2 \dots a'_n$ be a cycle on F obtained by an orthogonal projection of $A = a_1a_2 \dots a_n$ to F, where a'_i corresponds to a_i for each i. For $i = 1, 2, \dots, n$, let l_i be the straight line on F through a'_i and a'_{i+1}. If there exists a'_{i+2} lying on l_i for some i, then return *false*. Otherwise go to Step 2.

Step 2. Let F_i^L and F_i^R be the two half-planes on F separated by l_i, where we suppose that F_i^L contains a'_{i+2}. For each i, if there exists no a'_j with $j \neq i$ contained in F_i^R, then return *true*. Otherwise, return *false*.

Program 2. Let A and B be two point sets in $\mathbb{R}^3$. We check whether or not there exists a plane F containing all points of $A \cap B$ and separating $A' = A - (A \cap B) \neq \emptyset$ and $B' = B - (A \cap B) \neq \emptyset$ in distinct sides. If there exists such F, then we say that F is *admissible*.

Input: $\mathbb{R}^3$-coordinates of point sets $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$, and a plane F containing all points of $A \cap B$.

Output: *True* if F is admissible, and *false* otherwise.

The following is the procedures in Program 2.

Step 1. Set F^+ and F^- to be the two half-spaces in $\mathbb{R}^3$ separated by F, where some point $a_i \in A'$ is contained in F^+.

Step 2. If there is some $a_i \in F^-$, then return *false*. Otherwise go to Step 3.

Step 3. If there is some $b_i \in F^+$, then return *false*. Otherwise return *true*.

The following is an easy observation.

Proposition 2.3 *Let X be a map on a surface, and let $f_1, \ldots, f_n$ be the faces of X. Suppose that the coordinates of all vertices of X are given. If any two of f_i and f_j share at most one edge, then Programs* 1 *and* 2 *check whether the coordinates give an exhibition of X.* ■

Chapter 3

Geometric realizations of graphs on orientable surfaces

In this chapter, we show some results on geometric realizations of maps on orientable surfaces. In general, Grünbaum conjectured that every triangulation on any orientable closed surface has a geometric realization [50]. It has already proved that Grünbaum's conjecture is wrong. However, there exist surfaces such that Grünbaum's conjecture is true. In this chapter, first, we introduce positive results on this conjecture. After that, we introduce negative results on this conjecture, which gives counterexample on the conjecture. Finally, we show open problems.

3.1 Positive results on Grünbaum's conjecture

For orientable surfaces of genus at most five, there exist some positive results on Grünbaum's conjecture. In the spherical case, Steinitz proved that a spherical map G has a geometric realization as a convex polyhedron if and only if G is 3-connected (Theorem 0.1). By his result, we obtain the following since every triangulation is 3-connected.

Theorem 3.1 (Steinitz [97]) *Every triangulation on the sphere has a geometric realization.*

In the toroidal case, there exist some results which deal with *vertex-minimal triangulations*, i.e. triangulations with the minimal possible number of vertices. In [32], Császár constructed a geometric realization of the vertex-minimal triangulation on the torus. Note that there is exactly one vertex-minimal triangulation on the torus shown by Möbius [80] (i.e. Möbius' torus). Bokowski and Eggert extended Császár's result as follows.

Theorem 3.2 (Bokowski and Eggert [17]) *The vertex-minimal triangulation on the torus has exactly* 72 *different types of geometric realizations.*

Moreover, Fendrich proved a positive result for larger class of toroidal triangulations than Möbius' torus.

Theorem 3.3 (Fendrich [46]) *Every triangulation on the torus with up to eleven vertices has a geometric realization.*

Although above results give no complete answer for Grünbaum's conjecture, Császár's torus and Theorems 3.2 and 3.3 gives partial answer in toroidal case.

The *representativity* of G, denoted by $r(G)$, is the minimum number of intersecting points of G and l, where l ranges over all essential simple closed curves on F^2. We say that G is *r-representative* if $r(G) \geq r$. Archdeacon, Bonnington and Ellis-Monaghan gave a complete answer for Grünbaum's conjecture on the torus by using topological graph theoretic methods. In order to solve realizability problem on toroidal triangulations, they have proved a theorem about exhibitions of toroidal maps.

Theorem 3.4 (Archdeacon, Bonnington and Ellis-Monaghan [5]) *Let G be a map on the torus. If G is* 3*-connected and* 3*-representative, then G has an exhibition.*

By using Theorem 3.4, Archdeacon, Bonnington and Ellis-Monaghan have proved that every triangulation on the torus has a geometric realization (Theorem 0.3). Note that we prove our main results (i.e. Theorems 0.14 and 0.15) by using similar way of the proof of Theorem 3.4 in Chapters 4 and 5.

In the case for orientable surfaces of genus from two to five, there exist results which deal with geometric realizability of vertex-minimal triangulations. In [65, 86], the series of examples of vertex-minimal triangulations for all orientable and nonorientable surfaces were constructed.

Bokowski and Lutz [14, 71] proved that every vertex-minimal triangulation on the orientable surface of genus two has a geometric realization (Theorem 0.4). Hougardy, Luts and Zelke [59] proved that every vertex-minimal triangulation on the orientable surfaces of genus three and four has a geometric realization (Theorems 0.5 and 0.6). In [59], they also proved that there exist vertex-minimal triangulations on the orientable surface of genus five which have geometric realizations (Theorem 0.7). Moreover, Hougardy, Lutz and Zelke found small coordinates for geometric realizations of triangulations on the orientable surfaces of genus from one to three. Again, Bokowski and Eggert [17] constructed 72 different types of geometric realizations of Möbius' torus (Theorem 3.2). For their geometric realizations, the following theorem was proved.

Theorem 3.5 (Hougardy, Lutz and Zelke [62]) *For the* 72 *types of geometric realizations of Möbius' torus,* 13 *have coordinate-minimal integer realizations in the* $2 \times 2 \times 3$-*cuboid,* 18 *in the* $3 \times 3 \times 3$-*cube,* 32 *in the* $4 \times 4 \times 4$-*cube, and* 7 *in the* $5 \times 5 \times 5$-*cube, respectively. The* 2 *remaining of the* 72 *types of geometric realizations are realizable in the* $6 \times 6 \times 6$-*cube, but are not realizable in the* $4 \times 4 \times 4$-*cube.*

Moreover, in [62], Hougardy, Lutz and Zelke also proved the following theorems.

Theorem 3.6 (Hougardy, Lutz and Zelke [62]) *Let G be a triangulation on the torus. Then, G cannot be geometrically realized in general position in* $2 \times 2 \times 2$-*cube.*

Theorem 3.7 (Hougardy, Lutz and Zelke [62]) *Let G be a triangulation on the torus with up to* 10 *vertices. Then, G has a geometric realization in general position in the* $4 \times 4 \times 4$-*cube. In particular, other than eleven examples, G has a geometric realization in general position in the* $3 \times 3 \times 3$-*cube, and the eleven examples cannot be geometrically realized in general position in the* $3 \times 3 \times 3$-*cube.*

For vertex-minimal triangulations on the orientable surfaces of genus two and three, Hougardy, Lutz and Zelke proved the following.

Theorem 3.8 (Hougardy, Lutz and Zelke [60]) *Every vertex minimal triangulations on the orientable surface of genus two has a geometric realization in general position in the* $4 \times 4 \times 4$-*cube.*

Theorem 3.9 (Hougardy, Lutz and Zelke [60]) *Every vertex minimal triangulations on the orientable surface of genus two cannot be geometrically realized in general position in the* $3 \times 3 \times 3$*-cube.*

Theorem 3.10 (Hougardy, Lutz and Zelke [61]) *At least* 17 *of the* 20 *vertex minimal triangulation on the orientable surface of genus three have geometric realizations in general position* $5 \times 5 \times 5$*-cube.*

Theorem 3.11 (Hougardy, Lutz and Zelke [61]) *Every vertex minimal triangulation on the orientable surface of genus three cannot be geometrically realized in general position in the* $4 \times 4 \times 4$*-cube.*

3.2 Negative results on Grünbaum's conjecture

Bokowski and Guedes de Oliveira [18] proved that there exists a vertex-minimal triangulation on the orientable surface of genus six which has no geometric realization (Theorem 0.8, see also [13]). Schewe extended their result, that is, he prove that every vertex-minimal triangulations on the orientable surface of genus six has no geometric realization (Theorem 0.9). Schewe also found vertex-minimal triangulations on the orientable surface of genus five with no geometric realization (Theorem 0.10). Moreover, for at least one of triangulations in Theorem 0.10, it is possible to remove a triangle face from the triangulation while maintaining non-geometrically realizability. Therefore, we obtain the following.

Proposition 3.12 *Let* F^2 *be an orientable surface of genus at least five. Then, there is an infinite family of triangulations on* F^2 *with no geometric realizations.*

Proof. Suppose that a triangulation G on the orientable surface of genus five with no geometric realization has a face f such that $G - f$ maintain non-geometrically realizability. Let H be a triangulation on the orientable surface of genus at least one, and let H' be a triangulation obtained from H by removing the interior of a face. By pasting the boundaries of $G - f$ and H', we obtain a triangulation on the orientable surface of higher genus with no geometric realization whose order is $|V(G)| + |V(H)| - 3$. ■

Proposition 3.12 claims that we can find counter examples for Grünbaum's conjecture in the orientable surfaces of genus at least five.

3.3 Open problems

Grünbaum's conjecture is no longer true now, but it is still open for the orientable closed surfaces of genus from two to four. For geometric realizability of triangulations on orientable surfaces, there are some problems as follows.

Problem 1 *Does a triangulation on the orientable surface of genus from two to four have a geometric realization?*

In the case on orientable surface of genus two, Lutz [71] conjectured that we can obtain a positive result. Moreover, Hougardy, Lutz and Zelke [59] extended this conjecture, that is, they conjectured that "*every triangulation on the orientable surfaces of genus from two to four has a geometric realization*". By using exhibitions, Archdeacon, Bonnington and Ellis-Monaghan have proved Theorem 0.3. Using similar idea, we might solve Problem 1. So, the following problems help to solve the problem.

Problem 2 *Does a* 3*-connected and* 3*-representative map on the orientable surface of genus from two to four have an exhibition?*

Problem 3 *Let* G *be a* 3*-connected and* 3*-representative map on the orientable surface of genus from two to four. Then, are there obstructions when we construct an exhibition of* G*?*

By Proposition 3.12, we can find triangulations on the orientable surface of genus at least five which have no geometric realizations. However, there exist geometrically realizable triangulations on the orientable surface of genus at least five (Theorem 0.7). Therefore, we can consider the following problem.

Problem 4 *Which triangulations on the orientable surfaces of genus at least five have geometric realizations?*

Let G be a triangulation on the orientable surfaces of genus at least five. We do not know how many obstructions there are when we make a geometric realization of G. Therefore, if we solve the following problem, then we might obtain a complete answer for Problem 4.

Problem 5 *For triangulations on the orientable surfaces of genus at least five, can we characterize obstructions which break geometric realizability of them?*

Chapter 4

Geometric realizations of triangulations on the projective plane

In this chapter, we consider nonorientable surfaces, in particular, the projective plane. Since the projective plane itself is not embeddable in $\mathbb{R}^3$, no map on the projective plane has a geometric realization. However, the surface obtained from the projective plane by removing a disk (i.e. a *Möbius band*) is embeddable in $\mathbb{R}^3$, and hence we can expect that a Möbius triangulation might have a geometric realization. First, we put known results on geometric realizations of projective triangulations with one face removed. After that, we prove Theorem 0.14, which gives a necessary and sufficient condition for geometrically realizable projective triangulations with one face removed.

4.1 Known results

Let G be a projective triangulation and let f be a face of G. Let $G - f$ denote the Möbius triangulation obtained from G by removing the interior of f. By Fact 0.12, we have the following.

Fact 4.1 *Let G be a projective triangulation and let f be a face of G. If G has a nesting 3-cycle of f, then $G - f$ has no geometric realization.*

Bonnington and Nakamoto proved that, for any projective triangulation G, if we choose a face f of G carefully, then $G-f$ has a geometric realization [10] (Theorem 0.13). In this thesis, we consider which faces f of a given projective triangulation G can be chosen so that $G-f$ is geometrically realizable. Suppose that G has a face f with a nesting 3-cycle of f. Then, by Fact 4.1, $G-f$ has no geometric realization. In this chapter, we prove that a nesting 3-cycle of f is the only obstruction breaking geometrically realizability of $G-f$ (Theorem 0.14).

Theorem 0.14 immediately implies Theorem 0.13, since we can always choose a face f in a projective triangulation G with no nesting 3-cycle. (Suppose that a face g of G has nesting 3-cycles. Choosing C among them to *outermost*, i.e. C has no nesting 3-cycle, we can take a face f in the 2-cell region bounded by C and sharing an edge with C has no nesting 3-cycle.)

Moreover, Theorem 0.14 implies the following since, for any 3-cycle C of a 4-connected projective triangulation, Int C contains no 3-cycle other than C.

Corollary 4.2 *Let G be a 4-connected projective triangulation. Then, for any face f of G, $G-f$ has a geometric realization.*

A weaker version of Corollary 4.2 has already been proved by Chávez, Fijavž, Márquez, Nakamoto and Suárez [30].

Theorem 4.3 (Chávez, Fijavž, Márquez, Nakamoto and Suárez[30]) *If G is a 5-connected projective triangulation, then for any face f of G, $G-f$ has a geometric realization.*

Corollary 4.2 can be strengthened by introducing a notion of the cyclical k-connectivity. A graph H is said to be *cyclically k-connected* if H has no k-cut S such that at least two components of $H-S$ have a cycle. We have the following, which is conjectured in [30].

Corollary 4.4 *Let G be a projective triangulation and let f be a face of G. Then, for any face f of G, $G-f$ has a geometric realization if and only if G is cyclically 4-connected.*

Here we describe an outline of the proof of Theorem 0.14. The proof is split into two parts, in which the first part is graph theoretical and the second part is geometrical. Exhibitions of maps connect two different methods.

In order to prove Theorem 0.14, we first prove the following and Section 4.2 is devoted to it.

Lemma 4.5 *Let G be a projective triangulation and let f be a face of G with no nesting 3-cycle. Then G has a 4-cycle $C = v_1v_2v_3v_4$ separating G into a Möbius triangulation G_M and a near triangulation G_D both of whose boundaries are C such that*

(i) *G_M has a sub-map X with boundary cycle C which is isomorphic to a subdivision of one of the nine maps shown in Figure 4.1,*

(ii) *G_D has a diagonal v_iv_{i+2} for some i, and*

(iii) *G_D contains f such that $f = v_iv_{i+1}v_{i+2}$, or $v_iv_{i+2} \notin E(f)$ but $v_i \in V(f)$.*

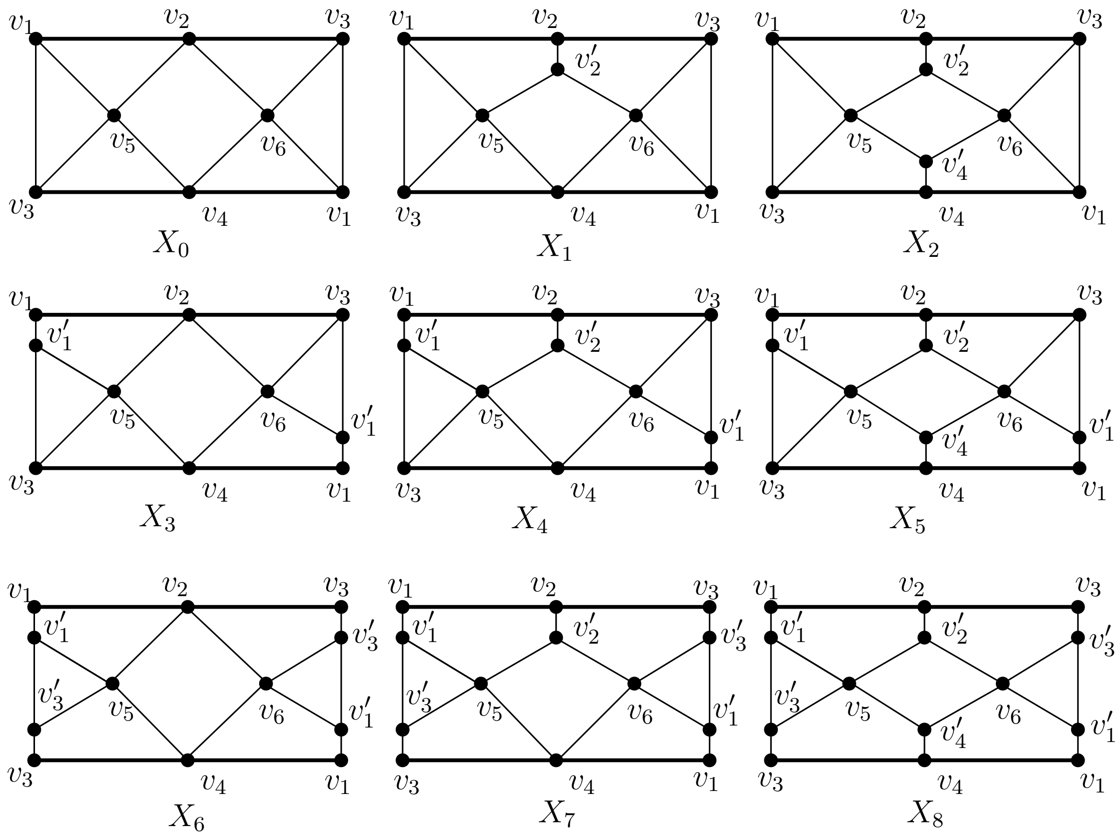


Figure 4.1: Nine maps on the projective plane.

In Section 4.3, we shall prove that G_M in Lemma 4.5 has a geometric realization, and so does $G_D - f$ in the lemma to complete a geometric realization of $G - f$. In order to prove that we use computer programing in Chapter 2.

In Section 4.4, we give a proof of Theorem 0.14.

4.2 Contraction in Möbius 4-triangulations

Let G be a triangulation on a surface F^2 and let e be an edge of G. *Contraction* of e is to remove e, identify the two endpoints of e and replace two pairs of parallel edges by two single edges. (For an edge e contained in ∂G, the contraction of e can be defined similarly except replacing one pair of parallel edges with a single edge.) Let G/e denote the graph obtained from G by contracting e. If G/e is simple, then e is said to be *contractible*. We always contract only a contractible edge. The inverse operation of the contraction of e is a *splitting* of v where v is a vertex obtained from e by the contraction of it.

Lemma 4.6 **([81])** *Let G be a triangulation on a surface F^2, and let Δ be a triangular region which is not a face. Then there is a contractible edge in the interior of Δ.*

Lemma 4.6 immediately implies the following.

Lemma 4.7 *Let G be a triangulation on a non-spherical surface. Then G has no contractible edge if and only if each edge of G is contained in an essential 3-cycle.* ■

Lemma 4.8 *Let G be a triangulation on a non-spherical surface and let $C = v_1v_2v_3v_4$ be a trivial 4-cycle of G. If G has no contractible edge, then the interior of C contains either a single diagonal or a single vertex. Furthermore, the latter happens if and only if the exterior of C has two edges v_1v_3 and v_2v_4.*

Proof. If Int C has a diagonal v_iv_{i+2}, C satisfies Lemma 4.8 by Lemma 4.7. So we suppose that C has no diagonal in the interior of C. Then it is easy to see that the subgraph of G induced by the inner vertices of C is connected. So, if the interior of C contains at least two vertices, then we can find an edge joining two inner vertices w_1 and w_2. By Lemma 4.7, w_1w_2 is contained in an essential 3-cycle $D = w_1w_2x$ since G has no contractible edge. However, since w_1 and w_2 are inner vertices of the interior of C, D must be contained in $\mathrm{Int}\, C$. This contradicts that D is essential. If the interior of C contains a single vertex v, then v is adjacent to v_1, v_2, v_3, v_4. Since vv_i is not contractible for $i = 1, 2, 3, 4$, v_i must be adjacent to v_{i+2} in the exterior of C. ■

Let R be a near triangulation with a boundary cycle C and let $x, y \in V(C)$. A path jointing x and y and intersecting C only at x and y is called an *inner* (x, y)*-path* or an *inner path*. We often use the following lemma proved in [10] to find a suitable subgraph in a near triangulation.

Lemma 4.9 ([10]) *Let R be a near triangulation with the boundary cycle C and let $x, y \in V(C)$ with $xy \notin E(C)$. There is an inner (x, y)-path in R if and only if there is no chord pq for any $p, q \in V(C) - \{x, y\}$ such that x, p, y and q appear on C in this cyclic order.*

We also use lemma proved in [78] to find a suitable subgraph. If two paths P and Q have no common vertex except for their endpoints, then these are said to be *internally disjoint.*

Lemma 4.10 (Menger [78]) *Let G be a graph and let $v, v_1, \cdots, v_k$ be distinct vertices of G. Then G has k internally disjoint paths from v to v_i, for $i = 1, \cdots, k$, if and only if G has no $S \subset V(G) - \{v, v_1, \cdots, v_k\}$ separating v and $\{v, v_1, \cdots, v_k\}$ in G such that $|S| < k$.*

The Möbius band admits two types of essential simple closed curves: One is homotopic to the boundary of the Möbius band, and the other is homotopic to a center line of the Möbius band. The former separates the Möbius band, and the latter is non-separating.

In this section, we deal with a Möbius triangulation M whose boundary cycle has length exactly 4, which is called a *Möbius* 4*-triangulation.* (We sometimes let a Möbius 4-triangulation express a map on the projective plane with only one quadrilateral face and all other faces are triangular.) Let ∂M denote the boundary 4-cycle of M. Let e be an *inner* edge of M, that is, an edge not lying on ∂M. We always consider the contractions of contractible inner edges. A Möbius 4-triangulation G is said to be *contractible* to another

Möbius 4-triangulation G' if G is transformed in G' by a sequence of contractions of inner edges.

Let X be the map on the Möbius band with six vertices and the boundary 4-cycle $v_1v_2v_3v_4$ such that each of two inner vertices is adjacent to all of v_1, v_2, v_3 and v_4 (See Figure 4.2). A Möbius 4-triangulation G with at least six vertices is said to be *X-framed* if G has two inner vertices each of which is adjacent to all four vertices in the boundary.

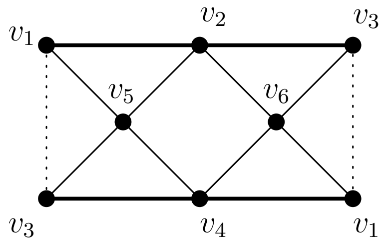


Figure 4.2: The map X.

Lemma 4.11 *Let M be a Möbius 4-triangulation with the boundary cycle C. If M has no 3-cycle homotopic to C, M is contractible to a X-framed Möbius 4-triangulation with the boundary cycle C.*

Proof. Let $C = v_1v_2v_3v_4$. Suppose that M has a trivial 3-cycle D which is not a face boundary. Then int D has a contractible edge e by Lemma 4.6. If M/e has a 3-cycle homotopic to C, e is contained in a 4-cycle homotopic to C in M, contrary to that M has no 3-cycle homotopic to C or M is simple. Therefore M/e has no 3-cycle homotopic to C and we may suppose that M has no trivial separating 3-cycle. Suppose that M has a 4-cycle $C' = v_1'v_2'v_3'v_4'$ homotopic to C but $C' \neq C$. If M has at least two such cycles, we take *outermost* one (i.e. no 4-cycle homotopic to C in the region bounded by C and C'). By Lemma 4.10, M has a path P_i from v_i to v_i' for $i = 1, 2, 3, 4$ such that each P_i is disjoint to P_j for $j = 1, 2, 3, 4$, but $j \neq i$, since M has no 3-cycle homotopic to C. There exists an edge $e = v_i'p$ on P_i adjacent to v_i' for some i since $C \neq C'$. If e is a non-contractible edge, e is contained in an essential 3-cycle $v_i'pq$ since M has no trivial separating 3-cycle. Moreover, since M has no 3-cycle homotopic to C, $v_i'pq$ is non-separating. So q must be contained in C' and $q \in V(P_{i-1})$ or $V(P_{i+1})$, contrary to the simpleness of M. Therefore we may suppose that e is a contractible edge. If M/e has a 3-cycle homotopic to C, e is contained in a 4-cycle $v_i'pqr$ homotopic to C in M. Let M' be a triangulation on the Möbius band bounded by C'. Since C' is outermost in M, r must be an inner vertex of M', contrary to that M has no 3-cycle homotopic to C or M is simple. Therefore if M has a 4-cycle C' homotopic to C we can contract edges of M so that $C' = C$ without 3-cycles homotopic to C. So we may suppose that M has no 4-cycle homotopic to C except for C and we shall prove that if M has no contractible edge, M is a X-framed Möbius 4-triangulation.

Observe that $\deg_M(v_i) \geq 3$ for $i = 1, 2, 3, 4$. (For otherwise, i.e. if $\deg_M(v_2) = 2$ for example, then M has a face $v_1v_2v_3$, and hence $v_1v_3v_4$ is a 3-cycle of M homotopic to C, a contradiction.) If $\deg_M(v_i) = 3$ for some i, say $N_M(v_2) = \{v_1, x, v_3\}$, then we have $x \notin \{v_2, v_4\}$. Hence we take $C' = v_1xv_3v_4$ as a 4-cycle homotopic to C in M, a contradiction. Therefore we may suppose that $\deg_M(v_i) \geq 4$ for $i = 1, 2, 3, 4$. So, if we let $v_iv_{i+1}l_i$ be the face of M incident to the edge v_iv_{i+1} for each i, then l_i and l_{i+1} are distinct. We shall prove that $l_1 = l_3$ and $l_2 = l_4$. So we may suppose $l_1 \neq l_3$ in M by symmetry. Since v_1l_1 is not contractible, it is contained in an essential 3-cycle, say $C_1 = v_1l_1x$, by Lemma 4.7. Moreover, since M has no 3-cycle homotopic to C, C_1 is non-separating. Similarly, since v_3l_3 is not contractible by symmetry, v_3l_3 is contained in an essential non-separating 3-cycle $C_2 = v_3l_3y$. Since C_1 and C_2 are non-separating cycles on the Möbius band, and since $l_1 \neq l_3$, $v_1 \neq l_3$ $(l_1 \neq v_3)$, $v_1 \neq v_3$, $x \neq v_3$ $(v_1 \neq y)$ and $x \neq l_3$ $(l_1 \neq y)$ we have $x = y$. See the left in Figure 4.3. By Lemma 4.8, each of the two quadrilateral regions $v_1xl_3v_4$ and $l_1v_2v_3y$ has a diagonal, which must be an edge xv_4 and yv_2 respectively. (For otherwise, we would have $\deg_M(v_2) = 3$ or $\deg_M(v_4) = 3$, contrary to that $\deg_M(v_i) \geq 4$ for $i = 1, 2, 3, 4$.) Moreover, we have $x = y = l_2 = l_4$. See the right in Figure 4.3.

Now we focus on the edges v_2l_1 and v_4l_3. Similarly to the edge v_1l_1, if they are not contractible, they are contained in essential non-separating 3-cycles v_2l_1x' and v_4l_3y', respectively. Since v_2 and l_1 have only one common neighbor other than v_1 and l_2, we have $x' = v_3$. By the same argument, we have $y' = v_1$. This is a contradiction since two 3-cycles $v_2l_1v_3$ and $v_4l_3v_1$ cannot exist simultaneously. So, we can contract inner edges until $l_1 = l_3$ and $l_2 = l_4$. See the left of Figure 4.4. Hence M is a X-framed Möbius

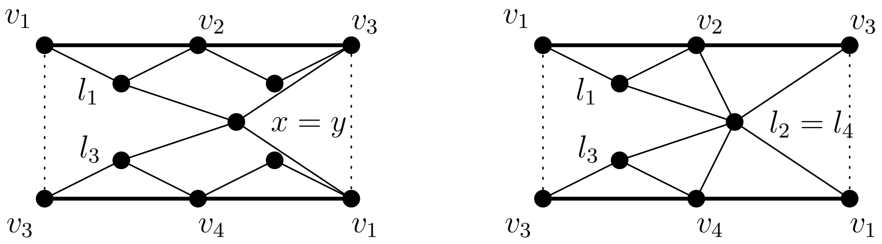


Figure 4.3: The structure of M.

4-triangulation. ■

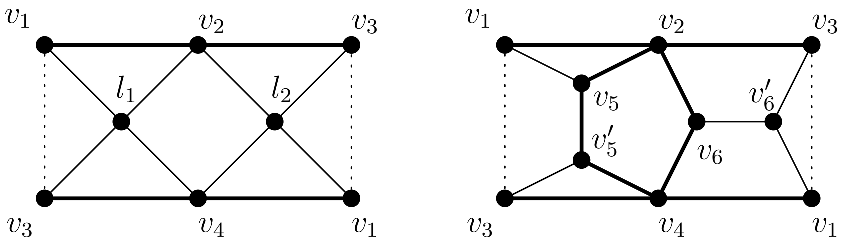


Figure 4.4: The structure of M (left) and an inner trivial cycle of H (right).

Let M_0 be a X-framed Möbius 4-triangulation with the boundary cycle $C = v_1v_2v_3v_4$ and inner vertices v_5, v_6 adjacent to each vertex on C. Let H_0 be a X-frame of M_0. Consider a splitting of v_i of H_0 into two vertices v_i and v'_i of degree 3. We always suppose that v_i, for $i = 1, 2, 3, 4$, lies on C. Let H, called a *split X-frame*, be a map with the boundary 4-cycle C on the Möbius band obtained from H_0 by applying a sequence of either splitting of each v_i or subdividing each edge. We call a vertex v of H whose degree is greater than 2 is a *node* and a path in H containing only two nodes as its endpoints a *segment*. Let $P(a, b)$ denote a path on a segment of H with two endpoints a, b. If $a = b$, we suppose that $P(a, b) = \{a\}$. Moreover, $\{v_i, v'_i\}$ is called a *boundary split pair* or *boundary nodes* if they arose by a splitting of v_i on the boundary, otherwise it is called an *inner split pair* or *inner nodes*. Let *d-segment* be a segment of a split X-frame whose endpoints are a boundary node and an inner node. A trivial cycle of the split X-frame which contains four d-segments without segments on the boundary is called an *inner trivial cycle*. (For example, see the right of Figure 4.4.) Observe that a split X-frame has two inner trivial cycles.

Proof of Lemma 4.5. Let D be a 3-cycle surrounding f. If G has at least two such 3-cycles, we choose the maximal one. Let $\tilde{f}$ denote a triangular region bounded by D. Observe that G has no 3-cycle surrounding $\tilde{f}$ except for $\partial\tilde{f}$ since G has no nesting 3-cycle of f. Since G is a triangulation, G has three faces neighboring to $\tilde{f}$ among which we choose a face f' as follows. If f has an edge e such that $e \in E(\partial\tilde{f})$, we choose f' so that e lies on $\partial(\tilde{f} \cup f')$. So we consider the case that $E(f) \cap E(\partial\tilde{f}) = \emptyset$. Let u_1, u_2, u_3 be vertices lying on $\partial\tilde{f}$. Since f has no nesting 3-cycle, there exists a vertex v of f such that $v \in V(\partial\tilde{f})$. We may suppose $v = u_1$. In this case, we choose f' whose boundary contains u_1u_3. Let C be $\partial(\tilde{f} \cup f')$ and let G_M (resp., G_D) be the Möbius triangulation (resp., the near triangulation) with boundary cycle C. Observe that C satisfies the conditions (ii) and (iii) of Lemma 4.5.

We shall prove that C satisfies the condition (i) of Lemma 4.5. Observe that, G_M has no 3-cycle homotopic to C by the definition of C. So G_M is contractible to a X-framed 4-triangulation G_{M0} with the boundary cycle $v_1v_2v_3v_4$ by Lemma 4.11. Therefore G_M has a split X-frame.

Claim 1 *There exists a split X-frame of G_M with the boundary cycle $v_1v_2v_3v_4$ and no inner split pair.*

Proof. Let H_0 be a X-frame of G_{M0} and let $V(H_0) - V(C) = \{v_5, v_6\}$. Let H be a split X-frame of G_M with the boundary cycle $C = v_1v_2v_3v_4$ obtained from H_0 by a sequence of splitting of a vertex v_i of H_0 into v_i, v_i' of degree 3 and subdividing edges. We may suppose that each segment of H has no chord in G_M. (Otherwise we can take a shorter segment in G_M.) By the simpleness of G_{M0}, we may suppose that the interior of $v_5v_4v_6v_2$ contains no chord from v_5 to v_6 in G_{M0}. (Otherwise the interior of $v_5v_3v_6v_1$ does not contain it.) Therefore we may suppose that the interior of one inner trivial cycle of H has no chord xy such that $x \in V(P(v_5, v_5'))$ and $y \in V(P(v_6, v_6'))$ in G_M. Let H' be a split X-frame of G_M with the boundary cycle $v_1v_2v_3v_4$. Let u_i, u_i' for $i = 1, 2, 3, 4$ be boundary nodes of H' and u_i, u_i' for $i = 5, 6$ be inner nodes of H'. We always take H' in G_M so that the interior of one inner trivial cycle has no chord from $x \in V(P(u_5, u_5'))$ to $y \in V(P(u_6', u_6))$ and $P(u_i, u_i')$ for $i = 5, 6$ is as short as possible.

Suppose that each of u_i for $i = 1, 2, 3, 4$ is a boundary split pair. (Otherwise we consider $u'_i = u_i$ for $i = 1, 2, 3, 4$.) We shall prove that neither u_5 nor u_6 is not a split pair of H'. Suppose that u_5 is a split pair of H'. Then, there are two cases of the splitting of u_5 as shown in Figure 4.5.

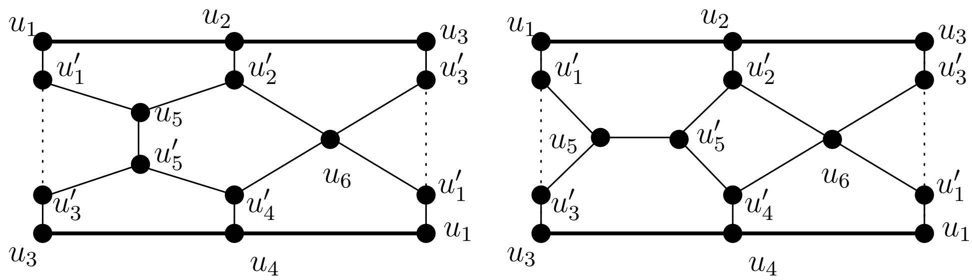


Figure 4.5: A splitting of u_5 in H'.

First we consider the case in the left-hand side of Figure 4.5. There exist two inner trivial cycles D_1, D_2 in H'. By the definition of H', we may suppose that $D_1 = P(u_5, u'_5)P(u'_5, u'_4)P(u'_4, u'_6)\ P(u'_6, u_6)P(u_6, u'_2)P(u'_2, u_5)$ and Int D_1 has no chord from $x \in V(P(u_5, u'_5))$ to $y \in V(P(u'_6, u_6))$ in G_M. Let us consider the neighbor of u_5 in Int D_1 on G_M. If Int D_1 has a chord from u_5 to l, where $l \in V(P(u'_4, u'_5)) - \{u'_5\}$ or $l \in V(P(u'_4, u_6)) - \{u_6\}$, we can take other split X-frames, contrary to the assumption of H'. (See Figure 4.6.)

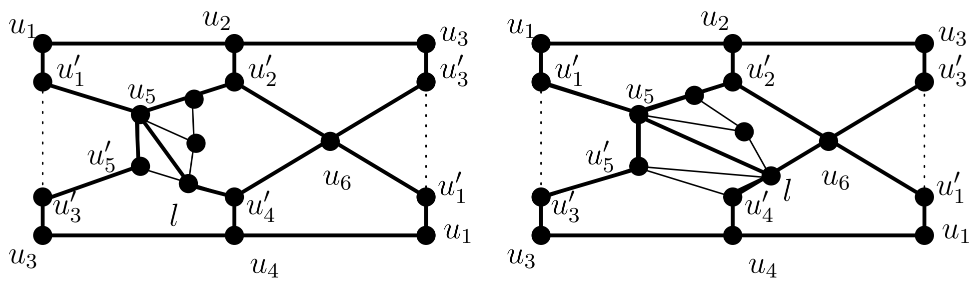


Figure 4.6: Other split X-frames.

If Int D_1 has a chord from u_5 to l, where $l \in V(P(u'_2, u_6)) - \{u_6\}$, then we can take an inner path from l to l', where l' is adjacent to u_5 on $P(u_5, u'_5)$, through the neighbor of u_5. If we regard l' as new u_5, we can take another split X-frame, contrary to the assumption of H'. (See the left of Figure 4.7.) If Int D_1 does not have such chords, we can take a path from u_5' to u_2' through the neighbor of u_5. So we can take another split X-frame, contrary to the assumption of H'. (See the right of Figure 4.7.)

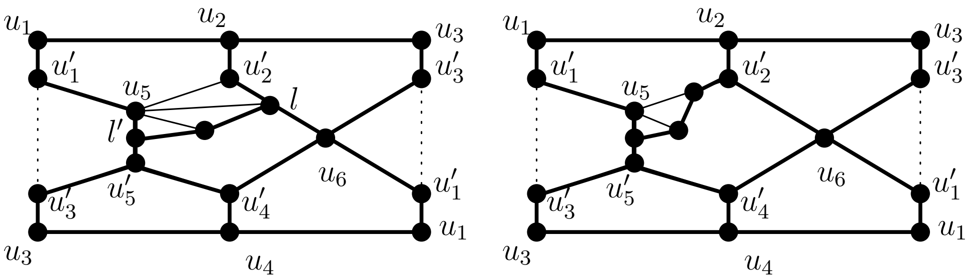


Figure 4.7: Other split X-frames.

In the right-hand side of Figure 4.5, we consider the neighbor of u_5 in the interior of $P(u_5, u_5')\ P(u_5', u_2')P(u_2', u_2)u_2u_1P(u_1, u_1')P(u_1', u_5)$. By the same argument, we can obtain a contradiction. So u_5 of H' is not a split node. We can deal with u_6 similarly. □

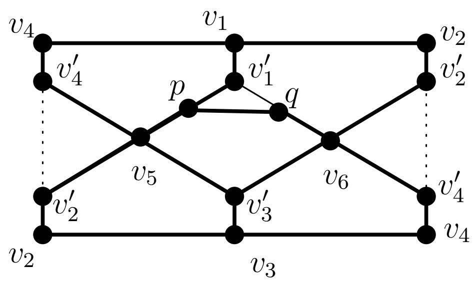
By Claim 1, G_M has a split X-frame with no inner split pair. Next we shall prove that there exists a split X-frame with no inner split pair such that an inner trivial cycle has an inner path whose endpoints are two boundary nodes of it.

Claim 2 *Let D_j for $j = 1, 2$ be an inner trivial cycle of a split X-frame in G_M. There exists a split X-frame with the boundary $v_1v_2v_3v_4$ and no inner split pair such that one of Int D_j has an inner path from a boundary node to another boundary node in G_M.*

Proof. Let H be a split X-frame of G_M with the boundary $v_1v_2v_3v_4$ and no inner split pair. We may suppose that each segment of H has no chord in G_M. (Otherwise we can take a shorter segment in G_M.) Let v_5, v_6 be inner nodes of H. We may suppose $D_1 = P(v_1', v_5)P(v_5, v_3')P(v_3', v_6)P(v_6, v_1')$. (If v_i is not a split node, we regard v_i as a node on D_j, for $j = 1, 2$.) We shall prove that one of Int D_j contains an inner (v_j', v_{j+2}')-path in G_M. By symmetry and simpleness of G_M, we may suppose that Int D_1 contains no chord v_5v_6. We take H so that Int D_1 contains few chords in G_M as possible. By Lemma 4.9 and symmetry, if Int D_1 contains no inner (v_1', v_3')-path we may suppose that Int D_1 contains a chord pq such that $p \in V(P(v_5, v_1')) - \{v_1', v_5\}$, $q \in V(P(v_1', v_6)) - \{v_1'\}$ or $q \in V(P(v_3', v_6)) - \{v_3', v_6\}$.

If $q \in V(P(v_1', v_6)) - \{v_1'\}$, we can regard p as a new boundary node v_1' and we can take another split X-frame with no chord v_5v_6 in Int D_1 so that the number of chords contained in Int D_1 is fewer, a contradiction. (See the left of Figure 4.8.) Therefore we may suppose that Int D_1 contains no chord xy such that $x \in V(P(v_5, v_1'))$, $y \in V(P(v_6, v_1'))$ or $x \in V(P(v_5, v_3'))$, $y \in V(P(v_6, v_3'))$.

If $q \in V(P(v_3', v_6)) - \{v_3', v_6\}$, let $P(v_3, v_3')P(v_3', v_6)P(v_6, v_4')P(v_4', v_4)v_4v_3 = D$. In this case, Int D has no chord rs such that $r \in V(P(v_6, q)) - \{q\}$, $s \in V(P(q, v_3')) - \{q\}$ or $s \in V(P(v_3, v_3'))$ in G_M. (Otherwise the segment from v_6 to v_3' has a chord or we can take another split X-frame with no chord v_5v_6 in Int D_1 so that the number of chords in Int D_1 is fewer, a contradiction.) Therefore Int D has a path from q to v_4 which does not intersect $V(P(v_3, v_3') \cup P(v_3', v_6) - \{q\})$. If we regard q as a v_6, we can take another split X-frame with no chord v_5v_6 in Int D_1 so that the number of chords in Int D_1 is fewer, a contradiction. (See the right of Figure 4.8.) Hence Int D_1 has no chord pq and we can take an inner (v_1', v_3')-path in Int D_1. □



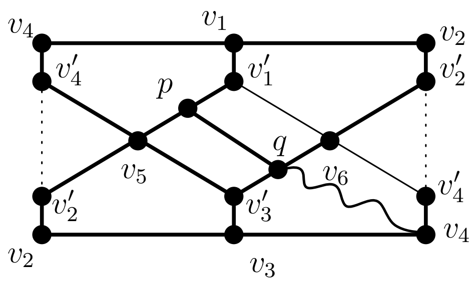


Figure 4.8: Other split X-frames.

By Claims 1 and 2, G_M has a split X-frame H with no inner split nodes and an inner (v_i, v_{i+2})-path P such that $P \cap H = P(v_i, v_i') \cup P(v_{i+2}, v_{i+2}')$. Then G_M has one of sixteen subgraphs since for each boundary node, there exist two possibilities depending on whether it has a split pair or not. By symmetry of $H \cup P$, G_M has a subdivision of one of the nine maps shown in Figure 4.1 as a subgraph. Therefore C satisfies the condition (i) of the lemma. ■

4.3 Constructions of geometric realizations

In this section, in order to prove Theorem 0.15, we shall construct geometric realizations and exhibitions of given maps. We first construct an exhibition of the nine maps $X_0, X_1, \ldots, X_8$ shown in Figure 4.1.

Lemma 4.12 *For $i = 0, \ldots, 8$, each X_i has an exhibition.*

Let $f_1, \ldots, f_n$ be the faces of X_i and let D_j be the boundary cycle of f_j, for each j. To prove Lemma 4.12, by the definition of exhibitions, we have to arrange the vertices of X_i in $\mathbb{R}^3$ so that

(i) the embedding of each D_j is projected to some plane as a convex polygon, and

(ii) for any distinct $j, k \in \{1, \ldots, n\}$, two convex hulls $\langle D_j \rangle$ and $\langle D_k \rangle$ are disjoint except for their common points in X_i.

In Proposition 2.3, suppose two faces f_1 and f_2 of X share two vertices x and y, but x and y are not adjacent in the boundary of f_1. Then, to start Program 2, we are given a plane π containing x and y such that all vertices of f_1 and those of f_2 except x and y are located in two half-spaces separated by π. Though f_1 and f_2 share only x and y, the convex polygon, say P_1, corresponding to f_1 and constructed by Program 1 share a segment xy on π with the convex polygon, say P_2, corresponding to f_2. We can see that P_1 and P_2 share a segment xy but f_1 and f_2 do not share an edge xy. Hence this is never an exhibition of X.

Proof of Lemma 4.12. For each map X_i, since any two faces share at most one edge, we can apply Proposition 2.3. We construct an exhibition of only X_2 since the remaining cases can be dealt similarly. (The cases for X_0 and X_1 are easier than that of X_2, since they have many triangular disks.)

Give the $\mathbb{R}^3$-coordinates of the vertices $v_1, v_2, v_2', v_3, v_4, v_4', v_5, v_6$ of X_2 as follows:

$$\begin{array}{lll} v_1 = (12, -10, 10), & v_2 = (-10, 10, -6), & v_2' = (-9, 10, -4), \\ v_3 = (-16, -10, 8), & v_4 = (5, 10, -12), & v_4' = (4, 10, -10), \\ v_5 = (0, 1, 0), & v_6 = (-1, 20, 0). & \end{array}$$

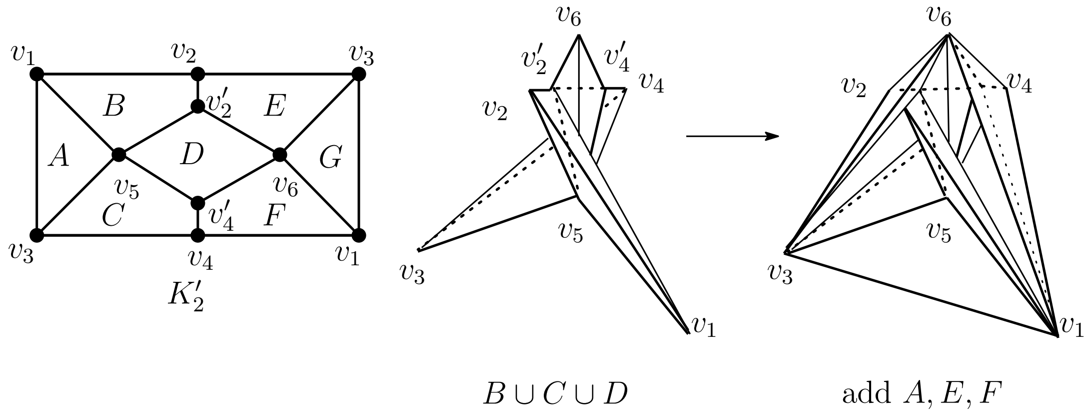


Figure 4.9: An exhibition of X_2.

See Figure 4.9. We put A, B, C, D, E, F, G to be the faces of X_2 as shown in the left. Then we can verify our coordinates of X_2 really gives an exhibition, as follows. We first see that by Program 1 in Chapter 2, each of the boundary cycles of A, B, C, D, E, F, G can be projected to some plane as a convex polygon:

The projection B to the plane $y = 0$ is a convex polygon,
the projection of C to the plane $z = 0$ is a convex polygon,
the projection of D to the plane $z = 0$ is a convex polygon,
the projection of E to the plane $-\frac{1}{1000}x - \frac{5017}{1000}y + \frac{8675}{1000}z = 0$ is a convex polygon,
and the projection of F to the plane $y = 0$ is a convex polygon.

Secondly, by taking some suitable plane F in Program 2 in Chapter 2, we can see that for any choice of two distinct faces, the convex hulls corresponding to them do not collide except their common points as in Table 6.2 in Appendix.

Therefore, our coordinates for X_2 can be verified to give an exhibition. The center of Figure 4.9 shows three polygons corresponding to the convex-hulls of B, C, D determined by our coordinates, and the right shows the body obtained from the convex hulls corresponding to A, B, C, D, E, F. Finally, to the body, we can easily add the disk corresponding to G to get an exhibition of X_2.

For other X_i's, we can check the following coordinates give an exhibition similarly to those for X_2: We put movies of each exhibition of X_i in the web site [100]. ■

For X_0, $v_1 = (12, -10, 10)$, $v_2 = (-10, 10, -6)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_1, $v_1 = (12, -10, 10)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_3, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_4, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_5, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_4' = (4, 10, -10)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_6, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_3 = (-16, -10, 8)$, $v_3' = (-14, -10, 10)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_7, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_3' = (-14, -10, 10)$ $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_8, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_3' = (-14, -10, 10)$, $v_4 = (5, 10, -12)$, $v_4' = (4, 10, -10)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

Lemma 4.13 *Let M be a Möbius triangulation with boundary cycle $v_1v_2v_3v_4$ containing one of the sub-maps $X_0, X_1, \ldots, X_8$. Unless M has an edge v_1v_3 and v_2v_4, then M has a geometric realization to which we can add a flat triangular disk $v_jv_{j+1}v_{j+2}$ with no intersection except their common points, for each $j = 1, 2, 3, 4$.*

Proof. Suppose that M contains one of the sub-maps $X_0, \ldots, X_8$, say X. By Lemma 4.12, each X_i has an exhibition ψ, and hence M has a geometric realization $\hat{M}$, where we take the coordinates of the vertices of X contained in $\hat{M}$ by ψ as the same in Lemma 4.12. Suppose that $X = X_0$ and we try to add a flat triangular disk $v_1v_3v_4$. Observe that the region, denoted by D, of X formed by two triangular faces $v_1v_5v_3$ and $v_1v_3v_6$, and a triangular disk $\Delta = v_1v_3v_4$ or $\Delta' = v_2v_3v_4$ added to it do not satisfy the assumption of Proposition 2.3. Let $\tilde{M}$ denote the map $M \cup \Delta$ or $M \cup \Delta'$.

We first consider Δ. We note that D and Δ do not share an edge v_1v_3 in $\tilde{M}$. In this case, since D has no chord v_1v_3 in M by the assumption, we construct D in a geometric realization of $\hat{M}$, as follows: Let P be a shortest path in D corresponding to an edge v_1v_3 of X. Then P must be chordless in M. (For otherwise, we can choose a shorter one.) Since D has no diagonal v_1v_3, P has an inner vertex, say v, in M. Hence, by Lemma 4.9, M has a path P_1 from v to a vertex x on the path corresponding to v_1v_5 or v_3v_5, and a path P_2 from v to a vertex y on the path corresponding to v_1v_6 or v_3v_6, where each of x and y is distinct from v_1 and v_3. Then D can divided into four regions, say D_1, D_2, D_3, D_4. Move a position of v in $\hat{M}$ slightly toward the interior of $\psi(D)$, where $\psi(D)$ is a tetrahedron in $\hat{M}$ corresponding to D. Note that a very small movement of v in $\mathbb{R}^3$ does not yield an intersection of faces in $\hat{M}$. Hence we can take an exhibition of each of the regions D_1, D_2, D_3 and D_4 to get $\hat{M}$, and we can get a geometric realization of $\tilde{G}$. (For example, see Figure 4.10.)

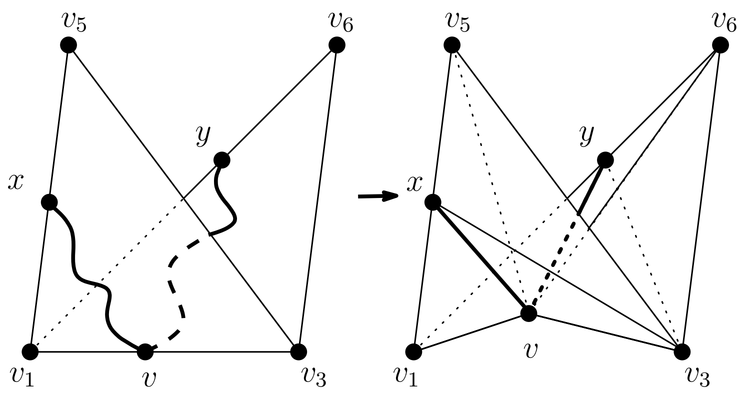


Figure 4.10: Moving v slightly to get $\hat{M}$.

Now we secondly consider $\Delta' = v_1v_2v_4$. Let D' be the region of X bounded by $v_2v_5v_4v_6$. If D' has an internal path between v_2 and v_4, then we can do similarly as in the previous case. Otherwise, by Lemma 4.9, D' has a chord uv where v_2, u, v_4 and v appear on the boundary of D' in this order. In this case, let D'_1 and D'_2 be the two regions obtained from D' by dividing along uv, and consider the exhibitions of D'_1 and D'_2. Since our coordinates of D' makes a tetrahedron in $\hat{M}$, these two exhibitions of D'_1 and D'_2 share only a segment uv, and the exhibition of $D'_1 \cup D'_2$ does not intersect a straight segment in $\mathbb{R}^3$ between v_2 and v_4. (For example, see Figure 4.11.)

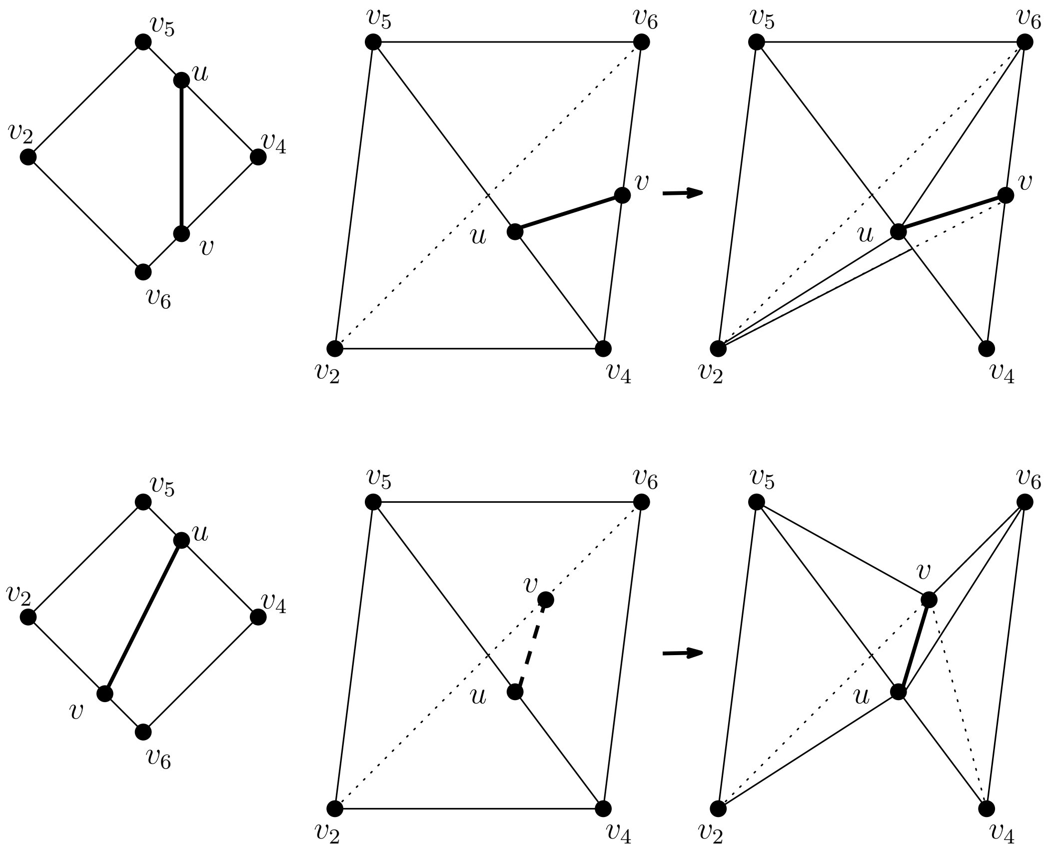


Figure 4.11: An exhibition of D' and two exhibitions $D'_1 \cup D'_2$.

Consequently, we can construct a geometric realization $\hat{M}$ avoiding a segment between v_1 and v_3 (or between v_2 and v_4) except their ends. Hence we can add each of Δ (or Δ') to X without intersections of faces, which can be checked by Programs 1 and 2.

However, we can not add each of the triangular disks $v_1v_2v_3$ and $v_1v_2v_4$ if we use the coordinates of each vertex of X_i as in Lemma 4.12. In this case, we use the following coordinates of X_i, each of which is basically obtained from those in the proof of Lemma 4.12 by replacing the coordinate of each boundary node v_j (resp v'_j), for $j = 1, 2, 3, 4$, with that of v_{j+2} (resp v'_{j+2}) and the coordinate of each inner node v_j, for $j = 5, 6$, with that of v_{j+1}. ■

For X_0, $v_1 = (-16, -10, 8)$, $v_2 = (5, 10, -12)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_1, $v_1 = (-16, -10, 8)$, $v_2 = (5, 10, -12)$, $v'_2 = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_2 $v_1 = (-16, -10, 8)$, $v_2 = (5, 10, -12)$, $v'_2 = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v'_4 = (-9, 10, -4)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_3, $v_1 = (-16, -10, 8)$, $v'_1 = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_4, $v_1 = (-16, -10, 8)$, $v'_1 = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v'_2 = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_5, $v_1 = (-16, -10, 8)$, $v'_1 = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v'_2 = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v'_4 = (-9, 10, -4)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_6, $v_1 = (-16, -10, 8)$, $v'_1 = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v_3 = (12, -10, 10)$, $v'_3 = (11, -10, 20)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_7, $v_1 = (-16, -10, 8)$, $v'_1 = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v'_2 = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v'_3 = (11, -10, 20)$ $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_8, $v_1 = (-16, -10, 8)$, $v'_1 = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v'_2 = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v'_3 = (11, -10, 20)$, $v_4 = (-10, 10, -6)$, $v'_4 = (-9, 10, -4)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

Lemma 4.14 *Let M be a Möbius triangulation with boundary cycle $v_1v_2v_3v_4$ containing one of the sub-maps $X_0, \ldots, X_8$. Let $\hat{M}_f$ be a geometric realization of $M \cup f$ constructed by the coordinates in Lemma 4.12 or Lemma 4.13, where $f = v_jv_{j+1}v_{j+2}$ is a triangular disk for some $j \in \{1, 2, 3, 4\}$. Then,*

(i) *for some point p in $\mathbb{R}^3$, we can add two triangular disks pv_jv_{j+2} and $pv_{i+2}v_{i+3}$ to the body of $\hat{M}_f$, and*

(ii) *for some points q and r in $\mathbb{R}^3$, we can add a tetrahedron $qrv_{i+2}v_{i+3}$, two triangular disks qv_iv_{i+3} and rv_iv_{i+2} to the body of $\hat{M}_f$.*

Proof. If $f = v_1v_2v_3$ or $v_1v_3v_4$, then we put $p = (0, 0, -40)$, $q = (-10, 10, -6 - \frac{1}{10})$ and $r = (0, 0, -40)$. If $f = v_2v_3v_4$ or $v_1v_2v_4$, then we put $p = (0, 0, -40)$, $q = (11, -10 + \frac{1}{10}, 8)$ and $r = (0, 0, -40)$. Programs 1 and 2 verifies Lemma 4.14. ■

4.4 Main result on projective triangulations

In this section, we shall prove Theorem 0.14, by using lemmas which we proved in Sections 4.2 and 4.3.

Proof of Theorem 0.14. Let G be a projective triangulation and let f be a face of G. If G has a nesting 3-cycle of f, then by Fact 4.1, G has no geometric realization, and so the necessity holds.

So we consider the sufficiency. Suppose that f has no nesting 3-cycle in G. Then G has a 4-cycle $C = v_1v_2v_3v_4$ satisfying (i), (ii) and (iii) of Lemma 4.5. Let G_M (resp., G_D) be a Möbius triangulation (resp., a near triangulation) with the boundary cycle C. Since C satisfies the condition (ii) of Lemma 4.5, G_D has a diagonal v_iv_{i+2} and the interior of $v_iv_{i+1}v_{i+2}$ contains f. For a simple notation, let R and R' denote the 2-cell regions of G bounded by $v_iv_{i+1}v_{i+2}$ and $v_iv_{i+2}v_{i+3}$, respectively. By Lemmas 4.12 and 4.13, $G_M \cup R'$ has a geometric realization.

If $f = R$, then we are done, and hence we suppose $f \neq R$. If $E(f) \cup E(C) \neq \emptyset$, then R has an inner vertex p with $p \in V(f)$. In this case, there exists a path from p to v_{i+2} but disjoint from v_i and v_{i+1}. If $E(f) \cup E(C) = \emptyset$, then R has two inner vertices q and r with $q, r \in V(f)$. In this case, there exist two disjoint paths, say P_1 and P_2, from q and r to v_{i+1} and v_{i+2}, not intersecting v_i. Without loss of generality, we may suppose that P_1 from q to v_{i+1} and P_2 is from r to v_{i+2}. Therefore, by Lemma 4.14, we can all faces in R except for f to the body of a geometric realization of $G_M \cup R'$ by taking suitable coordinates of p or q and r in $\mathbb{R}^3$. Therefore, $G - f$ has a geometric realization. ■

Chapter 5

Geometric realizations of triangulations on the Möbius band

In this chapter, we prove Theorem 0.15, which characterize geometrically realizable Möbius triangulations. In Chapter 4, we proved Theorem 0.14. By this theorem, we can characterize geometrically realizable Möbius triangulations whose boundary cycle is length 3. However, it does not characterize geometrically realizable Möbius triangulations with boundary cycle of length at least 4.

5.1 Structures for geometric realizations

In this section, we prove some lemmas in order to decide structures in Möbius triangulations.

Suppose that a Möbius triangulation M has a 3-cycle C homotopic to the boundary cycle B. If the Möbius triangulation M_C with boundary C has no 3-cycle homotopic to C (other than C itself), then C is said to be *maximal*. On the other hand, if the triangulation bounded by B and C has no 3-cycle homotopic to C but it is distinct from B and C, then C is said to be *minimal*. If C is disjoint from B, then C is called a *nesting cycle* in M. Note that let f be a new disk and let G denote the projective triangulation obtained from M and f by pasting their boundaries. Then, the nesting 3-cycle in M is a nesting 3-cycle of f in G.

If M has no nesting 3-cycle, we easily prove Theorem 0.15, as follows.

Lemma 5.1 *If a Möbius triangulation M has no nesting* 3*-cycle, then M has a geometric realization.*

Proof. Suppose that M has no nesting 3-cycle. Let $B = v_1v_2 \cdots v_l$ be the boundary cycle of M, where $l \geq 4$. Adding a new vertex v and faces $vv_1v_2, vv_2v_3, \cdots, vv_lv_1$ to M, we obtain a projective triangulation G. Suppose that M has a 3-cycle C homotopic to B, where we assume that C is maximal. By the assumption, C intersects B, say $v_1 \in V(C) \cap V(B)$. If we let f be the face of G bounded by vv_1v_2, then G has no nesting 3-cycle of f, by the maximality of C. Hence $G - f$ has a geometric realization, by Theorem 0.14. Even if M has no such C, then the same holds since $G - f$ has no nesting 3-cycle. ■

The proof of Theorem 0.14 proceeds, as follows: Let G be a projective triangulation and let f be a face of G with no nested 3-cycle. First, choose a face f' neighboring to f, and let G' be the Möbius triangulation with boundary 4-cycle obtained from G by removing the quadrilateral region $f \cup f'$. We secondly find a sub-map X of $G - f$ which is 2-cell embedded in the Möbius band. Then we finally proved the sub-map X has an exhibition, which should be extended to a geometric realization of $G - f$, by Lemma 2.2. (The choice of f' is sometimes complicated in the proof of Theorem 0.15, but we omit a description of the detailed argument.) The following two lemmas correspond to the second and third procedures in the proof.

A *Möbius* 4*-triangulation* means a triangulation on the Möbius band whose boundary cycle has length exactly four.

Lemma 5.2 *Let M be a Möbius* 4*-triangulation with boundary* 4*-cycle $B = v_1v_2v_3v_4$. If M has no* 3*-cycle homotopic to B, then G has a sub-map X which is isomorphic to a subdivision of some X_i shown in Figure* 4.1*, where the boundary cycle of X is $v_1v_2v_3v_4$.* ■

The four vertices v_1, v_2, v_3, v_4 of each X_i play an essential role in the proof, and hence we call them *boundary nodes* of X_i.

Lemma 5.3 *Let X be one of the nine maps shown in Figure* 4.1*. Let $i \in \{1, 2, 3, 4\}$ be any integer. Then the map X with a face $v_iv_{i+1}v_{i+2}$ added has an exhibition, where the subscripts are taken by modulo* 4*.* ■

In order to prove Lemma 5.3, we gave a $\mathbb{R}^3$-coordinate to each vertex of all X_i's. Moreover, by using the computer programs constructed in Chapter 2, we checked whether those $\mathbb{R}^3$-coordinates actually give an exhibition of H or not. That is, we checked the $\mathbb{R}^3$-coordinates of the vertices of X satisfies the conditions (i) and (ii) of exhibitions of maps.

Let M be a Möbius triangulation with boundary cycle B of length at least four, and suppose that M has a nesting 3-cycle but no two disjoint 3-cycles homotopic to B. Let $C = c_1c_2c_3$ be a maximal one. Then any 3-cycle of M homotopic to B other than C, if any, intersects C.

Lemma 5.4 *In M, we can take four distinct vertices v_1, v_2, v_3, v_4 in B such that for each i, there is a path P_i from v_i to a vertex in C satisfying the following condition:*

(i) *P_1, P_2, P_3, P_4 are disjoint, except at the end in C, and*

(ii) *each vertex on C is an endpoint of some P_i.*

Proof. Let C' be a minimal nesting 3-cycle. Note that we possibly have $C = C'$, and that C' intersects C, by the assumption on M. So we may suppose that $c_1 \in V(C) \cap V(C')$. Let R denote the region bounded by C' and B. Let x, y be vertices on C' with $x, y \neq c_1$, and let z be a vertex in R such that zyc_1 is a triangular face in R. By Lemma 4.10 and the minimality of C', if we introduce a new vertex p and join p to all vertices on B, then M has internally disjoint four paths from p to the four vertices c_1, x, y, z, meeting only at p. So we can find four distinct vertices on B which are connected to c_1, x, y and z by the four disjoint paths, say Q_1, Q_2, Q_3, Q_4, respectively. See Figure 5.1. (If z lies on B, then we regard the single vertex z as Q_4.) Moreover, by the same argument, we can take two disjoint paths from $\{x, y\}$ to $\{c_2, c_3\}$, say Q'_2, Q'_3, in the region bounded by C and C', which do not intersect c_1. (If $x = c_2$ and $y = c_3$, then we regard $Q'_2 = c_2$, $Q'_3 = c_3$.) Therefore, letting

$$P_1 = Q_1, \quad P_2 = Q_2 \cup Q'_2, \quad P_3 = Q_3 \cup Q'_3, \quad P_4 = Q_4 \cup zc_1,$$

we obtain required internally disjoint four paths. ■

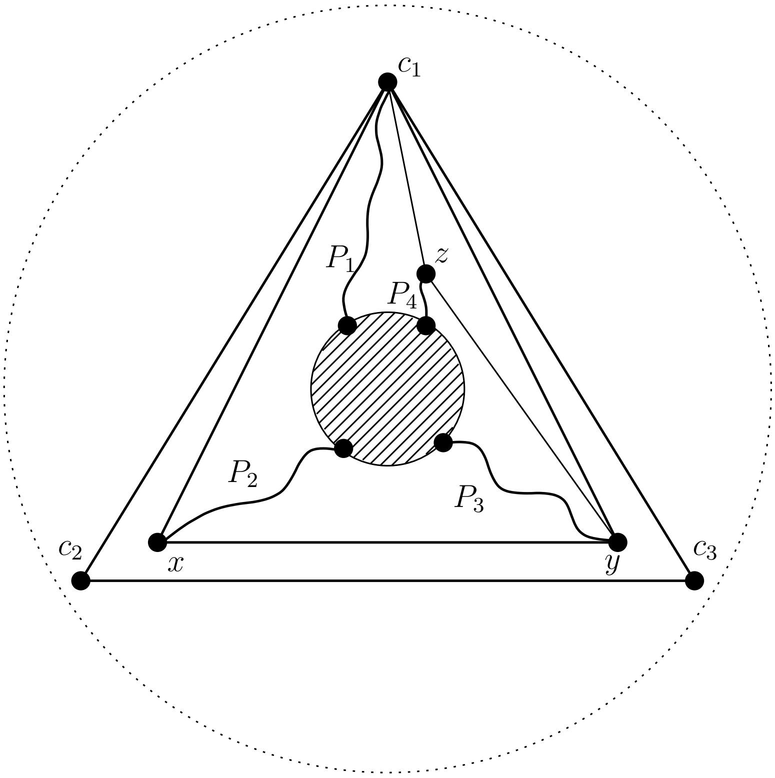


Figure 5.1: Four disjoint paths from four vertices on B to c_1, x, y, z.

By Lemma 5.4, M has four paths from the three vertices on C to some four distinct vertices on B. Let P_1, P_2, P_3 be the disjoint three paths from c_1, c_2, c_3 to three vertices on B, say p, w, q, respectively, and let P_4 be the path from c_1 to a vertex on B, say v, where p, w, q and v lie on B in this cyclic order but any two are not necessarily adjacent in B. Let c_1c_3x be a face of M contained in the Möbius triangulation with boundary C. (See Figure 5.2.) Let P be the path on B from p to q containing w, and let $D = P_1 \cup P \cup P_3 \cup c_3x \cup xc_1$. Let M_D be the Möbius triangulation with boundary D which is contained in M.

Lemma 5.5 *The Möbius triangulation M_D has a sub-map X' isomorphic to a subdivision of some X_i shown in Figure* 4.1 *such that the boundary cycle of X' coincides with D, and that the four boundary nodes of X' coincide with c_1, w, c_3 and x, respectively.*

Proof. By Lemma 5.2 and the maximality of C, the Möbius 4-triangulation bounded by $c_1c_2c_3x$ has a sub-map X which is isomorphic to a subdivision of some X_i shown in

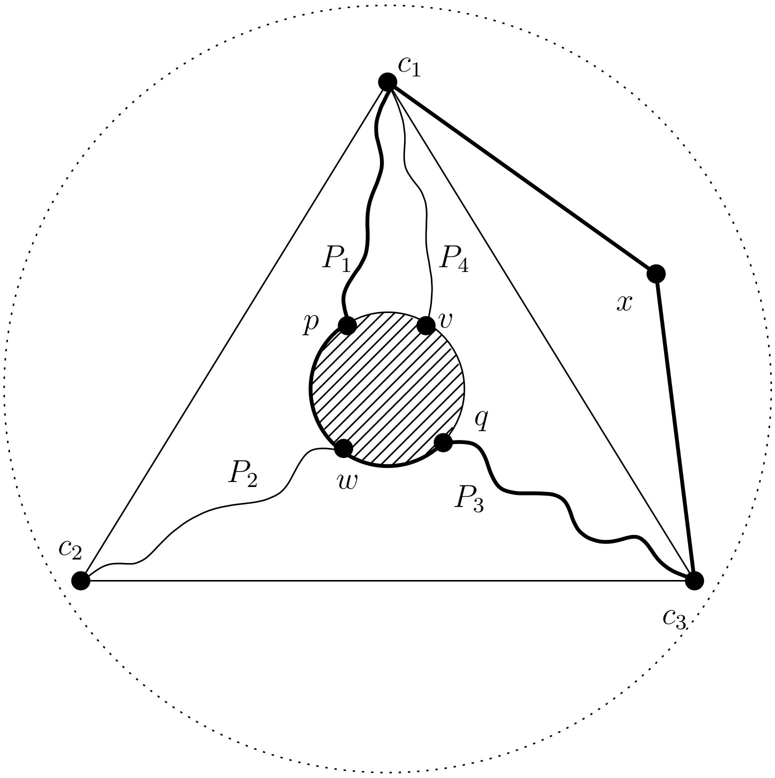


Figure 5.2: The cycle D in M.

Figure 4.1, where the boundary 4-cycle of X is $c_1c_2c_3x$. The sub-map X can be extended to a required sub-map X' of M_D by regarding $P \cup P_1 \cup P_3$ as a segment of the boundary of X', and w as its boundary node. (We note that X' might not be homeomorphic to X.) ■

Now we construct a geometric realization of M, using an exhibition of the sub-map X'.

Lemma 5.6 *Let M be a Möbius triangulation with boundary cycle B of length at least four. If M has a nesting 3-cycle C but M has no 3-cycle homotopic to B and disjoint from C, then M has a geometric realization.*

Proof. By Lemma 5.5, M_D has the sub-map X' which is isomorphic to a subdivision of some X_i in Figure 4.1, and whose boundary nodes are c_1, w, c_3 and x. By the assumption on D, c_1c_3x is a triangular face, say f, in M. By Lemmas 2.2 and 5.3, M_D with f added has a geometric realization, say $\hat{M}_D$. So, in order to make a geometric realization of M, it suffices to prove that an exhibition of a disk vqc_3c_1 and a triangular disk pvc_1 can be added to $\hat{M}_D$ without collisions of faces. Slightly modifying $\hat{M}_D$, we give the following $\mathbb{R}^3$-coordinates for p, q and v, in addition to the $\mathbb{R}^3$-coordinates of the vertices of X' given in Appendix.

For example, we suppose that X' is isomorphic to a subdivision of X_8, and that the vertices v_1, v_2, v_3, v_4 of X_8 coincide with the four vertices c_1, x, c_3, w, respectively. Then, X_8 with a new face $v_1v_2v_3$ added has an exhibition, by Lemma 5.3. In particular, we give the following $\mathbb{R}^3$-coordinates to the vertices of X_8:

$$
\begin{aligned}
&v_1 = (-16, -10, 8), \quad v_1' = (-14, -10, 10), \quad v_2 = (5, 10, -12), \quad v_2' = (4, 10, -10),\\
&v_3 = (12, -10, 10), \quad v_3' = (11, -10, 20), \quad v_4 = (-10, 10, -6), \quad v_4' = (-9, 10, -4),\\
&v_5 = (-1, 20, 0), \quad v_6 = (0, 1, 0).
\end{aligned}
$$

Moreover, letting

$$p = (-16 - \frac{1}{10}, -10 - \frac{1}{10},\ \ 8 + \frac{1}{5}), \quad q = (12 - \frac{1}{10}, -10, 10), \quad v = (0, -16, 0),$$

we can add an exhibition of a disk vqc_3c_1 and a triangular disk pvc_1. Then we get a geometric realization of M.

Even if X' is a subdivision of X_8, then there are three more cyclic permutations to identify v_1, v_2, v_3, v_4 and c_1, x, c_3, w. Moreover, we also have to consider the case when X is a subdivision of $X_0, \ldots, X_7$. In all of those cases, we can verify that M has a geometric realization, by giving the vertices of X' and p, q, v, as in Appendix. ■

5.2 Main result on Möbius triangulations

Now we shall prove Theorem 0.15 by using lemmas which we proved in Section 5.1.

Proof of Theorem 0.15. Let M be a Möbius triangulation with boundary cycle B. If B has length 3, then Theorem 0.14 proves the result.

So we suppose that B has length at least 4. If M has two disjoint 3-cycles C and C' homotopic to the boundary of M, then by Fact 0.12, M has no geometric realization, and so the necessity holds.

Let us consider the sufficiency. If M has no nesting 3-cycle, then M has a geometric realization, by Lemma 5.1. Now suppose that M has a nesting 3-cycle. Since M is assumed to have no two disjoint 3-cycles homotopic to B, we may suppose that M has no 3-cycle homotopic to B and disjoint from C. Then, by Lemma 5.6, M has a geometric realization. ■

Chapter 6

Appendix

6.1 List of $\mathbb{R}^3$ coordinates for Lemma 5.6

6.1.1 $v_1 = c_1$, $v_2 = x$, $v_3 = c_3$, $v_4 = w$

If v_1, v_2, v_3, v_4 coincide with c_1, x, c_3, w respectively, then we give the following $\mathbb{R}^3$-coordinates to each vertex of X_i and p, q, v.

For X_0, $v_1 = (-16, -10, 8)$, $v_2 = (5, 10, -12)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_1, $v_1 = (-16, -10, 8)$, $v_2 = (5, 10, -12)$, $v_2' = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_2 $v_1 = (-16, -10, 8)$, $v_2 = (5, 10, -12)$, $v_2' = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_4' = (-9, 10, -4)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_3, $v_1 = (-16, -10, 8)$, $v_1' = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_4, $v_1 = (-16, -10, 8)$, $v_1' = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v_2' = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_5, $v_1 = (-16, -10, 8)$, $v_1' = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v_2' = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_4' = (-9, 10, -4)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_6, $v_1 = (-16, -10, 8)$, $v_1' = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v_3 = (12, -10, 10)$, $v_3' = (11, -10, 20)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_7, $v_1 = (-16, -10, 8)$, $v_1' = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v_2' = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_3' = (11, -10, 20)$ $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_8, $v_1 = (-16, -10, 8)$, $v_1' = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v_2' = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_3' = (11, -10, 20)$, $v_4 = (-10, 10, -6)$, $v_4' = (-9, 10, -4)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For p, q, r, letting

$$p = (-16 - \frac{1}{10}, -10 - \frac{1}{10}, 8 + \frac{1}{5}), \quad q = (12 - \frac{1}{10}, -10, 10), \quad v = (0, -16, 0).$$

6.1.2 $v_1 = x$, $v_2 = c_3$, $v_3 = w$, $v_4 = c_1$

If v_1, v_2, v_3, v_4 coincide with x, c_3, w, c_1 respectively, then we give the following $\mathbb{R}^3$-coordinates to each vertex of X_i and p, q, v.

For X_0, $v_1 = (-16, -10, 8)$, $v_2 = (5, 10, -12)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_1, $v_1 = (-16, -10, 8)$, $v_2 = (5, 10, -12)$, $v_2' = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_2 $v_1 = (-16, -10, 8)$, $v_2 = (5, 10, -12)$, $v_2' = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_4' = (-9, 10, -4)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_3, $v_1 = (-16, -10, 8)$, $v_1' = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_4, $v_1 = (-16, -10, 8)$, $v_1' = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v_2' = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_5, $v_1 = (-16, -10, 8)$, $v_1' = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v_2' = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_4 = (-10, 10, -6)$, $v_4' = (-9, 10, -4)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_6, $v_1 = (-16, -10, 8)$, $v_1' = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v_3 = (12, -10, 10)$, $v_3' = (11, -10, 20)$, $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_7, $v_1 = (-16, -10, 8)$, $v_1' = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v_2' = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_3' = (11, -10, 20)$ $v_4 = (-10, 10, -6)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For X_8, $v_1 = (-16, -10, 8)$, $v_1' = (-14, -10, 10)$, $v_2 = (5, 10, -12)$, $v_2' = (4, 10, -10)$, $v_3 = (12, -10, 10)$, $v_3' = (11, -10, 20)$, $v_4 = (-10, 10, -6)$, $v_4' = (-9, 10, -4)$, $v_5 = (-1, 20, 0)$, $v_6 = (0, 1, 0)$.

For p, q, r, letting

$$p = (-10 + \frac{1}{5}, 10 - \frac{1}{10}, -6), \quad q = (5, 10 - \frac{1}{10}, -12), \quad v = (-1, -13, -1).$$

6.1.3 $v_1 = c_3$, $v_2 = w$, $v_3 = c_1$, $v_4 = x$

If v_1, v_2, v_3, v_4 coincide with c_3, w, c_1, x respectively, then we give the following $\mathbb{R}^3$-coordinates to each vertex of X_i and p, q, v.

For X_0, $v_1 = (12, -10, 10)$, $v_2 = (-10, 10, -6)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_1, $v_1 = (12, -10, 10)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_2, $v_1 = (12, -10, 10)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_4' = (4, 10, -10)$, $v_5 = (0, 1, 0)$ $v_6 = (-1, 20, 0)$.

For X_3, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_4, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_5, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_4' = (4, 10, -10)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_6, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_3 = (-16, -10, 8)$, $v_3' = (-14, -10, 10)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_7, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_3' = (-14, -10, 10)$ $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_8, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_3' = (-14, -10, 10)$, $v_4 = (5, 10, -12)$, $v_4' = (4, 10, -10)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For p, q, r, letting

$$p = (-16 - \frac{1}{10}, -10 - \frac{1}{10}, 8 + \frac{1}{5}), \quad q = (12 - \frac{1}{10}, -10, 10), \quad v = (0, -16, 0).$$

6.1.4 $v_1 = w$, $v_2 = c_1$, $v_3 = x$, $v_4 = c_3$

If v_1, v_2, v_3, v_4 coincide with w, c_1, x, c_3 respectively, then we give the following $\mathbb{R}^3$-coordinates to each vertex of X_i and p, q, v.

For X_0, $v_1 = (12, -10, 10)$, $v_2 = (-10, 10, -6)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_1, $v_1 = (12, -10, 10)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_2, $v_1 = (12, -10, 10)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_4' = (4, 10, -10)$, $v_5 = (0, 1, 0)$ $v_6 = (-1, 20, 0)$.

For X_3, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_4, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_5, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_4 = (5, 10, -12)$, $v_4' = (4, 10, -10)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_6, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_3 = (-16, -10, 8)$, $v_3' = (-14, -10, 10)$, $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_7, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_3' = (-14, -10, 10)$ $v_4 = (5, 10, -12)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For X_8, $v_1 = (12, -10, 10)$, $v_1' = (11, -10, 20)$, $v_2 = (-10, 10, -6)$, $v_2' = (-9, 10, -4)$, $v_3 = (-16, -10, 8)$, $v_3' = (-14, -10, 10)$, $v_4 = (5, 10, -12)$, $v_4' = (4, 10, -10)$, $v_5 = (0, 1, 0)$, $v_6 = (-1, 20, 0)$.

For p, q, r, letting

$$p = (-10 + \frac{1}{5}, 10 - \frac{1}{10}, -6), \quad q = (5, 10 - \frac{1}{10}, -12), \quad v = (-1, -13, -1).$$

6.2 Planes for Lemma 4.12

The plane	convex-hulls
$11x - 128y - 154z = -128$	A and B
$3x + 16y + 28z = 16$	A and C
$38x + 2y - 162z = 2$	A and D
$11x - 128y - 154z = -128$	A and E
$190x + 10y - 217z = 10$	A and F
$11x - 128y - 154z = -128$	A and G
$18x + 62y + 63z = 62$	B and C
$18x + 14y - 9z = 14$	B and D
$10x - 6y - 5z = -130$	B and E
$95x + 65y + 19z = 585$	B and F
$90x - 101y - 420z = -2110$	B and G
$54x + 106y + 117z = 106$	C and D
$28x - 48y - 28z = -192$	C and E
$8x + 7y + 4z = 62$	C and F
$30x - 127y - 420z = -257$	C and G
$11x - 6y - 5z = -130$	D and E
$30x + 2y + 13z = 10$	D and F
$30x - 127y - 840z = -2570$	D and G
$10x - 6y - 5z = -130$	E and F
$90x + 179y + 840z = 3490$	E and G
$90x - 101y - 420z = -2110$	F and G

Table 6.1: Planes distinguishing two point sets.

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