Classification of singularities of pedal curves in S^2

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Classification of singularities of pedal curves in S^2 *

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1 Introduction

Let I be an open interval, S^2 be the unit sphere in $\mathbf{R}^3$ and $\mathbf{r} : I \to S^2$ be a C^∞ map such that $\left\|\frac{d\mathbf{r}}{ds}(s)\right\| = 1$, which is called a *spherical unit speed curve* in S^2.

Let P be a point in $S^2 - \{\pm\mathbf{n}(s) \mid s \in I\}$, where $\mathbf{n}$ is the dual of $\mathbf{r}$. The *pedal curve* with the *pedal point* P for a given spherical unit speed curve $\mathbf{r}$ is a curve obtained by mapping $s \in I$ to the unique nearest point in $C_{\mathbf{n}(s)}$ from P, where $C_{\mathbf{n}(s)}$ is the great circle of S^2 which tangents to the vector $\frac{d\mathbf{r}}{ds}(s)$ at $\mathbf{r}(s)$. The pedal curve with the pedal point P for $\mathbf{r}$ is denoted by $Pe_{\mathbf{r},P}$. Note that since all points in $C_{\mathbf{n}(s)}$ are the nearest points from $\pm\mathbf{n}(s)$ the pedal point P must be outside $\{\pm\mathbf{n}(s) \mid s \in I\}$. We put

$$\mathbf{t}(s) = \frac{d\mathbf{r}}{ds}(s),\ \mathbf{n}(s) = \mathbf{r}(s) \times \mathbf{t}(s),$$

where $\mathbf{r}(s) \times \mathbf{t}(s)$ means the vector product of $\mathbf{r}(s)$ and $\mathbf{t}(s)$. These are called the *tangent vector* and the *normal vector* respectively. By definitions the vector $\mathbf{t}(s)$ is perpendicular to $\mathbf{r}(s)$ and the vector $\mathbf{n}(s)$ is perpendicular to both of $\mathbf{r}(s)$ and $\mathbf{t}(s)$. The map $\mathbf{n} : I \to S^2$, which is called the *dual* of $\mathbf{r}$, is relatively well understood (for instance, see [1], [5],[11]).

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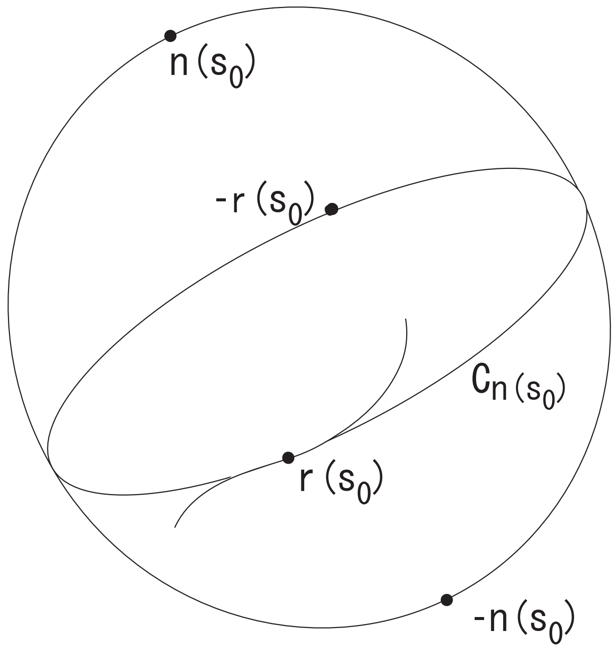

Figure 1: $\pm\mathbf{r}(s_0), \pm\mathbf{n}(s_0)$ and $C_{\mathbf{n}(s_0)}$

Let $\kappa_g(s)$ be the geodesic curvature of a spherical unit speed curve $\mathbf{r}(s)$ at s (for the definition of the geodesic curvature, see §2). In [10] the following has been shown.

Theorem 1 ([10]) *Let $\mathbf{r}$ be a spherical unit speed curve. Let $s_0 \in I$ be such that $\kappa_g(s_0) \neq 0$ and P be a point of $S^2 - \{\pm\mathbf{n}(s_0)\}$. Then the following hold.*

1. *If $P \in S^2 - \{\pm\mathbf{n}(s_0)\} - \{\pm\mathbf{r}(s_0)\}$, then the map-germ $Pe_{\mathbf{r},P} : (I, s_0) \to (S^2, Pe_{\mathbf{r},P}(s_0))$ is smooth, that is to say, it is C^∞ right-left equivalent to the map-germ $(\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ given by $\sigma \mapsto (\sigma, 0)$.*

2. *If $P \in \{\pm\mathbf{r}(s_0)\}$, then the map-germ $Pe_{\mathbf{r},P} : (I, s_0) \to (S^2, Pe_{\mathbf{r},P}(s_0))$ is C^∞ right-left equivalent to the map-germ $(\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ given by $\sigma \mapsto (\sigma^2, \sigma^3)$.*

The purpose of this paper is to classify the singularities of pedal curves $Pe_{\mathbf{r},P}$ for s_0 with $\kappa_g(s_0) = 0$.

Theorem 2 *Let $\mathbf{r} : I \to S^2$ be a spherical unit speed curve. Let $s_0 \in I$ such that $\kappa_g(s_0) = \kappa_g'(s_0) = \cdots = \kappa_g^{(k-1)}(s_0) = 0, \kappa_g^{(k)}(s_0) \neq 0$ $(k \geq 1)$ and P be a point of $S^2 - \{\pm\mathbf{n}(s_0)\}$. Then the following hold.*

1. *If* $P \in S^2 - \{\pm\mathbf{n}(s_0)\} - C_{\mathbf{n}(s_0)}$*, then the map-germ* $Pe_{\mathbf{r},P} : (I, s_0) \to (S^2, Pe_{\mathbf{r},P}(s_0))$ *is* C^1 *right-left equivalent to the map-germ given by* $\sigma \mapsto (\sigma^{k+1}, \sigma^{k+2})$.

2. *If* $P \in C_{\mathbf{n}(s_0)} - \{\pm\mathbf{r}(s_0)\}$*, then the map-germ* $Pe_{\mathbf{r},P} : (I, s_0) \to S^2$ *is* C^1 *right-left equivalent to the map-germ given by* $\sigma \mapsto (\sigma^{k+1}, \sigma^{2k+3})$.

3. *If* $P \in \{\pm\mathbf{r}(s_0)\}$*, then the map-germ* $Pe_{\mathbf{r},P} : (I, s_0) \to (S^2, Pe_{\mathbf{r},P}(s_0))$ *is* C^1 *right-left equivalent to the map-germ given by* $\sigma \mapsto (\sigma^{k+2}, \sigma^{2k+3})$.

Here, for $r = \infty$ or 1 two map-germs $f, g : (\mathbf{R}, 0) \to (\mathbf{R}^2, \mathbf{0})$ are said to be C^r *right-left equivalent* if there exist germs of C^r diffeomorphisms $h_s : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ and $h_t : (\mathbf{R}^2, \mathbf{0}) \to (\mathbf{R}^2, \mathbf{0})$ such that $h_t \circ f \circ h_s^{-1} = g$.

Combining theorems 1 and 2 yields a complete C^1 classification of singularities of pedal curves $Pe_{\mathbf{r},P}$ for spherical unit speed curves whose geodesic curvatures are nowhere flat.

In §2, we recall Serret-Frenet type formula for a spherical unit speed curve and give several applications of it. §3 is devoted to give an explicit formula for $Pe_{\mathbf{r},P}$ and the main tool to prove theorem 2. Proofs of 3, 2 and 1 of theorem 2 are given in §4, §5 and §6 respectively. Finally, in §7 we give a remark on possibility of improving C^1 right-left equivalence to C^∞ right-left equivalence in theorem 2.

The authors wish to thank S. Izumiya for sending his useful hand-written note ([8]).

2 Serret-Frenet type formula and its applications

Lemma 2.1 *For the orthogonal moving frame* $\{\mathbf{r}(s), \mathbf{t}(s), \mathbf{n}(s)\}$ *the following Serret-Frenet type formula holds.*

$$\left\{ \begin{array}{rcl} \mathbf{r}'(s) & = & \mathbf{t}(s) \\ \mathbf{t}'(s) & = & -\mathbf{r}(s) + \kappa_g(s)\mathbf{n}(s) \\ \mathbf{n}'(s) & = & -\kappa_g(s)\mathbf{t}(s). \end{array} \right.$$

Here prime means differentiating with respect to s and $\kappa_g(s)$ is called the *geodesic curvature* of $\mathbf{r}$ at s which is given by

$$\kappa_g(s) = \det(\mathbf{r}(s), \mathbf{t}(s), \mathbf{t}'(s)).$$

[*Proof of lemma 2.1*] We put $\mathbf{t}'(s) = a_1\mathbf{r}(s) + b_1\mathbf{t}(s) + c_1\mathbf{n}(s)$ and we show that $a_1 = -1,\ b_1 = 0$ and $c_1 = \kappa_g(s)$.

Since $\mathbf{t}(s)\cdot\mathbf{r}(s) = 0$, we have that $\mathbf{t}'(s)\cdot\mathbf{r}(s) = -1$, where $\mathbf{a}\cdot\mathbf{b}$ means the scalar product of two 3-dimensional vectors $\mathbf{a}, \mathbf{b}$. Thus, $a_1 = -1$. Since $\mathbf{t}(s)\cdot\mathbf{t}(s) = 1$, we have that $\mathbf{t}'(s)\cdot\mathbf{t}(s) = 0$ and thus $b_1 = 0$.
Finally,

$$\begin{aligned} \kappa_g(s) &= \det(\mathbf{r}(s), \mathbf{t}(s), \mathbf{t}'(s)) \\ &= \det(\mathbf{r}(s), \mathbf{t}(s), -\mathbf{r}(s) + c_1\mathbf{n}(s)) \\ &= c_1. \end{aligned}$$

Next, we show that $\mathbf{n}'(s) = -\kappa_g(s)\mathbf{t}(s)$. Since $\mathbf{n}(s) = \mathbf{r}(s)\times\mathbf{t}(s)$, we have

$$\begin{aligned} \mathbf{n}'(s) &= \mathbf{r}'(s)\times\mathbf{t}(s) + \mathbf{r}(s)\times\mathbf{t}'(s) \\ &= \mathbf{r}(s)\times(\kappa_g(s)\mathbf{n}(s)) \\ &= -\kappa_g(s)\mathbf{t}(s). \end{aligned}$$

$\square$

By lemma 2.1, we see that the dual $\mathbf{n}$ is non-singular at s if and only if $\kappa_g(s) \neq 0$.

Let s_0 be an element of I. For any i $(1 \le i)$ and any s such that $s + s_0 \in I$, we put

$$\varphi_i(s) = (\kappa_g(s+s_0), \kappa_g'(s+s_0), \cdot\cdot\cdot, \kappa_g^{(i-1)}(s+s_0)).$$

Let $\mathcal{E}_1$ (resp. $\mathcal{E}_i$) be the set of all C^∞ function-germs $(\mathbf{R}, s_0) \to \mathbf{R}$ (resp. $(\mathbf{R}^i, \varphi_i(s_0)) \to \mathbf{R}$), m_i be the subset of $\mathcal{E}_i$ consisting of all function-germs with zero constant terms. Then, $\varphi_i^* m_i \mathcal{E}_1$ is an ideal of $\mathcal{E}_1$ and we consider quotient $\mathcal{E}_1$ algebras of the following types:

$$\frac{\mathcal{E}_1}{\varphi_i^* m_i \mathcal{E}_1.}$$

Lemma 2.2 *Let s_0 be an element of I. Then the following hold for any i $(1 \le i)$.*

1. $\mathbf{r}^{(i+1)}(s+s_0)\cdot\mathbf{r}(s+s_0) \in \varphi_i^* m_i\mathcal{E}_1 + \mathbf{R}$.
2. $\mathbf{r}^{(i+1)}(s+s_0)\cdot\mathbf{t}(s+s_0) \in \varphi_i^* m_i\mathcal{E}_1 + \mathbf{R}$.
3. $\mathbf{r}^{(i+2)}(s+s_0)\cdot\mathbf{n}(s+s_0) + \varphi_i^* m_i\mathcal{E}_1 = \kappa_g^{(i)}(s+s_0) + \varphi_i^* m_i\mathcal{E}_1$.

Lemma 2.3 *Let s_0 be an element of I. Then the following hold for any i $(1 \le i)$.*

1. $\mathbf{n}^{(i+1)}(s+s_0)\cdot\mathbf{n}(s+s_0)\in\varphi_i^*m_i\mathcal{E}_1$.

2. $\mathbf{n}^{(i+1)}(s+s_0)\cdot\mathbf{t}(s+s_0)+\varphi_i^*m_i\mathcal{E}_1=-\kappa_g^{(i)}(s+s_0)+\varphi_i^*m_i\mathcal{E}_1$.

3. $\mathbf{n}^{(i+2)}(s+s_0)\cdot\mathbf{r}(s+s_0)+\varphi_i^*m_i\mathcal{E}_1=i\kappa_g^{(i)}(s+s_0)+\varphi_i^*m_i\mathcal{E}_1$.

For simplicity of notations, we let f mean the image $f(s+s_0)$ for any map $f:I\to\mathbf{R}^n$ in the proofs of lemmas 2.2 and 2.3.

[*Proof of lemma 2.2*] We prove lemma 2.2 by induction with respect to i. For $i=1$ it is enough to show the following three:

$$\mathbf{r}''\cdot\mathbf{r}=-1, \tag{1}$$
$$\mathbf{r}''\cdot\mathbf{t}=0, \tag{2}$$
$$\mathbf{r}'''\cdot\mathbf{n}=\kappa_g'. \tag{3}$$

By lemma 2.1 we see that $\mathbf{r}'\cdot\mathbf{r}=\frac{1}{2}(\mathbf{r}\cdot\mathbf{r})'=0$. Furthermore, by using lemma 2.1, we have that $\mathbf{r}''\cdot\mathbf{r}=(\mathbf{r}'\cdot\mathbf{r})'-\mathbf{r}'\cdot\mathbf{r}'=-1$. Thus, (1) holds. For (2), lemma 2.1 shows that $\mathbf{r}'\cdot\mathbf{t}=-1$. Further use of lemma 2.1 shows that $\mathbf{r}''\cdot\mathbf{t}=(\mathbf{r}'\cdot\mathbf{t})'-\mathbf{r}'\cdot\mathbf{t}'=0$. For (3), lemma 2.1 shows that $\mathbf{r}'\cdot\mathbf{n}=0$. Further use of lemma 2.1 shows that $\mathbf{r}''\cdot\mathbf{n}=(\mathbf{r}'\cdot\mathbf{n})'-\mathbf{r}'\cdot\mathbf{n}'=\kappa_g$. Once more use of lemma 2.1 shows that $\mathbf{r}'''\cdot\mathbf{n}=(\mathbf{r}''\cdot\mathbf{n})'-\mathbf{r}''\cdot\mathbf{n}'=\kappa_g'$.

Next, we prove lemma 2.2 for $i=j+1$ under the assumption that lemma 2.2 holds for $i\le j$. By differentiating

$$\mathbf{r}^{(j+1)}\cdot\mathbf{r}\in\varphi_j^*m_j\mathcal{E}_1+\mathbf{R},$$

we have

$$\mathbf{r}^{(j+2)}\cdot\mathbf{r}+\mathbf{r}^{(j+1)}\cdot\mathbf{r}'\in\varphi_{j+1}^*m_{j+1}\mathcal{E}_1.$$

Since 2 of lemma 2.2 for $i=j$ holds by the assumption, by using lemma 2.1 we see that 1 of lemma 2.2 for $i=j+1$ holds.

By differentiating

$$\mathbf{r}^{(j+1)}\cdot\mathbf{t}\in\varphi_j^*m_j\mathcal{E}_1+\mathbf{R},$$

we have

$$\mathbf{r}^{(j+2)}\cdot\mathbf{t}+\mathbf{r}^{(j+1)}\cdot\mathbf{t}'\in\varphi_{j+1}^*m_{j+1}\mathcal{E}_1.$$

By using lemma 2.1, 1 of lemma 2.2 for $i=j+1$ and 3 of lemma 2.2 for $i=j-1$, we see that 2 of lemma 2.2 for $i=j+1$ holds.

Finally, by differentiating

$$\mathbf{r}^{(j+2)}\cdot\mathbf{n}+\varphi_j^*m_j\mathcal{E}_1=\kappa_g^{(j)}+\varphi_j^*m_j\mathcal{E}_1,$$

we have

$$\mathbf{r}^{(j+3)} \cdot \mathbf{n} + \mathbf{r}^{(j+2)} \cdot \mathbf{n}' + \varphi_{j+1}^{*} m_{j+1} \mathcal{E}_1 = \kappa_g^{(j+1)} + \varphi_{j+1}^{*} m_{j+1} \mathcal{E}_1.$$

By using lemma 2.1 and 2 of lemma 2.2 for $i = j+1$, we see that 3 of lemma 2.2 for $i = j+1$ holds. □

[*Proof of lemma 2.3*] We prove lemma 2.3 by induction with respect to i.

For $i = 1$ it is enough to show the following three:

$$\mathbf{n}'' \cdot \mathbf{n} = -\kappa_g^2, \tag{4}$$

$$\mathbf{n}'' \cdot \mathbf{t} = -\kappa_g', \tag{5}$$

$$\mathbf{n}''' \cdot \mathbf{r} = \kappa_g' + \kappa_g. \tag{6}$$

By lemma 2.1 we see that $\mathbf{n}' \cdot \mathbf{n} = \frac{1}{2}(\mathbf{n} \cdot \mathbf{n})' = 0$. Furthermore, by using lemma 2.1, we have that $\mathbf{n}'' \cdot \mathbf{n} = (\mathbf{n}' \cdot \mathbf{n})' - \mathbf{n}' \cdot \mathbf{n}' = -\kappa_g^2$. Thus, (4) holds. For (5), lemma 2.1 shows that $\mathbf{n}' \cdot \mathbf{t} = -\kappa_g$. Further use of lemma 2.1 shows that $\mathbf{n}'' \cdot \mathbf{t} = (\mathbf{n}' \cdot \mathbf{t})' - \mathbf{n}' \cdot \mathbf{t}' = -\kappa_g'$. For (6), lemma 2.1 shows that $\mathbf{n}' \cdot \mathbf{r} = 0$. Further use of lemma 2.1 shows that $\mathbf{n}'' \cdot \mathbf{r} = (\mathbf{n}' \cdot \mathbf{r})' - \mathbf{n}' \cdot \mathbf{r}' = \kappa_g$. Once more use of lemma 2.1 shows that $\mathbf{n}''' \cdot \mathbf{r} = (\mathbf{n}'' \cdot \mathbf{r})' - \mathbf{n}'' \cdot \mathbf{r}' = \kappa_g' + \kappa_g$.

Next, we prove lemma 2.3 for $i = j+1$ under the assumption that lemma 2.3 holds for $i \leq j$. By differentiating

$$\mathbf{n}^{(j+1)} \cdot \mathbf{n} \in \varphi_j^{*} m_j \mathcal{E}_1,$$

we have

$$\mathbf{n}^{(j+2)} \cdot \mathbf{n} + \mathbf{n}^{(j+1)} \cdot \mathbf{n}' \in \varphi_{j+1}^{*} m_{j+1} \mathcal{E}_1.$$

Since 2 of lemma 2.3 for $i = j$ holds by the assumption, by using lemma 2.1 we see that 1 of lemma 2.3 for $i = j+1$ holds.

By differentiating

$$\mathbf{n}^{(j+1)} \cdot \mathbf{t} + \varphi_j^{*} m_j \mathcal{E}_1 = -\kappa_g^{(j)} + \varphi_j^{*} m_j \mathcal{E}_1,$$

we have

$$\mathbf{n}^{(j+2)} \cdot \mathbf{t} + \mathbf{n}^{(j+1)} \cdot \mathbf{t}' + \varphi_{j+1}^{*} m_{j+1} \mathcal{E}_1 = -\kappa_g^{(j+1)} + \varphi_{j+1}^{*} m_{j+1} \mathcal{E}_1.$$

By using lemma 2.1, 1 of lemma 2.3 for $i = j+1$ and 3 of lemma 2.3 for $i = j-1$ if $j \geq 1$, we see that 2 of lemma 2.3 for $i = j+1$ holds.

Finally, by differentiating

$$\mathbf{n}^{(j+2)} \cdot \mathbf{r} + \varphi_j^{*} m_j \mathcal{E}_1 = j\kappa_g^{(j)} + \varphi_j^{*} m_j \mathcal{E}_1,$$

we have

$$\mathbf{n}^{(j+3)} \cdot \mathbf{r} + \mathbf{n}^{(j+2)} \cdot \mathbf{r}' + \varphi_{j+1}^{*} m_{j+1} \mathcal{E}_1 = j\kappa_g^{(j+1)} + \varphi_{j+1}^{*} m_{j+1} \mathcal{E}_1.$$

By using lemma 2.1 and 2 of lemma 2.3 for $i = j+1$, we see that 3 of lemma 2.3 for $i = j+1$ holds. □

3 Explicit formula for a pedal curve

Let $\mathbf{r}$ be a spherical unit speed curve and $\mathbf{n}$ be its dual. Let P be any point in $S^2 - \{\pm\mathbf{n}(s) \mid s \in I\}$. To characterize the singularities of the pedal curve with the pedal point P we prepare an explicit formula for $Pe_{\mathbf{r},P}$.

Lemma 3.1

$$Pe_{\mathbf{r},P}(s) = \frac{1}{\sqrt{1-(P\cdot\mathbf{n}(s))^2}}(P-(P\cdot\mathbf{n}(s))\mathbf{n}(s)).$$

[*Proof of lemma 3.1*] For any $s \in S^1$, by subtracting $(P\cdot\mathbf{n}(s))\mathbf{n}(s)$ from P we obtain the vector $P-(P\cdot\mathbf{n}(s))\mathbf{n}(s)$ in $\mathbf{R}^3$ which is a positive scalar multiple of $Pe_{\mathbf{r},P}(s)$. Normalizing this vector gives the right hand side of the formula in lemma 3.1, which must be the vector $Pe_{\mathbf{r},P}(s)$. □

By this formula, we can characterize the singularities of the pedal curve with the pedal point P as follows.

Lemma 3.2

$$Pe'_{\mathbf{r},P}(s) = 0 \quad \Longleftrightarrow \kappa_g(s) = 0 \ \textit{or}\ P = \mathbf{r}(s).$$

[*Proof of lemma 3.2*] By differentiating $Pe_{\mathbf{r},P}$ and using lemma 2.1, we have the following.

$$\begin{aligned} Pe'_{\mathbf{r},P}(s) = & - \kappa_g(s)\frac{(P\cdot\mathbf{n}(s))(P\cdot\mathbf{t}(s))}{(1-(P\cdot\mathbf{n}(s)))^{\frac{3}{2}}}\Big((P\cdot\mathbf{r}(\mathbf{s}))\mathbf{r}(s)+(P\cdot\mathbf{t}(s))\mathbf{t}(s)\Big) \\ & + \kappa_g(s)\frac{1}{(1-(P\cdot\mathbf{n}(s)))^{\frac{1}{2}}}\Big((P\cdot\mathbf{n}(\mathbf{s}))\mathbf{t}(s)+(P\cdot\mathbf{t}(s))\mathbf{n}(s)\Big). \end{aligned}$$

Since $\{\mathbf{r}(s), \mathbf{t}(s), \mathbf{n}(s)\}$ is an orthogonal frame, we see that $Pe'_{\mathbf{r},P}(s) = 0$ if and only if $\kappa_g(s) = 0$ or $P = \mathbf{r}(s)$. □

Let P be a point of $S^2 - \{\pm\mathbf{n}(s) \mid s \in I\}$. We consider the following C^∞ map $\Psi_P : S^2 - \{\pm P\} \to S^2$.

$$\Psi_P(\mathbf{x}) = \frac{1}{\sqrt{1-(P\cdot\mathbf{x})^2}}(P-(P\cdot\mathbf{x})\mathbf{x}).$$

We see that the image $\Psi_P(S^2 - \{\pm P\})$ is inside the open hemisphere centered at P. Let this open hemisphere, the set $\pi(S^2 - \{\pm P\})$ be denoted by X_P, B_P respectively, where $\pi : S^2 \to P^2(\mathbf{R})$ is the canonical projection. Note that X_P is C^∞ diffeomorphic to the 2-dimensional open disc $\{(x,y) \mid x^2+y^2 < 1\}$ and B_P is C^∞ diffeomorphic to the open Möbius band.

Since $\Psi_P(\mathbf{x}) = \Psi_P(-\mathbf{x})$, Ψ_P induces the map $\widetilde{\Psi}_P : B_P \to X_P$.

We let B be the set

$$\{(x_1, x_2) \times [\xi_1 : \xi_2] \in \mathbf{R}^2 \times P^1(\mathbf{R}) \mid x_1\xi_2 = x_2\xi_1\}.$$

Let $p : \mathbf{R}^2 \times P^1(\mathbf{R}) \to \mathbf{R}^2$ be the canonical projection. In [10], we have constructed a concrete C^∞ diffeomorphism

$$h_1 : B_p \to B$$

which gives the equality

$$p \circ h_1 = q_p \circ \widetilde{\Psi}_P,$$

where $q_p : \mathbf{R}^3 \to \mathbf{R}^2$ is the orthogonal projection to the 2-dimensional linear subspace perpendicular to P. Since we need this construction to prove theorem 2, we recall arguments in [10] briefly.

First, by a suitable rotation of $\mathbf{R}^3$ around the origin, we may assume that $P = (0, 0, 1)$. We put

$$\begin{aligned} U_1 &= \{(x_1, x_2) \times [\xi_1 : \xi_2] \in \mathbf{R}^2 \times P^1(\mathbf{R}) \mid x_1\xi_2 = x_2\xi_1,\ \xi_1 \neq 0\}, \\ U_2 &= \{(x_1, x_2) \times [\xi_1 : \xi_2] \in \mathbf{R}^2 \times P^1(\mathbf{R}) \mid x_1\xi_2 = x_2\xi_1,\ \xi_2 \neq 0\} \end{aligned}$$

and

$$\begin{aligned} U_{P,1} &= \{\pi((x_1, x_2, x_3)) \mid x_1 \neq 0\ \}, \\ U_{P,2} &= \{\pi((x_1, x_2, x_3)) \mid x_2 \neq 0\ \}. \end{aligned}$$

Furthermore, we put as follows.

$$\begin{aligned} \varphi_1 : U_1 \to \mathbf{R}^2, &\qquad (x_1, x_2) \times [\xi_1 : \xi_2] \mapsto (u_1, u_2) = \left(x_1, \frac{\xi_2}{\xi_1}\right), \\ \varphi_2 : U_2 \to \mathbf{R}^2, &\qquad (x_1, x_2) \times [\xi_1 : \xi_2] \mapsto (u_1', u_2') = \left(\frac{\xi_1}{\xi_2}, x_2\right) \end{aligned}$$

and

$$\begin{aligned} \varphi_{P,1}(\pi((x_1, x_2, x_3))) &= \left(-\tan(\lambda)x_1, \frac{x_2}{x_1}\right), \\ \varphi_{P,2}(\pi((x_1, x_2, x_3))) &= \left(\frac{x_1}{x_2}, -\tan(\lambda)x_2\right), \end{aligned}$$

where $\lambda = \sin^{-1}(x_3)$ $(-\frac{\pi}{2} < \lambda < \frac{\pi}{2})$. Since $p : B \to \mathbf{R}^2$ is the blow up of the plane centered at the origin, it is well-known that $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ is an atlas for B and

$$\begin{aligned} p \circ \varphi_1^{-1}(u_1, u_2) &= (u_1, u_1 u_2), \\ p \circ \varphi_2^{-1}(u_1', u_2') &= (u_1' u_2', u_2'). \end{aligned}$$

For our $\{(U_{P,1}, \varphi_{P,1}), (U_{P,2}, \varphi_{P,2})\}$ and $\widetilde{\Psi}_P$, we can show the same results (for details, see [10]).

1. $\{(U_{P,1}, \varphi_{P,1}), (U_{P,2}, \varphi_{P,2})\}$is an atlas for $\pi(S^2 - \{\pm P\})$.
2.

$$\begin{aligned} q \circ \widetilde{\Psi}_P \circ \varphi_{P,1}^{-1}(u_1, u_2) &= (u_1, u_1 u_2), \\ q \circ \widetilde{\Psi}_P \circ \varphi_{P,2}^{-1}(u_1', u_2') &= (u_1' u_2', u_2'), \end{aligned}$$

where $q : \mathbf{R}^3 \to \mathbf{R}^2$ is the canonical projection taking first two coordinates (note that we have put $P = (0, 0, 1)$).

3. $\varphi_1^{-1} \circ \varphi_{P,1}(\pi(x_1, x_2, x_3)) = \varphi_2^{-1} \circ \varphi_{P,2}(\pi(x_1, x_2, x_3))$
for any $\pi(x_1, x_2, x_3) \in U_{P,1} \cap U_{P,2}$.

For general P, it suffices to compose suitable rotations of S^2 .

4 Proof of 3 of theorem 2

We would like to apply the argument in §3, thus we assume that $P = (0, 0, 1)$. By a suitable rotation of S^2, we may assume that $\mathbf{n}(s_0) = (1, 0, 0)$. Furthermore, in the case of 3 of theorem 2, $\mathbf{r}(s_0) = (0, 0, \pm 1)$ and $\mathbf{t}(s_0) = (0, \mp 1, 0)$. From lemma 2.3, we may put the map germ $\mathbf{n} : (I, s_0) \to (S^2, \mathbf{n}(s_0))$ as follows.

$$\mathbf{n}(s) = \begin{pmatrix} 1 - \alpha_2(s - s_0) \\ \pm \frac{1}{(k+1)!} \kappa_g^{(k)}(s_0)(s - s_0)^{k+1} + \beta_2(s - s_0) \\ \pm \frac{k}{(k+2)!} \kappa_g^{(k)}(s_0)(s - s_0)^{k+2} + \beta_3(s - s_0) \end{pmatrix}$$

where α_2, β_i are certain C^∞ function-germs $(\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that $\frac{d^j \alpha_i}{ds^j}(0)$ $= \frac{d^j \beta_i}{ds^j}(0) = 0$ $(j \le k + i - 1)$.

Since in §3 we have put

$$\varphi_{P,1}(\pi((x_1, x_2, x_3))) = \left(-\tan(\lambda) x_1, -\frac{x_2}{x_1}\right) = (u_1, u_2)$$

where $\sin(\lambda) = x_3$, we have

$$\varphi_{P,1}(\pi(\mathbf{n}(s))) = \begin{pmatrix} -\tan(\lambda)(1 - \alpha_2(s - s_0)) \\ -\frac{c_1(s - s_0)^{k+1} + \beta_2(s - s_0)}{1 - \alpha_2(s - s_0)} \end{pmatrix},$$

where c_1 is non-zero constant given by $c_1 = \pm\frac{1}{(k+1)!}\kappa_g^{(k)}(s_0)$.

Since

$$q \circ \widetilde{\Psi}_P \circ \varphi_{P,1}^{-1}(u_1, u_2) = (u_1, u_1u_2),$$

we see that the map-germ $\widetilde{\Psi}_P \circ \mathbf{n} : (I, s_0) \to (S^2, \widetilde{\Psi}_P \circ \mathbf{n}(s_0))$ is C^∞ right-left equivalent to the following.

$$\begin{pmatrix} -\tan(\lambda)(1 - \alpha_2(s - s_0)) \\ \tan(\lambda)(c_1(s - s_0)^{k+1} + \beta_2(s - s_0)) \end{pmatrix}.$$

This shows that the map germ $Pe_{\mathbf{r},P} : (I, s_0) \to (S^2, Pe_{\mathbf{r},P}(s_0))$ is C^∞ right-left equivalent to the following, where $\widetilde{\beta}_i$ is a certain C^∞ function-germs $(\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that $\frac{d^j\widetilde{\beta}_i}{ds^j}(0) = 0$ $(j \leq k + i + 1)$.

$$\begin{pmatrix} \mp\frac{k}{(k+2)!}\kappa_g^{(k)}(s_0)(s - s_0)^{k+2} + \widetilde{\beta}_1(s - s_0) \\ \pm\frac{k}{(k+2)!}\kappa_g^{(k)}(s_0)c_1(s - s_0)^{2k+3} + \widetilde{\beta}_{k+2}(s - s_0) \end{pmatrix}.$$

Lemma 4.1 (theorem 3.3 in [5]) *Let $f : (\mathbf{R}, 0) \to \mathbf{R}$ be a C^∞ function-germ. Suppose that $f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0$ and $f^{(k)}(0) \neq 0$. Then there exists a germ of C^∞ diffeomorphism $h : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that $f(h(s)) = \pm s^k$, where we have + or - according as $f^{(k)}(0)$ is > 0 or < 0.*

Since

$$\mp\frac{k}{(k+2)!}\kappa_g^{(k)}(s_0) \neq 0 \quad \text{and} \quad \frac{d^j\widetilde{\beta}_1}{ds^j}(0) = 0 \ (j \leq k + 2),$$

by using lemma 4.1 and by composing appropriate scales and reflections along coordinate axes of $\mathbf{R}^2$ if necessary, we see that $Pe_{\mathbf{r},P}$ is C^∞ right-left equivalent to the following form:

$$(\sigma^{k+2}, \sigma^{2k+3} + \gamma_{k+2}(\sigma)),$$

where γ_{k+2} is a C^∞ function-germ $(\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that $\frac{d^j\gamma_{k+2}}{d\sigma^j}(0) = 0$ $(j \leq 2k + 3)$.

To finish the proof of 3 of theorem 2, it is sufficient to show that

$$h_2(x_1, x_2) = (x_1, x_2 + \gamma_{k+2}(x_2^{\frac{1}{2k+3}}))$$

is a germ of C^1 diffeomorphism. Here, note that $2k + 3$ is odd. Thus, $x_2 \mapsto x_2^{\frac{1}{2k+3}}$ is well-defined and continuous even at $x_2 = 0$.

By Hadamard's lemma (lemma3.4 of [5]), there exists a C^∞ function-germ $\widetilde{\gamma}_{k+2} : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that $\gamma_{k+2}(\sigma) = \sigma^{2k+4}\widetilde{\gamma}_{k+2}(\sigma)$. Thus, we have the following:

$$\begin{aligned}
\frac{d\gamma_{k+2}(x_2^{\frac{1}{2k+3}})}{dx_2}(x_2) &= \lim_{h\to 0} \frac{\gamma_{k+2}((x_2+h)^{\frac{1}{2k+3}}) - \gamma_{k+2}(x_2^{\frac{1}{2k+3}})}{h} \\
&= \lim_{h\to 0} \frac{(x_2+h)^{\frac{2k+4}{2k+3}}\widetilde{\gamma}_{k+2}((x_2+h)^{\frac{1}{2k+3}}) - x_2^{\frac{2k+4}{2k+3}}\widetilde{\gamma}_{k+2}(x_2^{\frac{1}{2k+3}})}{h} \\
&= \lim_{h\to 0} \frac{(x_2+h)^{\frac{2k+4}{2k+3}} - x_2^{\frac{2k+4}{2k+3}}}{h}\widetilde{\gamma}_{k+2}((x_2+h)^{\frac{1}{2k+3}}) \\
&\quad + x_2^{\frac{2k+4}{2k+3}} \lim_{h\to 0} \frac{\widetilde{\gamma}_{k+2}((x_2+h)^{\frac{1}{2k+3}}) - \widetilde{\gamma}_{k+2}(x_2^{\frac{1}{2k+3}})}{h} \\
&= \frac{2k+4}{2k+3} x_2^{\frac{1}{2k+3}} \widetilde{\gamma}_{k+2}(x_2^{\frac{1}{2k+3}}) + \frac{1}{2k+3} x_2^{\frac{2}{2k+3}} \widetilde{\gamma}'_{k+2}(x_2^{\frac{1}{2k+3}}).
\end{aligned}$$

Thus, $x_2 \mapsto \frac{d\gamma_{k+2}(x_2^{\frac{1}{2k+3}})}{dx_2}(x_2)$ is well-defined and continuous even at $x_2 = 0$. Since we see $\frac{d\gamma_{k+2}(x_2^{\frac{1}{2k+3}})}{dx_2}(0) = 0$, the Jacobian matrix of h_2 at $(0,0)$ is the unit matrix and therefore h_2 is a germ of C^1 diffeomorphism. □

5 Proof of 2 of theorem 2

For the proof of 2 of theorem 2, we use similar arguments as in §4. We assume that $P = (0,0,1)$. By a suitable rotation of S^2, we may assume that $\mathbf{n}(s_0) = (1,0,0)$. Furthermore, in the case of 2 of theorem 2, $\mathbf{r}(s_0) = (0,a,b)$ and $\mathbf{t}(s_0) = (0,-b,a)$ $(a, b \in \mathbf{R}, a^2+b^2 = 1,\ a \neq 0)$. From lemma 2.3, we may put the map germ $\mathbf{n} : (I, s_0) \to (S^2, \mathbf{n}(s_0))$ as follows.

$$\mathbf{n}(s) = \begin{pmatrix} 1 - \alpha_2(s-s_0) \\ -b\gamma(s) + a\delta(s) \\ a\gamma(s) + b\delta(s) \end{pmatrix},$$

where

$$\begin{aligned}
\gamma(s) &= -\frac{1}{(k+1)!}\kappa_g^{(k)}(s_0)(s-s_0)^{k+1} + \beta_2(s-s_0) \\
\delta(s) &= \frac{k}{(k+2)!}\kappa_g^{(k)}(s_0)(s-s_0)^{k+2} + \beta_3(s-s_0)
\end{aligned}$$

and α_2, β_i are certain C^∞ function-germs $(\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that $\frac{d^j\alpha_i}{ds^j}(0) = \frac{d^j\beta_i}{ds^j}(0) = 0$ $(j \leq k+i-1)$. Thus, we have that

$$\varphi_{P,1}(\pi(\mathbf{n}(s))) = \begin{pmatrix} -\tan(\lambda)(1-\alpha_2(s-s_0)) \\ -\frac{-b\gamma(s)+a\delta(s)}{1-\alpha_2(s-s_0)} \end{pmatrix},$$

where $\sin(\lambda) = a\gamma(s) + b\delta(s)$. Note that since $a \neq 0$, the third component of $\mathbf{n}$(s) have the non-vanishing term of order $k+1$.

Lemma 5.1 *Let $h_1 : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear isomorphism given by $h_1(u_1, u_2) = (u_1, u_2 + \frac{b}{a}u_1)$ and $h_2 : \mathbf{R}^2 \to \mathbf{R}^2$ be the C^∞ diffeomprphism given by $h_2(U_1, U_2) = (U_1, U_2 + \frac{b}{a}U_1^2)$. Then,*

$$q \circ \widetilde{\Psi}_P \circ \varphi_{P,1}^{-1} \circ h_1(u_1, u_2) = h_2 \circ q \circ \widetilde{\Psi}_P \circ \varphi_{P,1}^{-1}(u_1, u_2).$$

A straight forward calculation gives the proof of lemma 5.1.

By using lemma 5.1 we see that for $u_1 = -\tan(\lambda)(1-\alpha_2(s-s_0))$ and $u_2 = -\frac{-b\gamma(s)+a\delta(s)}{1-\alpha_2(s-s_0)}$

$$q \circ \widetilde{\Psi}_P \circ \varphi_{P,1}^{-1} \circ h_1(u_1, u_2) \tag{7}$$

is C^∞ right-left equivalent to $Pe_{\mathbf{r},P}$ near s_0. On the other hand, by using Taylor expansions we see that for $u_1 = -\tan(\lambda)(1-\alpha_2(s-s_0))$ and $u_2 = -\frac{-b\gamma(s)+a\delta(s)}{1-\alpha_2(s-s_0)}$ (7) may be put as follows.

$$\begin{pmatrix} a\frac{1}{(k+1)!}\kappa_g^{(k)}(s_0)(s-s_0)^{k+1} + \widetilde{\beta}_0(s-s_0) \\ -a^2\frac{1}{(k+1)!}\kappa_g^{(k)}(s_0)\frac{k}{(k+2)!}\kappa_g^{(k)}(s_0)(s-s_0)^{2k+3} + \widetilde{\beta}_{k+2}(s-s_0) \end{pmatrix},$$

where $\widetilde{\beta}_i$ is a certain C^∞ function-germs $(\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that $\frac{d^j\widetilde{\beta}_i}{ds^j}(0) = 0$ $(j \leq k+i+1)$.

By using lemma 4.1 and by composing appropriate scales and reflections along coordinate axes of $\mathbf{R}^2$ if necessary, we see that $Pe_{\mathbf{r},P}$ is C^∞ right-left equivalent to the following form:

$$(\sigma^{k+1}, \sigma^{2k+3} + \gamma_{k+2}(\sigma)),$$

where γ_{k+2} is a C^∞ function-germ $(\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that $\frac{d^j\gamma_{k+2}}{d\sigma^j}(0) = 0$ $(j \leq 2k+3)$.

To finish the proof of 2 of theorem 2, it is sufficient to show that

$$h_2(x_1, x_2) = (x_1, x_2 + \gamma_{k+2}(x_2^{\frac{1}{2k+3}}))$$

is a germ of C^1 diffeomorphism, but it has been already proved in §4. $\square$

6 Proof of 1 of theorem 2

By the assumption of 1 of theorem 2, we see that $P \cdot \mathbf{n}(s_0) \neq 0$. We assume that $P = (0,0,1)$. From lemma 2.3, we see that

$$\begin{aligned}
\mathbf{n}(s)\cdot\mathbf{t}(s_0) &= -\frac{1}{(k+1)!}\kappa_g^{(k)}(s_0)(s-s_0)^{k+1} + \beta_2(s-s_0),\\
\mathbf{n}(s)\cdot\mathbf{r}(s_0) &= \frac{k}{(k+2)!}\kappa_g^{(k)}(s_0)(s-s_0)^{k+2} + \beta_3(s-s_0),\\
\mathbf{n}(s)\cdot\mathbf{n}(s_0) &= 1-\alpha_2(s-s_0),
\end{aligned}$$

where α_2, β_i are certain C^∞ function-germs $(\mathbf{R},0)\to(\mathbf{R},0)$ such that $\frac{d^j\alpha_i}{ds^j}(0) = \frac{d^j\beta_i}{ds^j}(0) = 0 \quad (j \leq k+i-1)$. Thus, we see that the map-germ $\mathbf{n} : (I, s_0) \to (S^2, \mathbf{n}(s_0))$ is C^∞ right-left equivalent to the map-germ of the following form:

$$\mathbf{n}(s) = \begin{pmatrix} -\frac{1}{(k+1)!}\kappa_g^{(k)}(s_0)(s-s_0)^{k+1} + \beta_2(s-s_0) \\ \frac{k}{(k+2)!}\kappa_g^{(k)}(s_0)(s-s_0)^{k+2} + \beta_3(s-s_0) \\ 1-\alpha_2(s-s_0) \end{pmatrix}.$$

By using lemma 4.1 and by composing appropriate scales and reflections along coordinate axes of $\mathbf{R}^2$ if necessary, we see that $Pe_{\mathbf{r},P}$ is C^∞ right-left equivalent to the following form:

$$(\sigma^{k+1} + \gamma_{-1}(\sigma), \sigma^{k+2} + \gamma_0(\sigma)),$$

where γ_i is a C^∞ function-germ $(\mathbf{R},0)\to(\mathbf{R},0)$ such that $\frac{d^j\gamma_i}{d\sigma^j}(0) = 0$ $(j \leq k+i+2)$.

If $k+1$ is odd, we consider

$$h_2(x_1,x_2) = (x_1 + \gamma_{-1}(x_1^{\frac{1}{k+1}}), x_2 + \gamma_0(x_1^{\frac{1}{k+1}})).$$

If $k+2$ is odd, we consider

$$h_2(x_1,x_2) = (x_1 + \gamma_{-1}(x_2^{\frac{1}{k+2}}), x_2 + \gamma_0(x_2^{\frac{1}{k+2}})).$$

In each case, $x_l \mapsto x_l^{\frac{1}{k+l}}$ is well-defined and continuous at $x_l = 0$, where $l = 1$ (resp. 2) if k is even (resp. odd).

By using the same notation $\widetilde{\gamma}_i$ as in §4, we have the following:

$$\begin{aligned}
\frac{d\gamma_i(x_l^{\frac{1}{k+l}})}{dx_l} &= \lim_{h\to 0}\frac{\gamma_i((x_l+h)^{\frac{1}{k+l}})-\gamma_i(x_l^{\frac{1}{k+l}})}{h} \\
&= \lim_{h\to 0}\frac{(x_l+h)^{\frac{k+i+3}{k+l}}\widetilde{\gamma}_i((x_l+h)^{\frac{1}{k+l}})-x_l^{\frac{k+i+3}{k+l}}\widetilde{\gamma}_i(x_l^{\frac{1}{k+l}})}{h} \\
&= \lim_{h\to 0}\frac{(x_l+h)^{\frac{k+i+3}{k+l}}-x_l^{\frac{k+i+3}{k+l}}}{h}\widetilde{\gamma}_i((x_l+h)^{\frac{1}{k+l}}) \\
&\quad + x_l^{\frac{k+i+3}{k+l}}\lim_{h\to 0}\frac{\widetilde{\gamma}_i((x_l+h)^{\frac{1}{k+l}})-\widetilde{\gamma}_i(x_l^{\frac{1}{k+l}})}{h} \\
&= \frac{k+i+3}{k+l}x_l^{\frac{i-l+3}{k+l}}\widetilde{\gamma}_i(x_l^{\frac{1}{k+l}})+\frac{1}{k+l}x_l^{\frac{i-l+4}{k+l}}\widetilde{\gamma}_i'(x_l^{\frac{1}{k+l}}).
\end{aligned}$$

Since $i \geq -1$ and $l \leq 2$, we have that $i-l+4 > i-l+3 \geq 0$. Thus, $x_l \mapsto \frac{d\gamma_i(x_l^{\frac{1}{k+l}})}{dx_l}(x_l)$ is well-defined and continuous even at $x_l = 0$. Furthermore, we have $\frac{d\gamma_{-1}(x_l^{\frac{1}{k+l}})}{dx_l}(0) = 0$ for $l = 1$ and $\frac{d\gamma_0(x_l^{\frac{1}{k+l}})}{dx_l}(0) = 0$ for $l = 2$. Therefore the Jacobian matrix of h_2 at $(0,0)$ is a triangular matrix whose diagonal elements are 1, and thus h_2 is a germ of C^1 diffeomorphism and therefore the map-germ $Pe_{\mathbf{r},P}$ is C^1 right-left equivalent to the map-germ given by $\sigma \mapsto (\sigma^{k+1}, \sigma^{k+2})$. □

7 Remark on C^1 right-left equivalence in theorem 2

It is natural to expect that C^1 right-left equivalence in theorem 2 is improvable to C^∞ right-left equibalence. However, by the following reason, it seems to be almost impossible to do so in general.

Let $i_1, i_2 \quad (i_1 < i_2)$ be positive integers. Let $\mathcal{S}(i_1, i_2)$ be the set consists of all C^∞ map-germs $(\mathbf{R}, 0) \to (\mathbf{R}^2, (0,0))$ which are C^∞ right-left equivalent to $s \mapsto (s^{i_1} + higher, s^{i_2} + higher)$. Then, for any (i_1, i_2) with $i_1 < i_2$ the set $\mathcal{S}(i_1, i_2)$ is contained in a single $\mathcal{K}$-orbit in the sense of Mather ([9]). On the other hand, the codimension of the $\mathcal{A}$-orbit of the map-germ $s \mapsto (s^{i_1}, s^{i_2})$ in $\mathcal{S}(i_1, i_2)$ is positive for any (i_1, i_2) in the following.

1. $\mathcal{S}(k+1, k+2)$ $(k \geq 3)$. These sets correspond to 1 of theorem 2. In the case that $1 \leq k \leq 2$, for each k the set $\mathcal{S}(k+1, k+2)$ coincides exceptionally with a single $\mathcal{A}$-orbit.

2. $\mathcal{S}(k+1, 2k+3)$ $(k \geq 2)$. These sets correspond to 2 of theorem 2. In the case that $k=1$, the set $\mathcal{S}(2,5)$ coincides exceptionally with a single $\mathcal{A}$-orbit.

3. $\mathcal{S}(k+2, 2k+3)$ $(k \geq 2)$. These sets correspond to 3 of theorem 2. In the case that $k=1$, the set $\mathcal{S}(3,5)$ coincides exceptionally with a single $\mathcal{A}$-orbit.

However, by the above exceptional cases we can see that there are several cases which we can improve C^1 right-left equivalence to C^∞ right-left equivalence.

Theorem 3 *Let $\mathbf{r} : I \to S^2$ be a spherical unit speed curve. Let $s_0 \in I$ such that $\kappa_g(s_0) = 0, \kappa_g'(s_0) \neq 0$ and P be a point of $S^2 - \{\pm\mathbf{n}(s_0)\}$. Then the following hold.*

1. *If $P \in S^2 - \{\pm\mathbf{n}(s_0)\} - C_{\mathbf{n}(s_0)}$, then the map-germ $Pe_{\mathbf{r},P} : (I, s_0) \to (S^2, Pe_{\mathbf{r},P}(s_0))$ is C^∞ right-left equivalent to the map-germ given by $\sigma \mapsto (\sigma^2, \sigma^3)$.*

2. *If $P \in C_{\mathbf{n}(s_0)} - \{\pm\mathbf{r}(s_0)\}$, then the map-germ $Pe_{\mathbf{r},P} : (I, s_0) \to S^2$ is C^∞ right-left equivalent to the map-germ given by $\sigma \mapsto (\sigma^2, \sigma^5)$.*

3. *If $P \in \{\pm\mathbf{r}(s_0)\}$, then the map-germ $Pe_{\mathbf{r},P} : (I, s_0) \to (S^2, Pe_{\mathbf{r},P}(s_0))$ is C^∞ right-left equivalent to the map-germ given by $\sigma \mapsto (\sigma^3, \sigma^5)$.*

Theorem 4 *Let $\mathbf{r} : I \to S^2$ be a spherical unit speed curve. Let $s_0 \in I$ such that $\kappa_g(s_0) = \kappa_g'(s_0) = 0, \kappa_g''(s_0) \neq 0$ and P be a point of $S^2 - \{\pm\mathbf{n}(s_0)\} - C_{\mathbf{n}(s_0)}$. Then, the map-germ $Pe_{\mathbf{r},P} : (I, s_0) \to (S^2, Pe_{\mathbf{r},P}(s_0))$ is C^∞ right-left equivalent to the map-germ given by $\sigma \mapsto (\sigma^3, \sigma^4)$.*

By combining theorems 1, 3 and 4 and observation of classification of $\mathcal{A}$-simple singularities of plane curves ([2], [4], [6]), we see that we can define the "genericity" of point pairs $(\mathbf{r}(s_0), P)$ precisely so that the set

$$\{\mathcal{A}\text{-equivalence class of } Pe_{\mathbf{r},P} : (I, s_0) \to S^2, (\mathbf{r}(s_0), P) \text{ is generic}\}$$

is equal to the set of all $\mathcal{A}$-equivalence classes of $\mathcal{A}$-simple map germs $f : (\mathbf{R}, 0) \to \mathbf{R}^2$ such that $T\mathcal{A}(f) = T\mathcal{K}(f)$.

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