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Dedicated to Professor Shyuichi Izumiya on the occasion of his sixtieth birthday

We introduce map germs of pedal unfolding type and the notion of normalized Legendrian map germs. We show that the fundamental theorem of calculus provides a natural one-to-one correspondence between Whitney umbrellas of pedal unfolding type and normalized swallowtails.

1. Introduction

The map germ

(1) $$f(x, y) = (xy, x^2, y)$$

is known as the normal form of Whitney umbrella, after Whitney's pioneering works [1943; 1944]. Compose the germ (1) with the coordinate transformations

$$h_s(x, y) = (x, x^2 + y) \quad \text{and} \quad h_t(X, Y, Z) = (X, -Z, -Y + Z),$$

where (X, Y, Z) are the standard coordinates of the target space $\mathbb{R}^3$. This leads to the map germ

(2) $$g(x, y) = h_t \circ f \circ h_s(x, y) = (x^3 + xy, -x^2 - y, y).$$

Set

(3) $$\begin{aligned} G(x, y) &= \left(\int_0^x (x^3 + xy)dx, \int_0^x (-x^2 - y)dx, y\right) \\ &= \left(\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2y, -\tfrac{1}{3}x^3 - xy, y\right). \end{aligned}$$

Compose the map germ (3) with the scaling transformations

$$H_s(x, y) = (x, \tfrac{1}{6}y) \quad \text{and} \quad H_t(X, Y, Z) = (12X, 12Y, 6Z)$$

to obtain the map germ

(4) $$H_t \circ G \circ H_s(x, y) = (3x^4 + x^2y, -4x^3 - 2xy, y),$$

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known as the normal form of the swallowtail [Bruce and Giblin 1992, page 129].

Two C^∞ map germs $\varphi, \psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ are said to be $\mathscr{A}$-equivalent if there exist germs of C^∞ diffeomorphisms

$$h_s : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \quad \text{and} \quad h_t : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0),$$

such that $\psi = h_t \circ \varphi \circ h_s$. A C^∞ map germ $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is called a *Whitney umbrella* if it is $\mathscr{A}$-equivalent to (1); it is called a *swallowtail* if it is $\mathscr{A}$-equivalent to (4). As seen above, the Whitney umbrella (1) produces the swallowtail (4) via (2) and (3). By the converse procedure, the swallowtail (4) produces the Whitney umbrella (1).

It is impossible to produce a swallowtail by integrating (1) directly. This is because the discriminant set of (4) is not diffeomorphic to the discriminant set of

$$(x, y) \mapsto \left(\int_0^x xy\,dx, \int_0^x x^2 dx, y \right). \tag{5}$$

Note that the form (2) may be written as follows:

$$g(x, y) = \big(x(x^2 + y), -(x^2 + y), y\big) = \big(b(-x, -(x^2 + y)), y\big),$$

where $b(X, Y) = (XY, Y)$ (b stands for "blowdown").

Definition 1.1. (i) A C^∞ map germ $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ having the following form is said to be of *pedal unfolding type*.

$$\varphi(x, y) = \big(n(x, y)p(x, y), p(x, y), y\big) = \big(b(n(x, y), p(x, y)), y\big), \tag{6}$$

where $n : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ is a C^∞ function germ, such that

$$\frac{\partial n}{\partial x}(0, 0) \neq 0 \quad \text{and} \quad p : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \text{ is a } C^\infty \text{ function germ.}$$

(ii) For a C^∞ map germ of pedal unfolding type

$$\varphi(x, y) = \big(n(x, y)p(x, y), p(x, y), y\big),$$

set

$$\mathscr{I}(\varphi)(x, y) = \left(\int_0^x n(x, y)p(x, y)dx, \int_0^x p(x, y)dx, y \right).$$

The map germ $\mathscr{I}(\varphi) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is called the *integration of* φ.

(iii) A C^∞ map germ $\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^{m+1}, 0)$ is called a *Legendrian map germ* if there exists a germ of C^∞ vector field $\nu_\Phi : (\mathbb{R}^m, 0) \to T_1\mathbb{R}^{m+1}$ along Φ such that

$$\frac{\partial \Phi}{\partial x_1}(x_1, \ldots, x_m) \cdot \nu_\Phi(x_1, \ldots, x_m) = \cdots = \frac{\partial \Phi}{\partial x_m}(x_1, \ldots, x_m) \cdot \nu_\Phi(x_1, \ldots, x_m) = 0$$

and the map germ $L_\Phi : (\mathbb{R}^m, 0) \to T_1\mathbb{R}^{m+1}$ defined by

$$L_\Phi(x_1, \dots, x_m) = \big(\Phi(x_1, \dots, x_m), \nu_\Phi(x_1, \dots, x_m)\big)$$

is nonsingular. L_Φ is called a *Legendrian lift of* Φ. (Here the dot stands for the scalar product of two vectors of $T_{\Phi(x,y)}\mathbb{R}^{m+1}$, and $T_1\mathbb{R}^{m+1}$ is the unit tangent bundle of $\mathbb{R}^{m+1}$.) The C^∞ vector field ν_Φ is called a *unit normal vector field* of Φ.

(iv) A Legendrian map germ $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is said to be *normalized* if it has the form

$$\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y) \tag{7}$$

with

$$\frac{\partial \Phi_2}{\partial x}(0, 0) = 0 \tag{8}$$

and if, furthermore,

$$\nu_\Phi(0, 0) = \frac{\partial}{\partial X} \quad \text{or} \quad \nu_\Phi(0, 0) = -\frac{\partial}{\partial X}. \tag{9}$$

(v) For a normalized Legendrian map germ $\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y)$, set

$$\mathfrak{D}(\Phi)(x, y) = \left(\frac{\partial \Phi_1}{\partial x}(x, y), \frac{\partial \Phi_2}{\partial x}(x, y), y\right).$$

The map germ $\mathfrak{D}(\Phi) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is called the *differential* of Φ.

We showed in [Nishimura 2010] that any germ $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ of a one-parameter pedal unfolding of a spherical pedal curve has the form (6). Hence, a map germ φ having the form (6) is said to be of pedal unfolding type. As shown in [Nishimura 2010], not only nonsingular map germs, but also Whitney umbrellas may be realized as germs of one-parameter pedal unfoldings of spherical pedal curves. For more information on Legendrian map germs, see [Arnold et al. 1985; Izumiya 1987; Zakalyukin 1976; 1983]. Note that both (3) and (10) are normalized Legendrian map germs.

Proposition 1.2. (i) *If* $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *is a* C^∞ *map germ of pedal unfolding type*, $\mathfrak{I}(\varphi)$ *is a normalized Legendrian map germ.*

(ii) *If* $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *is a normalized Legendrian map germ*, $\mathfrak{D}(\Phi)$ *is a map germ of pedal unfolding type.*

Set

$$\mathcal{W} = \{\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \text{ Whitney umbrella of pedal unfolding type}\},$$

$$\mathcal{S} = \{\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \text{ normalized swallowtail}\}.$$

The main purpose of this paper is to show the following:

Theorem 1.3. (i) *The map $\mathcal{I} : \mathcal{W} \to \mathcal{S}$ defined by $\mathcal{W} \ni \varphi \mapsto \mathcal{I}(\varphi) \in \mathcal{S}$ is well-defined and bijective.*

(ii) *The map $\mathcal{D} : \mathcal{S} \to \mathcal{W}$ defined by $\mathcal{S} \ni \Phi \mapsto \mathcal{D}(\Phi) \in \mathcal{W}$ is well-defined and bijective.*

Incidentally, we show Theorem 1.4. A C^∞ map germ $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is called a *cuspidal edge* if Φ is $\mathcal{A}$-equivalent to the following:

$$(x, y) \mapsto \left(\tfrac{1}{3}x^3, \tfrac{1}{2}x^2, y\right). \tag{10}$$

Set

$\mathcal{N} = \{\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ nonsingular map germ of pedal unfolding type$\}$,

$\mathcal{C} = \{\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ normalized cuspidal edge$\}$.

Theorem 1.4. (i) *The map $\mathcal{I} : \mathcal{N} \to \mathcal{C}$ defined by $\mathcal{N} \ni \varphi \mapsto \mathcal{I}(\varphi) \in \mathcal{C}$ is well-defined and bijective.*

(ii) *The map $\mathcal{D} : \mathcal{C} \to \mathcal{N}$ defined by $\mathcal{C} \ni \Phi \mapsto \mathcal{D}(\Phi) \in \mathcal{N}$ is well-defined and bijective.*

Any stable map germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is either a Whitney umbrella or nonsingular, and any Legendrian stable singularity $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is either a cuspidal edge or a swallowtail (see [Arnold et al. 1985], for example). Theorems 1.3 and 1.4 can thus be regarded as a "fundamental theorem of calculus" for stable map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ and Legendrian stable singularities $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$.

Based on Theorems 1.3 and 1.4, it is natural to ask:

Question 1.5. (i) Let $\varphi_1, \varphi_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be two C^∞ map germs of pedal unfolding type. Suppose that φ_1 is $\mathcal{A}$-equivalent to φ_2. Is $\mathcal{I}(\varphi_1)$ necessarily $\mathcal{A}$-equivalent to $\mathcal{I}(\varphi_2)$?

(ii) Let $\Phi_1, \Phi_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be two normalized Legendrian map germs. Suppose that Φ_1 is $\mathcal{A}$-equivalent to Φ_2. Is $\mathcal{D}(\Phi_1)$ necessarily $\mathcal{A}$-equivalent to $\mathcal{D}(\Phi_2)$?

In Section 2, several preparations for the proofs of Theorems 1.3 and 1.4 and the proof of Proposition 1.2 are given. Theorems 1.3 and 1.4 are proved in Section 3 and Section 4 respectively.

2. Preliminaries

Function germs with two variables and map germs with two variables. Let $\mathcal{E}_2$ be the set of C^∞ function germs $(\mathbb{R}^2, 0) \to \mathbb{R}$, and let m_2 be the subset of $\mathcal{E}_2$

consisting of C^∞ function germs $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$. The set $\mathscr{E}_2$ has a natural $\mathbb{R}$-algebra structure. For a C^∞ map germ $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, let $\varphi^* : \mathscr{E}_2 \to \mathscr{E}_2$ be the $\mathbb{R}$-algebra homomorphism defined by $\varphi^*(u) = u \circ \varphi$. Set $Q(\varphi) = \mathscr{E}_2/\varphi^* m_2 \mathscr{E}_2$. Then, $Q(\varphi)$ is an $\mathbb{R}$-algebra. A special case of [Mather 1969, Theorem 2.1] follows.

Proposition 2.1. *Let $p : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be a C^∞ function germ.*

(i) *The $\mathbb{R}$-algebra $Q(p(x, y), y)$ is isomorphic to $Q(x^2, y)$ if and only if*

$$\frac{\partial p}{\partial x}(0, 0) = 0 \quad \text{and} \quad \frac{\partial^2 p}{\partial x^2}(0, 0) \neq 0.$$

(ii) *The $\mathbb{R}$-algebra $Q(p(x, y), y)$ is isomorphic to $Q(x, y)$ if and only if*

$$(x, y) \mapsto (p(x, y), y)$$

is a germ of C^∞ diffeomorphism.

Definition 2.2 [Mond 1985]. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the form $T(s, \lambda) = (-s, \lambda)$. Two C^∞ function germs $p_1, p_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ are said to be $\mathscr{K}^T$*-equivalent* if there exists a germ of C^∞ diffeomorphism

$$h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$$

of the form $h \circ T = T \circ h$, and a C^∞ function germ $M : (\mathbb{R}^2, (0, 0)) \to \mathbb{R} - \{0\}$ of the form $M \circ T = M$, such that $p_1 \circ h(x, y) = M(x, y) p_2(x, y)$.

Theorem 2.3 [Mond 1985]. *Two C^∞ map germs $\varphi_i : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ $(i = 1, 2)$ of the form*

$$\varphi_i(x, y) = (x p_i(x^2, y), x^2, y), \quad \textit{where } p_i(x^2, y) \notin m_2^\infty \quad (i = 1,\ 2)$$

are $\mathscr{A}$-equivalent if and only if the function germs $p_i(x^2, y)$ are $\mathscr{K}^T$-equivalent. Here,

$$m_2^\infty = \left\{ q : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \;\middle|\; \frac{\partial^{i+j} q}{\partial x^i \partial y^j}(0, 0) = 0 \textit{ for all } i, j \in \{0\} \cup \mathbb{N} \right\}.$$

From this and the Malgrange preparation theorem [Arnold et al. 1985], we have:

Corollary 2.4. *Two C^∞ map germs $\varphi_i : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ $(i = 1, 2)$ of the form*

$$\varphi_i(x, y) = (n_i(x, y) p_i(x^2, y), x^2, y),$$

where $p_i(x^2, y) \notin m_2^\infty$ and $(\partial n_i/\partial x)(0, 0) \neq 0$ for $i = 1, 2$, are $\mathscr{A}$-equivalent if and only if the function germs $p_i(x^2, y)$ are $\mathscr{K}^T$-equivalent.

Map germs of pedal unfolding type. Let $\varphi : I \times J \to \mathbb{R}^3$ be a representative of a given C^∞ map germ of pedal unfolding type, where I, J are sufficiently small intervals containing the origin of $\mathbb{R}$. We may put $\varphi(x, y) = \big(n(x, y)p(x, y), p(x, y)\big)$. Set

$$\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y) = \left(\int_0^x n(x, y)p(x, y)dx, \int_0^x p(x, y)dx, y\right)$$

and

$$\tilde{\mu}_\Phi(x, y) = \frac{\partial}{\partial X} - n(x, y)\frac{\partial}{\partial Y}.$$

Since $\tilde{\mu}_\Phi(x, y) \neq 0$ for any $x \in I$ and $y \in J$, for any fixed $y \in J$ we may define the map germ $L_{\Phi,y} : (\mathbb{R}, 0) \to T_1\mathbb{R}^2$ as

$$L_{\Phi,y}(x) = \left((\Phi_1(x, y), \Phi_2(x, y)), \frac{\tilde{\mu}_\Phi(x, y)}{\|\tilde{\mu}_\Phi(x, y)\|}\right),$$

where $T_1\mathbb{R}^2$ is the unit tangent bundle of $\mathbb{R}^2$. Then, since φ is a representative of a map germ of pedal unfolding type, we have:

Lemma 2.5. *For any $y \in J$, $L_{\Phi,y} : (\mathbb{R}, 0) \to T_1\mathbb{R}^2$ is a Legendrian lift of the map germ $x \mapsto (\Phi_1(x, y), \Phi_2(x, y))$.*

This implies:

Lemma 2.6. *For any $y \in J$, the map germ $\widetilde{\Phi}_y : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ defined by $\widetilde{\Phi}_y(x) = (\Phi_1(x, y), \Phi_2(x, y))$ is a Legendrian map germ.*

Next, set

$$\tilde{\nu}_\Phi(x, y) = \tilde{\mu}_\Phi(x, y) - \left(\frac{\partial \Phi_1}{\partial y}(x, y) - n(x, y)\frac{\partial \Phi_2}{\partial y}(x, y)\right)\frac{\partial}{\partial Z}.$$

Lemma 2.7. *For any $x \in I$ and $y \in J$,*

$$\tilde{\nu}_\Phi(x, y) \cdot \frac{\partial \Phi}{\partial x}(x, y) = 0, \quad \tilde{\nu}_\Phi(x, y) \cdot \frac{\partial \Phi}{\partial y}(x, y) = 0.$$

Since $\tilde{\nu}_\Phi(x, y) \neq 0$ for any $x \in I$ and $y \in J$, we may define the map germ

$$L_\Phi : (\mathbb{R}^2, 0) \to T_1\mathbb{R}^3$$

as

$$L_\Phi(x, y) = \left(\Phi(x, y), \frac{\tilde{\nu}_\Phi(x, y)}{\|\tilde{\nu}_\Phi(x, y)\|}\right).$$

Then Lemma 2.7 implies successively:

Lemma 2.8. *$L_\Phi : (\mathbb{R}^2, 0) \to T_1\mathbb{R}^3$ is a Legendrian lift of Φ.*

Lemma 2.9. *$\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a Legendrian map germ.*

Normalized Legendrian map germs. Let $\Phi : U \to \mathbb{R}^3$ be a representative of a given normalized Legendrian map germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, where U is a sufficiently small neighborhood of the origin of $\mathbb{R}^2$. We assume that the origin of $\mathbb{R}^2$ is a singular point of Φ. By condition (7) of the definition of normalized Legendrian map germs, we may assume that Φ has the form

$$\Phi(x, y) = \bigl(\Phi_1(x, y), \Phi_2(x, y), y\bigr).$$

Since Φ is a representative of a Legendrian map germ, we have the following:

Lemma 2.10. *There exists a C^∞ vector field ν_Φ along Φ,*

$$\nu_\Phi(x, y) = n_1(x, y)\frac{\partial}{\partial X} + n_2(x, y)\frac{\partial}{\partial Y} + n_3(x, y)\frac{\partial}{\partial Z},$$

such that

(i) $$n_1(x, y)\frac{\partial \Phi_1}{\partial x}(x, y) + n_2(x, y)\frac{\partial \Phi_2}{\partial x}(x, y) = 0;$$

(ii) $$n_1(x, y)\frac{\partial \Phi_1}{\partial y}(x, y) + n_2(x, y)\frac{\partial \Phi_2}{\partial y}(x, y) + n_3(x, y) = 0;$$

(iii) *the map $L_\Phi : U \to T_1\mathbb{R}^3$ defined by $L_\Phi(x, y) = (\Phi(x, y), \nu_\Phi(x, y))$ is an immersion.*

Condition (9) in the definition of normalized Legendrian map germs gives:

Lemma 2.11. *For the vector field ν_Φ, $n_1(0, 0) \neq 0$ and $n_2(0, 0) = n_3(0, 0) = 0$.*

By Lemma 2.10(i) and Lemma 2.11, we have the following equality of function germs:

(11) $$\frac{\partial \Phi_1}{\partial x}(x, y) = -\frac{n_2(x, y)}{n_1(x, y)}\frac{\partial \Phi_2}{\partial x}(x, y).$$

This, together with condition (8) in the definition of normalized Legendrian maps, implies that

(12) $$\mathfrak{D}(\Phi)(0, 0) = (0, 0, 0).$$

The next lemma is clear:

Lemma 2.12. *The function germs n and p given by*

$$n(x, y) = -\frac{n_2(x, y)}{n_1(x, y)} \quad \textit{and} \quad p(x, y) = \frac{\partial \Phi_2}{\partial x}(x, y)$$

are of class C^∞, and satisfy $\mathfrak{D}(\Phi)(x, y) = \bigl(n(x, y)p(x, y), p(x, y), y\bigr)$.

Lemma 2.13. *The function germ n satisfies $n(0, 0) = 0$ and $\dfrac{\partial n}{\partial x}(0, 0) \neq 0$.*

Proof. By Lemma 2.11, we have $n(0,0)=0$. Assume, for a contradiction, that $\partial n/\partial x$ vanishes at the origin; then so does $\partial n_2/\partial x$. At the same time, by differentiating both sides of the equality in Lemma 2.10(ii) with respect to x, we have

$$n_1(0,0)\frac{\partial^2\Phi_1}{\partial x\partial y}(0,0)+\frac{\partial n_3}{\partial x}(0,0)=0. \tag{13}$$

Because Φ is a normalized Legendrian map germ such that the origin of $\mathbb{R}^2$ is a singular point of Φ, we obtain $(\partial n_3/\partial x)(0,0)\neq 0$, which together with (13) gives

$$\frac{\partial^2\Phi_1}{\partial x\partial y}(0,0)\neq 0.$$

From (8), (11), and Lemma 2.11 we have a contradiction. □

Definition 2.14. Let $\Phi:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a Legendrian map germ, and let ν_Φ be a unit normal vector field of Φ given in the definition of Legendrian map germs. The C^∞ function germ $LJ_\Phi:(\mathbb{R}^2,0)\to\mathbb{R}$ defined by

$$LJ_\Phi(x,y)=\det\left(\frac{\partial\Phi}{\partial x}(x,y),\ \frac{\partial\Phi}{\partial y}(x,y),\ \nu_\Phi(x,y)\right)$$

is called the *Legendrian Jacobian* of Φ.

Note that if ν_Φ satisfies the conditions of unit normal vector field of Φ, then $-\nu_\Phi$ also satisfies them. Thus, the sign of $LJ_\Phi(x,y)$ depends on the particular choice of unit normal vector field ν_Φ. The Legendrian Jacobian of Φ is also called the *signed area density function* [Saji et al. 2009b]. Although it is reasonable to call LJ_Φ the area density function from the viewpoint of investigating the singular surface $\Phi(U)$ (U is a sufficiently small neighborhood of the origin of $\mathbb{R}^2$), it is also reasonable to call it the Legendrian Jacobian from the viewpoint of investigating the singular map germ Φ.

Let $\Phi:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a normalized Legendrian map germ and ν_Φ a unit normal vector field of Φ. Write

$$\Phi(x,y)=\big(\Phi_1(x,y),\Phi_2(x,y),y\big),$$
$$\nu_\Phi(x,y)=n_1(x,y)\frac{\partial}{\partial X}+n_2(x,y)\frac{\partial}{\partial Y}+n_3(x,y)\frac{\partial}{\partial Z}.$$

By Lemma 2.11, we may set

$$\tilde{\nu}_\Phi(x,y)=\frac{\partial}{\partial X}+\frac{n_2(x,y)}{n_1(x,y)}\frac{\partial}{\partial Y}+\frac{n_3(x,y)}{n_1(x,y)}\frac{\partial}{\partial Z}.$$

We now give a formula for the Legendrian Jacobian. We start with the cross product (vector product)

$$\frac{\partial\Phi}{\partial x}(x,y)\times\frac{\partial\Phi}{\partial y}(x,y)=\frac{\partial\Phi_2}{\partial x}(x,y)\tilde{\nu}_\Phi(x,y).$$

This gives

$$LJ_\Phi(x, y) = \frac{(\partial\Phi_2/\partial x)(x, y)}{n_1(x, y)}. \tag{14}$$

Proof of Proposition 1.2. (i) Set

$$\mathcal{I}(\varphi) = \Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y).$$

By Lemma 2.9, Φ is a Legendrian map germ. Thus, it is sufficient to show that (8) and (9) in the definition of normalized Legendrian map germs are satisfied.

Set $\varphi(x, y) = \big(n(x, y)p(x, y), p(x, y), y\big)$. By the definition of map germs of pedal unfolding type, we have $n(0, 0) = 0$ and $p(0, 0) = 0$. It follows that

$$\frac{\partial\Phi_2}{\partial x}(0, 0) = p(0, 0) = 0.$$

Thus, condition (8) is satisfied. By Lemma 2.8, the germ L_Φ given by

$$L_\Phi(x, y) = \left(\Phi(x, y), \frac{\tilde{\nu}_\Phi(x, y)}{\|\tilde{\nu}_\Phi(x, y)\|}\right)$$

is a germ of Legendrian lift of Φ, where

$$\tilde{\nu}_\Phi(x, y) = \frac{\partial}{\partial X} - n(x, y)\frac{\partial}{\partial Y} - \left(\frac{\partial\Phi_1}{\partial y}(x, y) - n(x, y)\frac{\partial\Phi_2}{\partial y}(x, y)\right)\frac{\partial}{\partial Z}.$$

Since $n(0, 0) = 0$ and

$$\frac{\partial\Phi_1}{\partial y}(0, 0) = \int_0^0 \frac{\partial np}{\partial y}(x, 0)\, dx = 0,$$

we have

$$\frac{\tilde{\nu}_\Phi(0, 0)}{\|\tilde{\nu}_\Phi(0, 0)\|} = \frac{\partial}{\partial X}.$$

Thus, condition (9) is satisfied, proving part (i) of the proposition.

Proposition 1.2(ii) follows from (12), Lemma 2.12, and Lemma 2.13. □

3. Proof of Theorem 1.3

Suppose that both $\mathcal{I} : \mathcal{W} \to \mathcal{S}$ and $\mathcal{D} : \mathcal{S} \to \mathcal{W}$ are well-defined. By the fundamental theorem of calculus, we have $\mathcal{D} \circ \mathcal{I}(\varphi) = \varphi$ for all $\varphi \in \mathcal{W}$, and $\mathcal{I} \circ \mathcal{D}(\Phi) = \Phi$ for all $\Phi \in \mathcal{S}$. That is, both $\mathcal{I}$ and $\mathcal{D}$ are bijective. Therefore, in order to complete the proof, it is sufficient to show that both $\mathcal{I}$ and $\mathcal{D}$ are well-defined.

Proof that $\mathcal{I} : \mathcal{W} \to \mathcal{S}$ is well-defined. Let $\varphi(x, y) = (n(x, y)p_\varphi(x, y), p_\varphi(x, y), y)$ be an element of $\mathcal{W}$. Set $\Phi = \mathcal{I}(\varphi)$. Then Φ is a normalized Legendrian map germ by Proposition 1.2. Let g be the Whitney umbrella of pedal unfolding type — see (2) in Section 1:

$$g(x, y) = (xp_g(x, y), p_g(x, y), y) = (x(x^2 + y), -x^2 - y, y).$$

Lemma 3.1. *There exists a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that h has the form $h(x, y) = (h_1(x, y), h_2(y))$ and $p_\varphi \circ h(x, y)$ is $x^2 + y$ or $-(x^2 + y)$.*

Proof. Since φ is a Whitney umbrella of pedal unfolding type, we have

$$Q(p_\varphi(x, y), y) \cong Q(\varphi) \cong Q(g) \cong Q(x^2, y).$$

Thus, we may set $p_\varphi(x, 0) = a_2 x^2 + o(x^2)$ ($a_2 \neq 0$) by Proposition 2.1. By the Morse lemma with parameters [Bruce and Giblin 1992], there exists a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that h has the form

$$h(x, y) = (h_1(x, y), h_2(y)) \quad \text{and} \quad p_\varphi \circ h(x, y) = \pm(x^2 + q(y))$$

by a certain C^∞ function germ $q : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$. Since φ is $\mathcal{A}$-equivalent to g, by Corollary 2.4, $\pm(x^2 + q(y))$ is $\mathcal{K}^T$-equivalent to p_g and thus $q : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is a germ of C^∞ diffeomorphism. Lemma 3.1 follows. □

Set $G = \mathcal{I}(g)$. Then, G has the form of (3) from Section 1, which is a normalized swallowtail. Since G is normalized, $\partial/\partial x$ is the null vector field for G defined in [Kokubu et al. 2005; Saji et al. 2009a], that is,

$$\frac{\partial G}{\partial x}(x, y) = 0$$

holds for any (x, y) which is a singular point of G. Since G is a swallowtail, we have, by [Saji et al. 2009a, Corollary 2.5],

$$LJ_G(0, 0) = \frac{\partial LJ_G}{\partial x}(0, 0) = 0, \quad \frac{\partial^2 LJ_G}{\partial x^2}(0, 0) \neq 0, \quad Q\Big(LJ_G, \frac{\partial LJ_G}{\partial x}\Big) \cong Q(x, y).$$

On the other hand, by (14) and Lemma 3.1, there exists a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and a C^∞ function germ $\xi : (\mathbb{R}^2, 0) \to \mathbb{R}$, such that h has the form $h(x, y) = \big(h_1(x, y), h_2(y)\big)$, $\xi(0, 0) \neq 0$ and we have

$$LJ_\Phi \circ h(x, y) = \xi(x, y) LJ_G(x, y).$$

Because $\partial/\partial x$ is the null vector field for Φ (this is because Φ is normalized), LJ_Φ satisfies

$$LJ_\Phi(0, 0) = \frac{\partial LJ_\Phi}{\partial x}(0, 0) = 0, \quad \frac{\partial^2 LJ_\Phi}{\partial x^2}(0, 0) \neq 0, \quad Q\Big(LJ_\Phi, \frac{\partial LJ_\Phi}{\partial x}\Big) \cong Q(x, y).$$

Hence, Φ is a swallowtail by [Saji et al. 2009a, Corollary 2.5], and we have proved that $\mathcal{I} : \mathcal{W} \to \mathcal{S}$ is well-defined. □

Proof that $\mathfrak{D} : \mathcal{S} \to \mathcal{W}$ is well-defined.. Let Φ be an element of $\mathcal{S}$. Then, by Proposition 2.1, $\mathfrak{D}(\Phi)$ is of pedal unfolding type; we must show that it is a Whitney umbrella.

Lemma 3.2. *For the Legendrian Jacobian LJ_Φ, we have*

$$LJ_\Phi(0,0) = \frac{\partial LJ_\Phi}{\partial x}(0,0) = 0, \quad \frac{\partial^2 LJ_\Phi}{\partial x^2}(0,0) \neq 0, \quad Q\Big(LJ_\Phi, \frac{\partial LJ_\Phi}{\partial x}\Big) \cong Q(x, y).$$

Proof. Since Φ is normalized, $\partial/\partial x$ is the null vector field. Since Φ is a swallowtail, Lemma 3.2 follows from [Saji et al. 2009a, Corollary 2.5]. □

Since $\mathfrak{D}(\Phi)$ is of pedal unfolding type, there exists a C^∞ function germ $n : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ such that

$$\frac{\partial n}{\partial x}(0,0) \neq 0 \quad \text{and} \quad \frac{\partial \Phi_1}{\partial x}(x, y) = n(x, y)\frac{\partial \Phi_2}{\partial x}(x, y),$$

where $\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y)$. Set $p_\varphi = \partial\Phi_2/\partial x$. By (14), Lemma 3.2, and the Morse lemma with parameters, there is a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that h has the form $h(x, y) = (h_1(x, y), h_2(y))$ and $p_\varphi \circ h(x, y) = \pm(x^2 + y)$. Then, by Corollary 2.4, $\mathfrak{D}(\Phi)$ is $\mathcal{A}$-equivalent to g. □

4. Proof of Theorem 1.4

As with Theorem 1.3, it is sufficient to show that both $\mathfrak{I} : \mathcal{N} \to \mathcal{C}$ and $\mathfrak{D} : \mathcal{C} \to \mathcal{N}$ are well-defined.

Proof that $\mathfrak{I} : \mathcal{N} \to \mathcal{C}$ is well-defined. Let $\varphi(x, y) = (n(x, y)p_\varphi(x, y), p_\varphi(x, y), y)$ be an element of $\mathcal{N}$. Set $\Phi = \mathfrak{I}(\varphi)$. Then, since φ is of pedal unfolding type, Φ is a normalized Legendrian map germ by Proposition 1.2. Let g be the nonsingular map germ of pedal unfolding type defined by $g(x, y) = (x^2, x, y)$.

Lemma 4.1. *There exists a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ having the form $h(x, y) = (h_1(x, y), h_2(y))$ and such that $p_\varphi \circ h(x, y) = x$.*

Proof. Since φ is nonsingular and of pedal unfolding type, we have

$$Q(p_\varphi(x, y), y) \cong Q(\varphi) \cong Q(g) \cong Q(x, y).$$

Thus, $(p_\varphi(x, y), y)$ is a germ of C^∞ diffeomorphism by Proposition 2.1. From the form of $(p_\varphi(x, y), y)$, its inverse map germ $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ has the form $h(x, y) = (h_1(x, y), h_2(y))$. Since h is the inverse map germ of $(p_\varphi(x, y), y)$, it follows that $p_\varphi \circ h(x, y) = x$. □

Since Φ is normalized, $\partial/\partial x$ is the null vector field for Φ. By Lemma 4.1 and (14), we have

$$\frac{\partial LJ_\Phi}{\partial x}(0,0) \neq 0.$$

Thus, the null vector field $\partial/\partial x$ is transverse to $\{(x, y) \mid LJ_\Phi(x, y) = 0\}$ at $(0, 0) \in \mathbb{R}^2$. Hence, Φ is a cuspidal edge by [Kokubu et al. 2005, Proposition 1.3], showing that $\mathcal{I} : \mathcal{N} \to \mathcal{C}$ is well-defined. $\square$

Proof that $\mathfrak{D} : \mathcal{C} \to \mathcal{N}$ is well-defined. Let Φ be an element of $\mathcal{C}$. By Proposition 1.2, $\mathfrak{D}(\Phi)$ is of pedal unfolding type.

Lemma 4.2. *The Legendrian Jacobian LJ_Φ satisfies*

$$LJ_\Phi(0, 0) = 0 \quad \text{and} \quad \frac{\partial LJ_\Phi}{\partial x}(0, 0) \neq 0.$$

Proof. Since $\partial/\partial x$ is the null vector field for Φ and Φ is a cuspidal edge, Lemma 4.2 follows from [Saji et al. 2009a, Corollary 2.5]. $\square$

Since $\mathfrak{D}(\Phi)$ is of pedal unfolding type, there exists a C^∞ function germ $n : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ such that

$$\frac{\partial n}{\partial x}(0, 0) \neq 0 \quad \text{and} \quad \frac{\partial \Phi_1}{\partial x}(x, y) = n(x, y)\frac{\partial \Phi_2}{\partial x}(x, y),$$

where $\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y)$. Set $p_\varphi = \partial\Phi_2/\partial x$. By Lemma 4.2 and (14), the map germ $(x, y) \mapsto (p_\varphi(x, y), y)$ is a germ of a C^∞ diffeomorphism. Thus, $\mathfrak{D}(\Phi)$ is nonsingular. Since $\mathfrak{D}(\Phi)(0, 0) = (0, 0, 0)$, we have proved that $\mathfrak{D} : \mathcal{C} \to \mathcal{N}$ is well-defined. $\square$

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