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Virtual photon structure functions to the next-to-next-to-leading order in QCD

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We investigate the unpolarized virtual photon structure functions $F_2^\gamma(x, Q^2, P^2)$ and $F_L^\gamma(x, Q^2, P^2)$ in perturbative QCD for the kinematical region $\Lambda^2 \ll P^2 \ll Q^2$, where $-Q^2$ $(-P^2)$ is the mass squared of the probe (target) photon and Λ is the QCD scale parameter. Using the framework of the operator product expansion supplemented by the renormalization group method, we derive the definite predictions for the moments of $F_2^\gamma(x, Q^2, P^2)$ up to the next-to-next-to-leading order (NNLO) (the order $\alpha\alpha_s$) and for the moments of $F_L^\gamma(x, Q^2, P^2)$ up to the next-to-leading order (NLO) (the order $\alpha\alpha_s$). The NNLO corrections to the sum rule of $F_2^\gamma(x, Q^2, P^2)$ are negative and found to be 7%–10% of the sum of the LO and NLO contributions, when $P^2 = 1$ GeV^2 and $Q^2 = 30 \sim 100$ GeV^2 or $P^2 = 3$ GeV^2 and $Q^2 = 100$ GeV^2, and the number of active quark flavors n_f is three or four. The NLO corrections to F_L^γ are also negative. The moments are inverted numerically to obtain the predictions for $F_2^\gamma(x, Q^2, P^2)$ and $F_L^\gamma(x, Q^2, P^2)$ as functions of x.

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I. INTRODUCTION

The experiments at the CERN Large Hadron Collider (LHC) will begin shortly and it is much anticipated that signals for the new physics beyond the standard model (SM) will be discovered [1]. Once these signals are observed, they will be examined more closely in a proposed e^+e^- collider machine called the International Linear Collider (ILC) [2]. In analyzing these signals for the new physics, the knowledge of the SM, especially of QCD, will be more important than ever before. It is well known that, in e^+e^- collision experiments, the cross section for the two-photon processes $e^+e^- \to e^+e^- +$ hadrons shown in Fig. 1 dominates at high energies over other processes such as the annihilation process $e^+e^- \to \gamma^* \to$ hadrons. Here we consider the two-photon processes in the double-tag events, where both the outgoing e^+ and e^- are detected. In particular, we investigate the case in which one of the virtual photons is very far off shell (large $Q^2 \equiv -q^2$), while the other is close to the mass shell (small $P^2 \equiv -p^2$). This process can be viewed as a deep-inelastic electron-photon scattering where the target is a photon rather than a nucleon [3]. In the deep-inelastic scattering off a photon target, we can study the photon structure functions, which are the analogs of the nucleon structure functions. The photon structure functions are defined in the lowest order of the QED coupling constant $\alpha = e^2/4\pi$ and, in this paper, they are of order α.

The unpolarized (spin-averaged) photon structure functions $F_2^\gamma(x, Q^2)$ and $F_L^\gamma(x, Q^2)$ of the real photon ($P^2 = 0$) were first studied in the parton model (PM) [4] and then investigated in perturbative QCD (pQCD). A pioneering work was done by Witten [5] in which he derived the leading order (LO) QCD contributions to F_2^γ and F_L^γ. A few years later the next-to-leading order (NLO) corrections to F_2^γ were calculated [6]. These results were obtained in the framework based on the operator product expansion (OPE) [7] supplemented by the renormalization group (RG) method. The same results were rederived by the QCD improved PM powered by the parton evolution equations [8,9]. Recently, the lowest six even-integer Mellin moments of the photon-parton splitting functions were calculated to the next-to-next-to-leading order (NNLO) and the parton distributions of the real photon and the structure function F_2^γ were analyzed [10]. The same au-

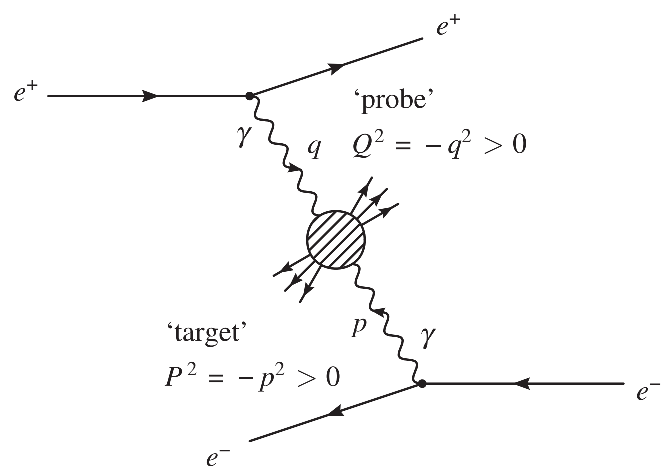

FIG. 1. Deep-inelastic scattering on a virtual photon in the e^+e^- collider experiments.

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thors later gave the compact and accurate parametrization of the photon-parton splitting functions up to the NNLO in Ref. [11].

When polarized beams are used in e^+e^- collision experiments, we can get information on the spin structure of the photon. The QCD analysis of the polarized structure function $g_1^\gamma(x, Q^2)$ for the real photon target was performed in the LO [12] and in the NLO [13,14]. For more information on the theoretical and experimental investigation of both unpolarized and polarized photon structure, see Ref. [15].

A unique and interesting feature of the photon structure functions is that, in contrast with the nucleon case, the target mass squared $-P^2$ is not fixed but can take various values and that the structure functions show different behaviors depending on the values of P^2. The photon has two characters: The photon couples directly to quarks (pointlike nature) and, also, it behaves as vector bosons (hadronic nature) [16]. Thus the structure function $F_2^\gamma(x, Q^2)$ of the real photon ($P^2 = 0$) may be decomposed as

$$F_2^\gamma(x, Q^2) = F_2^\gamma(x, Q^2)|_{\text{pointlike}} + F_2^\gamma(x, Q^2)|_{\text{hadronic}}. \quad (1.1)$$

The first term, a pointlike piece, can be calculated, in principle, in a perturbative method. On the other hand, the second term, a hadronic piece, can only be computed by some nonperturbative methods like lattice QCD, or estimated, for example, by the vector meson dominance model [16].

The moments of $F_2^\gamma(x, Q^2)|_{\text{pointlike}}$ and $F_2^\gamma(x, Q^2)|_{\text{hadronic}}$ for even n may be written, respectively, as

$$\int_0^1 dx x^{n-2} F_2^\gamma(x, Q^2)|_{\text{pointlike}} = \alpha\left\{\frac{1}{\alpha_s(Q^2)} a_n + b_n + \mathcal{O}(\alpha_s(Q^2))\right\}, \quad (1.2)$$

$$\int_0^1 dx x^{n-2} F_2^\gamma(x, Q^2)|_{\text{hadronic}} = \alpha h_n(\alpha_s(Q^2)), \quad (1.3)$$

where x is the Bjorken variable and $\alpha_s(Q^2) = g^2(Q^2)/4\pi$ is the QCD running coupling constant. Since $1/\alpha_s(Q^2)$ behaves as $\ln(Q^2/\Lambda^2)$ at large Q^2, where Λ is the QCD scale parameter, the first term $a_n/\alpha_s(Q^2)$ dominates over the b_n term and also over the hadronic term $h_n(\alpha_s(Q^2))$. The definite prediction for the LO contributions a_n was given in Ref. [5]. Meanwhile, the NLO corrections b_n were calculated only for $n > 2$ in Ref. [6]. For $n > 2$, the hadronic moments $h_n(\alpha_s(Q^2))$ vanish in the large-Q^2 limit and the b_n terms give finite contributions. However, at $n = 2$, the hadronic energy-momentum tensor operator comes into play. Because of the conservation of this operator, b_n shows a singularity at $n = 2$ and $h_{n=2}(\alpha_s(Q^2))$ does not vanish at large Q^2. Actually, $h_n(\alpha_s(Q^2))$ also develops a singularity at $n = 2$ which cancels out the one of b_n, and $h_n(\alpha_s(Q^2))$ and b_n in combination give a finite but perturbatively incalculable contribution at $n = 2$ [17]. The fact that definite information on the NLO second moment is missing prevents us from fully predicting the shape and magnitude of the structure function of $F_2^\gamma(x, Q^2)$ up to the order $\mathcal{O}(\alpha)$.

It was then pointed out in Ref. [17] that the situation changes significantly when we analyze the structure function of a virtual photon with P^2 much larger than the QCD parameter Λ^2. More specifically, we consider the following kinematical region,

$$\Lambda^2 \ll P^2 \ll Q^2. \quad (1.4)$$

In this region, the hadronic component of the photon can also be dealt with *perturbatively* and thus a definite prediction of the whole structure function, its shape and magnitude, may become possible. In fact, the virtual photon structure function $F_2^\gamma(x, Q^2, P^2)$ in the kinematical region (1.4) was calculated in the LO (the order α/α_s) [18] and in the NLO (the order α) [17,19], and the longitudinal structure function $F_L^\gamma(x, Q^2, P^2)$ in the LO (the order α) [17] without any unknown parameters. It is notable that the pathology of singularity, which appeared at $n = 2$ in the term b_n of Eq. (1.2) for the real photon target, disappeared from the moments of $F_2^\gamma(x, Q^2, P^2)$. The parton contents of the virtual photon for the case (1.4) were studied in Refs. [20–22].

In the same kinematical region (1.4), the polarized virtual structure function $g_1^\gamma(x, Q^2, P^2)$ was investigated up to the NLO in QCD in Ref. [23] and in the second paper of [14]. Moreover, the polarized parton distributions inside the virtual photon were analyzed in various factorization schemes [24]. Quite recently the first moment of $g_1^\gamma(x, Q^2, P^2)$ was calculated up to the NNLO [25].

In this paper we investigate the unpolarized virtual photon structure functions $F_2^\gamma(x,Q^2,P^2)$ and $F_L^\gamma(x,Q^2,P^2)$ in the kinematical region (1.4) in QCD. Here we neglect all the power corrections of the form $(P^2/Q^2)^k$ ($k = 1, 2, \ldots$) which may arise from target mass and higher-twist effects. We present definite predictions for $F_2^\gamma(x, Q^2, P^2)$ up to the NNLO (the order $\alpha\alpha_s$) and for $F_L^\gamma(x, Q^2, P^2)$ up to the NLO (the order $\alpha\alpha_s$). The recent calculations of the three-loop anomalous dimensions for the quark and gluon operators [26,27] and of the three-loop photon-quark and photon-gluon splitting functions [11] have paved the way for this investigation. Using the framework of the OPE supplemented by the RG method, we give, in the next section, an expression for the moments of $F_2^\gamma(x, Q^2, P^2)$ up to the NNLO corrections. In Sec. III we enumerate all the necessary QCD parameters to evaluate the NNLO corrections. In Sec. IV the second moment of $F_2^\gamma(x, Q^2, P^2)$ will be evaluated up to the NNLO. The numerical analysis of $F_2^\gamma(x, Q^2, P^2)$ as a function of x will be given in Sec. V. In Sec. VI the longitudinal virtual photon structure function $F_L^\gamma(x, Q^2, P^2)$ will be analyzed up to the NLO. The final section is devoted to the conclusions.

II. THEORETICAL FRAMEWORK BASED ON THE OPE AND THE NNLO CORRECTIONS TO $F_2^\gamma(x, Q^2, P^2)$

In this article we analyze the virtual photon structure functions $F_2^\gamma(x, Q^2, P^2)$ and $F_L^\gamma(x, Q^2, P^2)$ using the theoretical framework based on the OPE and RG method. Unless otherwise stated, we will follow the notation of Ref. [6]. Let us consider the forward virtual photon scattering amplitude for $\gamma(q) + \gamma(p) \rightarrow \gamma(q) + \gamma(p)$ illustrated in Fig. 2,

$$T_{\mu\nu\rho\tau}(p, q) = i \int d^4x d^4y d^4z e^{iq\cdot x} e^{ip\cdot(y-z)} \times \langle 0|T(J_\mu(x)J_\nu(0)J_\rho(y)J_\tau(z))|0\rangle, \quad (2.1)$$

where J_μ is the electromagnetic current. Its absorptive part is related to the structure tensor $W_{\mu\nu\rho\tau}(p, q)$ for the target photon with mass squared $p^2 = -P^2$ probed by the photon with $q^2 = -Q^2$:

$$W_{\mu\nu\rho\tau}(p, q) = \frac{1}{\pi} \mathrm{Im} T_{\mu\nu\rho\tau}(p, q). \quad (2.2)$$

Taking a spin average for the target photon, we get

$$\begin{aligned} W^\gamma_{\mu\nu}(p, q) &= \frac{1}{2}\sum_\lambda \epsilon^{\rho *}_{(\lambda)}(p) W_{\mu\nu\rho\tau}(p, q)\epsilon^\tau_{(\lambda)}(p) \\ &= -\frac{1}{2} g^{\rho\tau} W_{\mu\nu\rho\tau}(p, q) \\ &= \frac{1}{2}\int d^4x e^{iq\cdot x}\langle\gamma(p)|J_\mu(x)J_\nu(0)|\gamma(p)\rangle_{\text{spin av}}. \end{aligned} \quad (2.3)$$

Now $W^\gamma_{\mu\nu}(p, q)$ is expressed in terms of two independent structure functions $F_L^\gamma(x, Q^2, P^2)$ and $F_2^\gamma(x, Q^2, P^2)$:

$$\begin{aligned} W^\gamma_{\mu\nu}(p, q) &= \left\{g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right\}\frac{1}{x}F_L^\gamma(x, Q^2, P^2) \\ &+ \left\{-g_{\mu\nu} + \frac{q^\mu p^\nu + p^\mu q^\nu}{p\cdot q} - \frac{p_\mu p_\nu}{(p\cdot q)^2} q^2\right\} \\ &\times \frac{1}{x} F_2^\gamma(x, Q^2, P^2), \end{aligned} \quad (2.4)$$

where $x = Q^2/2p\cdot q$.

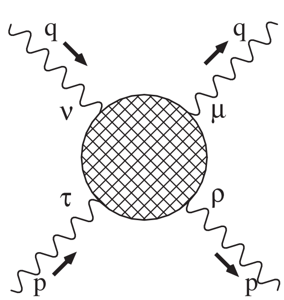

FIG. 2. Forward scattering of a virtual photon with momentum q and another virtual photon with momentum p. The Lorentz indices are denoted by μ, ν, ρ, τ.

Applying OPE for the product of two electromagnetic currents at short distance, we get

$$\begin{aligned} & i\int d^4x e^{iq\cdot x} T(J_\mu(x)J_\nu(0)) \\ &= \left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right]\sum_{\substack{n=0 \\ n=\text{even}}}\left(\frac{2}{Q^2}\right)^n q_{\mu_1}\cdots q_{\mu_n}\sum_i C^i_{L,n} O_i^{\mu_1\cdots\mu_n} \\ &+ [-g_{\mu\lambda}g_{\nu\sigma}q^2 + g_{\mu\lambda}q_\nu q_\sigma + g_{\nu\sigma}q_\mu q_\lambda - g_{\mu\nu}q_\lambda q_\sigma] \\ &\times \sum_{\substack{n=2 \\ n=\text{even}}}\left(\frac{2}{Q^2}\right)^n q_{\mu_1}\cdots q_{\mu_{n-2}}\sum_i C^i_{2,n} O_i^{\lambda\sigma\mu_1\cdots\mu_{n-2}} + \cdots, \end{aligned} \quad (2.5)$$

where $C^i_{L,n}$ and $C^i_{2,n}$ are the coefficient functions which contribute to the structure functions F_L^γ and F_2^γ, respectively, and $O_i^{\mu_1\cdots\mu_n}$ and $O_i^{\lambda\sigma\mu_1\cdots\mu_{n-2}}$ are spin-n twist-2 operators (hereafter we often refer to $O_i^{\mu_1\cdots\mu_n}$ as O_i^n). The sum on i runs over the possible twist-2 operators, and $\cdots$ represents other terms with irrelevant coefficient functions and operators. In fact, the relevant O_i^n are singlet quark (ψ), gluon (G), nonsinglet quark (NS), and photon (γ) operators as follows:

$$O_\psi^{\mu_1\cdots\mu_n} = i^{n-1}\bar\psi\gamma^{\{\mu_1}D^{\mu_2}\cdots D^{\mu_n\}}\mathbf{1}\psi - \text{trace terms}, \quad (2.6a)$$

$$O_G^{\mu_1\cdots\mu_n} = \tfrac{1}{2}i^{n-2}G^{\{\mu_1}_\alpha D^{\mu_2}\cdots D^{\mu_{n-1}}G^{\alpha\mu_n\}} - \text{trace terms}, \quad (2.6b)$$

$$O_{NS}^{\mu_1\cdots\mu_n} = i^{n-1}\bar\psi\gamma^{\{\mu_1}D^{\mu_2}\cdots D^{\mu_n\}} \times (Q^2_{ch} - \langle e^2\rangle\mathbf{1})\psi - \text{trace terms}, \quad (2.6c)$$

$$O_\gamma^{\mu_1\cdots\mu_n} = \tfrac{1}{2}i^{n-2}F^{\{\mu_1}_\alpha \partial^{\mu_2}\cdots\partial^{\mu_{n-1}}F^{\alpha\mu_n\}} - \text{trace terms}, \quad (2.6d)$$

where {} means complete symmetrization over the Lorentz indices $\mu_1\cdots\mu_n$ and D^μ denotes the covariant derivative. In quark operators O_ψ^n and O_{NS}^n given in Eqs. (2.6a) and (2.6c), $\mathbf{1}$ is an $n_f \times n_f$ unit matrix, Q^2_{ch} is the square of the $n_f\times n_f$ quark-charge matrix, with n_f being the number of active quark (i.e., the massless quark) flavors, and $\langle e^2\rangle = (\sum_i^{n_f} e_i^2)/n_f$ is the average charge squared where e_i is the electromagnetic charge of the active quark with flavor i in the unit of proton charge. It is noted that we have a relation $\mathrm{Tr}(Q^2_{\text{ch}} - \langle e^2\rangle\mathbf{1}) = 0$. The essential feature in the analysis of the photon structure functions, in contrast to the case of the nucleon counterparts, is the appearance of photon operators O_γ^n in addition to the familiar hadronic operators O_ψ^n, O_G^n, and O_{NS}^n [5].

The spin-averaged matrix elements of these operators sandwiched by the photon state with momentum p are expressed as

$$\langle\gamma(p)|O_i^{\mu_1\cdots\mu_n}|\gamma(p)\rangle_{\text{spin av}} = A_n^i(\mu^2, P^2)\{p^{\mu_1}\cdots p^{\mu_n} - \text{trace terms}\} \quad (2.7)$$

with $i = \psi, G, NS, \gamma$, and μ is the renormalization point. Then the moment sum rules for F_2^γ and F_L^γ are given as follows [7]:

$$\int_0^1 dx x^{n-2} F_2^\gamma(x, Q^2, P^2) = \sum_{i=\psi,G,NS,\gamma} C_{2,n}^i(Q^2/\mu^2, \bar{g}(\mu^2), \alpha) \times A_n^i(\mu^2, P^2), \quad (2.8a)$$

$$\int_0^1 dx x^{n-2} F_L^\gamma(x, Q^2, P^2) = \sum_{i=\psi,G,NS,\gamma} C_{L,n}^i(Q^2/\mu^2, \bar{g}(\mu^2), \alpha) \times A_n^i(\mu^2, P^2), \quad (2.8b)$$

with $\bar{g}(\mu^2)$ being the effective running QCD coupling constant at μ^2. Recall that in this article the photon structure functions are defined to be of order α. Since the coefficient functions $C_{2,n}^\gamma$ and $C_{L,n}^\gamma$ are $\mathcal{O}(\alpha)$, it is sufficient to evaluate A_n^γ at $\mathcal{O}(1)$. Thus we have

$$A_n^\gamma(\mu^2, P^2) = 1. \quad (2.9)$$

On the other hand, the matrix elements A_n^i $(i = \psi, G, NS)$ for the hadronic operators start at $\mathcal{O}(\alpha)$. For $-p^2 = P^2 \gg \Lambda^2$, we can calculate A_n^i $(i = \psi, G, NS)$ perturbatively in each power of g^2. When μ^2 is chosen at P^2, they are expressed as

$$A_n^i(\mu^2, P^2)|_{\mu^2=P^2} = \frac{\alpha}{4\pi} \tilde{A}_n^i(\bar{g}(P^2)), \quad \text{for} \quad i = \psi, G, NS. \quad (2.10)$$

Let us first analyze the structure function $F_2^\gamma(x, Q^2, P^2)$. We will evaluate its moment sum rule up to the NNLO. The Q^2 dependence of the coefficient functions $C_{2,n}^i(Q^2/\mu^2, \bar{g}(\mu^2), \alpha)$ in (2.8a) is governed by the RG equation. Putting $\mu^2 = -p^2 = P^2$, its solution is given by

$$C_{2,n}^i(Q^2/P^2, \bar{g}(P^2), \alpha) = \left(T \exp\left[\int_{\bar{g}(Q^2)}^{\bar{g}(P^2)} dg \frac{\gamma_n(g, \alpha)}{\beta(g)}\right]\right)_{ij} \times C_{2,n}^j(1, \bar{g}(Q^2), \alpha), \quad (2.11)$$

with $i, j = \psi, G, NS$, and γ. Here $\beta(g)$ is the beta function and $\gamma_n(g^2, \alpha)$ is the anomalous dimension matrix. To the lowest order in α, this matrix has the following form:

$$\gamma_n(g, \alpha) = \begin{pmatrix} \hat{\gamma}_n(g) & \mathbf{0} \\ \mathbf{K}_n(g, \alpha) & 0 \end{pmatrix}, \quad (2.12)$$

where $\hat{\gamma}_n(g^2)$ is the usual 3×3 anomalous dimension matrix in the hadronic sector,

$$\hat{\gamma}_n(g) = \begin{pmatrix} \gamma_{\psi\psi}^n(g) & \gamma_{G\psi}^n(g) & 0 \\ \gamma_{\psi G}^n(g) & \gamma_{GG}^n(g) & 0 \\ 0 & 0 & \gamma_{NS}^n(g) \end{pmatrix}, \quad (2.13)$$

and $\mathbf{K}_n(g, \alpha)$ is the three-component row vector

$$\mathbf{K}_n(g, \alpha) = (K_\psi^n(g, \alpha), K_G^n(g, \alpha), K_{NS}^n(g, \alpha)), \quad (2.14)$$

which represents the mixing between the photon operator and the remaining three hadronic operators. Then the evolution factor in (2.11) is expressed as [6]

$$T \exp\left[\int_{\bar{g}(Q^2)}^{\bar{g}(P^2)} dg \frac{\gamma_n(g, \alpha)}{\beta(g)}\right] = \begin{pmatrix} M_n & \mathbf{0} \\ \mathbf{X}_n & 1 \end{pmatrix}, \quad (2.15)$$

where

$$M_n(Q^2/P^2, \bar{g}(P^2)) = T \exp\left[\int_{\bar{g}(Q^2)}^{\bar{g}(P^2)} dg \frac{\hat{\gamma}_n(g)}{\beta(g)}\right], \quad (2.16)$$

$$\mathbf{X}_n(Q^2/P^2, \bar{g}(P^2), \alpha) = \int_{\bar{g}(Q^2)}^{\bar{g}(P^2)} dg \frac{\mathbf{K}_n(g, \alpha)}{\beta(g)} T \exp\left[\int_{\bar{g}(Q^2)}^{g} dg' \frac{\hat{\gamma}_n(g')}{\beta(g')}\right]. \quad (2.17)$$

Thus using (2.9), (2.10), (2.11), (2.15), (2.16), and (2.17), we get

$$\begin{aligned}\int_0^1 dx x^{n-2} F_2^\gamma(x, Q^2, P^2) &= \frac{\alpha}{4\pi} \tilde{\mathbf{A}}_n(\bar{g}(P^2)) \cdot M_n(Q^2/P^2, \bar{g}(P^2)) \cdot \mathbf{C}_{2,n}(1, \bar{g}(Q^2)) \\ &\quad + \mathbf{X}_n(Q^2/P^2, \bar{g}(P^2), \alpha) \cdot \mathbf{C}_{2,n}(1, \bar{g}(Q^2)) \\ &\quad + C_{2,n}^\gamma(1, \bar{g}(Q^2), \alpha), \end{aligned} \quad (2.18)$$

with

$$\tilde{\mathbf{A}}_n(\bar{g}) = (\tilde{A}_n^\psi(\bar{g}), \tilde{A}_n^G(\bar{g}), \tilde{A}_n^{NS}(\bar{g})), \quad (2.19)$$

and

$$\mathbf{C}_{2,n}(1, \bar{g}) = \begin{pmatrix} C_{2,n}^\psi(1, \bar{g}) \\ C_{2,n}^G(1, \bar{g}) \\ C_{2,n}^{NS}(1, \bar{g}) \end{pmatrix}. \quad (2.20)$$

In order to evaluate $M_n(Q^2/P^2, \bar{g}(P^2))$ in (2.16) up to the NNLO, we first expand $\hat{\gamma}_n(g)$ in powers of g^2 up to the three-loop level as

$$\begin{aligned}\hat{\gamma}_n(g) &= \hat{\gamma}_n^{(0)}(g) + \hat{\gamma}_n^{(1)}(g) + \hat{\gamma}_n^{(2)}(g) + \cdots \\ &= \frac{g^2}{16\pi^2} \hat{\gamma}_n^{(0)} + \frac{g^4}{(16\pi^2)^2} \hat{\gamma}_n^{(1)} + \frac{g^6}{(16\pi^2)^3} \hat{\gamma}_n^{(2)} + \cdots. \end{aligned} \quad (2.21)$$

Then, putting $\bar{g}_1 = \bar{g}(P^2)$ and $\bar{g}_2 = \bar{g}(Q^2)$, we find that $M_n(Q^2/P^2, \bar{g}(P^2))$ is expanded as

$$M_n(Q^2/P^2,\bar g(P^2)) = T\exp\left[\int_{\bar g_2}^{\bar g_1} dg \frac{\hat{\boldsymbol\gamma}_n(g)}{\beta(g)}\right]$$
$$= \exp\left[\int_{\bar g_2}^{\bar g_1} dg \frac{\hat{\boldsymbol\gamma}_n^{(0)}(g)}{\beta(g)}\right] + \int_{\bar g_2}^{\bar g_1} dg \exp\left[\int_{g}^{\bar g_1} dg' \frac{\hat{\boldsymbol\gamma}_n^{(0)}(g')}{\beta(g')}\right]\frac{\hat{\boldsymbol\gamma}_n^{(1)}(g)}{\beta(g)}\exp\left[\int_{\bar g_2}^{g} dg'' \frac{\hat{\boldsymbol\gamma}_n^{(0)}(g'')}{\beta(g'')}\right]$$
$$+ \int_{\bar g_2}^{\bar g_1} dg \exp\left[\int_{g}^{\bar g_1} dg' \frac{\hat{\boldsymbol\gamma}_n^{(0)}(g')}{\beta(g')}\right]\frac{\hat{\boldsymbol\gamma}_n^{(2)}(g)}{\beta(g)}\exp\left[\int_{\bar g_2}^{g} dg'' \frac{\hat{\boldsymbol\gamma}_n^{(0)}(g'')}{\beta(g'')}\right] + \int_{\bar g_2}^{\bar g_1} dg_a \exp\left[\int_{g_a}^{\bar g_1} dg' \frac{\hat{\boldsymbol\gamma}_n^{(0)}(g')}{\beta(g')}\right]\frac{\hat{\boldsymbol\gamma}_n^{(1)}(g_a)}{\beta(g_a)}$$
$$\times \int_{\bar g_2}^{g_a} dg_b \exp\left[\int_{g_b}^{g_a} dg'' \frac{\hat{\boldsymbol\gamma}_n^{(0)}(g'')}{\beta(g'')}\right]\frac{\hat{\boldsymbol\gamma}_n^{(1)}(g_b)}{\beta(g_b)}\exp\left[\int_{\bar g_2}^{g_b} dg''' \frac{\hat{\boldsymbol\gamma}_n^{(0)}(g''')}{\beta(g''')}\right] + \cdots. \tag{2.22}$$

To evaluate the integrals, we make full use of the projection operators obtained from the one-loop anomalous dimension matrix $\hat\gamma_n^{(0)}$ in (2.21) [6]:

$$\hat\gamma_n^{(0)} = \sum_{i=+,-,NS} \lambda_i^n P_i^n, \tag{2.23}$$

where $\lambda_i^n (i = +, -, NS)$ and P_i^n are eigenvalues of $\hat\gamma_n^{(0)}$ and the corresponding projection operators, respectively. The explicit forms of λ_i^n and P_i^n are given in Appendix A. Expanding $\beta(g)$ in powers of g^2 up to the three-loop level as

$$\beta(g) = -\frac{g^3}{16\pi^2}\beta_0 - \frac{g^5}{(16\pi^2)^2}\beta_1 - \frac{g^7}{(16\pi^2)^3}\beta_2 + \cdots, \tag{2.24}$$

we perform integration in (2.22). The final form of $M_n(Q^2/P^2, \bar g(P^2))$ up to the NNLO is given in (A7) in Appendix A.

Similarly, expanding $\boldsymbol{K}_n(g,\alpha)$ in powers of g^2 up to the three-loop level as

$$\boldsymbol{K}_n(g,\alpha) = -\frac{e^2}{16\pi^2}\boldsymbol{K}_n^{(0)} - \frac{e^2 g^2}{(16\pi^2)^2}\boldsymbol{K}_n^{(1)} - \frac{e^2 g^4}{(16\pi^2)^3}\boldsymbol{K}_n^{(2)} + \cdots, \tag{2.25}$$

we can evaluate $\boldsymbol{X}_n(Q^2/P^2, \bar g(P^2), \alpha)$ in (2.17) up to the NNLO. The result is given in (A8) in Appendix A.

Finally, expansions are made for the photon matrix elements of hadronic operators $\tilde{\boldsymbol{A}}_n(\bar g(P^2))$ in (2.19) as well as the coefficient functions $\boldsymbol{C}_{2,n}(1,\bar g(Q^2))$ in (2.20) and $C_{2,n}^\gamma(1,\bar g(Q^2),\alpha)$ in (2.18) up to the two-loop level as follows:

$$\tilde{\boldsymbol{A}}_n(\bar g(P^2)) = \tilde{\boldsymbol{A}}_n^{(1)} + \frac{\bar g^2(P^2)}{16\pi^2}\tilde{\boldsymbol{A}}_n^{(2)} + \cdots, \tag{2.26}$$

$$\boldsymbol{C}_{2,n}(1,\bar g(Q^2)) = \boldsymbol{C}_{2,n}^{(0)} + \frac{\bar g^2(Q^2)}{16\pi^2}\boldsymbol{C}_{2,n}^{(1)} + \frac{\bar g^4(Q^2)}{(16\pi^2)^2}\boldsymbol{C}_{2,n}^{(2)} + \cdots, \tag{2.27}$$

$$C_{2,n}^\gamma(1,\bar g(Q^2),\alpha) = \frac{e^2}{16\pi^2}C_{2,n}^{\gamma(1)} + \frac{e^2\bar g^2(Q^2)}{(16\pi^2)^2}C_{2,n}^{\gamma(2)} + \cdots. \tag{2.28}$$

Then putting (2.26), (2.27), (2.28), (A7), and (A8) into (2.18), we obtain the expression for the moment sum rule of $F_2^\gamma(x,Q^2,P^2)$ up to the NNLO $(\alpha\alpha_s)$ corrections as follows:

$$\int_0^1 dx x^{n-2}F_2^\gamma(x,Q^2,P^2) = \frac{\alpha}{4\pi}\frac{1}{2\beta_0}\left\{\frac{4\pi}{\alpha_s(Q^2)}\sum_i \mathcal{L}_i^n\left[1-\left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n+1}\right] + \sum_i \mathcal{A}_i^n\left[1-\left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n}\right]\right.$$
$$+\sum_i \mathcal{B}_i^n\left[1-\left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n+1}\right] + \mathcal{C}^n + \frac{\alpha_s(Q^2)}{4\pi}\left(\sum_i \mathcal{D}_i^n\left[1-\left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n-1}\right]\right.$$
$$\left.\left.+\sum_i \mathcal{E}_i^n\left[1-\left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n}\right] + \sum_i \mathcal{F}_i^n\left[1-\left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n+1}\right] + \mathcal{G}^n\right) + \mathcal{O}(\alpha_s^2)\right\}, \quad \text{with } i=+,-,NS, \tag{2.29}$$

where $d_i^n = \frac{\lambda_i^n}{2\beta_0}$. The coefficients $\mathcal{L}_i^n$, $\mathcal{A}_i^n$, $\mathcal{B}_i^n$, $\mathcal{C}^n$, $\mathcal{D}_i^n$, $\mathcal{E}_i^n$, $\mathcal{F}_i^n$, and $\mathcal{G}^n$ are given by

$$\mathcal{L}_i^n = \boldsymbol{K}_n^{(0)}P_i^n\boldsymbol{C}_{2,n}^{(0)}\frac{1}{d_i^n+1}, \tag{2.30}$$

$$\mathcal{A}_i^n = -\boldsymbol{K}_n^{(0)}\sum_j\frac{P_j^n\hat\gamma_n^{(1)}P_i^n}{\lambda_j^n-\lambda_i^n+2\beta_0}\boldsymbol{C}_{2,n}^{(0)}\frac{1}{d_i^n} - \boldsymbol{K}_n^{(0)}P_i^n\boldsymbol{C}_{2,n}^{(0)}\frac{\beta_1}{\beta_0}\frac{1-d_i^n}{d_i^n} + \boldsymbol{K}_n^{(1)}P_i^n\boldsymbol{C}_{2,n}^{(0)}\frac{1}{d_i^n} - 2\beta_0\tilde{\boldsymbol{A}}_n^{(1)}P_i^n\boldsymbol{C}_{2,n}^{(0)}, \tag{2.31}$$

$$\mathcal{B}_i^n = \boldsymbol{K}_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{1+d_i^n} + \boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(1)} \frac{1}{1+d_i^n} - \boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{d_i^n}{1+d_i^n}, \tag{2.32}$$

$$\mathcal{C}^n = 2\beta_0 (C_{2,n}^{\gamma(1)} + \tilde{\boldsymbol{A}}_n^{(1)} \cdot \boldsymbol{C}_{2,n}^{(0)}), \tag{2.33}$$

$$\begin{aligned}\mathcal{D}_i^n = & -\boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \frac{1}{1-d_i^n}\right)\left(1 - \frac{d_i^n}{2}\right) - \boldsymbol{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{1-d_j^n}{1-d_i^n} \\ & - \boldsymbol{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 4\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \left(\frac{1 - d_i^n + d_j^n}{1-d_i^n}\right) + \boldsymbol{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(2)} P_i^n}{\lambda_j^n - \lambda_i^n + 4\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{1-d_i^n} \\ & - \boldsymbol{K}_n^{(0)} \sum_{j,k} \frac{P_k^n \hat{\gamma}_n^{(1)} P_j^n \hat{\gamma}_n^{(1)} P_i^n}{(\lambda_j^n - \lambda_i^n + 2\beta_0)(\lambda_k^n - \lambda_i^n + 4\beta_0)} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{1-d_i^n} + \boldsymbol{K}_n^{(1)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} + \boldsymbol{K}_n^{(1)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{1-d_i^n} \\ & - \boldsymbol{K}_n^{(2)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{1}{1-d_i^n} + 2\beta_0 \tilde{\boldsymbol{A}}_n^{(1)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} - 2\beta_0 \tilde{\boldsymbol{A}}_n^{(1)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} d_i^n - 2\beta_0 \tilde{\boldsymbol{A}}_n^{(2)} P_i^n \boldsymbol{C}_{2,n}^{(0)},\end{aligned} \tag{2.34}$$

$$\begin{aligned}\mathcal{E}_i^n = & -\boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(1)} \frac{\beta_1}{\beta_0} \frac{1-d_i^n}{d_i^n} - \boldsymbol{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(1)} \frac{1}{d_i^n} + \boldsymbol{K}_n^{(1)} P_i^n \boldsymbol{C}_{2,n}^{(1)} \frac{1}{d_i^n} + \boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1^2}{\beta_0^2}(1 - d_i^n) \\ & - \boldsymbol{K}_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{1-d_i^n}{d_i^n} + \boldsymbol{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \\ & - \boldsymbol{K}_n^{(0)} \sum_{j,k} \frac{P_j^n \hat{\gamma}_n^{(1)} P_i^n \hat{\gamma}_n^{(1)} P_k^n}{(\lambda_i^n - \lambda_k^n + 2\beta_0)(\lambda_j^n - \lambda_i^n + 2\beta_0)} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{d_i^n} - \boldsymbol{K}_n^{(1)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} + \boldsymbol{K}_n^{(1)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{d_i^n} \\ & - 2\beta_0 \tilde{\boldsymbol{A}}_n^{(1)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} + 2\beta_0 \tilde{\boldsymbol{A}}_n^{(1)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} d_i^n - 2\beta_0 \tilde{\boldsymbol{A}}_n^{(1)} P_i^n \boldsymbol{C}_{2,n}^{(1)},\end{aligned} \tag{2.35}$$

$$\begin{aligned}\mathcal{F}_i^n = & \boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(2)} \frac{1}{1+d_i^n} - \boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(1)} \frac{\beta_1}{\beta_0} \frac{d_i^n}{1+d_i^n} + \boldsymbol{K}_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(1)} \frac{1}{1+d_i^n} \\ & + \boldsymbol{K}_n^{(0)} P_i^n \boldsymbol{C}_{2,n}^{(0)} \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \frac{1}{1+d_i^n}\right) \frac{d_i^n}{2} - \boldsymbol{K}_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{d_j^n}{1+d_i^n} \\ & - \boldsymbol{K}_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n}{\lambda_i^n - \lambda_j^n + 4\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \frac{1 + d_i^n - d_j^n}{1+d_i^n} + \boldsymbol{K}_n^{(0)} \sum_j \frac{P_i^n \hat{\gamma}_n^{(2)} P_j^n}{\lambda_i^n - \lambda_j^n + 4\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{1}{1+d_i^n} \\ & + \boldsymbol{K}_n^{(0)} \sum_{j,k} \frac{P_i^n \hat{\gamma}_n^{(1)} P_j^n \hat{\gamma}_n^{(1)} P_k^n}{(\lambda_j^n - \lambda_k^n + 2\beta_0)} \boldsymbol{C}_{2,n}^{(0)} \left(\frac{1}{\lambda_i^n - \lambda_j^n + 2\beta_0} - \frac{1}{\lambda_i^n - \lambda_k^n + 4\beta_0}\right) \frac{1}{1+d_i^n},\end{aligned} \tag{2.36}$$

$$\mathcal{G}^n = 2\beta_0 (C_{2,n}^{\gamma(2)} + \tilde{\boldsymbol{A}}_n^{(1)} \cdot \boldsymbol{C}_{2,n}^{(1)} + \tilde{\boldsymbol{A}}_n^{(2)} \cdot \boldsymbol{C}_{2,n}^{(0)}), \tag{2.37}$$

with $i, j, k = +, -, NS$. The LO term $\mathcal{L}_i^n$ was obtained by Witten [5]. The NLO (α) corrections $\mathcal{A}_i^n$, $\mathcal{B}_i^n$, and $\mathcal{C}^n$ without terms with $\tilde{\boldsymbol{A}}_n^{(1)}$ were first derived by Bardeen and Buras [6] for the case of the *real* photon target (i.e. $P^2 = 0$). Later, authors in Ref. [17] analyzed the NLO (α) corrections for the case of the *virtual* photon target ($P^2 \gg \Lambda^2$) and the terms with $\tilde{\boldsymbol{A}}_n^{(1)}$ were added to $\mathcal{A}_i^n$ and $\mathcal{C}^n$. The coefficients $\mathcal{D}_i^n$, $\mathcal{E}_i^n$, $\mathcal{F}_i^n$, and $\mathcal{G}^n$ are the NNLO ($\alpha\alpha_s$) corrections and they are new.

For $n = 2$, one of the eigenvalues, $\lambda_-^{n=2}$, in Eq. (2.23) vanishes and we have $d_-^{n=2} = 0$. This is due to the fact that the corresponding operator is the hadronic energy-momentum tensor and is, therefore, conserved with a null anomalous dimension [6]. The coefficients $\mathcal{A}_-^{n=2}$ and $\mathcal{E}_-^{n=2}$ have terms which are proportional to $\frac{1}{d_-^{n=2}}$ and thus diverge. However, we see from (2.29) that these coefficients are multiplied by a factor $[1-(\alpha_s(Q^2)/\alpha_s(P^2))^{d_-^{n=2}}]$ which vanishes. In the end, the coefficients $\mathcal{A}_-^{n=2}$ and $\mathcal{E}_-^{n=2}$ multiplied by this factor remain finite [17].

III. PARAMETERS IN THE $\overline{\rm MS}$ SCHEME

All the quantities necessary to evaluate the NNLO $(\alpha\alpha_s)$ corrections to the moments of $F_2^\gamma(x,Q^2,P^2)$ have been calculated and most of them are presented in the literature, except for the two-loop photon matrix elements of hadronic operators $\tilde{A}_n^{(2)\psi}$, $\tilde{A}_n^{(2)G}$, and $\tilde{A}_n^{(2)NS}$. Also for the three-loop anomalous dimensions $K_\psi^{(2),n}$, $K_G^{(2),n}$, and $K_{NS}^{(2),n}$, we only have approximate expressions in the form of photon-quark and photon-gluon splitting functions. In the following we will enumerate all these necessary parameters. The expressions are the ones calculated in the modified minimal subtraction ($\overline{\rm MS}$) scheme [28].

A. Quark-charge factors and β function parameters

The following quark-charge factors are often used below:

$$\delta_\psi=\langle e^2\rangle=\sum_{i=1}^{n_f}e_i^2/n_f,\qquad \delta_{NS}=1,\qquad \delta_\gamma=3n_f\langle e^4\rangle=3\sum_{i=1}^{n_f}e_i^4. \tag{3.1}$$

The β function parameters β_0, β_1, and β_2 [29] are given by

$$\beta_0=\frac{11}{3}C_A-\frac{2}{3}n_f, \tag{3.2}$$

$$\beta_1=\frac{34}{3}C_A^2-\frac{10}{3}C_An_f-2C_Fn_f, \tag{3.3}$$

$$\beta_2=\frac{2857}{54}C_A^3-\frac{1415}{54}C_A^2n_f-\frac{205}{18}C_AC_Fn_f+\frac{79}{54}C_An_f^2+C_F^2n_f+\frac{11}{9}C_Fn_f^2, \tag{3.4}$$

with $C_A=3$ and $C_F=\frac{4}{3}$ in QCD.

B. Coefficient functions

As shown in (2.27) and (2.28), we need the hadronic coefficient functions $C_{2,n}^i(1,\bar{g}(Q^2))$ with $i=\psi, G$ and NS, and the photon coefficient function $C_{2,n}^\gamma(1,\bar{g}(Q^2),\alpha)$ up to the two-loop level. At tree level, we have

$$C_{2,n}^{\psi(0)}=\delta_\psi,\qquad C_{2,n}^{G(0)}=0,\qquad C_{2,n}^{NS(0)}=\delta_{NS},\qquad C_{2,n}^{\gamma(0)}=0. \tag{3.5}$$

The one-loop coefficient functions were calculated in the minimal subtraction (MS) scheme in Refs. [28,30]. The $\overline{\rm MS}$ results are written as

$$C_{2,n}^{\psi(1)}=\delta_\psi\bar{B}_\psi^n,\qquad C_{2,n}^{G(1)}=\delta_\psi\bar{B}_G^n,\qquad C_{2,n}^{NS(1)}=\delta_{NS}\bar{B}_{NS}^n,\qquad C_{2,n}^{\gamma(1)}=\delta_\gamma\bar{B}_\gamma^n, \tag{3.6}$$

where $\bar{B}_\psi^n=\bar{B}_{NS}^n$ and $\bar{B}_G^n$ are obtained, for example, from the MS-scheme results for $B_\psi^n=B_{NS}^n$ and B_G^n given in Eqs. (4.10) and (4.11) of Ref. [6] by discarding the terms proportional to $\ln(4\pi-\gamma_E)$. $\bar{B}_\gamma^n$ is related to $\bar{B}_G^n$ by $\bar{B}_\gamma^n=(2/n_f)\bar{B}_G^n$.

The two-loop coefficient functions corresponding to the hadronic operators were calculated in the $\overline{\rm MS}$ scheme in Refs. [31,32]. They were expressed in fractional momentum space as functions x. The results in Mellin space as functions of n are found, for example, in Ref. [33]:

$$C_{2,n}^{\psi(2)}=\delta_\psi\{c_{2,q}^{(2),+{\rm ns}}(n)+c_{2,q}^{(2),-{\rm ns}}(n)+c_{2,q}^{(2),{\rm ps}}(n)\}, \tag{3.7}$$

$$C_{2,n}^{G(2)}=\delta_\psi c_{2,g}^{(2)}(n), \tag{3.8}$$

$$C_{2,n}^{NS(2)}=\delta_{NS}\{c_{2,q}^{(2),+{\rm ns}}(n)+c_{2,q}^{(2),-{\rm ns}}(n)\}, \tag{3.9}$$

where $c_{2,q}^{(2),+{\rm ns}}(n)$, $c_{2,q}^{(2),-{\rm ns}}(n)$, $c_{2,q}^{(2),{\rm ps}}(n)$, and $c_{2,g}^{(2)}(n)$ are given in Eqs. (197), (198), (201), and (202) in Appendix B of Ref. [33], respectively, with N being replaced by n. The two-loop photon coefficient function $C_{2,n}^{\gamma(2)}$ is expressed as

$$C_{2,n}^{\gamma(2)}=\delta_\gamma c_{2,\gamma}^{(2)}(n), \tag{3.10}$$

where $c_{2,\gamma}^{(2)}(n)$ is obtained from $c_{2,g}^{(2)}(n)$ in (3.8) by replacing $C_A\to 0$ and $\frac{n_f}{2}\to 1$ [10].

C. Anomalous dimensions

The one-loop anomalous dimensions for the hadronic sector were calculated a long time ago [34,35]. The expressions of $\gamma_{\psi\psi}^{(0),n}=\gamma_{NS}^{(0),n}$, $\gamma_{\psi G}^{(0),n}$, $\gamma_{G\psi}^{(0),n}$, and $\gamma_{GG}^{(0),n}$ are given, for example, in Eqs. (4.1), (4.2), (4.3), and (4.4) of Ref. [6], respectively, with f being replaced by n_f. As for the one-loop anomalous dimension row vector $\boldsymbol{K}_n^{(0)}=(K_\psi^{(0),n},K_G^{(0),n},K_{NS}^{(0),n})$, we have $K_G^{(0),n}=0$, and $K_\psi^{(0),n}$ and $K_{NS}^{(0),n}$ are given, respectively, in Eqs. (4.5) and (4.6) of Ref. [6] with f being replaced by n_f again.

The two-loop anomalous dimensions for the hadronic sector were calculated in Ref. [30] and recalculated using a different method and a different gauge in Ref. [36]. The results by the two groups agreed with each other except in the part of $\gamma_{GG}^{(1),n}$ proportional to C_G^2, but this discrepancy was solved later [37]. They are given by

$$\gamma_{NS}^{(1),n}=2\gamma_{\rm ns}^{(1)+}(n), \tag{3.11}$$

$$\gamma_{\psi\psi}^{(1),n}=2(\gamma_{\rm ns}^{(1)+}(n)+\gamma_{\rm ps}^{(1)}(n)), \tag{3.12}$$

$$\gamma_{\psi G}^{(1),n} = 2\gamma_{\rm qg}^{(1)}(n), \tag{3.13}$$

$$\gamma_{G\psi}^{(1),n} = 2\gamma_{\rm gq}^{(1)}(n), \tag{3.14}$$

$$\gamma_{GG}^{(1),n} = 2\gamma_{\rm gg}^{(1)}(n), \tag{3.15}$$

where $\gamma_{\rm ns}^{(1)+}(n)$ is given in Eq. (3.5) of Ref. [26], and $\gamma_{\rm ps}^{(1)}(n)$, $\gamma_{\rm qg}^{(1)}(n)$, $\gamma_{\rm gq}^{(1)}(n)$, and $\gamma_{\rm gg}^{(1)}(n)$ are given, respectively, in Eqs. (3.6), (3.7), (3.8), and (3.9) of Ref. [27], with N being replaced by n. The factor of 2 in (3.11), (3.12), (3.13), (3.14), and (3.15) appears since, in Refs. [26,27], the anomalous dimension γ of the renormalized operator O is defined as $dO/d\ln\mu^2 = -\gamma O$ instead of $dO/d\ln\mu = -\gamma O$.

The two-loop anomalous dimensions $K_\psi^{(1),n}$, $K_{NS}^{(1),n}$, and $K_G^{(1),n}$ can be obtained from $\gamma_{\psi G}^{(1),n}$ and $\gamma_{GG}^{(1),n}$ by replacing color factors with relevant charge factors [6]. Moreover we need an additional procedure for $K_G^{(1),n}$. They are given by

$$K_\psi^{(1),n} = -3n_f\langle e^2\rangle C_F D_{\psi G}(n), \tag{3.16}$$

$$K_{NS}^{(1),n} = -3n_f(\langle e^4\rangle - \langle e^2\rangle^2) C_F D_{\psi G}(n), \tag{3.17}$$

$$K_G^{(1),n} = -3n_f\langle e^2\rangle C_F(D_{GG}(n) - 8), \tag{3.18}$$

where $D_{\psi G}(n)$ and $D_{GG}(n)$ are obtained from $\gamma_{\psi G}^{(1),n}$ and $\gamma_{GG}^{(1),n}$, respectively, by replacing $C_A \to 0$ and $C_F n_f \to 2$. The number 8 in (3.18) is due to the gluon self-energy contribution to $\gamma_{GG}^{(1),n}$, which should be dropped for $K_G^{(1),n}$ [38,39].

The three-loop anomalous dimensions for the hadronic sector have been calculated recently in Refs. [26,27]. They are expressed as

$$\gamma_{NS}^{(2),n} = 2\gamma_{\rm ns}^{(2)+}(n), \tag{3.19}$$

$$\gamma_{\psi\psi}^{(2),n} = 2(\gamma_{\rm ns}^{(2)+}(n) + \gamma_{\rm ps}^{(2)}(n)), \tag{3.20}$$

$$\gamma_{\psi G}^{(2),n} = 2\gamma_{\rm qg}^{(2)}(n), \tag{3.21}$$

$$\gamma_{G\psi}^{(2),n} = 2\gamma_{\rm gq}^{(2)}(n), \tag{3.22}$$

$$\gamma_{GG}^{(2),n} = 2\gamma_{\rm gg}^{(2)}(n), \tag{3.23}$$

where $\gamma_{\rm ns}^{(2)+}(n)$ is given in Eq. (3.7) of Ref. [26], and $\gamma_{\rm ps}^{(2)}(n)$, $\gamma_{\rm qg}^{(2)}(n)$, $\gamma_{\rm gq}^{(2)}(n)$, and $\gamma_{\rm gg}^{(2)}(n)$ are given, respectively, in Eqs. (3.10), (3.11), (3.12), and (3.13) of Ref. [27], with N being replaced by n.

Concerning the three-loop anomalous dimensions $K_\psi^{(2),n}$, $K_{NS}^{(2),n}$, and $K_G^{(2),n}$, the exact expressions have not appeared in the literature yet. In fact, the lowest six even-integer Mellin moments, $n = 2, \ldots, 12$, of these anomalous dimensions were calculated and given in Ref. [10]. Quite recently, the authors of Ref. [10] have presented compact parametrizations of the three-loop photon-nonsinglet quark and photon-gluon splitting functions, $P_{\rm ns\gamma}^{(2)}(x)$ and $P_{\rm g\gamma}^{(2)}(x)$, instead of providing the exact analytic results [11]. It is remarked there that their parametrizations deviate from the lengthy full expressions by about 0.1% or less. They also gave in Ref. [11] the analytic expression of the three-loop photon-pure-singlet quark splitting function $P_{\rm ps\gamma}^{(2)}(x)$. It is true that we can infer the analytic expressions for some parts of $K_\psi^{(2),n}$, $K_{NS}^{(2),n}$, and $K_G^{(2),n}$ from the known three-loop results of $\gamma_{\psi G}^{(2),n}$ and $\gamma_{GG}^{(2),n}$. For instance, the expressions of $K_\psi^{(2),n}$ and $K_{NS}^{(2),n}$ which have the color factor C_F^2 are obtained from $\gamma_{\psi G}^{(2),n}$ by taking the terms which are proportional to the color factor $n_f C_F^2$. Also, the terms of $K_G^{(2),n}$ which have the color factors $n_f C_F$ and C_F^2 are related to the ones of $\gamma_{GG}^{(2),n}$ with the color factors $n_f^2 C_F$ and $n_f C_F^2$, respectively. But, at present, we do not have the exact analytic expressions of $K_\psi^{(2),n}$, $K_{NS}^{(2),n}$, and $K_G^{(2),n}$ as a whole.

Under these circumstances we are reconciled to the use of approximate expressions for $K_\psi^{(2),n}$, $K_{NS}^{(2),n}$, and $K_G^{(2),n}$. They are obtained by taking the Mellin moments of the parametrizations for $P_{\rm ns\gamma}^{(2)}(x)$ and $P_{\rm g\gamma}^{(2)}(x)$, and of the exact result for $P_{\rm ps\gamma}^{(2)}(x)$, which are presented in Ref. [11]. Then we have

$$K_{NS}^{(2),n} \approx K_{NS\,\rm approx}^{(2),n} \equiv -3n_f(\langle e^4\rangle - \langle e^2\rangle^2) 2E_{\rm ns\gamma}^{\rm approx}(n), \tag{3.24}$$

$$K_\psi^{(2),n} \approx K_{\psi\,\rm approx}^{(2),n} \equiv -3n_f\langle e^2\rangle 2\{E_{\rm ns\gamma}^{\rm approx}(n) + E_{\rm ps\gamma}(n)\}, \tag{3.25}$$

$$K_G^{(2),n} \approx K_{G\,\rm approx}^{(2),n} \equiv -3n_f\langle e^2\rangle 2E_{G\gamma}^{\rm approx}(n), \tag{3.26}$$

where the explicit expressions of $E_{\rm ns\gamma}^{\rm approx}(n)$, $E_{\rm ps\gamma}(n)$, and $E_{G\gamma}^{\rm approx}(n)$ are given in Appendix B. Again the appearance of the factor of 2 in (3.24), (3.25), and (3.26) is due to the difference in definition of the anomalous dimensions. As mentioned earlier, the lowest six even-integer Mellin moments, $n = 2, \ldots, 12$, of $K_{NS}^{(2),n}$, $K_\psi^{(2),n}$, and $K_G^{(2),n}$ were given in Ref. [10]. When we write $K_{NS}^{(2),n}$ and $K_G^{(2),n}$ as

$$K_{NS}^{(2),n} \equiv -3n_f(\langle e^4\rangle - \langle e^2\rangle^2) 2E_{\rm ns\gamma}(n), \tag{3.27}$$

$$K_G^{(2),n} \equiv -3n_f\langle e^2\rangle 2E_{G\gamma}(n), \tag{3.28}$$

then we get the exact results of $E_{\rm ns\gamma}(n)$ and $E_{G\gamma}(n)$ for even $n = 2, \ldots, 12$. We give in Table I the results of $E_{\rm ns\gamma}(n)$, $E_{\rm ns\gamma}^{\rm approx}(n)$, $E_{\rm G\gamma}(n)$, and $E_{\rm G\gamma}^{\rm approx}(n)$ in numerical form for the lowest six even-integer values of n. We see the deviations

TABLE I. Numerical values of $E_{\mathrm{ns}\gamma}(n)$, $E_{\mathrm{ns}\gamma}^{\mathrm{approx}}(n)$, $E_{G\gamma}(n)$, and $E_{G\gamma}^{\mathrm{approx}}(n)$ for the lowest six even-integer values of n. The values for $E_{\mathrm{ns}\gamma}(n)$ [$E_{G\gamma}(n)$] are found in Eq. (3.1) [Eq. (3.3)] or obtained by evaluating Eqs. (A.1)–(A.6) [Eqs. (A.7)–(A.12)] of Ref. [10]. The values of $E_{\mathrm{ns}\gamma}^{\mathrm{approx}}(n)$ and $E_{G\gamma}^{\mathrm{approx}}(n)$ are obtained from the expressions given in (B2) and (B3) in Appendix B, respectively.

	$E_{\mathrm{ns}\gamma}(n)$	$E_{\mathrm{ns}\gamma}^{\mathrm{approx}}(n)$	$E_{G\gamma}(n)$	$E_{G\gamma}^{\mathrm{approx}}(n)$
$n=2$	$-86.9753+1.47051n_f$	$-86.9844+1.47104n_f$	$31.4197+5.15775n_f$	$31.4155+5.15803n_f$
$n=4$	$-102.831+1.47737n_f$	$-102.848+1.47787n_f$	$23.9427+1.10886n_f$	$23.9419+1.10888n_f$
$n=6$	$-109.278+1.65653n_f$	$-109.299+1.65699n_f$	$15.6517+0.695953n_f$	$15.6507+0.695944n_f$
$n=8$	$-111.167+1.69550n_f$	$-111.192+1.69592n_f$	$10.9661+0.498196n_f$	$10.9651+0.498178n_f$
$n=10$	$-111.035+1.67061n_f$	$-111.062+1.67099n_f$	$8.16031+0.379060n_f$	$8.15953+0.379038n_f$
$n=12$	$-109.943+1.61908n_f$	$-109.972+1.61943n_f$	$6.34829+0.300274n_f$	$6.34777+0.300250n_f$

of $E_{\mathrm{ns}\gamma}^{\mathrm{approx}}(n)$ from $E_{\mathrm{ns}\gamma}(n)$ and $E_{G\gamma}^{\mathrm{approx}}(n)$ from $E_{G\gamma}(n)$ are both far less than 0.1% for these values of n.

D. Photon matrix elements

The two-loop operator matrix elements have been calculated up to the finite terms by Matiounine, Smith, and van Neerven (MSvN) [40]. Using their results and changing color-group factors, we obtain the photon matrix elements of hadronic operators up to the two-loop level.

First we clear up a subtle issue which appears in the calculation of the photon matrix elements of the hadronic operators. The one-loop gluon coefficient function $\bar{B}_G^n$ in (3.6) was calculated by two groups, BBDM and FRS (we have taken initials of the authors of Refs. [28,30], respectively). Both groups evaluated one-loop diagrams contributing to the forward virtual photon-gluon scattering as well as those contributing to the matrix element of the quark operator between gluon states, and they took a difference between the two to obtain $\bar{B}_G^n$. But actually BBDM calculated the gluon spin-averaged contributions, i.e., multiplying $g_{\rho\tau}$ and contracting pairs of Lorentz indices ρ and τ, whereas FRS picked up the parts which are proportional to $g_{\rho\tau}$. Thus the BBDM results on the contributions to the forward virtual photon-gluon scattering and the gluon matrix element of the quark operator are different from those by FRS, but the difference between the two contributions, i.e., $\bar{B}_G^n$, is the same, as it should be.

We have defined the photon structure functions F_2^γ and F_L^γ in (2.3) and (2.4), taking a spin average of the target photon for the structure tensor $W_{\mu\nu\rho\tau}(p,q)$. We, therefore, adopt the BBDM result rather than that of FRS and convert it to the photon case. Then, for the photon matrix elements of the hadronic operators at one-loop level, we get

$$\tilde{A}_n^{(1)\psi}=3n_f\langle e^2\rangle H_q^{(1)}(n),\qquad \tilde{A}_n^{(1)G}=0,\qquad \tilde{A}_n^{(1)NS}=3n_f(\langle e^4\rangle-\langle e^2\rangle^2)H_q^{(1)}(n), \tag{3.29}$$

where

$$H_q^{(1)}(n)=4\left[-\frac{1}{n}+\frac{1}{n^2}-\frac{4}{(n+1)^2}+\frac{4}{(n+2)^2}+\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}\right)S_1(n)\right], \tag{3.30}$$

with $S_1(n)=\sum_{j=1}^n\frac{1}{j}$. Actually, $H_q^{(1)}(n)$ is related to the BBDM result on the one-loop gluon matrix element of the quark operator $A_{nG}^{(2)\psi}$ given in Eq. (6.2) of Ref. [28] as $A_{nG}^{(2)\psi}=\frac{\alpha_s}{4\pi}\frac{n_f}{2}H_q^{(1)}(n)$.

MSvN have presented in Appendix A of Ref. [40] full expressions for the two-loop corrected operator matrix elements which are unrenormalized and include external self-energy corrections. The expressions are given in parton momentum fraction space, i.e., in z space. Taking the moments, the unrenormalized matrix elements of the (flavor-singlet) quark operators between gluon states are written as [see Eq. (2.18) of Ref. [40]]

$$\hat{A}_{qg,\rho\tau}\left(n,\frac{-p^2}{\mu^2},\frac{1}{\epsilon}\right)=\hat{A}_{qg}^{\mathrm{PHYS}}(n)T_{\rho\tau}^{(1)}+\hat{A}_{qg}^{\mathrm{EOM}}(n)T_{\rho\tau}^{(2)}+\hat{A}_{qg}^{\mathrm{NGI}}(n)T_{\rho\tau}^{(3)}, \tag{3.31}$$

where

$$\hat{A}_{qg}^{k}(n)=\int_0^1 dz z^{n-1}\hat{A}_{qg}^{k}\left(z,\frac{-p^2}{\mu^2},\frac{1}{\epsilon}\right),\qquad k=\mathrm{PHYS},\ \mathrm{EOM}\quad\text{and}\quad\mathrm{NGI}, \tag{3.32}$$

and the expressions of $\hat{A}_{qg}^{\mathrm{PHYS}}(z,\frac{-p^2}{\mu^2},\frac{1}{\epsilon})$, $\hat{A}_{qg}^{\mathrm{EOM}}(z,\frac{-p^2}{\mu^2},\frac{1}{\epsilon})$, and $\hat{A}_{qg}^{\mathrm{NGI}}(z,\frac{-p^2}{\mu^2},\frac{1}{\epsilon})$ are given in Eqs. (A7), (A8), and (A9) of Ref. [40], respectively. Refer to Ref. [40] for the explanation of the "PHYS," "EOM," and "NGI" parts. The tensors $T_{\rho\tau}^{(i)}(i=1,2,3)$ are given by [see Eqs. (2.19)–(2.21) of Ref. [40] and note that we have changed the Lorentz indices of gluon fields from $\mu\nu$ to $\rho\tau$]

$$T_{\rho\tau}^{(1)}=\left[g_{\rho\tau}-\frac{p_\rho\Delta_\tau+\Delta_\rho p_\tau}{\Delta\cdot p}+\frac{\Delta_\rho\Delta_\tau p^2}{(\Delta\cdot p)^2}\right](\Delta\cdot p)^n, \tag{3.33}$$

$$T_{\rho\tau}^{(2)}=\left[\frac{p_\rho p_\tau}{p^2}-\frac{p_\rho\Delta_\tau+\Delta_\rho p_\tau}{\Delta\cdot p}+\frac{\Delta_\rho\Delta_\tau p^2}{(\Delta\cdot p)^2}\right](\Delta\cdot p)^n, \tag{3.34}$$

$$T_{\rho\tau}^{(3)}=\left[-\frac{p_\rho\Delta_\tau+\Delta_\rho p_\tau}{2\Delta\cdot p}+\frac{\Delta_\rho\Delta_\tau p^2}{(\Delta\cdot p)^2}\right](\Delta\cdot p)^n, \tag{3.35}$$

where Δ_μ is a lightlike vector ($\Delta^2=0$). The renormalization of $\hat{A}_{qg,\rho\tau}(n,\frac{-p^2}{\mu^2},\frac{1}{\epsilon})$ proceeds as follows: First the coupling constant and gauge constant renormalizations are performed. Then the remaining ultraviolet divergences are removed by multiplication of the operator renormalization constants. We get the finite expression at $\mu^2=-p^2$ as

$$A_{qg,\rho\tau}(n)|_{\mu^2=-p^2} = \left\{\frac{\alpha_s}{4\pi}a_{qg}^{(1)}(n) + \left(\frac{\alpha_s}{4\pi}\right)^2 a_{qg}^{(2)}(n)\right\}T_{\rho\tau}^{(1)} + \left\{\frac{\alpha_s}{4\pi}b_{qg}^{(1)}(n) + \left(\frac{\alpha_s}{4\pi}\right)^2 b_{qg}^{(2)}(n)\right\}T_{\rho\tau}^{(2)} + \left(\frac{\alpha_s}{4\pi}\right)^2 a_{qA}^{(2)}(n)T_{\rho\tau}^{(3)}. \tag{3.36}$$

The expressions of $a_{qg}^{(i)}(n)$ and $b_{qg}^{(i)}(n)(i=1,2)$ are given in Appendix C, while $a_{qA}^{(2)}(n)$ is made up of the terms proportional to $C_A\frac{n_f}{2}$ and is, therefore, irrelevant to the photon matrix element of the quark operator. Now multiplying $g^{\rho\tau}$ and contracting pairs of indices ρ and τ, we get

$$\frac{1}{2}g^{\rho\tau}\frac{1}{(\Delta\cdot p)^n}A_{qg,\rho\tau}(n)|_{\mu^2=-p^2} = \frac{\alpha_s}{4\pi}\left\{a_{qg}^{(1)}(n) - \frac{1}{2}b_{qg}^{(1)}(n)\right\} + \left(\frac{\alpha_s}{4\pi}\right)^2\left\{a_{qg}^{(2)}(n) - \frac{1}{2}b_{qg}^{(2)}(n) - \frac{1}{2}a_{qA}^{(2)}(n)\right\}. \tag{3.37}$$

We can see from the expressions of $a_{qg}^{(1)}(n)$ and $b_{qg}^{(1)}(n)$ in (C2) and (C3), respectively, that the FRS result for the one-loop gluon matrix element of the quark operator corresponds to $a_{qg}^{(1)}(n)$, while the BBDM result corresponds to the combination $\{a_{qg}^{(1)}(n) - \frac{1}{2}b_{qg}^{(1)}(n)\}$. Indeed we find that $H_q^{(1)}(n)$ in (3.30) is written as $\frac{n_f}{2}H_q^{(1)}(n) = \{a_{qg}^{(1)}(n) - \frac{1}{2}b_{qg}^{(1)}(n)\}$.

The two-loop photon matrix elements of the quark operators are derived from the combination $\{a_{qg}^{(2)}(n) - \frac{1}{2}b_{qg}^{(2)}(n) - \frac{1}{2}a_{qA}^{(2)}(n)\}$ in (3.37) with the following replacements: $C_A\rightarrow 0$, $(\frac{n_f}{2})^2\rightarrow 0$, and $C_F\frac{n_f}{2}\rightarrow[C_F\times\text{charge factor}]$. The terms proportional to $(\frac{n_f}{2})^2$ in $a_{qg}^{(2)}(n)$ and $b_{qg}^{(2)}(n)$ come from the external gluon self-energy corrections and should be discarded for the photon case. Thus we obtain

$$\begin{aligned}\tilde{A}_n^{(2)\psi} &= 3n_f\langle e^2\rangle H_q^{(2)}(n),\\ \tilde{A}_n^{(2)NS} &= 3n_f(\langle e^4\rangle - \langle e^2\rangle^2)H_q^{(2)}(n),\end{aligned} \tag{3.38}$$

where

$$\begin{aligned}H_q^{(2)}(n) = C_F\bigg\{&\left(\frac{1}{n} - \frac{2}{n+1} + \frac{2}{n+2}\right)\left(-\frac{4}{3}S_1(n)^3 - 4S_2(n)S_1(n) + \frac{64}{3}S_3(n) - 16S_{2,1}(n) - 48\zeta_3\right) + S_1(n)^2\left(\frac{6}{n} - \frac{8}{n+1}\right.\\ &\left. + \frac{16}{n+2} - \frac{16}{n^2} + \frac{40}{(n+1)^2} - \frac{32}{(n+2)^2}\right) + S_1(n)\left(\frac{4}{n} - \frac{80}{n+1} + \frac{56}{n+2} + \frac{16}{n^2} - \frac{48}{(n+1)^2} + \frac{64}{(n+2)^2} - \frac{32}{n^3}\right.\\ &\left. + \frac{176}{(n+1)^3} - \frac{128}{(n+2)^3}\right) + S_2(n)\left(\frac{6}{n} + \frac{8}{n+1} - \frac{16}{n^2} + \frac{40}{(n+1)^2} - \frac{32}{(n+2)^2}\right) + \frac{38}{n} - \frac{70}{n+1} + \frac{56}{n+2} + \frac{56}{n^2}\\ &- \frac{198}{(n+1)^2} + \frac{144}{(n+2)^2} - \frac{22}{n^3} - \frac{40}{(n+1)^3} + \frac{128}{(n+2)^3} + \frac{20}{n^4} + \frac{88}{(n+1)^4}\bigg\}.\end{aligned} \tag{3.39}$$

Similarly the renormalized matrix elements of the gluon operators between gluon states at $\mu^2=-p^2$ are written as [the unrenormalized version is given in Eq. (2.33) of Ref. [40]]

$$A_{gg,\rho\tau}(n)|_{\mu^2=-p^2} = \left\{\frac{\alpha_s}{4\pi}a_{gg}^{(1)}(n) + \left(\frac{\alpha_s}{4\pi}\right)^2 a_{gg}^{(2)}(n)\right\}T_{\rho\tau}^{(1)} + \left\{\frac{\alpha_s}{4\pi}b_{gg}^{(1)}(n) + \left(\frac{\alpha_s}{4\pi}\right)^2 b_{gg}^{(2)}(n)\right\}T_{\rho\tau}^{(2)} + \left\{\frac{\alpha_s}{4\pi}a_{gA}^{(1)}(n) + \left(\frac{\alpha_s}{4\pi}\right)^2 a_{gA}^{(2)}(n)\right\}T_{\rho\tau}^{(3)}. \tag{3.40}$$

The one-loop results $a_{gg}^{(1)}(n)$, $b_{gg}^{(1)}(n)$, and $a_{gA}^{(1)}(n)$ are all proportional to the color factor C_A and thus they are irrelevant to the photon matrix elements. Also, the two-loop result $a_{gA}^{(2)}(n)$ is made up of the terms proportional to C_A^2 or $C_A\frac{n_f}{2}$ and is irrelevant. The expressions of $a_{gg}^{(2)}(n)$ and $b_{gg}^{(2)}(n)$ are given by (C6) and (C7), respectively, in Appendix C. Then, we take the combination $\{a_{gg}^{(2)}(n) - \frac{1}{2}b_{gg}^{(2)}(n)\}$ and make replacements, $C_A\rightarrow 0$, $(\frac{n_f}{2})^2\rightarrow 0$, and $C_F\frac{n_f}{2}\rightarrow[C_F\times\text{charge factor}]$. Furthermore, we realize that the last two terms in parentheses of (C6) have also resulted from the external gluon self-energy corrections and are thus irrelevant for the photon case. In the end, we obtain for the photon matrix elements of the gluon operators

$$\tilde{A}_n^{(2)G} = 3n_f\langle e^2\rangle H_G^{(2)}(n), \tag{3.41}$$

where

$$H_G^{(2)}(n) = C_F\left\{(S_1(n)^2 + S_2(n))\left(\frac{16}{3(n-1)} + \frac{4}{n} - \frac{4}{n+1} - \frac{16}{3(n+2)} - \frac{8}{n^2} - \frac{8}{(n+1)^2}\right) + S_1(n)\left(-\frac{32}{9(n-1)} - \frac{32}{n} + \frac{32}{n+1} + \frac{32}{9(n+2)} + \frac{32}{n^2} + \frac{8}{(n+1)^2} - \frac{64}{3(n+2)^2} - \frac{32}{n^3} - \frac{48}{(n+1)^3}\right) + S_{-2}(n)\left(\frac{32}{3(n-1)} - \frac{32}{n} + \frac{32}{n+1} - \frac{32}{3(n+2)}\right) + \frac{872}{27(n-1)} - \frac{80}{n} + \frac{16}{n+1} + \frac{856}{27(n+2)} - \frac{40}{n^2} + \frac{104}{(n+1)^2} + \frac{64}{9(n+2)^2} + \frac{44}{n^3} + \frac{28}{(n+1)^3} - \frac{128}{3(n+2)^3} - \frac{40}{n^4} - \frac{88}{(n+1)^4}\right\}. \quad (3.42)$$

With all these necessary parameters at hand, we are now ready to analyze the moments of $F_2^\gamma(x, Q^2, P^2)$ up to the NNLO. First we evaluate the coefficients $\mathcal{L}_i^n$, $\mathcal{A}_i^n$, $\mathcal{B}_i^n$, $\mathcal{C}^n$, $\mathcal{D}_i^n$, $\mathcal{E}_i^n$, $\mathcal{F}_i^n$, and $\mathcal{G}^n$ with $i = +, -, NS$, the expressions of which are given in Eqs. (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), and (2.37), for $n = 2, 4, \ldots, 12$ in the cases of $n_f = 3$ and $n_f = 4$. The results are listed in Table II (for $n_f = 3$) and Table III (for $n_f = 4$). In Table 1 of Ref. [17], the numerical values of the seven NLO coefficients, $\mathcal{A}_i^n$, $\mathcal{B}_i^n$, and $\mathcal{C}^n$ with $i = +, -, NS$ for $n = 2, 4, \ldots, 20$ in the case of $n_f = 4$, were already given. Our results for $\mathcal{A}_i^n$, $\mathcal{B}_i^n$, and $\mathcal{C}^n$ in Table III are consistent with those in Ref. [17] except for the values of $\mathcal{A}_+^n$ and $\mathcal{A}_-^n$. The discrepancy in the values of $\mathcal{A}_+^n$ and $\mathcal{A}_-^n$ arises from the term -8 in the parentheses of Eq. (3.18). See the discussion below Eq. (3.18). The numerical calculation of the NNLO coefficients $\mathcal{D}_+^n$, $\mathcal{D}_-^n$, and $\mathcal{D}_{NS}^n$ for $n = 2, 4, \ldots, 12$ in Tables II and III was performed by using the "exact" values of the three-loop anomalous dimensions, $K_{NS}^{(2),n}$, $K_\psi^{(2),n}$, and $K_G^{(2),n}$, for $n = 2, \ldots, 12$ given in Ref. [10] and also by using the approximate expressions $K_{NS\,\mathrm{approx}}^{(2),n}$, $K_{\psi\,\mathrm{approx}}^{(2),n}$, and $K_{G\,\mathrm{approx}}^{(2),n}$ defined in Eqs. (3.24), (3.25), and (3.26) (in parentheses). The coefficients $\mathcal{A}_-^n$ and $\mathcal{E}_-^n$ cannot be evaluated at $n = 2$ since they become singular there. More details concerning this singularity will be discussed in the next section.

The coefficients $\mathcal{D}_-^n$ and $\mathcal{D}_{NS}^n$ in Table II take extremely large values at $n = 6$. The values of $\mathcal{D}_-^{n=6}$ and $\mathcal{D}_{NS}^{n=6}$ in Table III are also large. This is due to the fact that $\mathcal{D}_-^n$ and $\mathcal{D}_{NS}^n$ have terms with the factors $\frac{1}{1-d_-^n}$ and $\frac{1}{1-d_{NS}^n}$, respectively, and that d_-^n and d_{NS}^n happen to be very close to 1 at $n = 6$. Actually, we obtain $d_-^{n=6} = 0.995\,846$ and $d_{NS}^{n=6} = 1.000\,35$ for $n_f = 3$ (Table II), and $d_-^{n=6} = 1.074\,27$ and $d_{NS}^{n=6} = 1.080\,38$ for $n_f = 4$ (Table III). But we see from (2.29) that $\mathcal{D}_-^n$ and $\mathcal{D}_{NS}^n$ are multiplied, respectively, by the factors $[1 - (\frac{\alpha_s(Q^2)}{\alpha_s(P^2)})^{d_-^n - 1}]$ and $[1 - (\frac{\alpha_s(Q^2)}{\alpha_s(P^2)})^{d_{NS}^n - 1}]$ which become very small when d_-^n and d_{NS}^n are close to 1. Thus the contributions of the parts with $\mathcal{D}_-^{n=6}$ and $\mathcal{D}_{NS}^{n=6}$ to the 6th moment of $F_2^\gamma(x, Q^2, P^2)$ do not stand out from the others.

IV. SUM RULE OF $F_2^\gamma(x, Q^2, P^2)$

The sum rule of the structure function F_2^γ,

$$\int_0^1 dx F_2^\gamma(x, Q^2, P^2), \quad (4.1)$$

can be studied by taking the $n \to 2$ limit of Eq. (2.29). At $n = 2$ one of the eigenvalues of $\hat{\gamma}_{n=2}(g)$, the anomalous

TABLE II. Numerical values of $\mathcal{L}_i^n$, $\mathcal{A}_i^n$, $\mathcal{B}_i^n$, $\mathcal{D}_i^n$, $\mathcal{E}_i^n$, $\mathcal{F}_i^n (i = +, -, NS)$, and $\mathcal{C}^n$ and $\mathcal{G}^n$ for $n = 2, 4, \ldots, 12$ in the case of $n_f = 3$. The calculation of $\mathcal{D}_+^n$, $\mathcal{D}_-^n$, and $\mathcal{D}_{NS}^n$ was performed by using the exact values of $K_{NS}^{(2),n}$, $K_\psi^{(2),n}$, and $K_G^{(2),n}$ given in Ref. [10] and also by using the approximate expressions $K_{NS\,\mathrm{approx}}^{(2),n}$, $K_{\psi\,\mathrm{approx}}^{(2),n}$, and $K_{G\,\mathrm{approx}}^{(2),n}$ defined in Eqs. (3.24), (3.25), and (3.26) (in parentheses).

n	$\mathcal{L}_+^n$	$\mathcal{L}_-^n$	$\mathcal{L}_{NS}^n$	$\mathcal{A}_+^n$	$\mathcal{A}_-^n$	$\mathcal{A}_{NS}^n$	$\mathcal{B}_+^n$	$\mathcal{B}_-^n$	$\mathcal{B}_{NS}^n$	$\mathcal{C}^n$
2	0.4690	0.4267	0.4248	−2.8403	⋯	−5.5940	1.7481	−1.8535	0.8290	−9.3333
4	0.004 336	0.3639	0.1836	−0.5543	−2.6267	−1.3299	0.073 53	3.3149	1.4607	−10.7467
6	0.000 5428	0.2324	0.1164	0.061 33	−1.8806	−0.9403	0.016 52	2.9783	1.5349	−9.1088
8	0.000 149 3	0.1689	0.084 51	0.009 544	−1.6566	−0.8277	0.006 245	2.9612	1.4906	−7.7504
10	0.000 058 03	0.1318	0.065 91	0.002 817	−1.5336	−0.7664	0.002 993	2.8263	1.4169	−6.7116
12	0.000 027 48	0.1075	0.053 75	0.001 087	−1.4425	−0.7210	0.001 652	2.6744	1.3390	−5.9074

n	$\mathcal{D}_+^n$	$\mathcal{D}_-^n$	$\mathcal{D}_{NS}^n$	$\mathcal{E}_+^n$	$\mathcal{E}_-^n$	$\mathcal{E}_{NS}^n$	$\mathcal{F}_+^n$	$\mathcal{F}_-^n$	$\mathcal{F}_{NS}^n$	$\mathcal{G}^n$
2	60.5098 (60.5014)	32.9286 (32.9251)	63.1965 (63.1909)	−10.5867	⋯	−10.9168	6.9729	−13.7973	3.7817	−251.3619
4	7.9871 (7.9873)	25.9791 (25.9222)	11.6147 (11.5840)	−9.3990	−23.9288	−10.5807	1.3106	48.8620	20.6599	−204.5836
6	0.018 77 (0.018 77)	−4007.0415(−4011.3304)	24 025.6303 (24 050.8845)	1.8667	−24.0991	−12.4017	0.4596	56.4575	29.3881	−176.9466
8	0.032 22 (0.032 21)	165.7976 (165.9277)	82.2116 (82.2758)	0.3993	−29.0367	−14.5993	0.2217	67.5380	34.0579	−157.4181
10	−0.001 732(−0.001 738)	109.3285 (109.4090)	54.5447 (54.5847)	0.1453	−32.8877	−16.4753	0.1249	73.0111	36.6197	−142.6108
12	−0.018 25(−0.018 26)	86.8381 (86.9019)	43.3780 (43.4098)	0.06532	−35.8891	−17.9598	0.07764	75.9024	38.0024	−130.8717

TABLE III. Numerical values of $\mathcal{L}_i^n$, $\mathcal{A}_i^n$, $\mathcal{B}_i^n$, $\mathcal{D}_i^n$, $\mathcal{E}_i^n$, $\mathcal{F}_i^n (i = +, -, NS)$, and $\mathcal{C}^n$ and $\mathcal{G}^n$ for $n = 2, 4, \ldots, 12$ in the case of $n_f = 4$. The calculation of $\mathcal{D}_+^n$, $\mathcal{D}_-^n$, and $\mathcal{D}_{NS}^n$ was performed by using the exact values of $K_{NS}^{(2),n}$, $K_\psi^{(2),n}$, and $K_G^{(2),n}$ given in Ref. [10] and also by using the approximate expressions $K_{NS\,\mathrm{approx}}^{(2),n}$, $K_{\psi\,\mathrm{approx}}^{(2),n}$, and $K_{G\,\mathrm{approx}}^{(2),n}$ defined in Eqs. (3.24), (3.25), and (3.26) (in parentheses).

n	$\mathcal{L}_+^n$	$\mathcal{L}_-^n$	$\mathcal{L}_{NS}^n$	$\mathcal{A}_+^n$	$\mathcal{A}_-^n$	$\mathcal{A}_{NS}^n$	$\mathcal{B}_+^n$	$\mathcal{B}_-^n$	$\mathcal{B}_{NS}^n$	$\mathcal{C}^n$
2	0.8078	1.0582	0.6231	2.7608	···	−6.0944	3.8774	−8.5894	1.3076	−16.3237
4	0.009 356	0.7327	0.2661	5.1244	−3.7321	−1.3858	0.1688	0.4820	2.1599	−18.7956
6	0.001 235	0.4656	0.1679	0.095 29	−2.9038	−1.0480	0.039 09	6.0485	2.2395	−15.9311
8	0.000 346 5	0.3374	0.1215	0.019 53	−2.7046	−0.9735	0.014 95	5.9573	2.1613	−13.5552
10	0.000 136 2	0.2627	0.09461	0.006 354	−2.5904	−0.9322	0.007 216	5.6671	2.0468	−11.7384
12	0.000 064 97	0.2140	0.07704	0.002 598	−2.4906	−0.8963	0.003 999	5.3501	1.9293	−10.3319

n	$\mathcal{D}_+^n$	$\mathcal{D}_-^n$	$\mathcal{D}_{NS}^n$	$\mathcal{E}_+^n$	$\mathcal{E}_-^n$	$\mathcal{E}_{NS}^n$	$\mathcal{F}_+^n$	$\mathcal{F}_-^n$	$\mathcal{F}_{NS}^n$	$\mathcal{G}^n$
2	−84.4549(−84.4748)	64.6182 (64.6102)	63.5804 (63.5722)	13.2519	···	−12.7900	7.0067	−68.4928	0.8275	−439.6247
4	−17.3048(−17.3044)	−140.7574(−140.9078)	−64.7231(−64.7847)	92.4633	−2.4550	−11.2489	2.6666	−28.0786	24.5807	−357.8108
6	0.4163 (0.4163)	894.6070 (895.0955)	301.7867 (301.9492)	3.0154	−37.7241	−13.9828	1.0159	99.2476	37.1197	−309.4744
8	−0.047 47(−0.047 49)	326.5791 (326.7480)	116.7791 (116.8392)	0.8428	−47.7463	−17.3117	0.5046	121.0094	43.9510	−275.3197
10	−0.2306(−0.2306)	228.9242 (229.0460)	82.2373 (82.2809)	0.3366	−55.8735	−20.1683	0.2888	132.4677	47.7946	−249.4221
12	−0.7548(−0.7548)	183.5002 (183.6024)	65.9952 (66.0319)	0.1600	−62.2711	−22.4456	0.1813	139.2236	49.9533	−228.8909

dimension matrix in the hadronic sector given in (2.13), vanishes, due to the conservation of the energy-momentum tensor. Thus we have a zero eigenvalue, $\lambda_-^{n=2} = 0$, for the one-loop anomalous dimension matrix $\hat{\gamma}_{n=2}^{(0)}$ and, therefore, we get $d_-^{n=2} = \frac{\lambda_-^{n=2}}{2\beta_0} = 0$. Among the coefficients which appeared in (2.29), two of them, namely, $\mathcal{A}_-^n$ and $\mathcal{E}_-^n$, would develop singularities at $n = 2$, since those coefficients have terms with the factor $\frac{1}{d_-^n}$. However, as we see from (2.29), both $\mathcal{A}_-^n$ and $\mathcal{E}_-^n$ are multiplied by a factor $[1 - (\frac{\alpha_s(Q^2)}{\alpha_s(P^2)})^{d_-^n}]$ which also vanishes at $n = 2$. Provided that we regard the expression $\frac{1}{\epsilon}(1 - x^\epsilon)$ as its limiting value for $\epsilon \to 0$, $-\ln x$, then the $\mathcal{A}_-^n$ and $\mathcal{E}_-^n$ parts of (2.29) give finite contributions as

$$\lim_{n\to 2} \mathcal{A}_-^n \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)} \right)^{d_-^n} \right] = -\bar{\mathcal{A}}_-^{n=2} \ln \frac{\alpha_s(Q^2)}{\alpha_s(P^2)}, \quad (4.2)$$

$$\lim_{n\to 2} \mathcal{E}_-^n \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)} \right)^{d_-^n} \right] = -\bar{\mathcal{E}}_-^{n=2} \ln \frac{\alpha_s(Q^2)}{\alpha_s(P^2)}, \quad (4.3)$$

where

$$\bar{\mathcal{A}}_-^{n=2} = \left[-\boldsymbol{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_-^n}{\lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} - \boldsymbol{K}_n^{(0)} P_-^n \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} + \boldsymbol{K}_n^{(1)} P_-^n \boldsymbol{C}_{2,n}^{(0)} \right]_{n=2}, \quad (4.4)$$

$$\begin{aligned} \bar{\mathcal{E}}_-^{n=2} = \Bigg[& -\boldsymbol{K}_n^{(0)} P_-^n \boldsymbol{C}_{2,n}^{(1)} \frac{\beta_1}{\beta_0} - \boldsymbol{K}_n^{(0)} \sum_j \frac{P_j^n \hat{\gamma}_n^{(1)} P_-^n}{\lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(1)} \\ & + \boldsymbol{K}_n^{(1)} P_-^n \boldsymbol{C}_{2,n}^{(1)} - \boldsymbol{K}_n^{(0)} \sum_j \frac{P_-^n \hat{\gamma}_n^{(1)} P_j^n}{-\lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \frac{\beta_1}{\beta_0} \\ & - \boldsymbol{K}_n^{(0)} \sum_{j,k} \frac{P_j^n \hat{\gamma}_n^{(1)} P_-^n \hat{\gamma}_n^{(1)} P_k^n}{(-\lambda_k^n + 2\beta_0)(\lambda_j^n + 2\beta_0)} \boldsymbol{C}_{2,n}^{(0)} \\ & + \boldsymbol{K}_n^{(1)} \sum_j \frac{P_-^n \hat{\gamma}_n^{(1)} P_j^n}{-\lambda_j^n + 2\beta_0} \boldsymbol{C}_{2,n}^{(0)} \Bigg]_{n=2}. \end{aligned} \quad (4.5)$$

The coefficient functions, anomalous dimensions, and photon matrix elements at $n = 2$ are given in Appendix D. Using these values we obtain $\bar{\mathcal{A}}_-^{n=2} = -1.3274\times(-2.2857)$ and $\bar{\mathcal{E}}_-^{n=2} = 5.7664\,(18.553)$ for $n_f = 3\,(4)$. The numerical values of $\mathcal{L}_i^{n=2}$, $\mathcal{A}_i^{n=2}$, $\mathcal{B}_i^{n=2}$, $\mathcal{D}_i^{n=2}$, $\mathcal{E}_i^{n=2}$, $\mathcal{F}_i^{n=2}$ $(i = +, -, NS)$, and $\mathcal{C}^{n=2}$ and $\mathcal{G}^{n=2}$, except for $\mathcal{A}_-^{n=2}$ and $\mathcal{E}_-^{n=2}$, were already given in Table II (for $n_f = 3$) and Table III (for $n_f = 4$).

Let us express the sum rule in the following form:

$$\int_0^1 dx F_2^\gamma(x, Q^2, P^2) = \frac{\alpha}{4\pi} \frac{1}{2\beta_0} \left\{ \frac{4\pi}{\alpha_s(Q^2)} c_{\mathrm{LO}} + c_{\mathrm{NLO}} + \frac{\alpha_s(Q^2)}{4\pi} c_{\mathrm{NNLO}} + \mathcal{O}(\alpha_s^2) \right\}, \quad (4.6)$$

where the first, second, and third terms in the curly brackets correspond to the LO, NLO, and NNLO contributions, respectively. The coefficients c_{LO}, c_{NLO}, and c_{NNLO} depend on the number of the active quark flavors n_f, and also on $\alpha_s(Q^2)$ and $\alpha_s(P^2)$. For the QCD running coupling constant $\alpha_s(Q^2)$, we use the following formula which takes into account the β function parameters up to the three-loop level [41],

TABLE IV. The numerical values of the coefficients $c_{\rm LO}$, $c_{\rm NLO}$, and $c_{\rm NNLO}$ in Eq. (4.6), and the NLO and NNLO corrections relative to LO for the sum rule of $F_2^\gamma(x, Q^2, P^2)$ in several cases of Q^2 and P^2. The ratios of the NNLO to the sum of the LO and NLO contributions are also listed. For the QCD running coupling constant α_s, we have used the formula given in Eq. (4.7) with $\Lambda = 0.2$ GeV.

	Q^2(GeV^2)	P^2(GeV^2)	$c_{\rm LO}$	$c_{\rm NLO}$	$c_{\rm NNLO}$	LO	NLO	NNLO	NNLO/(LO + NLO)
$n_f = 3$	30	1	0.7631	−11.66	−331.2	1	−0.2063	−0.0791	−0.0997
	100	1	0.8613	−12.21	−355.3	1	−0.1649	−0.0558	−0.0668
	100	3	0.6690	−11.22	−313.8	1	−0.1949	−0.0634	−0.0787
$n_f = 4$	30	1	1.429	−18.90	−525.7	1	−0.1950	−0.0800	−0.0993
	100	1	1.614	−19.59	−551.4	1	−0.1541	−0.0551	−0.0651
	100	3	1.257	−18.38	−507.5	1	−0.1855	−0.0650	−0.0798

$$\frac{\alpha_s(Q^2)}{4\pi} = \frac{1}{\beta_0 L} - \frac{1}{(\beta_0 L)^2}\frac{\beta_1}{\beta_0}\ln L + \frac{1}{(\beta_0 L)^3}\left(\frac{\beta_1}{\beta_0}\right)^2 \times \left[\left(\ln L - \frac{1}{2}\right)^2 + \frac{\beta_0\beta_2}{\beta_1^2} - \frac{5}{4}\right] + \mathcal{O}\left(\frac{1}{L^4}\right), \quad (4.7)$$

where $L = \ln(Q^2/\Lambda^2)$, and β_0, β_1, and β_2 are given in Eqs. (3.2), (3.3), and (3.4). Taking $\Lambda = 0.2$ GeV, we get, for example, $\alpha_s(Q^2 = 100\ \mathrm{GeV}^2) = 0.1461\,(0.1595)$ and $\alpha_s(Q^2 = 3\ \mathrm{GeV}^2) = 0.2487\,(0.2717)$ for the case $n_f = 3\,(4)$.

We list in Table IV the numerical values of the coefficients $c_{\rm LO}$, $c_{\rm NLO}$, and $c_{\rm NNLO}$ for the cases $n_f = 3$ and 4. We have studied three cases: $(Q^2, P^2) = (30\ \mathrm{GeV}^2, 1\ \mathrm{GeV}^2)$, $(100\ \mathrm{GeV}^2, 1\ \mathrm{GeV}^2)$, and $(100\ \mathrm{GeV}^2, 3\ \mathrm{GeV}^2)$. We already know that $c_{\rm NLO}$ takes negative values [17]. We find that the coefficient $c_{\rm NNLO}$ also takes negative values which are rather large in magnitude compared with those of $c_{\rm LO}$ and $c_{\rm NLO}$. Also listed in Table IV are the NLO ($\alpha\alpha_s$) and NNLO ($\alpha\alpha_s^2$) corrections relative to LO (α) and the ratios of the NNLO to the sum of the LO and NLO contributions for the sum rule of $F_2^\gamma(x, Q^2, P^2)$. We see that the NNLO corrections give negative contributions to the sum rule. In fact, we will see in the next section that the NNLO corrections reduce $F_2^\gamma(x, Q^2, P^2)$ at larger x. For the kinematical region of Q^2 and P^2 which we have studied, the NNLO corrections are found to be rather large. When $P^2 = 1\ \mathrm{GeV}^2$ and $Q^2 = 30 \sim 100\ \mathrm{GeV}^2$ or $P^2 = 3\ \mathrm{GeV}^2$ and $Q^2 = 100\ \mathrm{GeV}^2$, and n_f is 3 or 4, the NNLO corrections are 7%–10% of the sum of the LO and NLO contributions.

V. NUMERICAL ANALYSIS OF $F_2^\gamma(x, Q^2, P^2)$

We now perform the inverse Mellin transform of (2.29) to obtain F_2^γ as a function of x. The nth moment is denoted as

$$M_2^\gamma(n, Q^2, P^2) = \int_0^1 dx x^{n-1}\frac{F_2^\gamma(x, Q^2, P^2)}{x}. \quad (5.1)$$

Then by inverting the moments (5.1) we get

$$\frac{F_2^\gamma(x, Q^2, P^2)}{x} = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} dn x^{-n} M_2^\gamma(n, Q^2, P^2), \quad (5.2)$$

where the integration contour runs to the right of all singularities of $M_2^\gamma(n, Q^2, P^2)$ in the complex n plane. In order to have better convergence of the numerical integration, we change the contour from the vertical line connecting $C - i\infty$ with $C + i\infty$ (C is an appropriate positive constant), introducing a small positive constant ε, to

$$n = C - \varepsilon|y| + iy, \qquad -\infty < y < \infty. \quad (5.3)$$

Hence we have

$$\frac{F_2^\gamma(x, Q^2, P^2)}{x} = \frac{1}{\pi}\int_0^\infty [\mathrm{Re}\{M_2^\gamma(z, Q^2, P^2)e^{-z\ln(x)}\} - \varepsilon\,\mathrm{Im}\{M_2^\gamma(z, Q^2, P^2)e^{-z\ln(x)}\}]dy, \quad (5.4)$$

where $z = C - \varepsilon y + iy$.

As we see from Eqs. (2.29), (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), and (2.37), the nth moment $M_2^\gamma(n, Q^2, P^2)$ is written in terms of coefficient functions, anomalous dimensions, and photon matrix elements, which in turn are expressed by the rational functions of integer n and also by the various harmonic sums [42]. Thus we need to make an analytic continuation of these harmonic sums from integer n to complex n. There are several proposals for this continuation [43,44]. The method we adopted here is to use the asymptotic expansions of the harmonic sums and their translation relations. The details are explained in Appendix E.

In Fig. 3 we plot the virtual photon structure function $F_2^\gamma(x, Q^2, P^2)$ predicted by pQCD for the case of $n_f = 4$, $Q^2 = 30\ \mathrm{GeV}^2$, and $P^2 = 1\ \mathrm{GeV}^2$ with the QCD scale parameter $\Lambda = 0.2$ GeV. The vertical axis corresponds to

$$F_2^\gamma(x, Q^2, P^2)/\frac{3\alpha}{\pi}n_f\langle e^4\rangle\ln\frac{Q^2}{P^2}. \quad (5.5)$$

Here we show four curves: the LO, NLO, and NNLO QCD results and the box (tree) diagram contribution including nonleading corrections. The box contribution is expressed by [17]

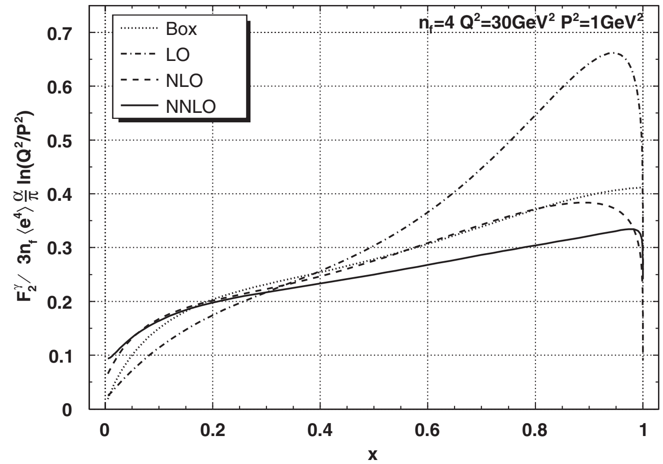

FIG. 3. Virtual photon structure function $F_2^\gamma(x, Q^2, P^2)$ in units of $(3\alpha n_f\langle e^4\rangle/\pi)\ln(Q^2/P^2)$ for $Q^2 = 30\ \mathrm{GeV}^2$ and $P^2 = 1\ \mathrm{GeV}^2$ with $n_f = 4$ and the QCD scale parameter $\Lambda = 0.2$ GeV. We plot the box (tree) diagram contribution including nonleading corrections (short-dashed line), the QCD LO (dash-dotted line), the NLO (long-dashed line), and the NNLO (solid line) results.

$$F_2^{\gamma(\mathrm{box})}(x, Q^2, P^2) = \frac{3\alpha}{\pi} n_f\langle e^4\rangle\bigg\{x[x^2 + (1-x)^2]\ln\frac{Q^2}{P^2} - 2x[1 - 3x + 3x^2 + (1 - 2x + 2x^2)\ln x]\bigg\}, \tag{5.6}$$

where power corrections P^2/Q^2 and quark mass effects are ignored. It is noted that, in these analyses, even for the LO and NLO QCD curves, we have used the QCD running coupling constant $\alpha_s(Q^2)$ which is valid up to the three-loop level and is governed by the formula (4.7), and we have put $\Lambda = 0.2$ GeV.

The LO and NLO QCD results with the same values of n_f, Q^2, and P^2, as well as the box contribution, were already given in Fig. 6 of Ref. [17]. But in Ref. [17] the formula for $\alpha_s(Q^2)$ which is valid in the one-loop level was used to obtain the LO curve, while the two-loop-level formula for $\alpha_s(Q^2)$ was applied for the NLO graph, and the QCD scale parameter Λ was set to be 0.1 GeV in both cases. The LO result in Fig. 3 has a similar shape as the corresponding one in Ref. [17] but is different in magnitude; the former is slightly larger than the latter for almost the whole x region. This is due to the fact that the one-loop-level formula for $\alpha_s(Q^2)$ was used for the LO curve in Ref. [17], while we applied the three-loop-level formula even for the LO result. On the other hand, the NLO curve in Fig. 3 is similar to the corresponding one in Ref. [17] in shape and magnitude.

Now we observe in Fig. 3 that there exist notable NNLO QCD corrections at larger x. The corrections are negative and the NNLO curve comes below the NLO one in the region $0.3 \lesssim x < 1$. This is expected from the $n = 2$ moment analysis in Sec. IV. From Table IV we see that the ratio of the NNLO to the sum of the LO and NLO contributions for the sum rule of $F_2^\gamma(x, Q^2, P^2)$ is -0.099 for the case of $n_f = 4$, $Q^2 = 30\ \mathrm{GeV}^2$, and $P^2 = 1\ \mathrm{GeV}^2$. At the lower x region, $0.05 \lesssim x \lesssim 0.3$, the NNLO corrections to the NLO results are found to be negligibly small.

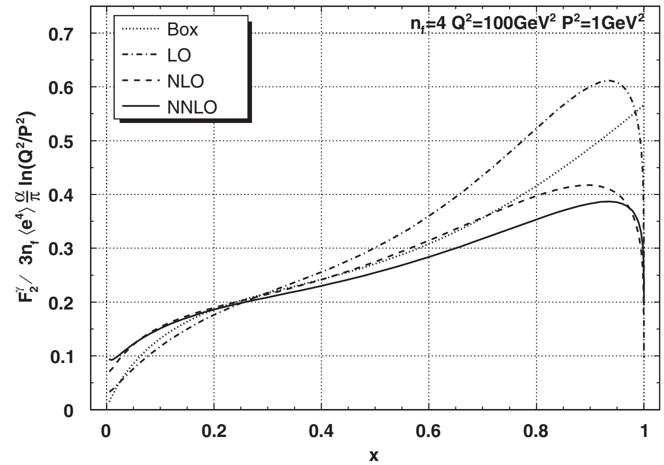

FIG. 4. Virtual photon structure function $F_2^\gamma(x, Q^2, P^2)$ for $Q^2 = 100\ \mathrm{GeV}^2$ and $P^2 = 1\ \mathrm{GeV}^2$ with $n_f = 4$ and $\Lambda = 0.2$ GeV.

We have also studied the QCD corrections to $F_2^\gamma(x, Q^2, P^2)$ with different Q^2 and P^2 but with $n_f = 4$. In Fig. 4 we plot the case for $Q^2 = 100\ \mathrm{GeV}^2$ and $P^2 = 1\ \mathrm{GeV}^2$. Another case for $Q^2 = 100\ \mathrm{GeV}^2$ and $P^2 = 3\ \mathrm{GeV}^2$ is shown in Fig. 5. We have not seen any sizable change for the normalized structure function (5.5) for these different values of Q^2 and P^2. In both cases the NNLO corrections reduce $F_2^\gamma(x, Q^2, P^2)$ at larger x. We have examined the $n_f = 3$ case as well. It is observed that the normalized structure function (5.5) is insensitive to the number of active flavors.

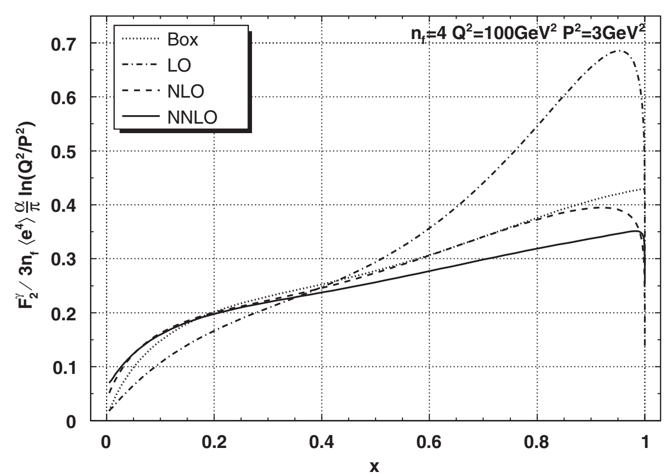

FIG. 5. Virtual photon structure function $F_2^\gamma(x, Q^2, P^2)$ for $Q^2 = 100\ \mathrm{GeV}^2$ and $P^2 = 3\ \mathrm{GeV}^2$ with $n_f = 4$ and $\Lambda = 0.2$ GeV.

Finally we should note that the spin-averaged structure function directly accessible in the experiment is not F_2^γ but rather the so-called effective structure function $F_{\rm eff}^\gamma \simeq F_2^\gamma + (3/2)F_L^\gamma$ as discussed in Refs. [15,21,45].

VI. LONGITUDINAL STRUCTURE FUNCTION $F_L^\gamma(x, Q^2, P^2)$

We have considered the structure function F_2^γ so far. Regarding another structure function F_L^γ, its LO contribution, which is of order α, was calculated in QCD for the real photon ($P^2 = 0$) target in Refs. [5,6]. The analysis was extended to the case of the virtual photon ($\Lambda^2 \ll P^2 \ll Q^2$) target [17]. We will now derive a formula for the moment sum rule of $F_L^\gamma(x, Q^2, P^2)$ up to the NLO ($\mathcal{O}(\alpha\alpha_s)$) corrections. Comparing Eq. (2.8b) with (2.8a) and examining the form of (2.11), we see that the formula for $F_L^\gamma(x, Q^2, P^2)$ is obtained from (2.18) only by replacing $\boldsymbol{C}_{2,n}(1, \bar{g}(Q^2))$ and $C_{2,n}^\gamma(1, \bar{g}(Q^2), \alpha)$ with $\boldsymbol{C}_{L,n}(1, \bar{g}(Q^2))$ and $C_{L,n}^\gamma(1, \bar{g}(Q^2), \alpha)$, respectively. An expansion is made for $\boldsymbol{C}_{L,n}(1, \bar{g}(Q^2))$ and $C_{L,n}^\gamma(1, \bar{g}(Q^2), \alpha)$ up to the two-loop level as

$$\boldsymbol{C}_{L,n}(1, \bar{g}(Q^2)) = \boldsymbol{C}_{L,n}^{(0)} + \frac{\bar{g}^2(Q^2)}{16\pi^2}\boldsymbol{C}_{L,n}^{(1)} + \frac{\bar{g}^4(Q^2)}{(16\pi^2)^2}\boldsymbol{C}_{L,n}^{(2)} + \cdots, \tag{6.1}$$

$$C_{L,n}^\gamma(1, \bar{g}(Q^2), \alpha) = \frac{e^2}{16\pi^2}C_{L,n}^{\gamma(1)} + \frac{e^2\bar{g}^2(Q^2)}{(16\pi^2)^2}C_{L,n}^{\gamma(2)} + \cdots. \tag{6.2}$$

Here we note that there is no contribution of the tree diagrams to the longitudinal coefficient functions and thus we have $\boldsymbol{C}_{L,n}^{(0)} = \boldsymbol{0}$.

The moments of $F_L^\gamma(x, Q^2, P^2)$ are then given as follows [see Eqs. (2.29), (2.30), (2.31), (2.32), (2.33), (2.34), (2.35), (2.36), and (2.37) for comparison]:

$$\int_0^1 dx x^{n-2}F_L^\gamma(x, Q^2, P^2) = \frac{\alpha}{4\pi}\frac{1}{2\beta_0}\left\{\sum_i \mathcal{B}_{(L),i}^n\left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n+1}\right] + \mathcal{C}_{(L)}^n + \frac{\alpha_s(Q^2)}{4\pi}\left(\sum_i \mathcal{E}_{(L),i}^n\left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n}\right] + \sum_i \mathcal{F}_{(L),i}^n\left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(P^2)}\right)^{d_i^n+1}\right] + \mathcal{G}_{(L)}^n\right) + \mathcal{O}(\alpha_s^2)\right\}, \quad \text{with} \quad i = +, -, NS, \tag{6.3}$$

where the coefficients $\mathcal{B}_{(L),i}^n$, $\mathcal{C}_{(L)}^n$, $\mathcal{E}_{(L),i}^n$, $\mathcal{F}_{(L),i}^n$, and $\mathcal{G}_{(L)}^n$ are

$$\mathcal{B}_{(L),i}^n = \boldsymbol{K}_n^{(0)}P_i^n\boldsymbol{C}_{L,n}^{(1)}\frac{1}{1+d_i^n}, \tag{6.4}$$

$$\mathcal{C}_{(L)}^n = 2\beta_0 C_{L,n}^{\gamma(1)}, \tag{6.5}$$

$$\mathcal{E}_{(L),i}^n = -\boldsymbol{K}_n^{(0)}P_i^n\boldsymbol{C}_{L,n}^{(1)}\frac{\beta_1}{\beta_0}\frac{1-d_i^n}{d_i^n} - \boldsymbol{K}_n^{(0)}\sum_j\frac{P_j^n\hat{\gamma}_n^{(1)}P_i^n}{\lambda_j^n - \lambda_i^n + 2\beta_0}\boldsymbol{C}_{L,n}^{(1)}\frac{1}{d_i^n} + \boldsymbol{K}_n^{(1)}P_i^n\boldsymbol{C}_{L,n}^{(1)}\frac{1}{d_i^n} - 2\beta_0\tilde{\boldsymbol{A}}_n^{(1)}P_i^n\boldsymbol{C}_{L,n}^{(1)}, \tag{6.6}$$

$$\mathcal{F}_{(L),i}^n = \boldsymbol{K}_n^{(0)}P_i^n\boldsymbol{C}_{L,n}^{(2)}\frac{1}{1+d_i^n} - \boldsymbol{K}_n^{(0)}P_i^n\boldsymbol{C}_{L,n}^{(1)}\frac{\beta_1}{\beta_0}\frac{d_i^n}{1+d_i^n} + \boldsymbol{K}_n^{(0)}\sum_j\frac{P_i^n\hat{\gamma}_n^{(1)}P_j^n}{\lambda_i^n - \lambda_j^n + 2\beta_0}\boldsymbol{C}_{L,n}^{(1)}\frac{1}{1+d_i^n}, \tag{6.7}$$

$$\mathcal{G}_{(L)}^n = 2\beta_0(C_{L,n}^{\gamma(2)} + \tilde{\boldsymbol{A}}_n^{(1)}\cdot\boldsymbol{C}_{L,n}^{(1)}), \tag{6.8}$$

with $i, j = +, -, NS$. The coefficients $\mathcal{B}_{(L),i}^n$ and $\mathcal{C}_{(L)}^n$ represent the LO terms [5,6,17], while the terms with $\mathcal{E}_{(L),i}^n$, $\mathcal{F}_{(L),i}^n$, and $\mathcal{G}_{(L)}^n$ are the NLO ($\alpha\alpha_s$) corrections and they are new. It is noted that, among these coefficients, $\mathcal{E}_{(L),-}^n$ becomes singular at $n = 2$ since it has terms with the factor $\frac{1}{d_-^n}$ and d_-^n vanishes as $n \to 2$. But again, as in the case of the moments of $F_2^\gamma(x, Q^2, P^2)$, this coefficient is multiplied by a factor $[1 - (\alpha_s(Q^2)/\alpha_s(P^2))^{d_-^n}]$, and thus the product remains finite at $n = 2$.

The one-loop longitudinal coefficient functions are well known [28,46,47]. They are written as

$$C_{L,n}^{\psi(1)} = \delta_\psi B_{\psi,L}^n, \qquad C_{L,n}^{G(1)} = \delta_\psi B_{G,L}^n, \qquad C_{L,n}^{NS(1)} = \delta_{NS}B_{NS,L}^n, \qquad C_{L,n}^{\gamma(1)} = \delta_\gamma B_{\gamma,L}^n, \tag{6.9}$$

where $B_{\psi,L}^n = B_{NS,L}^n$, $B_{G,L}^n$, and $B_{\gamma,L}^n$ are given, for example, in Eqs. (6.2)–(6.4) of Ref. [6]. The two-loop longitudinal coefficient functions corresponding to the hadronic operators were calculated in the $\overline{\rm MS}$ scheme in Refs. [31,32].[1] The results in Mellin space as functions of n are found, for example, in Ref. [33]:

$$C_{L,n}^{\psi(2)} = \delta_\psi\{c_{L,q}^{(2),{\rm ns}}(n) + c_{L,q}^{(2),{\rm ps}}(n)\}, \tag{6.10}$$

$$C_{L,n}^{G(2)} = \delta_\psi c_{L,g}^{(2)}(n), \tag{6.11}$$

[1]The earlier calculations [48–51] were found to be partly incorrect. For quark coefficient functions $c_{L,q}^{(2),{\rm ns}}$ and $c_{L,q}^{(2),{\rm ps}}$ in Eq. (6.10), there is a complete agreement between Ref. [51] and Refs. [31,32,52], while for gluon coefficient $c_{L,g}^{(2)}$ in Eq. (6.11) the result of Ref. [49] was corrected in Ref. [53].

$$C_{L,n}^{NS(2)} = \delta_{NS} c_{L,q}^{(2),\rm ns}(n), \tag{6.12}$$

where $c_{L,q}^{(2),\rm ns}(n)$, $c_{L,q}^{(2),\rm ps}(n)$, and $c_{L,g}^{(2)}(n)$ are given in Eqs. (203), (204), and (205) in Appendix B of Ref. [33], respectively, with N being replaced by n. The two-loop photon longitudinal coefficient function $C_{L,n}^{\gamma(2)}$ is expressed as

$$C_{L,n}^{\gamma(2)} = \delta_\gamma c_{L,\gamma}^{(2)}(n), \tag{6.13}$$

and $c_{L,\gamma}^{(2)}(n)$ is obtained from $c_{L,g}^{(2)}(n)$ in (6.11) by replacing $C_A \to 0$ and $\frac{n_f}{2} \to 1$.

Inverting the moments (6.3), we plot in Fig. 6 the longitudinal virtual photon structure function $F_L^\gamma(x, Q^2, P^2)$ predicted by pQCD for the case of $n_f = 4$, $Q^2 = 30\ \mathrm{GeV}^2$, and $P^2 = 1\ \mathrm{GeV}^2$ with the QCD scale parameter $\Lambda = 0.2$ GeV. The vertical axis is in units of $F_L^\gamma(x, Q^2, P^2)/\frac{3\alpha}{\pi} n_f\langle e^4\rangle$. Here we show three curves: the LO and NLO QCD results and the box (tree) diagram contribution, which is expressed by

$$F_L^{\gamma(\rm box)}(x, Q^2, P^2) = \frac{3\alpha}{\pi} n_f\langle e^4\rangle\{4x^2(1-x)\}. \tag{6.14}$$

The LO result in Fig. 6 is consistent with the corresponding one in Fig. 5 of Ref. [17], although the formulas used for $\alpha_s(Q^2)$ differ in detail. We see from Fig. 6 that the NLO QCD corrections are negative and the NLO curve comes below the LO one in the region $0.2 \lesssim x < 1$.

The QCD corrections to $F_L^\gamma(x, Q^2, P^2)$ for different values of Q^2, P^2, and n_f are also studied. The case for $Q^2 = 100\ \mathrm{GeV}^2$ and $P^2 = 1\ \mathrm{GeV}^2$ with $n_f = 4$ is shown in Fig. 7. The LO curve has hardly changed from the one for $Q^2 = 30\ \mathrm{GeV}^2$ and $P^2 = 1\ \mathrm{GeV}^2$. The NLO corrections get smaller. The LO and NLO QCD curves for $Q^2 = 100\ \mathrm{GeV}^2$ and $P^2 = 3\ \mathrm{GeV}^2$ with $n_f = 4$ appear to be almost the same as those in the case of $Q^2 = 30\ \mathrm{GeV}^2$ and $P^2 = 1$. The cases for $n_f = 3$ are examined as well and we find that the normalized function $F_L^\gamma(x, Q^2, P^2)/\frac{3\alpha}{\pi} n_f\langle e^4\rangle$ is insensitive to the number of active flavors.

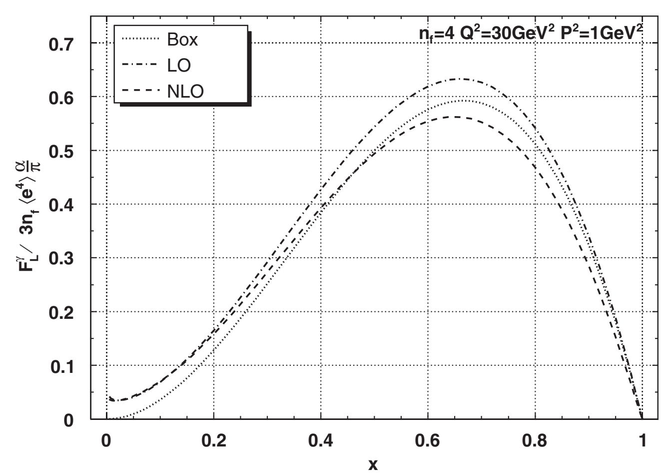

FIG. 6. Longitudinal photon structure function $F_L^\gamma(x, Q^2, P^2)$ in units of $(3\alpha n_f\langle e^4\rangle/\pi)$ for $Q^2 = 30\ \mathrm{GeV}^2$ and $P^2 = 1\ \mathrm{GeV}^2$ with $n_f = 4$ and the QCD scale parameter $\Lambda = 0.2$ GeV. We plot the box (tree) diagram contribution (short-dashed line), the QCD LO (dash-dotted line), and the NLO (long-dashed line) results.

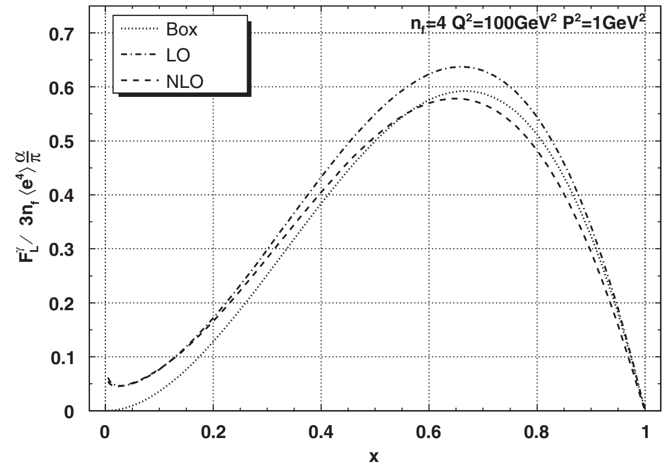

FIG. 7. Longitudinal photon structure function $F_L^\gamma(x, Q^2, P^2)$ for $Q^2 = 100\ \mathrm{GeV}^2$ and $P^2 = 1\ \mathrm{GeV}^2$ with $n_f = 4$ and $\Lambda = 0.2$ GeV.

VII. CONCLUSIONS

We have investigated the unpolarized virtual photon structure functions $F_2^\gamma(x, Q^2, P^2)$ and $F_L^\gamma(x, Q^2, P^2)$ for the kinematical region $\Lambda^2 \ll P^2 \ll Q^2$ in QCD. In the framework of the OPE supplemented by the RG method, we gave the definite predictions for the moments of $F_2^\gamma(x, Q^2, P^2)$ up to the NNLO (the order $\alpha\alpha_s$) and for the moments of $F_L^\gamma(x, Q^2, P^2)$ up to the NLO (the order $\alpha\alpha_s$). In the course of our evaluation, we utilized the recently calculated results of the three-loop anomalous dimensions for the quark and gluon operators. Also we derived the photon matrix elements of hadronic operators up to the two-loop level.

The sum rule of $F_2^\gamma(x, Q^2, P^2)$, i.e., the second moment, was numerically examined. The NNLO corrections are found to be 7%–10% of the sum of the LO and NLO contributions, when $P^2 = 1\ \mathrm{GeV}^2$ and $Q^2 = 30 \sim 100\ \mathrm{GeV}^2$ or $P^2 = 3\ \mathrm{GeV}^2$ and $Q^2 = 100\ \mathrm{GeV}^2$, and n_f is 3 or 4.

The inverse Mellin transform of the moments was performed to express the structure functions $F_2^\gamma(x, Q^2, P^2)$ and $F_L^\gamma(x, Q^2, P^2)$ as functions of x. We found that there exist sizable NNLO contributions for F_2^γ at larger x. The corrections are negative and the NNLO curve comes below the NLO one in the region $0.3 \lesssim x < 1$. At the lower x region, $0.05 \lesssim x \lesssim 0.3$, the NNLO corrections to the NLO results are found to be negligibly small. Concerning F_L^γ, the NLO corrections reduce the magnitude in the region $0.2 \lesssim x < 1$.

The comparison of the present NNLO theoretical prediction for the virtual photon structure functions with the existing experimental data will be discussed elsewhere.

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APPENDIX A: EVALUATION OF $M_n(Q^2/P^2, \bar{g}(P^2))$ AND $X_n(Q^2/P^2, \bar{g}(P^2), \alpha)$

In order to evaluate the integrals for $M_n(Q^2/P^2, \bar{g}(P^2))$ given in (2.22), we employ the same method that was used by Bardeen and Buras in Ref. [6] and make full use of the projection operators obtained from the one-loop anomalous dimension matrix $\hat{\gamma}_n^0$:

$$\hat{\gamma}_n^{(0)} = \sum_{i=+,-,NS} \lambda_i^n P_i^n, \tag{A1}$$

where $\lambda_i^n (i = +, -, NS)$ are eigenvalues of $\hat{\gamma}_n^0$ and are expressed as

$$\lambda_\pm^n = \frac{1}{2}\{\gamma_{\psi\psi}^{(0),n} + \gamma_{GG}^{(0),n} \pm [(\gamma_{\psi\psi}^{(0),n} - \gamma_{GG}^{(0),n})^2 + 4\gamma_{\psi G}^{(0),n}\gamma_{G\psi}^{(0),n}]^{1/2}\}, \tag{A2}$$

$$\lambda_{NS}^n = \gamma_{NS}^{(0),n}, \tag{A3}$$

and P_i^n are the corresponding projection operators,

$$P_\pm^n = \frac{1}{\lambda_\pm^n - \lambda_\mp^n}\begin{pmatrix} \gamma_{\psi\psi}^{(0),n} - \lambda_\mp^n & \gamma_{G\psi}^{(0),n} & 0 \\ \gamma_{\psi G}^{(0),n} & \gamma_{GG}^{(0),n} - \lambda_\mp^n & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A4}$$

$$P_{NS}^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A5}$$

With an expansion of $\beta(g)$ up to the three-loop level in (2.24), we get its inverse as follows:

$$\frac{1}{\beta(g)} = -\frac{16\pi^2}{\beta_0}\frac{1}{g^3}\left\{1 - \frac{g^2}{16\pi^2}\frac{\beta_1}{\beta_0} + \frac{g^4}{(16\pi^2)^2}\left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0}\right) + \cdots\right\}. \tag{A6}$$

Then, using (A1) and (A6), we perform integration in (2.22).

The result is

$$\begin{aligned} M_n(Q^2/P^2, \bar{g}(P^2)) &= \sum_i P_i^n\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_i^n} + \frac{1}{16\pi^2}\sum_i P_i^n\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_i^n} d_i^n\frac{\beta_1}{\beta_0}(\bar{g}_1^2 - \bar{g}_2^2) - \frac{1}{16\pi^2}\sum_{i,j}P_i^n\hat{\gamma}_n^{(1)}P_j^n\frac{1}{\lambda_i^n - \lambda_j^n + 2\beta_0}\left[\bar{g}_1^2\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_j^n}\right. \\ &\left.- \bar{g}_2^2\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_i^n}\right] + \frac{1}{(16\pi^2)^2}\sum_i P_i^n\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_i^n}\frac{1}{2}\left\{-d_i^n\left[\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0}\right](\bar{g}_1^4 - \bar{g}_2^4) + \left(d_i^n\frac{\beta_1}{\beta_0}\right)^2(\bar{g}_1^2 - \bar{g}_2^2)^2\right\} \\ &- \frac{1}{(16\pi^2)^2}\sum_{i,j}P_i^n\hat{\gamma}_n^{(1)}P_j^n\frac{1}{\lambda_i^n - \lambda_j^n + 2\beta_0}\frac{\beta_1}{\beta_0}(d_i^n\bar{g}_1^2 - d_j^n\bar{g}_2^2)\left[\bar{g}_1^2\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_j^n} - \bar{g}_2^2\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_i^n}\right] \\ &+ \frac{1}{(16\pi^2)^2}\sum_{i,j}P_i^n\hat{\gamma}_n^{(1)}P_j^n\frac{(1 + d_i^n - d_j^n)}{\lambda_i^n - \lambda_j^n + 4\beta_0}\frac{\beta_1}{\beta_0}\left[(\bar{g}_1^2)^2\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_j^n} - (\bar{g}_2^2)^2\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_i^n}\right] \\ &- \frac{1}{(16\pi^2)^2}\sum_{i,j}\frac{P_i^n\hat{\gamma}_n^{(2)}P_j^n}{\lambda_i^n - \lambda_j^n + 4\beta_0}\left[(\bar{g}_1^2)^2\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_j^n} - (\bar{g}_2^2)^2\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_i^n}\right] + \frac{1}{(16\pi^2)^2}\sum_{i,j,k}P_i^n\hat{\gamma}_n^{(1)}P_j^n\hat{\gamma}_n^{(1)}P_k^n \\ &\times\frac{1}{(\lambda_j^n - \lambda_k^n + 2\beta_0)}\left[\frac{1}{\lambda_i^n - \lambda_k^n + 4\beta_0}\left\{(\bar{g}_1^2)^2\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_k^n} - (\bar{g}_2^2)^2\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_i^n}\right\}\right. \\ &\left.- \frac{1}{\lambda_i^n - \lambda_j^n + 2\beta_0}\left\{\bar{g}_1^2\bar{g}_2^2\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_j^n} - (\bar{g}_2^2)^2\left(\frac{\bar{g}_2^2}{\bar{g}_1^2}\right)^{d_i^n}\right\}\right], \end{aligned} \tag{A7}$$

where $\bar{g}_1^2 = \bar{g}^2(P^2)$ and $\bar{g}_2^2 = \bar{g}^2(Q^2)$. The first term is the leading, and the second and third terms are the next-to-leading terms. The rest are the next-to-next-to-leading terms.

Once we get the above expression for $M_n(Q^2/P^2, \bar{g}(P^2))$ expanded up to the NNLO, we use an expansion of $\boldsymbol{K}_n(g, \alpha)$ in (2.25) up to the three-loop level and we can evaluate $X_n(Q^2/P^2, \bar{g}(P^2), \alpha)$ in (2.17) up to the NNLO. The result is

$$
\begin{aligned}
X_n(Q^2/P^2,\bar g(P^2),\alpha) &= \frac{e^2}{2\beta_0}\frac{1}{\bar g_2^2}K_n^{(0)}\sum_i P_i^n\frac{1}{d_i^n+1}\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n+1}\right]+\frac{e^2}{2\beta_0}\frac{1}{16\pi^2}K_n^{(0)}\frac{\beta_1}{\beta_0}\sum_i P_i^n\left\{-\frac{d_i^n}{d_i^n+1}\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n+1}\right]\right.\\
&\left.+\frac{d_i^n-1}{d_i^n}\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n}\right]\right\}+\frac{e^2}{2\beta_0}\frac{1}{16\pi^2}K_n^{(0)}\sum_i\left\{\sum_j\frac{P_i^n\hat\gamma_n^{(1)}P_j^n}{\lambda_i^n-\lambda_j^n+2\beta_0}\frac{1}{d_i^n+1}\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n+1}\right]\right.\\
&\left.-\sum_j\frac{P_j^n\hat\gamma_n^{(1)}P_i^n}{\lambda_j^n-\lambda_i^n+2\beta_0}\frac{1}{d_i^n}\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n}\right]\right\}+\frac{e^2}{2\beta_0}\frac{1}{16\pi^2}K_n^{(1)}\sum_i P_i^n\frac{1}{d_i^n}\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n}\right]\\
&+\frac{e^2}{2\beta_0}\frac{\bar g_2^2}{(16\pi^2)^2}K_n^{(0)}\sum_i P_i^n\left\{\frac{d_i^n}{2}\left(\frac{\beta_1^2}{\beta_0^2}-\frac{\beta_2}{\beta_0}\frac{1}{d_i^n+1}\right)\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n+1}\right]+\frac{\beta_1^2}{\beta_0^2}(1-d_i^n)\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n}\right]\right.\\
&\left.+\left(\frac{d_i^n}{2}-1\right)\left(\frac{\beta_1^2}{\beta_0^2}+\frac{\beta_2}{\beta_0}\frac{1}{d_i^n-1}\right)\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n-1}\right]\right\}+\frac{e^2}{2\beta_0}\frac{\bar g_2^2}{(16\pi^2)^2}K_n^{(0)}\frac{\beta_1}{\beta_0}\\
&\times\sum_i\left\{\left(-\sum_j\frac{P_i^n\hat\gamma_n^{(1)}P_j^n}{\lambda_i^n-\lambda_j^n+2\beta_0}\frac{d_j^n}{d_i^n+1}-\sum_j\frac{P_i^n\hat\gamma_n^{(1)}P_j^n}{\lambda_i^n-\lambda_j^n+4\beta_0}\frac{1+d_i^n-d_j^n}{d_i^n+1}\right)\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n+1}\right]\right.\\
&+\left(\sum_j\frac{P_i^n\hat\gamma_n^{(1)}P_j^n}{\lambda_i^n-\lambda_j^n+2\beta_0}\left(1-\frac{1}{d_i^n}\right)+\sum_j\frac{P_j^n\hat\gamma_n^{(1)}P_i^n}{\lambda_j^n-\lambda_i^n+2\beta_0}\right)\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n}\right]+\left(\sum_j\frac{P_j^n\hat\gamma_n^{(1)}P_i^n}{\lambda_j^n-\lambda_i^n+2\beta_0}\frac{1-d_j^n}{d_i^n-1}\right.\\
&\left.\left.+\sum_j\frac{P_j^n\hat\gamma_n^{(1)}P_i^n}{\lambda_j^n-\lambda_i^n+4\beta_0}\frac{1+d_j^n-d_i^n}{d_i^n-1}\right)\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n-1}\right]\right\}+\frac{e^2}{2\beta_0}\frac{\bar g_2^2}{(16\pi^2)^2}K_n^{(0)}\\
&\times\sum_i\left\{\sum_j\frac{P_i^n\hat\gamma_n^{(2)}P_j^n}{\lambda_i^n-\lambda_j^n+4\beta_0}\frac{1}{d_i^n+1}\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n+1}\right]-\sum_j\frac{P_j^n\hat\gamma_n^{(2)}P_i^n}{\lambda_j^n-\lambda_i^n+4\beta_0}\frac{1}{d_i^n-1}\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n-1}\right]\right\}\\
&+\frac{e^2}{2\beta_0}\frac{\bar g_2^2}{(16\pi^2)^2}K_n^{(0)}\sum_i\left\{\sum_{j,k}\frac{P_i^n\hat\gamma_n^{(1)}P_j^n\hat\gamma_n^{(1)}P_k^n}{\lambda_j^n-\lambda_k^n+2\beta_0}\left(\frac{1}{\lambda_i^n-\lambda_j^n+2\beta_0}-\frac{1}{\lambda_i^n-\lambda_k^n+4\beta_0}\right)\frac{1}{d_i^n+1}\right.\\
&\times\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n+1}\right]-\sum_{j,k}\frac{P_j^n\hat\gamma_n^{(1)}P_i^n\hat\gamma_n^{(1)}P_k^n}{\lambda_i^n-\lambda_k^n+2\beta_0}\frac{1}{\lambda_j^n-\lambda_i^n+2\beta_0}\frac{1}{d_i^n}\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n}\right]\\
&\left.+\sum_{j,k}\frac{P_k^n\hat\gamma_n^{(1)}P_j^n\hat\gamma_n^{(1)}P_i^n}{\lambda_j^n-\lambda_i^n+2\beta_0}\frac{1}{\lambda_k^n-\lambda_i^n+4\beta_0}\frac{1}{d_i^n-1}\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n-1}\right]\right\}+\frac{e^2}{2\beta_0}\frac{\bar g_2^2}{(16\pi^2)^2}K_n^{(1)}\frac{\beta_1}{\beta_0}\\
&\times\sum_i P_i^n\left\{-\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n}\right]+\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n-1}\right]\right\}+\frac{e^2}{2\beta_0}\frac{\bar g_2^2}{(16\pi^2)^2}K_n^{(1)}\\
&\times\sum_i\left\{\sum_j\frac{P_i^n\hat\gamma_n^{(1)}P_j^n}{\lambda_i^n-\lambda_j^n+2\beta_0}\frac{1}{d_i^n}\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n}\right]-\sum_j\frac{P_j^n\hat\gamma_n^{(1)}P_i^n}{\lambda_j^n-\lambda_i^n+2\beta_0}\frac{1}{d_i^n-1}\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n-1}\right]\right\}\\
&+\frac{e^2}{2\beta_0}\frac{\bar g_2^2}{(16\pi^2)^2}K_n^{(2)}P_i^n\left\{\frac{1}{d_i^n-1}\left[1-\left(\frac{\bar g_2^2}{\bar g_1^2}\right)^{d_i^n-1}\right]+\mathcal{O}(\bar g_2^2)\right\},
\end{aligned}
\tag{A8}
$$

where $\bar g_1^2=\bar g^2(P^2)$ and $\bar g_2^2=\bar g^2(Q^2)$. The first term is the leading, and the second through fourth terms are the next-to-leading terms. The rest are the next-to-next-to-leading terms.

APPENDIX B: $E_{ns\gamma}^{approx}(n)$, $E_{G\gamma}^{approx}(n)$, AND $E_{ps\gamma}(n)$

We give the explicit expressions of $E_{ns\gamma}^{approx}(n)$, $E_{G\gamma}^{approx}(n)$, and $E_{ps\gamma}(n)$ which have appeared in (3.24), (3.25), and (3.26). They are obtained by taking the Mellin moments of the parametrizations for $P_{ns\gamma}^{(2)}(x)$ and $P_{g\gamma}^{(2)}(x)$ and of the exact result for $P_{ps\gamma}^{(2)}(x)$, which are presented in Eqs. (6)–(8) of Ref. [11]. Using a single harmonic sum $S_m(n)$ defined by

$$S_m(n)=\sum_{j=1}^n\frac{1}{j^m},\tag{B1}$$

they are expressed as

$$\begin{aligned}
E_{\rm ns\gamma}^{\rm approx}(n) \equiv& -\int_0^1 dx x^{n-1}\{\text{the r.h.s. of Eq.(6) in Ref.[11]}\} \\
=& -\frac{128S_1(n)^4}{27n} + \frac{62.5244S_1(n)^3}{n} - \frac{50.08S_1(n)^3}{n+1} - \frac{256S_2(n)S_1(n)^2}{9n} - \frac{175.3S_1(n)^2}{n} - \frac{195.4S_1(n)^2}{n^2} \\
& - \frac{150.24S_1(n)^2}{(n+1)^2} - \frac{203.227S_2(n)S_1(n)}{n} - \frac{150.24S_2(n)S_1(n)}{n+1} - \frac{1024S_3(n)S_1(n)}{27n} - \frac{128S_2(n)^2}{9n} + \frac{785.14S_1(n)}{n} \\
& - \frac{325.4S_1(n)}{n^3} - \frac{300.48S_1(n)}{(n+1)^3} - \frac{175.3S_2(n)}{n} - \frac{520.8S_2(n)}{n^2} - \frac{150.24S_2(n)}{(n+1)^2} - \frac{591.151S_3(n)}{n} - \frac{100.16S_3(n)}{n+1} \\
& - \frac{256S_4(n)}{9n} - \frac{492.087}{n} + \frac{1262}{n+1} - \frac{449.2}{n+2} + \frac{1445}{n+3} + \frac{1279.86}{n^2} + \frac{1169}{(n+1)^2} - \frac{403.2}{n^3} + \frac{160}{n^4} - \frac{300.48}{(n+1)^4} \\
& - \frac{512}{9n^5} + n_f\Big\{ -\frac{32S_1(n)^3}{27n} - \frac{258.142S_1(n)^2}{n} + \frac{270S_1(n)^2}{n+1} + \frac{269.4S_1(n)^2}{n^2} + \frac{535.244S_2(n)S_1(n)}{n} \\
& - \frac{905.06S_1(n)}{n} + \frac{540S_1(n)}{(n+1)^2} + \frac{17.046S_1(n)}{n^3} - \frac{258.142S_2(n)}{n} + \frac{270S_2(n)}{n+1} + \frac{286.446S_2(n)}{n^2} + \frac{553.476S_3(n)}{n} \\
& - \frac{628.124}{n} - \frac{114.4}{n+1} + \frac{24.86}{n+2} + \frac{53.39}{n+3} - \frac{49.5895}{n^2} - \frac{26.63}{(n+1)^2} + \frac{21.984}{n^3} + \frac{540}{(n+1)^3} - \frac{64}{9n^4}\Big\},
\end{aligned} \tag{B2}$$

$$\begin{aligned}
E_{G\gamma}^{\rm approx}(n) \equiv& -\int_0^1 dx x^{n-1}\{\text{the r.h.s. of Eq.(7) in Ref.[11]}\} \\
=& \frac{32S_1(n)^3}{27n} - \frac{32S_1(n)^3}{27(n+1)} + \frac{79.13S_1(n)^2}{n} - \frac{79.13S_1(n)^2}{n+1} - \frac{433.2S_1(n)^2}{n^2} + \frac{429.644S_1(n)^2}{(n+1)^2} \\
& - \frac{862.844S_2(n)S_1(n)}{n} + \frac{862.844S_2(n)S_1(n)}{n+1} + \frac{1512.39S_1(n)}{n} - \frac{1512.39S_1(n)}{n+1} + \frac{549.5S_1(n)}{n^2} - \frac{707.76S_1(n)}{(n+1)^2} \\
& - \frac{2460S_1(n)}{n^3} + \frac{4185.69S_1(n)}{(n+1)^3} + \frac{628.63S_2(n)}{n} - \frac{628.63S_2(n)}{n+1} - \frac{2893.2S_2(n)}{n^2} + \frac{3756.04S_2(n)}{(n+1)^2} \\
& - \frac{3324.03S_3(n)}{n} + \frac{3324.03S_3(n)}{n+1} + \frac{73.1409}{n-1} - \frac{1673.57}{n} + \frac{3180.43}{n+1} - \frac{1420}{n+2} + \frac{406.7}{n+3} - \frac{566.7}{n+4} \\
& + \frac{128}{3(n-1)^2} + \frac{6400}{3n^2} - \frac{3688.39}{(n+1)^2} - \frac{2247.4}{n^3} + \frac{990.14}{(n+1)^3} + \frac{1600}{3n^4} + \frac{9438.76}{(n+1)^4} - \frac{3584}{9n^5} + \frac{3584}{9(n+1)^5} \\
& + \left(\frac{2460}{n} - \frac{2460}{n+1}\right)\zeta_3 + \left(\frac{2460}{n^2} - \frac{2460}{(n+1)^2}\right)\zeta_2 + n_f\Big\{\frac{32S_1(n)^2}{9n} - \frac{32S_1(n)^2}{9(n+1)} - \frac{9.133S_1(n)^2}{n^2} + \frac{9.133S_1(n)^2}{(n+1)^2} \\
& - \frac{18.266S_2(n)S_1(n)}{n} + \frac{18.266S_2(n)S_1(n)}{n+1} + \frac{46.4264S_1(n)}{n} - \frac{46.4264S_1(n)}{n+1} + \frac{16.18S_1(n)}{n^2} - \frac{23.2911S_1(n)}{(n+1)^2} \\
& - \frac{76.66S_1(n)}{n^3} + \frac{113.192S_1(n)}{(n+1)^3} + \frac{19.7356S_2(n)}{n} - \frac{19.7356S_2(n)}{n+1} - \frac{85.793S_2(n)}{n^2} + \frac{104.059S_2(n)}{(n+1)^2} \\
& - \frac{94.926S_3(n)}{n} + \frac{94.926S_3(n)}{n+1} + \frac{40.5597}{n-1} - \frac{21.1683}{n} + \frac{17.0286}{n+1} - \frac{93.37}{n+2} + \frac{101.05}{n+3} - \frac{44.1}{n+4} + \frac{115.341}{n^2} \\
& - \frac{161.767}{(n+1)^2} - \frac{52.82}{n^3} + \frac{13.3489}{(n+1)^3} - \frac{128}{9n^4} + \frac{299}{(n+1)^4}\Big\},
\end{aligned} \tag{B3}$$

$$\begin{aligned}
E_{\rm ps\gamma}(n) \equiv& -\int_0^1 dx x^{n-1}\{\text{the r.h.s. of Eq.(8) in Ref.[11]}\} \\
=& n_f C_F\Big\{ -\frac{2464}{81(n-1)} + \frac{432}{n} + \frac{72}{n+1} - \frac{38\,360}{81(n+2)} - \frac{344}{n^2} - \frac{368}{(n+1)^2} - \frac{3584}{27(n+1)^2} + \frac{288}{n^3} + \frac{208}{(n+1)^3} \\
& + \frac{448}{9(n+2)^3} - \frac{96}{n^4} + \frac{96}{(n+1)^4} + \frac{256}{3(n+2)^4} + \frac{64}{n^5} - \frac{128}{(n+1)^5}\Big\}.
\end{aligned} \tag{B4}$$

Note that $E_{\rm ps\gamma}(n)$ is an exact result.

APPENDIX C: MELLIN MOMENTS $a_{qg}^{(i)}(n)$ AND $b_{qg}^{(i)}(n)$ WITH $i=1,2$ AND $a_{gg}^{(2)}(n)$ AND $b_{gg}^{(2)}(n)$

The expressions of $a_{qg}^{(i)}(n)$ and $b_{qg}^{(i)}(n)$ with $i=1$ and 2 are obtained by taking the moments of the functions $a_{qg}^{(i)}(z)$ and $b_{qg}^{(i)}(z)$ as

$$a_{qg}^{(i)}(n)=\int_0^1 dz z^{n-1}a_{qg}^{(i)}(z), \qquad b_{qg}^{(i)}(n)=\int_0^1 dz z^{n-1}b_{qg}^{(i)}(z), \quad \text{with} \quad i=1,2 \tag{C1}$$

where $a_{qg}^{(i)}(z)$ and $b_{qg}^{(i)}(z)$ are extracted from the ϵ-independent terms of $\hat{A}_{qg}^{\rm PHYS}(z,\frac{-p^2}{\mu^2},\frac{1}{\epsilon})$ given in Eq. (A7) and of $\hat{A}_{qg}^{\rm EOM}(z,\frac{-p^2}{\mu^2},\frac{1}{\epsilon})$ in Eq. (A8) of Ref. [40], respectively. See also Eqs. (2.27) and (2.28) of Ref. [40].

The one-loop results are

$$a_{qg}^{(1)}(n)=\frac{n_f}{2}4\left\{\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}\right)(S_1(n)-1)+\frac{1}{n^2}-\frac{4}{(n+1)^2}+\frac{4}{(n+2)^2}\right\}, \tag{C2}$$

$$b_{qg}^{(1)}(n)=\frac{n_f}{2}16\left(\frac{1}{n+1}-\frac{1}{n+2}\right). \tag{C3}$$

The two-loop results are

$$\begin{aligned}a_{qg}^{(2)}(n)=C_F\frac{n_f}{2}\Bigg\{&\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}\right)\left(-\frac{4}{3}S_1(n)^3-4S_2(n)S_1(n)+\frac{64}{3}S_3(n)-16S_{2,1}(n)-48\zeta_3\right)\\&+S_1(n)^2\left(\frac{6}{n}-\frac{24}{n+1}+\frac{32}{n+2}-\frac{16}{n^2}+\frac{40}{(n+1)^2}-\frac{32}{(n+2)^2}\right)+S_1(n)\left(\frac{12}{n}+\frac{8}{n+1}-\frac{40}{n+2}+\frac{16}{n^2}-\frac{128}{(n+1)^2}\right.\\&\left.+\frac{128}{(n+2)^2}-\frac{32}{n^3}+\frac{176}{(n+1)^3}-\frac{128}{(n+2)^3}\right)+S_2(n)\left(\frac{6}{n}-\frac{8}{n+1}+\frac{16}{n+2}-\frac{16}{n^2}+\frac{40}{(n+1)^2}-\frac{32}{(n+2)^2}\right)+\frac{14}{n}\\&-\frac{38}{n+1}+\frac{48}{n+2}+\frac{64}{n^2}-\frac{126}{(n+1)^2}+\frac{80}{(n+2)^2}-\frac{22}{n^3}-\frac{88}{(n+1)^3}+\frac{128}{(n+2)^3}+\frac{20}{n^4}+\frac{88}{(n+1)^4}\Bigg\}\\&+\text{terms proportional to } C_A\frac{n_f}{2} \quad \text{or} \quad \left(\frac{n_f}{2}\right)^2,\end{aligned} \tag{C4}$$

$$\begin{aligned}b_{qg}^{(2)}(n)=16C_F\frac{n_f}{2}\Bigg\{&(S_1(n)^2+S_2(n))\left(-\frac{2}{n+1}+\frac{2}{n+2}\right)+S_1(n)\left(\frac{1}{n}+\frac{11}{n+1}-\frac{12}{n+2}-\frac{10}{(n+1)^2}+\frac{8}{(n+2)^2}\right)\\&-\frac{3}{n}+\frac{4}{n+1}-\frac{1}{n+2}+\frac{1}{n^2}+\frac{9}{(n+1)^2}-\frac{8}{(n+2)^2}-\frac{6}{(n+1)^3}\Bigg\}+\text{terms proportional to } C_A\frac{n_f}{2} \quad \text{or} \quad \left(\frac{n_f}{2}\right)^2,\end{aligned} \tag{C5}$$

where $S_{2,1}(n)=\sum_{j=1}^{n}\frac{1}{j^2}S_1(j)$. The terms proportional to $(\frac{n_f}{2})^2$ in $a_{qg}^{(2)}(n)$ and $b_{qg}^{(2)}(n)$ come from the external gluon self-energy corrections and should be discarded for the photon case.

Similarly, the expressions of $a_{gg}^{(2)}(n)$ and $b_{gg}^{(2)}(n)$ are obtained by taking the moments of the functions $a_{gg}^{(2)}(z)$ and $b_{gg}^{(2)}(z)$ which are extracted from the ϵ-independent terms of $\hat{A}_{gg}^{\rm PHYS}(z,\frac{-p^2}{\mu^2},\frac{1}{\epsilon})$ given in Eq. (A12) and of $\hat{A}_{gg}^{\rm EOM}(z,\frac{-p^2}{\mu^2},\frac{1}{\epsilon})$ in Eq. (A13) of Ref. [40], respectively. See also Eqs. (2.34) and (2.35) of Ref. [40].[2]

The two-loop results for $a_{gg}^{(2)}(n)$ and $b_{gg}^{(2)}(n)$ are

[2]Two terms, $\gamma_{gg}^{(0)}b_{gg}^{\epsilon,(1)}$ and $\gamma_{gq}^{(0)}b_{qg}^{\epsilon,(1)}$, are missing in the ϵ-independent terms of Eq. (2.35) of Ref. [40]. They both are needed in order to extract $b_{gg}^{(2)}(z)$ correctly.

$$a_{gg}^{(2)}(n) = C_F\frac{n_f}{2}\bigg\{(S_1(n)^2+S_2(n))\bigg(\frac{16}{3(n-1)}+\frac{4}{n}-\frac{4}{n+1}-\frac{16}{3(n+2)}-\frac{8}{n^2}-\frac{8}{(n+1)^2}\bigg)+S_1(n)\bigg(\frac{16}{9(n-1)}-\frac{48}{n}+\frac{32}{n+1}+\frac{128}{9(n+2)}+\frac{32}{n^2}+\frac{24}{(n+1)^2}-\frac{64}{3(n+2)^2}-\frac{32}{n^3}-\frac{48}{(n+1)^3}\bigg)+S_{-2}(n)\bigg(\frac{32}{3(n-1)}-\frac{32}{n}+\frac{32}{n+1}-\frac{32}{3(n+2)}\bigg)+\frac{680}{27(n-1)}-\frac{48}{n}+\frac{16}{n+1}+\frac{184}{27(n+2)}-\frac{56}{n^2}+\frac{56}{(n+1)^2}+\frac{256}{9(n+2)^2}+\frac{44}{n^3}+\frac{76}{(n+1)^3}-\frac{128}{3(n+2)^3}-\frac{40}{n^4}-\frac{88}{(n+1)^4}-\frac{55}{3}+16\zeta_3\bigg\}+\text{terms proportional to } C_A^2 \quad\text{or}\quad C_A\frac{n_f}{2}\quad\text{or}\quad\bigg(\frac{n_f}{2}\bigg)^2, \tag{C6}$$

$$b_{gg}^{(2)}(n) = C_F\frac{n_f}{2}\bigg\{S_1(n)\bigg(\frac{32}{3(n-1)}-\frac{32}{n}+\frac{64}{3(n+2)}+\frac{32}{(n+1)^2}\bigg)-\frac{128}{9(n-1)}+\frac{64}{n}-\frac{448}{9(n+2)}-\frac{32}{n^2}-\frac{96}{(n+1)^2}+\frac{128}{3(n+2)^2}+\frac{96}{(n+1)^3}\bigg\}+\text{terms proportional to } C_A^2\quad\text{or}\quad C_A\frac{n_f}{2}. \tag{C7}$$

The terms proportional to $(\frac{n_f}{2})^2$ in $a_{gg}^{(2)}(n)$ again come from the external gluon self-energy corrections and should be discarded for the photon case. Furthermore, the contribution of the last two terms in the curly brackets of (C6), more explicitly, $C_F\frac{n_f}{2}(-\frac{55}{3}+16\zeta_3)$, also results from the external gluon self-energy corrections and is thus irrelevant.

APPENDIX D: VALUES AT $n = 2$

1. Coefficient functions

With $\delta_\psi = \langle e^2\rangle$, $\delta_{NS} = 1$, and $\delta_\gamma = 3n_f\langle e^4\rangle$, we have the following:

(i) At tree level

$$C_{2,n=2}^{\psi(0)} = \delta_\psi, \qquad C_{2,n=2}^{G(0)} = 0, \qquad C_{2,n=2}^{NS(0)} = \delta_{NS}, \qquad C_{2,n=2}^{\gamma(0)} = 0. \tag{D1}$$

(ii) At one-loop level

$$C_{2,n=2}^{\psi(1)} = \delta_\psi C_F(\tfrac{1}{3}), \qquad C_{2,n=2}^{G(1)} = \delta_\psi n_f(-\tfrac{1}{2}), \qquad C_{2,n=2}^{NS(1)} = \delta_{NS}C_F(\tfrac{1}{3}), \qquad C_{2,n=2}^{\gamma(1)} = \delta_\gamma(-1). \tag{D2}$$

(iii) At two-loop level

$$C_{2,n=2}^{\psi(2)} = \delta_\psi\bigg\{C_AC_F\bigg(\frac{3677}{135}-\frac{128}{5}\zeta_3\bigg)+C_Fn_f\bigg(-\frac{457}{81}\bigg)+C_F^2\bigg(-\frac{4189}{810}+\frac{96}{5}\zeta_3\bigg)\bigg\},$$
$$C_{2,n=2}^{G(2)} = \delta_\psi\bigg\{C_Fn_f\bigg(-\frac{4799}{810}+\frac{16}{5}\zeta_3\bigg)+C_An_f\bigg(\frac{115}{324}-2\zeta_3\bigg)\bigg\},$$
$$C_{2,n=2}^{NS(2)} = \delta_{NS}\bigg\{C_AC_F\bigg(\frac{3677}{135}-\frac{128}{5}\zeta_3\bigg)+C_Fn_f(-4)+C_F^2\bigg(-\frac{4189}{810}+\frac{96}{5}\zeta_3\bigg)\bigg\}, \qquad C_{2,n=2}^{\gamma(2)} = \delta_\gamma C_F\bigg(-\frac{4799}{405}+\frac{32}{5}\zeta_3\bigg). \tag{D3}$$

2. Anomalous dimensions

(i) At one-loop level

$$\gamma_{NS}^{(0),n=2} = \gamma_{\psi\psi}^{(0),n=2} = C_F\bigg(\frac{16}{3}\bigg), \qquad \gamma_{\psi G}^{(0),n=2} = n_f\bigg(-\frac{4}{3}\bigg), \qquad \gamma_{G\psi}^{(0),n=2} = C_F\bigg(-\frac{16}{3}\bigg), \qquad \gamma_{GG}^{(0),n=2} = n_f\bigg(\frac{4}{3}\bigg), \tag{D4}$$

and

$$K_\psi^{(0),n=2} = 3n_f\langle e^2\rangle(\tfrac{8}{3}), \qquad K_G^{(0),n=2} = 0, \qquad K_{NS}^{(0),n=2} = 3n_f(\langle e^4\rangle-\langle e^2\rangle^2)(\tfrac{8}{3}). \tag{D5}$$

(ii) At two-loop level

$$\begin{aligned}
\gamma_{NS}^{(1),n=2} &= C_A C_F\left(\frac{752}{27}\right) + C_F n_f\left(-\frac{128}{27}\right) + C_F^2\left(-\frac{224}{27}\right),\\
\gamma_{\psi\psi}^{(1),n=2} &= C_A C_F\left(\frac{752}{27}\right) + C_F n_f\left(-\frac{208}{27}\right) + C_F^2\left(-\frac{224}{27}\right),\\
\gamma_{\psi G}^{(1),n=2} &= C_A n_f\left(-\frac{70}{27}\right) + C_F n_f\left(-\frac{148}{27}\right),\\
\gamma_{G\psi}^{(1),n=2} &= C_A C_F\left(-\frac{752}{27}\right) + C_F n_f\left(\frac{208}{27}\right) + C_F^2\left(\frac{224}{27}\right),\\
\gamma_{GG}^{(1),n=2} &= C_A n_f\left(\frac{70}{27}\right) + C_F n_f\left(\frac{148}{27}\right),
\end{aligned} \tag{D6}$$

and

$$K_{NS}^{(1),n=2} = 3n_f(\langle e^4\rangle - \langle e^2\rangle^2)C_F\left(\frac{296}{27}\right), \qquad K_{\psi}^{(1),n=2} = 3n_f\langle e^2\rangle C_F\left(\frac{296}{27}\right), \qquad K_{G}^{(1),n=2} = 3n_f\langle e^2\rangle C_F\left(-\frac{80}{27}\right). \tag{D7}$$

(iii) At three-loop level, we get from [26,27]

$$\begin{aligned}
\gamma_{NS}^{(2),n=2} &= C_A C_F n_f\left(-\frac{6256}{243}-\frac{128}{3}\zeta_3\right) + C_F C_A^2\left(\frac{41840}{243}+\frac{128}{3}\zeta_3\right) + C_F n_f^2\left(-\frac{448}{243}\right) + C_F^2 C_A\left(-\frac{17056}{243}-128\zeta_3\right)\\
&\quad + C_F^2 n_f\left(-\frac{6824}{243}+\frac{128}{3}\zeta_3\right) + C_F^3\left(-\frac{1120}{243}+\frac{256}{3}\zeta_3\right),\\
\gamma_{\psi\psi}^{(2),n=2} &= C_A C_F n_f\left(-\frac{44}{9}-\frac{256}{3}\zeta_3\right) + C_F C_A^2\left(\frac{41840}{243}+\frac{128}{3}\zeta_3\right) + C_F n_f^2\left(-\frac{568}{81}\right) + C_F^2 C_A\left(-\frac{17056}{243}-128\zeta_3\right)\\
&\quad + C_F^2 n_f\left(-\frac{14188}{243}+\frac{256}{3}\zeta_3\right) + C_F^3\left(-\frac{1120}{243}+\frac{256}{3}\zeta_3\right),\\
\gamma_{\psi G}^{(2),n=2} &= C_A C_F n_f\left(\frac{278}{9}-\frac{208}{3}\zeta_3\right) + C_A n_f^2\left(\frac{2116}{243}\right) + C_A^2 n_f\left(-\frac{3589}{81}+48\zeta_3\right) + C_F n_f^2\left(-\frac{346}{243}\right)\\
&\quad + C_F^2 n_f\left(-\frac{4310}{243}+\frac{64}{3}\zeta_3\right),\\
\gamma_{G\psi}^{(2),n=2} &= C_A C_F n_f\left(\frac{44}{9}+\frac{256}{3}\zeta_3\right) + C_A C_F^2\left(\frac{17056}{243}+128\zeta_3\right) + C_A^2 C_F\left(-\frac{41840}{243}-\frac{128}{3}\zeta_3\right) + C_F n_f^2\left(\frac{568}{81}\right)\\
&\quad + C_F^2 n_f\left(\frac{14188}{243}-\frac{256}{3}\zeta_3\right) + C_F^3\left(\frac{1120}{243}-\frac{256}{3}\zeta_3\right),\\
\gamma_{GG}^{(2),n=2} &= C_A C_F n_f\left(-\frac{278}{9}+\frac{208}{3}\zeta_3\right) + C_A n_f^2\left(-\frac{2116}{243}\right) + C_A^2 n_f\left(\frac{3589}{81}-48\zeta_3\right) + C_F n_f^2\left(\frac{346}{243}\right) + C_F^2 n_f\left(\frac{4310}{243}-\frac{64}{3}\zeta_3\right),
\end{aligned} \tag{D8}$$

and from [10]

$$\begin{aligned}
K_{NS}^{(2),n=2} &= -3n_f(\langle e^4\rangle - \langle e^2\rangle^2)\left\{C_F n_f\left(\frac{536}{243}\right) + C_F C_A\left(-\frac{6044}{243}-\frac{64}{3}\zeta_3\right) + C_F^2\left(-\frac{8620}{243}+\frac{128}{3}\zeta_3\right)\right\},\\
K_{\psi}^{(2),n=2} &= -3n_f\langle e^2\rangle\left\{C_F n_f\left(-\frac{692}{243}\right) + C_F C_A\left(-\frac{6044}{243}-\frac{64}{3}\zeta_3\right) + C_F^2\left(-\frac{8620}{243}+\frac{128}{3}\zeta_3\right)\right\},\\
K_{G}^{(2),n=2} &= -3n_f\langle e^2\rangle\left\{C_F n_f\left(\frac{1880}{243}\right) + C_F C_A\left(-\frac{1138}{243}+\frac{64}{3}\zeta_3\right) + C_F^2\left(\frac{9592}{243}-\frac{128}{3}\zeta_3\right)\right\}.
\end{aligned} \tag{D9}$$

Note that a relation $\gamma_{\psi\psi}^{(i),n=2}\gamma_{GG}^{(i),n=2} - \gamma_{\psi G}^{(i),n=2}\gamma_{G\psi}^{(i),n=2} = 0$ indeed holds for $i = 0, 1, 2$.

3. Photon matrix elements of quark and gluon operators

(i) At one-loop level

$$\tilde{A}^{(1)\psi}_{n=2} = 3n_f\langle e^2\rangle(\tfrac{2}{9}), \qquad \tilde{A}^{(1)G}_{n=2} = 0,$$
$$\tilde{A}^{(1)NS}_{n=2} = 3n_f(\langle e^4\rangle - \langle e^2\rangle^2)(\tfrac{2}{9}). \tag{D10}$$

(ii) At two-loop level

$$\tilde{A}^{(2)\psi}_{n=2} = 3n_f\langle e^2\rangle C_F\left(\frac{616}{81} - 16\zeta_3\right),$$
$$\tilde{A}^{(2)G}_{n=2} = 3n_f\langle e^2\rangle C_F\left(\frac{383}{81}\right),$$
$$\tilde{A}^{(2)NS}_{n=2} = 3n_f(\langle e^4\rangle - \langle e^2\rangle^2)C_F\left(\frac{616}{81} - 16\zeta_3\right). \tag{D11}$$

APPENDIX E: ANALYTIC CONTINUATION OF THE HARMONIC SUMS

The moments of $F_2^\gamma(x, Q^2, P^2)$ given in (2.29) are expressed by the rational functions of integer n and the various harmonic sums. The single harmonic sums are defined by

$$S_k(n) = \sum_{j=1}^{n} \frac{[\mathrm{sgn}(k)]^j}{j^{|k|}}, \tag{E1}$$

where $k = \pm 1, \pm 2, \ldots$, and the higher harmonic sums are defined recursively as

$$S_{k,m_1,\ldots,m_p}(n) = \sum_{j=1}^{n} \frac{[\mathrm{sgn}(k)]^j}{j^{|k|}} S_{m_1,\ldots,m_p}(j), \tag{E2}$$

where indices k and $m_1, \ldots, m_p$ take nonzero integers. In order to invert the moments so that we get F_2^γ as a function of x, we need to make an analytic continuation of these harmonic sums from integer n to complex n. Since the moment sum rules of the s-u-crossing-even structure function F_2^γ are defined for even integer n, the continuation should be performed from even n. Thus whenever a factor $(-1)^n$ appears, it should be replaced by $(+1)$. The method we adopted here for the analytic continuation is to use the asymptotic expansions of the harmonic sums and their translation relations. Choosing the following two harmonic sums,

$$S_1(n) = \sum_{j=1}^{n} \frac{1}{j}, \tag{E3}$$

$$S_{1,1,-2,1}(n) = \sum_{i=1}^{n} \frac{1}{i} \sum_{j=1}^{i} \frac{1}{j} \sum_{k=1}^{j} \frac{(-1)^k}{k^2} \sum_{l=1}^{k} \frac{1}{l}, \tag{E4}$$

as examples, we explain how we get approximate analytic formulas for these sums.

The asymptotic expansion of $S_1(n)$ for large n is well known:

$$S_1(n) = \ln(n) + \gamma_E + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + \cdots. \tag{E5}$$

The right-hand side has a simple analytic property. On the other hand, $S_1(n)$ satisfies the following translation relation:

$$S_1(n) = S_1(n+1) - \frac{1}{n+1}. \tag{E6}$$

This relation is valid not only for integer n, but also for complex n. Therefore, our algorithm to evaluate $S_1(n)$ at arbitrary complex n is as follows: (i) If $|n| \geq n_0$, where n_0 is some positive integer at which the asymptotic expansion (E5) holds at a desired accuracy, then we use the expansion (E5) to evaluate $S_1(n)$. (ii) For $|n| < n_0$, we apply the translation relation (E6) and shift the argument $n \to n + 1$ repeatedly, until the shifted new $\tilde{n}$ satisfies the condition $|\tilde{n}| \geq n_0$ so that the asymptotic expansion (E5) for $S_1(\tilde{n})$ may be used with a desired accuracy. Then $S_1(n)$ is evaluated by the formula

$$S_1(n) = S_1(\tilde{n}) - \sum_{i=1}^{\tilde{n}-n} \frac{1}{n+i}, \tag{E7}$$

where the expansion (E5) is used for $S_1(\tilde{n})$.

In the case of a more complicated higher harmonic sum $S_{1,1,-2,1}(n)$ with *even* integer n, its asymptotic expansion for large n is given by

$$S^{\mathrm{even}}_{1,1,-2,1}(n) = c_{0,2}\ln^2(n) + c_{0,1}\ln(n) + c_{0,0} + \frac{c_{1,1}\ln(n)}{n} + \frac{c_{1,0}}{n} + \frac{c_{2,1}\ln(n)}{n^2} + \frac{c_{2,0}}{n^2} + \cdots, \tag{E8}$$

where

$$c_{0,2} = c_{1,1} = -\frac{5}{16}\zeta(3) = -0.375\,642\,78\cdots, \qquad c_{0,1} = -\frac{5}{8}\gamma_E\zeta(3) - \frac{3}{40}\zeta^2(2) = -0.636\,589\,40\cdots,$$

$$c_{0,0} = -\frac{3}{40}\gamma_E\zeta^2(2) - \frac{5}{16}\gamma_E^2\zeta(3) - \ln(2)\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{7}{16}\ln^2(2)\zeta(3) + \frac{1}{6}\ln^3(2)\zeta(2) - \frac{1}{30}\ln^5(2) + \frac{1}{8}\zeta(2)\zeta(3) + \frac{1}{8}\zeta(5) - \mathrm{Li}_5\left(\frac{1}{2}\right) = -0.899\,307\,22\cdots, \tag{E9}$$

$$c_{1,0} = -\frac{5}{16}\gamma_E\zeta(3) - \frac{3}{80}\zeta^2(2) + \frac{5}{16}\zeta(3) = 0.057\,348\,080\cdots, \qquad c_{2,1} = \frac{5}{96}\zeta(3) = 0.062\,607\,130\cdots,$$

$$c_{2,0} = \frac{5}{96}\gamma_E\zeta(3) + \frac{1}{160}\zeta^2(2) - \frac{15}{64}\zeta(3) = -0.228\,682\,96\cdots.$$

Also $S^{\mathrm{even}}_{1,1,-2,1}(n)$ satisfies the following translation relation:

$$S^{\mathrm{even}}_{1,1,-2,1}(n) = S^{\mathrm{even}}_{1,1,-2,1}(n+2) - \left(\frac{1}{n+1} + \frac{1}{n+2}\right) \times S^{\mathrm{even}}_{1,-2,1}(n+2) + \frac{1}{(n+1)(n+2)} \times S^{\mathrm{even}}_{-2,1}(n+2). \tag{E10}$$

Note that the right-hand side of (E10) is written in terms of the same harmonic sum $S^{\mathrm{even}}_{1,1,-2,1}$ and lower harmonic sums $S^{\mathrm{even}}_{1,-2,1}$ and $S^{\mathrm{even}}_{-2,1}$ but with a larger argument $n+2$. When $|n| \geq n_0$, the asymptotic expansion (E8) is used to evaluate $S^{\mathrm{even}}_{1,1,-2,1}(n)$. For $|n| < n_0$, we apply (E10) and shift the argument $n \to n+2$ repeatedly, until the shifted new $\tilde{n}$ satisfies the condition $|\tilde{n}| \geq n_0$ so that the asymptotic expansion (E8) of $S^{\mathrm{even}}_{1,1,-2,1}(\tilde{n})$ can be used with a desired accuracy. The lower harmonic sums $S^{\mathrm{even}}_{1,-2,1}(n)$ and $S^{\mathrm{even}}_{-2,1}(n)$ are evaluated in a similar fashion.

In practice, we take $n_0 = 16$ for all harmonic sums. Then the asymptotic expansion formula for each harmonic sum is derived so as to ensure double precision accuracy (15 significant figures) at $n = n_0$. For example, $S_1(n)$ and $S^{\mathrm{even}}_{1,1,-2,1}(n)$ are expanded up to the terms with $1/n^{10}$ and $1/n^{18}$, respectively.

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