Equilibrium Term Structure of Interest Rates under Gaussian Endowment Processes

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DEDICATION

This dissertationis is dedicated to my family, Kuniko, Yoshiyasu, and Maho. Without their support, encouragement, tolerance and patience, this effort would not have been worthwhile.

Contents

[For Memorandum]

Chapter 1

General Introduction

Today, interest rates are key economic instruments. As interest rate markets continue to innovate and expand, it is becoming increasingly important to remain up to date with the latest practical developments.

As a matter of fact, interest rate models are well-developed after forty years since the most basic term structure model is pioneered by Merton (1973). Meanwhile, empirical challenges emerged as the many types of empirical analysis were conducted. In this dissertation, we especially focus on the term structure of interest rates and try to solve some problems provided as the empirical challenges stated below:

- the relation between the nominal interest rates and excess consumption
- the expectation puzzle
- the humped shape of the term structure of volatility
- the nonnegativity of nominal interest rates.

There are already well-developed models which explain the relation between the *real* interest rates and excess consumption. For instance, Wachter (2006) and Burashi and Jiltsov (2007) take consumption-based approaches to the term structure of interest rates and show the negative relation between the real short term interest rate and excess consumption. The excess consumption in these literatures is defined by the consumption excess to the consumption habit. While these models are attractive, they also imply the negative relation between the *nominal* interest rates and excess consumption whereas the empirical studies found the positive relation. In this sense, the existing models are not necessarily successful in explaining the observed phenomenon.

The second challenge, the expectation puzzle, is the puzzle which was found by Campbell and Shiller (1991). While the standard expectation hypothesis asserts that the change in spot rates is positively correlated with the slope of the spot rate curve, the persistent negative correlation is found. An explanation to this puzzle is provided by Dai and Singleton (2002) and Duffee (2002). They extend the traditional affine term structure models to the models which allow the market price of risk to move stochastically and show that the puzzle can be solved by these models. The question addressed in this dissertationis whether there is another economic story which explains the puzzle other than the time-varying market price of risk.

The third challenge is concerned with the volatility. The actual term structure of volatility is time-varying. Furthermore, the volatility curve is reported to be humped shaped. These phenomena are not explained by one-factor term structure models. Is there any single factor model which is possible to explain those phenomena?

Finally, the nominal interest rates are considered to be nonnegative. But, many of the standard models allow that the interest rates are negative. When we use the term structure models to price fixed income derivatives based on nominal interest rates, this may be a serious problem.

This dissertation tries to give some solutions to solve these modern challenges to the theory of term structure of interest rates. To give economic stories behind the challenges, we adopt consumption-based approaches in which the preference of representative agent and the process of endowment are specified. The models considered here are the equilibrium models in pure exchange economies. More specifically, we consider the pure exchange economies in which the aggregate endowment follows a Gaussian process (that is, the economies where the unconditional distribution of future endowment is normal). As is shown in the next chapter, when the preference of representative agent is defined by the time-additive CRRA (constant relative risk aversion) utility, the equilibrium generates the standard term structure model provided by Vasicek (1979). We depart from Vasicek model by the following extensions:

- to assume that economic variables are partially observable
- to extend the preference of representative agent to more general preference than CRRA utility.

By these extensions, we obtain the following results. First, by assuming the partial observability of economic variables, we explain the positive relation between the nominal interest rates and excess consumption. Second, the paper founds that the expectation puzzle can be explained even without time-varying market price of risk. The key to solve the puzzle here is the time-varying elasticity of intertemporal substitution. Third, this dissertation shows that under a more general preference than CRRA utility, the interest rates are nonnegative even under the assumption that the single factor follows a Gaussian process. And it is shown that the equilibrium in this one-factor economy can generate humped shaped volatility curve. Again, the main reason for the humped shape is that the elasticity of intertemporal substitution is time-varying.

The paper is organized as follows. In the next chapter, the term structure models and the consumption-based approach are reviewed briefly and the challenges to the term structure models to focus in this dissertation are provided in detail. In chapter 3, we examine the equilibrium in an economy where economic variables are partially observable. In chapter 4, another term structure model explaining the expectation puzzle is explored. In chapter 5, an economy with single factor is considered and the nonnegativity of interest rates and the shape of the volatility curve are examined.

[**For Memorandum**]

Chapter 2

Review of Term Structure Models

2.1 General Set Up

In the following sections, we review the literature on the term structure models. Before this, the general set up on the time span and the probability space is given in this section. Throughout this dissertation, the time span is assumed to be a continuous closed interval $[t_0, \tau]$. Let $(\Omega, \mathcal{F}, Q)$ be a complete probability space. A n-dimensional Wiener process is denoted as $\{W_t : t \in [t_0, \tau]\}$ where $W_t^\top = [W_{1t}, \cdots, W_{nt}]$. We assume that W_{it} and $W_{jt}(i \neq j)$ are independent. Filteration $\{\mathcal{F}_t : t \in [t_0, \tau]\}$ is chosen to be the $Q-$augmentation of the natural filteration generated by W_t.

Let us denote the pure discount bond price at time as $P(t, s)$ where s is the date of maturity. It is assumed that the face value of any bond is unity. The time t spot rate with matutiry s is defined by,

$$y(t, s) = -\frac{1}{s-t} \ln P(t, s). \tag{2.1}$$

The short rate is the short end of the spot rate curve and defined by

$$r_t = \lim_{s \to t} y(t, s). \tag{2.2}$$

The time t instantaneous forward rate of interest rate with maturity date is defined by

$$f(t, s) = -\frac{\partial \ln P(t, s)}{\partial s}. \tag{2.3}$$

It is straightforward that the following equation holds,

$$r_t = \lim_{s \to t} f(t, s). \tag{2.4}$$

It is well known that no risk-free arbitrage opportunity exists if an equivalent martingale measure Q^* exists, and the arbitrage free bond price is given by

$$P(t,s) = E^{Q^*}\left[\exp\left(-\int_t^s r_u du\right)\middle|\mathcal{F}_t\right], \tag{2.5}$$

where $E^{Q^*}[\cdot|\mathcal{F}_t]$ is the operator for conditional expectation under the equivalent martingale measure. Since Radon-Nikodym derivative follows a martingale process, from the martingale representation theorem, (2.5) is expressed as

$$\begin{aligned} P(t,s) &= E\left[\exp\left(-\frac{1}{2}\int_t^\tau \Lambda_u^\top \Lambda_u du - \int_t^\tau \Lambda_u^\top dW_u\right)\exp\left(-\int_t^s r_u du\right)\middle|\mathcal{F}_t\right] \\ &= E\left[\exp\left(-\frac{1}{2}\int_t^s \Lambda_u^\top \Lambda_u du - \int_t^s \Lambda_u^\top dW_u\right)\exp\left(-\int_t^s r_u du\right)\middle|\mathcal{F}_t\right], \end{aligned} \tag{2.6}$$

where $E[\cdot|\mathcal{F}_t]$ is the operator for conditional expectation under Q and $\Lambda_t^\top = [\Lambda_{1t}, \cdots, \Lambda_{nt}]$ is the vector of $\mathcal{F}_t-$measurable random variables. It is well known that this vector has an interpretation as the vector of market price of risk.

In the above equation, the determinants of bond price can be considered as two processes, $\{r_t : t \in [t_0, \tau]\}$ and $\{\Lambda_t : t \in [t_0, \tau]\}$, that is, the process of short rate and the process of market price of risk. Thus, specifying these two stochastic processes means that we specify a term structure model. In the next section, we review the term structure models from this viewpoint.

2.2 Term Structure Models

2.2.1 Affine term structure models

Affine term structure models are the models in which the instantaneous forward rate of interest rate is an affine function of state variables. Since the spot rate is the average of instantaneous forward rate of interest rates, the spot rate is also an affine function of state variables. Affine term structure models are now widely used for empirical analysis, because the interest rates are affine functions of state variables and does not require complicated estimation procedures.

General arguement for the affine term structure models was first given by Duffie and Kan (1996). Let n-dimensional stochastic process $\{X_t : t \in [t_0, \tau]\}$ where $X_t = [X_{1t}, \cdots, X_{nt}]^\top$

as the vector of state variables. And $\{W_t^* : t \in [t_0, \tau]\}$ is defined by $dW_t^* = dW_t + \Lambda_t dt,\ t \in [t_0, \tau]$. Obviously this process is the n-dimensional Wiener process under the equivalent martingale measure. The gradients of affine term structure models are:

1. the process of state variables:

$$\begin{aligned} dX_t &= \mu^*_{X,t}dt + \sigma^\top_{X,t}dW_t^*, \\ \mu^*_{X,t} &= g_0 + g_X X_t, \\ \sigma^\top_{X,t}\sigma_{X,t} &= h_0 + \sum_{i=1}^{n} h_i X_{it}, \end{aligned}$$

where g_0 is an $n \times 1$ vector and g_X and $h_i,\ i = 0, 1, \cdots, n$ are $n \times n$ matrices of constants.

2. short rate: the short rate is given by an affine function of state variables,

$$r_t = \delta_0 + \delta^\top X_t,$$

where δ_0 is a real number and δ is a $n \times 1$ vector of constants.

Note that the vector of market price of risk is specified through the equation $\mu^*_{X,t} = \mu_{X,t} - \sigma^\top_{X,t}\Lambda_t$, once the drift of state variables under the physical measure, $\mu_{X,t}$ is specified.

Duffie and Kan (1996) showed that under this specification, the instantaneous forward rate of interest rate is an affine function of state variables. Converse is also true if we assume that the term structure models are time homogeneous. Before Duffie and Kan (1996), many affine term structure models are provided. Let us briefly review some of these models which are the critical benchmarks for subsequent chapters.

Merton (1973) introduced the most simple term structure model calles as Merton toy model. It assumes that $n = 1$ and the process of short rate is given by

$$dr_t = \mu_r dt - \sigma_r dW_t,$$

where μ_r and σ_r are constants. And it is assumed that $\Lambda_t = \lambda,\ \forall t \in [t_0, \tau]$ where λ is a constant. The specification as an affine term structure model is: $g_0 = \mu_r + \lambda\sigma_r, g_X = 0, h_0 =$

$\sigma_r^2, h_1 = 0, \delta_0 = 0, \delta = 1$. Under this specification, the model implies that the instantaneous forward rate of interest rate is given by

$$f(t,s) \;=\; r_t + (\mu_r + \lambda\sigma_r)(s-t) - \frac{1}{2}\sigma_r(s-t)^2.$$

Cleary, the instantaneous forward rate of interest rate is an affine function of X_{1t}.

The serious drawback of Merton toy model is the process of the short rate is non-stationary. Vasicek(1977) provided the term structure models where the process of interest rate is stationary. He assumes that $n = 1$ and the short rate follows an Ornstein-Uhlenbeck process,

$$dr_t = \kappa(\theta - r_t)dt - \sigma_r dW_t,$$

where κ and θ are positive constant. To ensure that the short process is stationary, it is usually assumed that $\kappa > 0$. As Merton toy model, he assumes that $\Lambda_t = \lambda, \ \forall t \in [t_0, \tau]$. The specification as an affine term structure model is: $g_0 = \kappa\theta + \lambda\sigma_r, g_X = -\kappa, h_0 = \sigma_r^2, h_i = 0, \delta_0 = 0, \delta = 1$. Under this specification, the instantaneous forward rate is given by

$$f(t,s) \;=\; e^{-\kappa(s-t)}r_t + \left(1 - e^{-\kappa(s-t)}\right)\theta + \lambda\sigma_r\frac{1-e^{-\kappa(s-t)}}{\kappa} - \frac{1}{2}\sigma_r^2\left(\frac{1-e^{-\kappa(s-t)}}{\kappa}\right)^2.$$

Note that when $\kappa = 0$, then Vasicek model is reduced to Merton toy model. And also in this model, the instantaneous forward rate of interest rate is an affine function of X_{1t}.

The common property which is shared in the above models is that the compound interest rates follow normal distribution. Although this property is convenient when we obtain the rational price of fixed income derivatives, there is a serious shortcoming that interest rates are negative with positive probability. Cox, Ingersoll and Ross (1985b) overcomes this problem. In their model, n is set to be unity and the short rate is assumed to follow a square root process,

$$dr_t = \kappa(\theta - r_t)dt - \sigma_r\sqrt{r_t}dW_t,$$

where κ and θ are positive constant. The market price of risk is defined by $\Lambda_t = \lambda\sqrt{r_t}$. Note that the short rate is positive with probability one, because there is a reflecting barrier at zero.

The specification as an affine term structure model is: $g_0 = \kappa\theta + \lambda\sigma_r, g_X = -\kappa, h_0 = 0, h_1 = \sigma_r^2, \delta_0 = 0, \delta = 1$. Under this specification, the instantaneous forward rate of interest rate is also given as an affine function of X_{1t},

$$\begin{aligned}
f(t,s) &= \left(\frac{\partial}{\partial s}b(t,s)\right) r_t - \frac{\partial}{\partial s}A(t,s) \\
\text{where} & \\
A(t,s) &= \left(\frac{2\gamma e^{\frac{(\kappa+\lambda+\gamma)(s-t)}{2}}}{(\gamma+\kappa+\lambda)(e^{\gamma(s-t)}-1)+2\gamma}\right)^{\frac{2\kappa\theta}{\sigma_r^2}} \\
b(t,s) &= \frac{2(e^{\gamma(s-t)}-1)}{(\gamma+\kappa+\lambda)(e^{\gamma(s-t)}-1)+2\gamma} \\
\gamma &= \sqrt{(\kappa+\lambda)^2+2\sigma_r^2}.
\end{aligned}$$

All of the above models has only one state variable which drives spot rate curves. Litterman and Sheinkman (1991) showed that there must at least two factors to explain the actual movements of spot rate curves. In fact, many affine term structure models with more than one factor are provided in the last four decades. Langetieg (1980) extends Vasicek model. He assumed that the short rate is the sum of more than two state varibles which follow Ornstein-Uhlenbeck processes and the market price of risk is expressed as a vector of constants. Longstaff and Schwartz (1992) provide two-factor affine term structure model which is a natural extension of Cox, Ingersoll and Ross model. There are other models which can be categorized as the member of the affine term structure models such as Brown and Dybvig (1986), Hull and White (1987), Chen and Scott (1993), Brown and Schaefer (1994), Pearson and Sun (1994) for instance.

It is natural to extend the above models to the affine term structure models in which some state variabels follow square root processes and the others follow Gaussian processes. To construct this kind of model, restrictions must be imposed. Going back to the specification of affine term structure models, let us express $\sigma_{X,t}$ as $\sigma_{X,t}^\top = \Sigma\sqrt{S_t}$ where $S_{ii,t} = c_i + d_i^\top X_t$, $S_{ij} = 0, \imath \neq j$, $1 \leq i,j, \leq n$ and Σ is an $n \times n$ matrix of constants. To ensure that $S_{ii,t}$ is nonnegative, Dai and Singleton(2000) introduced the "canonical" model $\mathcal{A}_M(n)$ with $S_{ii,t} = \sqrt{X_{i,t}}$, $i = 1, \cdots, M$ and the remaining $n - M$ $S_{ii,t}$ being affine function of $(X_{1,t}, \cdots, X_{M,t})$. They provide an easily verifiable set of sufficient restrictions on the paremeters of $\mathcal{A}_M(n)$ to gurantee "admissibility". In this specification, Merton toy model, Vasicek model, Langtieg model are the models of $\mathcal{A}_0(n)$, $n \geq 1$. These models are called as *purely Gaussian term structure models*, since all of the state variables follow Gaussian processes. Cox, Ingersoll and Ross model and Longstaff and Schwartz model are identified as the models of $\mathcal{A}_n(n),\ n=1,2$.

2.2.2 Determinants of instantaneous forward rate of interest rate in purely Gaussian term structure models

As is shown in the previous section, the instantaneous forward rate in each model has a complicated form with many parameters. Even the standard model of Vasicek (1977) has not a simple form. In this section, we give intuitive interpretation to the instantaneous forward rate in purely Gaussian term structure models. At first, we define forward martingale measure.

Forward martingale measure with respect to maturity s is defined by Radon-Nikodym derivative,

$$\frac{dF_s}{dQ^*}=\frac{\exp\left(-\int_{t_0}^s r_u du\right)}{E^{Q^*}\left[\exp\left(-\int_{t_0}^s r_u du\right)\right]}.$$

By differentiating (2.5) with respect to s and using the definition of instantaneous forward rate (2.3), we obtain

$$\begin{aligned}
f(t,s) &= -\frac{\partial}{\partial s}\ln P(t,s)\\
&= -\frac{1}{P(t,s)}\left(E^{Q^*}\left[-r_s\exp\left(-\int_t^s r_u du\right)\middle|\mathcal{F}_t\right]\right)\\
&= E^{Q^*}\left[\left(\frac{\exp\left(-\int_t^s r_u du\right)}{E^{Q^*}\left[\exp\left(-\int_t^s r_u du\right)\middle|\mathcal{F}_t\right]}\right)r_s\middle|\mathcal{F}_t\right]\\
&= E^{F_s}[r_s|\mathcal{F}_t]
\end{aligned}$$

Thus, the instantaneous forward rate is equal to the expectation of future short rate under the forward martingale measure. We will use the forward martingale measure for the decomposition of instantaneous forward rate.

For convenience, let us define $f^*(t,s)$ as

$$f^*(t,s)=E^{Q^*}[r_s|\mathcal{F}_t].$$

Then, it is clear that

$$\begin{aligned}
& f(t,s) - f^*(t,s) \\
= & \left(-\frac{\partial}{\partial s}\ln E^{Q^*}\left[\exp\left(-\int_t^s r_u du\right)\bigg|\mathcal{F}_t\right]\right) \\
& -\left(-\frac{\partial}{\partial s}E^{Q^*}\left[\ln\left\{\exp\left(-\int_t^s r_u du\right)\right\}\bigg|\mathcal{F}_t\right]\right) \\
= & -\frac{\partial}{\partial s}\left(\ln E^{Q^*}\left[\exp\left(-\int_t^s r_u du\right)\bigg|\mathcal{F}_t\right] - E^{Q^*}\left[\ln\left\{\exp\left(-\int_t^s r_u du\right)\right\}\bigg|\mathcal{F}_t\right]\right).
\end{aligned}$$

So, the difference $f(t,s) - f^*(t,s)$ captures the effect of Jensen's inequality. Note that this difference diminishes when there is no interest rate risk. And this difference persistently remains even when $Q^* = Q$. Thus the difference is interpreted as the pure effect of interest rate uncertainty or convexity. Hereafter we will call this Jensen's inequality effect. The instantaneous forward rate of interest rate can be decomposed into three components as

$$\begin{aligned}
f(t,s) &= E[r_s|\mathcal{F}_t] + (E^{Q^*}[r_s|\mathcal{F}_t] - E[r_s|\mathcal{F}_t]) + (E^{F_s}[r(s)|\mathcal{F}_t] - E^{Q^*}[r_s|\mathcal{F}_t]) \\
&= E[r_s|\mathcal{F}_t] + (f^*(t,s) - E[r_s|\mathcal{F}_t]) + (f(t,s) - f^*(t,s)).
\end{aligned}$$

The first term in the right hand side is the expectation of future short rate. Since the third term is the pure effect of interest rate uncertainty, the remaining term which is the second term(the first parentheses) is the term of risk premium. This term diminishes when the vector of market price of risk is the zero vector.

In Merton toy model, it is clear that $E[r_s|\mathcal{F}_t] = r_t + \mu_r(s-t)$ and $E^{Q^*}[r_s|\mathcal{F}_t] = r_t + (\mu_r + \lambda\sigma_r)(s-t)$. Thus, Jensen's inequality effect is given by the quadratic function with respect to time to maturity $-\frac{1}{2}\sigma_r(s-t)^2$ and the risk premium is given by a linear function $\lambda\sigma_r(s-t)$.

In Vasicek model, $E[r_s|\mathcal{F}_t] = e^{-\kappa(s-t)}r_t + \left(1 - e^{-\kappa(s-t)}\right)\theta$ and $E^{Q^*}[r_s|\mathcal{F}_t] = e^{-\kappa(s-t)}r_t + \left(1 - e^{-\kappa(s-t)}\right)\theta + \lambda\sigma_r\frac{1-e^{-\kappa(s-t)}}{\kappa}$. Thus, in Vasicek model, Jensen's inequality effect is given by $-\frac{1}{2}\sigma_r^2\left(\frac{1-e^{-\kappa(s-t)}}{\kappa}\right)^2$ and the risk premium is given by $\lambda\sigma_r\frac{1-e^{-\kappa(s-t)}}{\kappa}$.

2.3 Consumption-based approach to the Term Structure of Interest Rates

As is explained in the previous section, when we restrict the class of term sturucture models to purely Gaussian term structure models, we can give an interpretation to each term in the instantaneous forward rate of interest rate: the expectation of future short rate, the risk premium, and Jensen's inequality effect. While this interpretation is convenient in discussing the determinants of interest rates, it does not work when we analyze the relation between interest rates and the economic parameters concerning with the risk preference of agents and the expectation or variance of change in macroeconomic indicators. The consumption-based approach to the term structure of interest rates allows us to do this. In this section, we briefly review the literature on this approach.

Consumption-based approach with pure exchange economies is the most widely used in analysing the property of asset price. In this approach, it is usually assumed that only one type of perishable consumption goods is consumed and the process of aggregate endowment is given exogeneously. The risky asset is assumed to be in zero net supply and as a result, the consumption level of representative agent is equal to the level of aggregate endowment in equilibrium.

The asset pricing in a pure exchange economy was first considered in Lucas (1978). He considered a discrete time model and provided the general formula for the pricing of risky assets. The continuous time counterpart of Lucas's model is considered by Duffie and Zame (1989). They define the entire economy by a collection $((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q), (U, \{y_t\}), \{D_t\})$ where U is the preference of representative agent and $\{y_t\} = \{y_t : t \in [t_0, \tau]\}$ is the process of aggregate endowment, and $\{D_t\} = \{D_t : t \in [t_0, \tau]\}$ is the n-dimensional stochastic process of cumulative dividends which are paid by n securities[1]. By assuming that the preference of representative agent is defined over the stream of consumption

$$U(\{c_s : s \in [t_0, \tau]\}) = E\left[\int_{t_0}^{\tau} v(c_s, s)ds\right],$$

where $v(\cdot, s)$ is strictly concave increasing with the first derivative $v_c(\cdot, s)$ satisfying $\lim_{x\to 0} v_c(x, s) =$

[1]Duffie and Zame (1989) considered a multi-agent and complete market environment. In this dissertation, we start with an economy where the representative agent exists and the market is complete

$+\infty$ and the aggregate endowment follows an Ito process, they showed that there exists an equilibrium such that security price at time t, S_t^k, $(t \in [t_0, \tau))$ satisfies

$$S_t^k = E\left[\int_t^\tau \frac{v_c(y_s, s)}{v_c(y_t, t)} dD_s^k \right| \mathcal{F}_t\Bigg], \;\; k = 1, \cdots, n.$$

Supoose that the cumulative dividend process of security k is defined by

$$D_u^k = \begin{cases} 0 & u < s \\ 1 & u \geq s \end{cases}.$$

Then, this security is the pure discount bond with maturity s and its price is given by

$$P(t,s) = S_t^k = E\left[\frac{v_c(y_s, s)}{v_c(y_t, t)} \right| \mathcal{F}_t\Bigg]. \tag{2.7}$$

As is explained in the previous section, in determining a term structure model, the two components, the short rate and the market price of risk should be specified. In the consumption-based approach, we can give some expressions to these components in terms of marginal utility of representative agent. From (2.4), the short rate in the equilibrium is given by

$$\begin{aligned} r_t &= \lim_{s \to t} f(t,s) \\ &= -\lim_{s \to t} \frac{\partial \ln P(t,s)}{\partial s} \\ &= -\lim_{s \to t} \frac{\partial E[v_c(y_s, s)|\mathcal{F}_t]/\partial s}{E[v_c(y_s, s)|\mathcal{F}_t]} \\ &= -\frac{\lim_{\Delta \to 0} \frac{E[v_c(y_{t+\Delta}, t+\Delta) - v_c(y_t, t)|\mathcal{F}_t]}{\Delta}}{v_c(y_t, t)}. \end{aligned}$$

That is, the short rate is the negative of instantaneous expected rate of change in the marginal utility. By Ito's lemma, we can have the following expression,

$$d \ln v_c(y_t, t) = -r_t - \frac{1}{2}\sigma_{v,t}^\top \sigma_{v,t} dt + \sigma_{v,t}^\top dW_t,$$

or

$$v_c(y_s, s) = v_c(y_t, t) \exp\left(-\int_t^s r_u du - \int_t^s \frac{1}{2}\sigma_{v,u}^\top \sigma_{v,u} du + \int_t^s \sigma_{v,u}^\top dW_u\right).$$

Substituting above equation into (2.7), we obtain

$$P(t,s) = E\left[\exp\left(-\int_t^s \frac{1}{2}\sigma_{v,u}^\top \sigma_{v,u} du + \int_t^s \sigma_{v,u}^\top dW_u\right) \exp\left(-\int_t^s r_u du\right) \right| \mathcal{F}_t\Bigg].$$

Finally, comparing this equation with (2.6), we can conclude that $\Lambda_t = -\sigma_{v,t}$. That is, the market price of risk is given by the negative of volatility of marginal utility.

As the most simple example, let us consider a pure exchange economy where $n = 1$ and the aggregate endowment follows a process,

$$\frac{dy_t}{y_t} = \mu_t\, dt + \sigma dW_t,$$

where μ_t follows a deterministic process and σ is a constant. It is clear that the unconditional distribution of $\ln y_t$ is normal. Throughout this dissertation, we will call the process of aggregate endowment is a *Gaussian endowment process* when the aggregate endowment has this property.

Next, let us assume that the preference of the representative agent is given by the standard CRRA utility,

$$v(c_t, t) = e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma},\ \gamma > 0.$$

It is straightforward to show that the short rate is given by

$$r_t = \rho + \gamma\mu_t - \frac{1}{2}\gamma^2\sigma^2,\ \forall t \in [t_0, \tau). \tag{2.8}$$

The market price of risk is given by $\gamma\sigma$. When $\sigma = 0$, then there is no uncertainty. In this special case, the short rate is given by $r_t = \rho + \gamma\mu_t$. The parameter γ is the coefficient of relative risk aversion. At the same time, it is also the reciprocal of the elasticity of intertemporal substitution since the preference of representative agent is a time-additive utility. When there is no uncertainty, this parameter is better interpreted as the second one and the second term of right hand side of (2.8) is usually interpreted as the effect of intertemporal substitution. The third term which appears when $\sigma \neq 0$ is usually interpreted as the effect of precautionary saving effect. In this term, γ should be interpreted as the coefficient of relative risk aversion.

While the above simple model provides some insight for the determination of interest rates, there is no interest rate uncertainty in this model. To investigate the movements of term structure of interest rates, we should extend the model. Natural extension is to modify the process of μ_t to some stochastic process. Goldstein and Zapatero (1996) considered this extension. They assume that the process of μ_t is given by an Ornstein-Uhlenbeck process

$$d\mu_t = \kappa(\bar{\mu} - \mu_t)dt + bdW_t,$$

where $\bar{\mu}$, κ, and b are constants. Under this extension, the expression of the short rate is still the same as (2.8). But the short rate process is stochastic and given by

$$dr_t = \kappa(\theta - r_t)dt - \sigma_r dW_t,$$

where θ and σ_r are given by

$$\begin{aligned}\theta &= \rho + \gamma\bar{\mu} - \frac{1}{2}\gamma^2\sigma^2, \\ \sigma_r &= -\gamma b.\end{aligned}$$

Since the market price of risk is $\gamma\sigma$ in this model, this model generates Vasicek model. Clearly when $\kappa = 0$, the model generates Merton toy model. Thus, Goldstein and Zapatero (1996) provide some insights for the determinants of term structure model. For instance, the speed of mean-reversion is identical among the short rate and μ_t. Thus, when the market considers that the (instantaneous) expected rate of growth of endowment is persistent, then the speed of mean-reversion of the short rate is slow and vice versa. The volatility of the short rate is determined by the ratio of the volatility of μ_t to the elasticity of intertemporal substitution. The interest rate risk becomes large when the volatility of income growth becomes large, but also when the elasticity of intertemporal substitution becomes low.

There is another formulation of pure exchange economy to generate Vasicek model other than the framework of Goldstein and Zapatero (1996). Assume that $n = 1$ and the endowment process is given by

$$\frac{dy_t}{y_t} = \mu dt + \sigma dW_t,$$

where μ and σ are constants. And assume that the representative agent has consumption habit and denote it as z_t. It is defined by the following equation,

$$z_t = \exp\left(\kappa_c \int_{t_0}^{t} e^{-\kappa_c(t-u)} \ln c_u du\right),$$

where κ_c is a positive constant. Apparently κ_c is the parameter for weights which are put on the past consumption streams when we define the weighted average of past consumption. Applying Ito's lemma, the process of consumption habit is given by,

$$d\ln z_t = \kappa_c(\ln y_t - \ln z_t)dt.$$

Finally, assume that the preference of representative agent is defined by

$$v(c_t, z_t, t) = e^{-\rho t}\frac{(c_t/z_t)^{1-\gamma}}{1-\gamma}, \ \gamma > 0.$$

Under the presence of consumption habit, the short rate which is the negative of instantaneous expected change in the marginal utility of consumption is given by

$$r_t = \rho + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sigma^2 + (1-\gamma)\kappa_c(\ln y_t - \ln z_t).$$

Applying Ito's lemma, we can obtain the process of the short rate as

$$dr_t = \kappa_c(\theta - r_t)dt - \sigma_r dW_t,$$

where θ and σ_r are given by

$$\begin{aligned} \theta &= \rho + \mu - \frac{1}{2}\gamma(1+\gamma)\sigma^2, \\ \sigma_r &= -(1-\gamma)\kappa_c\sigma. \end{aligned}$$

Since the market price of risk is given by $\gamma\sigma$, this model also generates Vasicek model. An interesting message from this model is that the mean reversion of the short rate can occur even in the absence of mean reversion of instantaneous expected rate of growth of endowment μ. Specifically, the speed of mean reversion of the short rate is determined by κ_c. Thus, the interest rate is persistent when the representative agent puts less weight on the most recent consumption level. Conversely, when the agent puts more weight on the most recent consumption, the speed of mean reversion of the short rate is fast. This model shares common property with Wachter (2006) who considered a pure exchange economy in a discrete time model[2]. Wachter's model is reviewed in the next section.

[2]Since the definitions of consumption habit are slightly different between the above model and Wachter's model, the short rates in these two models have different forms.

2.4 Empirical Challenges to Term Structure Models and Research Questions

2.4.1 Relation between nominal interest rates and excess consumption

A growing literature analyzes the relation between the nominal spot rate curve and real economy based on factor models with no-arbitrage restrictions. For instance, Ang and Piazzesi (2003) construct a real economic factor by extracting the first principal component from real activity measures, including the index of Help Wanted Advertising in Newspapers (HELP) and industrial production growth. Along with many other findings, they demonstrate that nominal interest rates positively react to the real economic factor shocks. In addition, Bikbov and Chernov (2010) use HELP as a proxy for the real activity and suggest a positive relationship between the nominal interest rates and real activity. Further, Ang, Don, and Piazzesi (2007) estimate the Taylor rule with no-arbitrage restrictions, indicating that the nominal short rate increases after a positive shock to the output gap.

While these studies provide evidence of positive relation between the nominal interest rates and real activity, the models which take consumption-based approach cannot successfully explain this positive relationship. For instance, following Campbell and Cochrane (1999), Wachter (2006) explores a discrete time consumption-based model of term structure of interest rates with habit persistence. In her model, utility function at each time is defined by $v(c_t, z_t) = e^{-\rho t}\frac{(c_t - z_t)^{1-\gamma}-1}{1-\gamma}$ where z_t is the level of external habit. Assuming that the logarithm of aggregate endowment follows a random walk process and denoting consumption excess to habit (excess consumption, hereafter) as $s_t = \ln\frac{c_t - z_t}{c_t}$, she derives the real short rate as (using her notations),

$$r_{f,t+1} = \rho + \gamma g - \frac{\gamma(1-\phi) - b}{2} + b(\bar{s} - s_t),$$

where g is the instantaneous expected rate of change in endowment and ϕ and b are parameters to be estimated. After estimating parameters, she finds that b is significantly positive. This is consistent with the result of simple regression which shows the negative relation between the real interest rates and excess consumption.

The nominal short rate in her model is given by,

$$r^{\$}_{f,t+1} = r_{f,t+1} + E_t[\Delta\pi_{t+1}] - \frac{1}{2}\sigma_\pi^\top\sigma_\pi - \sigma_\pi^\top\sigma_c\gamma(\lambda(s_t)) + 1),$$

where $\Delta\pi_t$ is the rate of inflation at time t and $\lambda(s_t)$ is defined by,

$$\lambda(s_t) = (1/\bar{S})\sqrt{1 - 2(s_t - \bar{s})} - 1.$$

σ_π is the vector of volatility of inflation and σ_c is the vector of volatility of consumption. Under her estimation, the covariance $\sigma_\pi^\top\sigma_c$ is negative. This means that the nominal short rate is also negatively related with excess consumption, since $\lambda(s_t)$ is decreasing in excess consumption. Thus, in Wachter's model, the positive relation between the nominal short rate and real activity cannot be explained with the parameters estimated.

Buraschi and Jiltsov (2007) propose a new class of non-affine models that link the macro economic variables and the spot rate curve when preferences are subject to habit persistence. Similar to Wachter's model, their model can reproduce many characteristics of the term structures of interest rates, but suggests a negative correlation between the excess consumption and nominal interest rates.

One may think that there is no exact linkage between the excess consumption and other macroeconomic indicators such as HELP index and it is possible that the nominal interest rate is positively related with HELP index whereas it is negatively related with the excess consumption. Let us affirm that there is positive correlation between the actual nominal interest rates and excess consumption by U.S. data. We conduct a simple regression analysis motivated by Wachter (2006)[3]. Wachter regresses the ex post real interest rate on surplus consumption proxy to see the negative relation between interest rates and surplus consumption. Following this idea, we regress the nominal interest rates for several maturities on excess consumption and price level proxies. Thus, the regressions we estimated are

$$r_{t+1}(n) = \alpha_0 + \alpha_1 \sum_{j=1}^{40} \phi^j \Delta \ln y_{t-j} + \alpha_2 \sum_{j=1}^{40} \phi^j \Delta \ln p_{t-j} + \varepsilon_{t+1}, \tag{2.9}$$

[3] Our regression is based on the quarterly data on consumption and price level from the first quarter of 1952 to the second quarter of 2007. The nominal yield data are quarterly treasury constant maturity rates with maturities of one, two, three years. These data are from the second quarter of 1962 for all maturities. Interest rates are obtained from the Global Financial Data and other data are taken from the Federal Reserve Economic Data (FRED).

where $r_{t+1}(n)$ is the nominal yield with maturity n. y_t and p_t are real consumption and price level respectively. Following Wachter (2006), ϕ is set to equal 0.97.

Maturity	Estimate of α_1	Std. error of α_1	Estimate of α_2	Std. error of α_2
1 year	0.192^{**}	0.082	0.205^{**}	0.024
2 years	0.146^{**}	0.041	0.208^{**}	0.012
3 years	0.122^{*}	0.071	0.205^{**}	0.020

Table 2.1: Estimates for the coefficient of the excess consumption in the regression (2.9). * and ** indicate results are significant at the 5% and 1% significance levels, respectively.

Table 2.1 reports the estimates and Newey-West standard errors for several maturities. In contrast with Wachter's result of negative estimates for α_1, the parameter α_1 is estimated to be positive and statistically significant for all short maturities, one, two, and three years. This result suggests that if we consider the nominal interest rate as a dependent variable and treat inflation as an explanatory variable, we can find a positive relation between the nominal short term interest rate and the excess consumption.

Overall, although the models with habit persistence show attractive features, they are not successful in explaining the comovements between nominal interest rates and excess consumption. Thus, we set two research questions:

- *Is there an equilibrium model which can generate positive relation between the nominal interest rates and excess consumption?*

- *If there is, what is the role for excess consumption in that model?*

In chapter 3, we consider a new model which takes consumption-based approach. We consider a pure exchange economy with Gaussian endowment process, but the economic variables are assumed to be partially observable. We investigate the equilibrium in this model and examine whether the nominal interest rates are positively related to the excess consumption.

2.4.2 Expectation puzzle

Let us denote R_i^n as time i spot rate with time to maturity n in months. Consider a simple regression,

$$R_{i+1}^{n-1} - R_i^n = \text{constant} + \phi_n \frac{R_i^n - R_i^1}{n-1} + \text{error term}.$$

The dependent variable is the change in yield in the next month and the dependent variable is the slope of current spot rate curve. The traditional expectation hypothesis which states that the slope of curve contains the information regarding the market expectation of future interest rates implies the regression coefficients are unity for all n.

On the contrary, Campbell and Shiller(1991) documented that this implication has been consistently rejected. The regression coefficient is significantly far from unity and even negative[4]. This situation becomes severe for longer maturities. This is called as "expectation puzzle". Table 2.2 shows the results reported in Dai and Singleton(2002).

time to maturity(in months)	12	24	48	84	120
regression coefficient ϕ_n	-1.425	-1.705	-2.147	-3.096	-4.173

Table 2.2: Regression coefficients reported in Dai and Singleton(2002). The regression coefficients are all negative and the absolute value is larger for longer maturities.

Since the puzzle implies that the slope of current curve and the change in yields in the next month are negatively correlated, it can be considered that the spot rate curve rotates under the mean reversion of short term interest rates. In other words, the expectation puzzle implies that short term interest rates and long term interest rates move in the opposite direction. For instance, when the short term interest rates are relatively high, it is likely that the spot rate curve has negative slope. After we observe the negative slope of the curve, the short term interest rates tend to fall by mean reversion. On the other hand, from the negative correlation with the slope, the long term interest rates rise in average. Thus, when the expectation puzzle occurs, it seems that the rotation of the curve is likely to occur.

There are many researches which successfully explain the expectation puzzle. For instance, Dai and Singleton(2002) and Duffee(2002) explain the puzzle by essentially affine

[4] Hereafter we will call this regression as Campbell-Shiller test

term structure models in which the market price of risk is time-varying. Li and Song(2012) introduce jumps into the affine term structure models and explain the puzzle and the humped shape of volatility curve simultaneously. There are also utility based models of term structure which explain the expectation puzzle. Wachter(2006) shows that the puzzle can occur in a pure exchange economy with external habit. Buraschi and Jiltsov(2007) show the puzzle can occur by introducing the money in the utility function.

The important point here is that all of the above models explain the puzzle by the time variation of market price of risk. The intuition behind these models is as follows. Suppose that the market price of risk is given as a decreasing function of the short term interest rate. When the short term interest rate is relatively high, it will fall in the next period by mean reversion. On the other hand, the risk premium of long term bonds is small or even negative because the market price of risk is negatively related to the short term interest rate. Thus, the price of long term bonds tends to decline in the next period. Since the short term interest rates fall and long term interest rates rise in the next period, the spot rate curve shifts as if it rotates and the expectation puzzle occurs.

In chapter 4, we consider a pure exchange economy with Gaussian endowment process. And we assume that the coefficient of relative risk aversion of representative agent is not constant. The research questions are as follows.

- *Is there an economic story which explains the expectation puzzle other than the story with time-varying market price of risk?*

- *What kind of affine term structure model is corresponding to this story?*

- *Is it possible that an affine term structure model with constant market price of risk explains the expectation puzzle?*

- *What is the difference between our affine term structure model which solves the puzzle and the existing affine term structure models?*

2.4.3 Humped shape of volatiity curve

There is substantial evidence that spot rates exhibit time-varying conditional volatility. On the other hand, in most of one-factor term structure models, the conditional volatility is constant through time. For instance, in Vasicek model, the volatility of the forward rate is given by

$$\frac{\partial f(t,s)}{\partial r_t} = \sigma_r e^{-\kappa(s-t)}. \tag{2.10}$$

Another important evidence is that the term structure of unconditional volatilities of spot rates tends to be hump-shaped. It is reported that the curve shows a hump that peaks around two to three years in time to maturity. Vasicek model does not have this property. (2.10) shows that volatility of forward rates is monotone decreasing in time to maturity. This feature is widely shared among the one-factor affine term structure models[5]. The questions addressed in this dissertation are:

- *Is there any one-factor term structure model in which the volatility curve is humped shape?*

- *What is the story behind the volatility hump from the perspective of consumption based approach?*

[5]Even the one-factor term structure models which are more general than affine term structure models do not exhibit humped-shaped volatility curve. For instance, the volatility curve is monotone in the one-factor quadratic term structure models. For more details, see Dai and Singleton (2004).

Chapter 3

Term Structure when Economic Varibles are Partially Observable

3.1 Introduction

As is explained in chapter 2, the relation between the excess consumption and the nominal interest rates is not successfully explained by equilibrium models. The main objective of this chapter is to provide a new equilibrium model that naturally generates a positive correlation between the nominal interest rates and excess consumption. Throughout this chapter, we focus on the partial observability of economic variables in a pure exchange economy where the aggregate endowment and its price follow a system of Gaussian processes. In a complete information model of pure exchange economy where the aggregate endowment follows a Gaussian process, the instantaneous expected rate of change in endowments is the state variable of the term structure of interest rates. On the other hand, in the real world, agents in the economy cannot observe the expected change in income. Rather, they infer the expected change in income based upon available information including the past income stream. This is also true for price level. From this point of view, we set up a model where the level of aggregate endowment and its price are observable but their instantaneous expected growth rates are not.

There are several studies that explore the role of partially observed income on consumption. For instance, Wang (2004) considers the optimal consumption rule when the agent can only observe her total income, not individual components. Guvenen (2007) proposes two stochastic income processes for life-cycle consumption behavior, and provides a systematic comparison of these implications to the US data. In addition, Wang (2009) investigates an individual's optimal consumption-saving and portfolio choice problem with unobservable income growth. Unlike these works, this chapter examines the market equilibrium interest rates when some economic variables are partially observable.

The effects of an unobservable factor on the market equilibrium have been studied also by a number of researchers. For instance, Detemple (1986), Dothan and Feldman (1986), and Björk, Davis, and Landen (2010) study the economy where an unobservable state variables exists, but these works concentrate on methodological issues. Other studies focus on the term structure of interest rates and investigate the functional relation between the interest rates and economic agents' estimates of an unobservable factor. Those examples include Feldman (1989), Feldman (2003), and Riedel (2000a, 2000b). This chapter differs from these previous studies in two important ways: First, our model includes two unobservable factors, but we still derive a closed-form solution for the equilibrium interest rates. As a consequence, our model can include two most important economic variables, income (consumption) growth and inflation rates. This is an important extension, since it makes the term structure behaviors more realistic without losing tractability. Second, this study investigates the functional relation between the estimates of unobservable factors and observable variables in detail to obtain the equilibrium interest rates as a function of observable variables. This is relevant as we can directly see the relation between the interest rates and macro economic variables.

The contributions of the chapter are summarized as follows. First, we derive closed-form solutions for nominal equilibrium interest rates. In our model of partial observability, the resulting nominal term structure model turns out to be a two-factor purely Gaussian affine model in which state variables are the economic agents' estimates of instantaneous expected growth rates of endowment and its price. In addition, with the stationary error process assumption, we show these state variables can be expressed as weighted sums of excess consumptions and price levels. This characterization of the agents' estimates is not found in the relevant literature to the best of our knowledge.

Second, with the characterization mentioned above, our model with partial observability provides a quite different interpretation to the role of excess consumption in determining interest rates from the consumption habit models. For economic agents engaging in the Bayesian inference, the excess consumption is not the surplus consumption giving them felicity. Rather, it is an economic indicator helping guess the current trend in income growth. Naturally, the economic agents' estimate of income growth, hence the equilibrium interest rates, can be increasing in excess consumption under some mild conditions on parameters. This forms a striking contrast to the consumption habit models in which the intertemporal substitution effect induces negative correlation between the excess consumption and interest rates.

Third, we conduct the empirical analysis based on our model. This is an important contribution, since the previous papers about incomplete information equilibrium mostly focus on theoretical issues, but not empirical implementations. The parameters for the system of stochastic differential equations are estimated from real consumption and inflation data, while the preference parameters are estimated using time series of interest rates. These results indicate reasonable values for all parameters and, more importantly, the positive correlation between the implied interest rates and excess consumption. In addition, the time series of the nominal yield implied by the model captures many of the short- and long-run fluctuations in the actual data for all maturities, in particular the short-term. Indeed, correlations between the time series of model-implied yields and the actual data are higher than those obtained by Wachter (2006).

This chapter is organized as follows. Section 2 describes a pure exchange economy where the representative agent exists. Section 3 shows how the representative agent infers the unobservable variables. Section 4 investigates the property of the equilibrium forward rate of interest rates and show that the nominal interest rates can be positively correlated with excess consumption. Section 5 explains data and discusses the empirical results, and then Section 6 concludes.

3.2 The Model

Consider a pure exchange economy of a single perishable consumption good. The time span of this economy is $[t_0, \tau]$. Let $(\Omega, \mathcal{F}, Q)$ be a complete probability space. Filtration $\{\mathcal{F}_t : t \in [t_0, \tau]\}$ denotes the $Q-$augmentation of natural filtration generated by four Brownian motions, $W_{y,t}$, $W_{p,t}$, $\widehat{W}_{y,t}$, and $\widehat{W}_{p,t}$. These Brownian motions are mutually independent except that $\widehat{W}_{y,t}$, $\widehat{W}_{p,t}$ are correlated. The correlation between $\widehat{W}_{y,t}$ and $\widehat{W}_{p,t}$ is described by $E\left(d\widehat{W}_{y,t}d\widehat{W}_{p,t}\middle|\mathcal{F}_t\right) = \hat{\rho}dt$ where $\hat{\rho}$ is constant.

The economy is endowed with a flow of the consumption good. The rate of aggregate endowment flow in real term and its price are denoted as y_t and p_t, $t \in [t_0, \tau]$. In this chapter, it is assumed that y_t, p_t, and their instantaneous expected change $\mu_{i,t}$ $(i = y, p)$ follow the system of stochastic differential equations,

$$
\begin{aligned}
\frac{dy_t}{y_t} &= \mu_{y,t}dt + \sigma_y dW_{y,t}, && (3.1)\\
d\mu_{y,t} &= \kappa_y(\bar{\mu}_y - \mu_{y,t})dt + \upsilon_y dW_{y,t} + \hat{\upsilon}_y d\widehat{W}_{y,t}, && (3.2)\\
\frac{dp_t}{p_t} &= \mu_{p,t}dt + \sigma_p dW_{p,t}, && (3.3)\\
d\mu_{p,t} &= \kappa_p(\bar{\mu}_p - \mu_{p,t})dt + \upsilon_p dW_{p,t} + \hat{\upsilon}_p d\widehat{W}_{p,t}, && (3.4)
\end{aligned}
$$

where κ_i, $\bar{\mu}_i$, σ_i, $\hat{\upsilon}_i$ $(i = y, p)$ are positive constants. We do not restrict the sign of constants, υ_i $(i = y, p)$, allowing for negative correlation between changes in level and instantaneous expected change. Although infinitesimal changes in endowment and price are independent, this does not mean that discrete time changes in both variables are independent since $\mu_{y,t}$ and $\mu_{p,t}$ are correlated.[1]

Throughout this chapter, we consider a model where y_t and p_t are observable but $\mu_{y,t}$ and $\mu_{p,t}$ are not. It is assumed that the true value for each parameter is known to all of the agents. Thus, the agents infer $\mu_{y,t}$ and $\mu_{p,t}$, given the history of endowment and price levels up to time t.

Let us denote the $Q-$augmentation of natural filtration generated by y_t and p_t as $\{\mathcal{F}_t^{y,p} : t \in [t_0, \tau]\}$. It is assumed that the distribution of μ_{y,t_0} and μ_{p,t_0} conditional on $\mathcal{F}_{t_0}^{y,p}$ is normal. This is an important assumption for optimal filtering used in this chapter. The estimates of $\mu_{y,t}$ and $\mu_{p,t}$ inferred by economic agents are denoted as $m_{y,t}$ and $m_{p,t}$. That is, $m_{i,t} = E\left(\mu_{i,t}\middle|\mathcal{F}_t^{y,p}\right)$ $(i = y, p)$. Covariances of filtering errors are denoted as $\phi_{ij,t} = E\left[(\mu_{i,t} - m_{i,t})(\mu_{j,t} - m_{j,t})\middle|\mathcal{F}_t^{y,p}\right]$ $(i, j = y, p)$.

It is assumed that the representative agent exists and she is assumed to have preference

[1]Under the general local covariance structure, we can only numerically solve the matrix Riccati equation satisfied by the filtering error defined below. Our local covariance structure allows us to solve the matrix Riccati equation analytically and give an economic interpretation to bond prices.

over the consumption flows,

$$E\left[\int_{t_0}^{\tau} v(c_s, s)ds \middle| \mathcal{F}_{t_0}^{y,p}\right],$$

where the utility at each time is defined by $v(c_s, s) = e^{-\delta s}\frac{c_s^{1-\gamma}}{1-\gamma}$, $\gamma > 0$. It is also assumed that the market is frictionless and securities are traded continuously in time. $P(t, s)$ denotes time t price of pure discount bond which promises to pay one unit of currency at time $s \in (t_0, \tau]$.

3.3 Estimation Process for the Representative Agent

The representative agent draws inferences about $\mu_{i,t}$ $(i = y, p)$ by observing y_t and p_t. She forms a posterior distribution and continuously update it. Her updating process is described by [2]

$$\begin{aligned} dm_{y,t} &= \kappa_y(\bar{\mu}_y - m_{y,t})dt + \left(\upsilon_y + \frac{\phi_{yy,t}}{\sigma_y}\right) d\overline{W}_{y,t} + \frac{\phi_{yp,t}}{\sigma_p} d\overline{W}_{p,t}, & (3.5) \\ dm_{p,t} &= \kappa_p(\bar{\mu}_p - m_{p,t})dt + \frac{\phi_{py,t}}{\sigma_y} d\overline{W}_{y,t} + \left(\upsilon_p + \frac{\phi_{pp,t}}{\sigma_p}\right) d\overline{W}_{p,t}, & (3.6) \end{aligned}$$

where mutually independent Wiener processes $\{\overline{W}_{y,t}\}$ and $\{\overline{W}_{p,t}\}$ are defined as

$$\begin{aligned} d\overline{W}_{y,t} &= \left(\frac{1}{\sigma_y}\right)\left(d\ln y_t - \left(m_{y,t} - (1/2)\sigma_y^2\right)dt\right), \\ d\overline{W}_{p,t} &= \left(\frac{1}{\sigma_p}\right)\left(d\ln p_t - \left(m_{p,t} - (1/2)\sigma_p^2\right)dt\right), \end{aligned}$$

and $\Phi_t = [\phi_{ij,t}]$ $(i, j = y, p)$ is the filtering error process . The filtering error process satisfies the following matrix Riccati equation,

$$\frac{d}{dt}\Phi_t = K\Phi_t + \Phi_t K^\top - \Phi_t G \Phi_t^\top + H, \tag{3.7}$$

where 2×2 matrices $K = [k_{ij}], H = [h_{ij}], G = [g_{ij}]$ $(i, j = 1, 2)$ are defined by

$$\begin{aligned} k_{11} &= -\kappa_y - \upsilon_y/\sigma_y,\ k_{12} = k_{21} = 0,\ k_{22} = -\kappa_p - \upsilon_p/\sigma_p, \\ h_{11} &= \hat{\upsilon}_y^2,\ h_{12} = h_{21} = \hat{\upsilon}_y\hat{\upsilon}_p\hat{\rho},\ h_{22} = \hat{\upsilon}_p^2, \\ g_{11} &= 1/\sigma_y^2,\ g_{12} = g_{21} = 0,\ g_{22} = 1/\sigma_p^2. \end{aligned}$$

[2] See Proposition 12.6 in Liptser and Shiryaev (1977).

By solving matrix Riccati equation, we obtain the stationary level of filtering error $\lim_{t\to\infty}\Phi_t = \overline{\Phi} = [\bar{\phi}_{ij}]$ $(i,j = y,p)$ as[3]

$$\begin{aligned}\bar{\phi}_{yy} &= (\kappa_y^* - \kappa_y)\sigma_y^2 - \upsilon_y\sigma_y, &(3.8)\\ \bar{\phi}_{pp} &= (\kappa_p^* - \kappa_p)\sigma_p^2 - \upsilon_p\sigma_p, &(3.9)\\ \bar{\phi}_{yp} &= \bar{\phi}_{py} = \frac{\hat{\upsilon}_y\hat{\upsilon}_p\hat{\rho}}{\lambda_1+\lambda_2}, &(3.10)\end{aligned}$$

where[4]

$$\begin{aligned}\kappa_y^* &= \frac{k_{11}^2 + h_{11}/\sigma_y^2 + \lambda_1\lambda_2}{\lambda_1+\lambda_2}, \kappa_p^* = \frac{k_{22}^2 + h_{22}/\sigma_p^2 + \lambda_1\lambda_2}{\lambda_1+\lambda_2},\\ \lambda_1 &= \left(\frac{(k_{11}^2 + h_{11}/\sigma_y^2) + (k_{22}^2 + h_{22}/\sigma_p^2) + D^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}},\\ \lambda_2 &= \left(\frac{(k_{11}^2 + h_{11}/\sigma_y^2) + (k_{22}^2 + h_{22}/\sigma_p^2) - D^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}},\\ D &= \left(\left(k_{11}^2 + h_{11}/\sigma_y^2\right) - \left(k_{22}^2 + h_{22}/\sigma_p^2\right)\right)^2 + \frac{4h_{12}^2}{\sigma_y^2\sigma_p^2}.\end{aligned}$$

It is clear that κ_y^* and κ_p^* are strictly positive.

3.4 Stationary Model

In the sequel, we impose stationarity to our model. This is in part for simplicity and in part because we want to give an economic interpretation to $m_{y,t}$ and $m_{p,t}$. We set the following assumption.

Assumption 1 *The initial value of filtering error process is given by its stationary level. That is,* $\phi_{ij,t_0} = \bar{\phi}_{ij}$ *for* $i,j = y,p$.

Obviously, the assumption implies that $\phi_{ij,t} = \bar{\phi}_{ij}, (i,j = y,p)$ for any $t \geq t_0$. Under this assumption, we can express the equilibrium bond price as follows.

Proposition 1 *The equilibrium bond prices are*

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right),$$

[3]The derivation of the solution and its limit is given in the appendix.

[4]A sufficient condition for λ_2 to be a real number is $h_{11}h_{22} - h_{12}^2 \geq 0$. But this inequality always holds. Thus, λ_2 is a real number.

where the nominal instantaneous forward rate of interest rate is

$$
\begin{aligned}
f(t,s) \;=\;& \delta+\gamma\left(m_{y,t}e^{-\kappa_y(s-t)}+\bar{\mu}_y\left(1-e^{-\kappa_y(s-t)}\right)-\sigma_y^2/2\right)-\gamma^2\sigma_y^2/2\\
&+m_{p,t}e^{-\kappa_p(s-t)}+\bar{\mu}_p\left(1-e^{-\kappa_p(s-t)}\right)-\sigma_p^2\\
&-\left(\frac{1-e^{-\kappa_y(s-t)}}{\kappa_y}\right)\left(\gamma^2\sigma_y^2(\kappa_y^*-\kappa_y)+\gamma\bar{\phi}_{py}\right)\\
&-\left(\frac{1-e^{-\kappa_p(s-t)}}{\kappa_p}\right)\left(\sigma_p^2(\kappa_p^*-\kappa_p)+\gamma\bar{\phi}_{yp}\right)\\
&-\frac{1}{2}\gamma^2\left(\frac{1-e^{-\kappa_y(s-t)}}{\kappa_y}\right)^2\left(\sigma_y^2(\kappa_y^*-\kappa_y)^2+\left(\frac{\bar{\phi}_{yp}}{\sigma_p}\right)^2\right)\\
&-\frac{1}{2}\left(\frac{1-e^{-\kappa_p(s-t)}}{\kappa_p}\right)^2\left(\sigma_p^2(\kappa_p^*-\kappa_p)^2+\left(\frac{\bar{\phi}_{py}}{\sigma_y}\right)^2\right)\\
&-\gamma\left(\frac{1-e^{-\kappa_y(s-t)}}{\kappa_y}\right)\left(\frac{1-e^{-\kappa_p(s-t)}}{\kappa_p}\right)\\
&\quad\times\left((\kappa_y^*-\kappa_y)\bar{\phi}_{py}+(\kappa_p^*-\kappa_p)\bar{\phi}_{yp}\right). \qquad (3.11)
\end{aligned}
$$

Proof. See appendix.

Thus, our equilibrium model can be identified as a two-factor completely affine term structure model where state variables $m_{y,t}$ and $m_{p,t}$ follow (3.5) and (3.6) with each filtering error replaced by its stationary level and the market prices of risk for $\overline{W}_{y,t}$ and $\overline{W}_{p,t}$ are $\gamma\sigma_y$ and σ_p.[5] As in Feldman (1989), we can decompose the instantaneous forward rate into three parts. The sum of the first two lines in (3.11) is the expectation of future short rate, and the sum of the next two lines is the risk premium. The remaining terms correspond to the Jensen's inequality bias.

The processes of state variables $m_{y,t}$ and $m_{p,t}$ are not time-homogeneous in the sense that they are affected by the initial values m_{y,t_0}, m_{p,t_0}, y_{t_0}, and p_{t_0}. To remove these effects, we take the limit of the state variables by letting t_0 go to $-\infty$. The following proposition shows the result.

Proposition 2 *The stationary processes of $m_{y,t}$ and $m_{p,t}$ are given by*

$$\lim_{t_0\to-\infty} m_{y,t} \;=\; \frac{\kappa_p^* c_y-(\bar{\phi}_{yp}/\sigma_p^2)c_p}{a_1a_2}$$

[5] The term structure in our model does not explode as in Riedel (2000a). This confirms the results of Feldman (2003). Also, since the market prices of risk are constant, the equilibrium avoids the pitfalls identified in Kraft (2009).

$$
\begin{aligned}
&+\frac{(a_1-\kappa_y)(\kappa_y^*-a_2)}{a_1-a_2}\left(\ln y_t-\int_{-\infty}^t a_1 e^{-a_1(t-s)}\ln y_s\,ds\right)\\
&-\frac{(a_2-\kappa_y)(\kappa_y^*-a_1)}{a_1-a_2}\left(\ln y_t-\int_{-\infty}^t a_2 e^{-a_2(t-s)}\ln y_s\,ds\right)\\
&+\frac{(a_1-\kappa_p)(\bar\phi_{yp}/\sigma_p^2)}{a_1-a_2}\left(\ln p_t-\int_{-\infty}^t a_1 e^{-a_1(t-s)}\ln p_s\,ds\right)\\
&-\frac{(a_2-\kappa_p)(\bar\phi_{yp}/\sigma_p^2)}{a_1-a_2}\left(\ln p_t-\int_{-\infty}^t a_2 e^{-a_2(t-s)}\ln p_s\,ds\right), \qquad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{t_0\to-\infty} m_{p,t} = &\frac{\kappa_y^* c_p-(\bar\phi_{py}/\sigma_y^2)c_y}{a_1a_2}\\
&+\frac{(a_1-\kappa_p)(\kappa_p^*-a_2)}{a_1-a_2}\left(\ln p_t-\int_{-\infty}^t a_1 e^{-a_1(t-s)}\ln p_s\,ds\right)\\
&-\frac{(a_2-\kappa_p)(\kappa_p^*-a_1)}{a_1-a_2}\left(\ln p_t-\int_{-\infty}^t a_2 e^{-a_2(t-s)}\ln p_s\,ds\right)\\
&+\frac{(a_1-\kappa_y)(\bar\phi_{py}/\sigma_y^2)}{a_1-a_2}\left(\ln y_t-\int_{-\infty}^t a_1 e^{-a_1(t-s)}\ln y_s\,ds\right)\\
&-\frac{(a_2-\kappa_y)(\bar\phi_{py}/\sigma_y^2)}{a_1-a_2}\left(\ln y_t-\int_{-\infty}^t a_2 e^{-a_2(t-s)}\ln y_s\,ds\right), \qquad (3.13)
\end{aligned}
$$

where a_1 and a_2 are defined as

$$
\begin{aligned}
a_1 &= \frac{\kappa_y^*+\kappa_p^*+\sqrt{(\kappa_y^*-\kappa_p^*)^2+4(\bar\phi_{yp}/\sigma_p^2)(\bar\phi_{py}/\sigma_y^2)}}{2},\\
a_2 &= \frac{\kappa_y^*+\kappa_p^*-\sqrt{(\kappa_y^*-\kappa_p^*)^2+4(\bar\phi_{yp}/\sigma_p^2)(\bar\phi_{py}/\sigma_y^2)}}{2}.
\end{aligned}
$$

Proof. See appendix.

Since a_1 and a_2 are strictly positive, the first and second integral in the right hand side of (3.12) (or the third and fourth integral in (3.13)) can be interpreted as weighted averages of the past (natural log of) aggregate consumptions where heavy weights are put on the recent consumptions. Thus, if time t is sufficiently distant from the initial date of the economy t_0, then the estimation value of $\mu_{y,t}$ can be approximated by an affine function of excess aggregate consumption.

Are the interest rates positively related to the excess consumption? Since instantaneous forward rates are increasing in $m_{y,t}$ and $m_{p,t}$, this question is equivalent to asking whether $m_{y,t}$ and $m_{y,t}$ are increasing in the excess consumption. To answer this question, let us consider the special case of $\hat\rho = 0$ at first. This argument helps us understand the general case of non-zero correlation. If $\hat{\rho} = 0$, then (3.12) and (3.13) are reduced to

$$\lim_{t_0 \to -\infty} m_{y,t} = \frac{c_y}{\kappa_y^*} + (\kappa_y^* - \kappa_y)\left(\ln y_t - \int_{-\infty}^{t} \kappa_y^* e^{-\kappa_y^*(t-s)} \ln y_s ds\right), \tag{3.14}$$

$$\lim_{t_0 \to -\infty} m_{p,t} = \frac{c_p}{\kappa_p^*} + (\kappa_p^* - \kappa_p)\left(\ln p_t - \int_{-\infty}^{t} \kappa_p^* e^{-\kappa_p^*(t-s)} \ln p_s ds\right). \tag{3.15}$$

Thus, the interest rates depend on the excess consumption only through $\lim_{t_0 \to -\infty} m_{y,t}$ in the case of $\hat{\rho} = 0$. Let us define the correlation coefficient between changes in y_t and $\mu_{y,t}$ as ρ_{y,μ_y}, that is, $E\left[d \ln y_t \, d\mu_{y,t} \middle| \mathcal{F}_t\right] = \rho_{y,\mu_y} dt$. Then we can establish the following proposition.

Proposition 3 *If $\hat{\rho} = 0$, $\lim_{t_0 \to -\infty} m_{y,t}$ is increasing in the excess consumption if and only if the following inequality holds*

$$\rho_{y,\mu_y} \geq -\frac{\left(\upsilon_y^2 + \hat{\upsilon}_y^2\right)^{\frac{1}{2}}}{2\kappa_y \sigma_y}. \tag{3.16}$$

Proof. See appendix.

Note that the numerator of the right hand side of (3.16) is the volatility of $\mu_{y,t}$. The interesting property of the condition (3.16) is that the lower bound for the correlation coefficient is strictly negative. This is the important effect of unobservability of $\mu_{y,t}$ on the equilibrium interest rates. To understand this, let us consider the case in which the inequalities

$$-\frac{\upsilon_y^2 + \hat{\upsilon}_y^2}{2\kappa_y} < \upsilon_y \sigma_y < 0$$

hold. The second inequality means that changes in μ_t and y_t are locally negatively correlated. But changes in $m_{y,t}$ is increasing in the excess consumption since the condition in Proposition 3 is met by the first inequality. This interesting result holds, because $\mu_{y,t}$ is unobservable and an increase in the excess consumption, for instance, makes agents infer that $\mu_{y,t}$ has become high even under the negative correlation between changes in $\mu_{y,t}$ and y_t.

As a corollary, we can state a sufficient condition for $m_{y,t}$ to be increasing in the excess consumption for any correlation coefficient ρ_{y,μ_y}.

Corollary 1 *Suppose that $\hat{\rho} = 0$ and the following condition is met;*

$$\kappa_y \leq \frac{1}{2}\left(\upsilon_y^2 + \hat{\upsilon}_y^2\right)^{\frac{1}{2}} / \sigma_y. \tag{3.17}$$

Then, $\lim_{t_0 \to -\infty} m_{y,t}$ is increasing in the excess consumption for any correlation coefficient $\rho_{y,\mu_y} \in [-1, 1]$.

Proof. See appendix.

The fraction in the right hand side of (3.17) is the volatility ratio which measures the relative size of the volatility of $\mu_{y,t}$ to the volatility of y_t. When this ratio is large, it is likely that the variation in the level of consumption is mainly caused by the changes in $\mu_{y,t}$. Hence, the estimates are likely to be increasing in the level of consumption. Of course, when the speed of mean reversion is very fast, the past observed consumption levels are not informative, because the drift converges to its long run mean immediately. Hence, κ_y emerges in the left hand side of (3.17) as the bound for the volatility ratio.

One of the major approaches to asset pricing today is pricing risky assets via consumption habit. In this approach, consumption habit is incorporated into the preference on consumption flows, and prices of risky assets in equilibrium are derived. One of the important properties common to these models is that equilibrium asset returns or interest rates are determined by excess consumption. As for interest rates, the consumption habit normally induces a negative correlation between interest rates and the excess consumption.[6] Thus, if interest rates exhibit a positive correlation with the excess consumption, then it is difficult to explain the interest rates by consumption habit. In this case, partial observability of economic variables may become a candidate which explains interest rates through excess consumption.

In the general case of $\hat{\rho} \neq 0$, both state variables $\lim_{t_0 \to -\infty} m_{y,t}$ and $\lim_{t_0 \to -\infty} m_{p,t}$ are expressed as the weighted sum of excess consumption and excess price. Each excess level has two types, which are in excess of the weighted average with weights a_1 and a_2. Thus to see how the level of interest rates depends on excess consumptions, we should check the sign of four coefficients of excess consumptions in (3.12) and (3.13).

Table 3.1 and Table 3.2 show the sign of each coefficient of excess consumption in m_y and m_p. As is easily seen, it never happens that all of four coefficients are positive or negative. When $\kappa_y < a_2$, three coefficients are positive. Conversely, when $a_1 < \kappa_y$, only one coefficient is positive. When κ_y is between a_1 and a_2, the number of positive coefficients varies depending on the sign of covariance component of filtering error $\bar{\phi}_{py}$. Therefore, the overall effect of excess consumptions should be examined empirically. Nonetheless three things deserve to

[6] For example, see Wachter (2006) and Buraschi and Jiltsov (2007).

estimate	type of ex. con.	$\kappa_y < a_2$	$a_2 < \kappa_y < a_1$	$a_1 < \kappa_y$
$\lim_{t_0 \to -\infty} m_{y,t}$	ex. con. with weight a_1	$+$	$+$	$-$
	ex. con. with weight a_2	$+$	$-$	$-$
$\lim_{t_0 \to -\infty} m_{p,t}$	ex. con. with weight a_1	$+$	$+$	$-$
	ex. con. with weight a_2	$-$	$+$	$+$

Table 3.1: Sign of each coefficient of excess consumption when $\hat{\rho} > 0$.

estimate	type of ex. con.	$\kappa_y < a_2$	$a_2 < \kappa_y < a_1$	$a_1 < \kappa_y$
$\lim_{t_0 \to -\infty} m_{y,t}$	ex. con. with weight a_1	$+$	$+$	$-$
	ex. con. with weight a_2	$+$	$-$	$-$
$\lim_{t_0 \to -\infty} m_{p,t}$	ex. con. with weight a_1	$-$	$-$	$+$
	ex. con. with weight a_2	$+$	$-$	$-$

Table 3.2: Sign of each coefficient of excess consumption when $\hat{\rho} < 0$.

be mentioned. First, when covariance component $\bar{\phi}_{yp}$ is relatively small, $a_2 \approx \min\{\kappa_y^*, \kappa_p^*\}$, $a_1 \approx \max\{\kappa_y^*, \kappa_p^*\}$, and $\kappa_y^* \approx (k_{11}^2 + h_{11}/\sigma_y^2)^{\frac{1}{2}}$. In this case, the condition (3.17) in corollary 1 is helpful in understanding each case intuitively. When κ_y is sufficiently small compared to the volatility ratio $(\upsilon_y^2 + \hat{\upsilon}_y^2)^{\frac{1}{2}}/\sigma_y$, $\kappa_y < \kappa_y^*$ is likely to hold and the estimate of $\mu_{y,t}$ is likely to be increasing in both types of excess consumption. Conversely, when the local correlation between $\mu_{y,t}$ and y_t is negative and κ_y is relatively large, $\kappa_y > \kappa_y^*$ is likely to hold and excess consumptions have negative impacts on the estimate of $\mu_{y,t}$, thus the level of interest rates. Again, note that negative local correlation between $\mu_{y,t}$ and y_t does not necessarily imply that $\kappa_y > (k_{11}^2 + h_{11}/\sigma_y)^{\frac{1}{2}} \approx \kappa_y^*$.

Second, the similar result can be obtained by examining the sign of $\partial \lim_{t_0 \to -\infty} m_{y,t}/\partial \ln y_t$. Simple calculation shows that the necessary and sufficient condition for strict positivity of the derivative is $\kappa_y < \kappa_y^*$. It is clear that $\kappa_y < a_2$ is sufficient for this inequality.[7]

[7] This argument can apply to the comparative statistics with respect to excess consumptions as far as the excess consumptions are nearly zero at time t, since the dynamics of excess consumptions is given by

$$d \ln y_t - a_i \left(\ln y_t - \int_{-\infty}^{t} a_i e^{-a_i(t-s)} \ln y_s ds \right) \ (i = 1, 2).$$

Third, the necessary and sufficient condition for $\partial \lim_{t_0 \to -\infty} m_{p,t} / \partial \ln y_t > 0$ is $\hat{\rho} > 0$. When the drift of aggregate consumption process is negatively correlated with the drift of price level, an increase in the aggregate consumption makes the estimate of expected inflation rate updated to the lower level. The magnitude of this effect is determined by the relative size of covariance component in the filtering error $\bar{\phi}_{yp}/\sigma_y$. When this filtering error is small, changes in $\ln y_t$ mainly affect the level of interest rates via the estimate of consumption growth.

3.5 Empirical Analysis

In the last section, we showed that under the stationary error process assumption the nominal bond yields can be expressed as a function of weighted sums of past excess consumptions and price levels. The analysis also indicated that the bond yields heavily depends on model parameters which we have to estimate from the data. In this section, we estimate the parameters for the economy represented by the system of stochastic differential equations (3.1)-(3.4) from the real consumption and CPI data. Then we estimate the preference parameters by minimizing the distance between theoretical and observed time series of interest rates.

Our empirical analysis is based on the quarterly data on consumption and price level from the first quarter of 1952 to the second quarter of 2007. Data on the real per-capita consumption are constructed by adding the seasonally adjusted real consumption of nondurables and services, then dividing by the population. For the price level, we use the seasonally adjusted consumer price index (CPI). The nominal yield data are quarterly treasury constant maturity rates with maturities of one, two, three, five, seven, and ten years. These data are from the second quarter of 1962 for all maturities. Interest rates are obtained from the Global Financial Data and other data are taken from the Federal Reserve Economic Data (FRED).

3.5.1 The state-space representation and Kalman filter

To estimate the parameters for the economy represented by the system of stochastic differential equations (3.1)-(3.4), we employ the Kalman filter to treat unobservable state variables. To this end, we first derive the state-space representation for the discretized versions of the system of stochastic differential equations, (3.1)-(3.4). By the Euler approximation,[8] the system can be discretized as

$$\begin{aligned}
\log(y_{t+1}/y_t) &= -\frac{\sigma_y^2}{2} + \mu_{y,t} + \sigma_y u_{y,t}, && (3.18)\\
\mu_{y,t+1} &= \kappa_y \bar{\mu}_y + (1-\kappa_y)\mu_{y,t} + \upsilon_y u_{y,t} + \hat{\upsilon}_y \hat{u}_{y,t}, && (3.19)\\
\log(p_{t+1}/p_t) &= -\frac{\sigma_p^2}{2} + \mu_{p,t} + \sigma_p u_{p,t}, && (3.20)\\
\mu_{p,t+1} &= \kappa_p \bar{\mu}_p + (1-\kappa_p)\mu_{p,t} + \upsilon_p u_{p,t} + \hat{\upsilon}_p \hat{u}_{p,t}, && (3.21)
\end{aligned}$$

where $u_{y,t}, u_{p,t}, \hat{u}_{y,t}$, and $\hat{u}_{p,t}$ are mutually independently distributed as standard normal distribution except that $\hat{u}_{y,t}$ and $\hat{u}_{p,t}$ have correlation $\hat{\rho}$. Also we set $\Delta t = 1$ for notational simplicity. Since local trends of endowment and price level, $\mu_{y,t}$ and $\mu_{p,t}$, are not observable, we employ the Kalman filter to estimate this system, rewriting this system as a state-space model as follows. The transition equation consists of equations (3.19) and (3.21), and can be written in vector notation as

$$x_t = c_x + F x_{t-1} + G w_{t-1} + v_t, \tag{3.22}$$

where $x_t = \begin{bmatrix} \mu_{y,t} \\ \mu_{p,t} \end{bmatrix}$, $c_x = \begin{bmatrix} \bar{\mu}_y \kappa_y \\ \bar{\mu}_p \kappa_p \end{bmatrix}$, $F = \begin{bmatrix} 1-\kappa_y & 0 \\ 0 & 1-\kappa_p \end{bmatrix}$, $G = \begin{bmatrix} \upsilon_y/\sigma_y & 0 \\ 0 & \upsilon_p/\sigma_p \end{bmatrix}$, $w_t = \begin{bmatrix} \sigma_y u_{y,t} \\ \sigma_p u_{p,t} \end{bmatrix}$, and $v_t = \begin{bmatrix} \hat{\upsilon}_y \hat{u}_{y,t-1} \\ \hat{\upsilon}_p \hat{u}_{p,t-1} \end{bmatrix}$. The other two equations, (3.18) and (3.20), form the observation (measurement) equation, which can be expressed in vector notation as

$$z_t = c_z + x_t + w_t, \tag{3.23}$$

where $z_t = \begin{bmatrix} \log(y_{t+1}/y_t) \\ \log(p_{t+1}/p_t) \end{bmatrix}$, $c_z = \begin{bmatrix} -\sigma_y^2/2 \\ -\sigma_p^2/2 \end{bmatrix}$. Note that the disturbances $[v_t', \ w_t']'$ are jointly Gaussian with mean 0 and covariance matrix $\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$, where $Q = \begin{bmatrix} \hat{\upsilon}_y^2 & \hat{\rho}\hat{\upsilon}_y\hat{\upsilon}_p \\ \hat{\rho}\hat{\upsilon}_y\hat{\upsilon}_p & \hat{\upsilon}_p^2 \end{bmatrix}$, $R = \begin{bmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_p^2 \end{bmatrix}$.

Given the state-space representation, (3.22) and (3.23), the Kalman filter can be used to construct a likelihood function for the observed data. From some initial conditions, the

[8] See Kloeden and Platen (1995) for the Euler approximation of the stochastic differential equations.

filter iterates between the prediction equations,

$$\begin{aligned}
x_{t|t-1} &= c_x + F\hat{x}_{t-1} + G(z_{t-1} - c_z - H\hat{x}_{t-1}), \\
P_{t|t-1} &= (F - GH)\hat{P}_{t-1}(F - GH) + Q
\end{aligned}$$

and the updating equations,

$$\begin{aligned}
\hat{x}_t &= x_{t|t-1} + P_{t|t-1}H(HP_{t|t-1}H + R)^{-1}(z_t - c_z - Hx_{t|t-1}), \\
\hat{P}_t &= P_{t|t-1} - P_{t|t-1}H(HP_{t|t-1}H + R)^{-1}HP_{t|t-1},
\end{aligned}$$

where

$$\begin{aligned}
x_{t|t-1} &= E(x_t|\mathcal{F}_{t-1}), \\
\hat{x}_t &= E(x_t|\mathcal{F}_t), \\
P_{t|t-1} &= E[(x_t - x_{t|t-1})(x_t - x_{t|t-1})'|\mathcal{F}_{t-1}], \\
\hat{P}_t &= E[(x_t - \hat{x}_t)(x_t - \hat{x}_t)'|\mathcal{F}_t].
\end{aligned}$$

The derivation of the Kalman filter can be found in the appendix. To start the Kalman filter, we use the unconditional mean $E(x_t) = [\bar{\mu}_y,\ \bar{\mu}_p]'$ for the initial values of state vector $\hat{x}_0$. Since we have analyzed the equilibrium interest rates in detail under Assumption 1, we also impose this assumption to estimate the model. Thus, we use the stationary value of error process $\overline{\Phi}$ for $\hat{P}_t$ for all t including the initial value $\hat{P}_0$.

3.5.2 Estimation results

Estimation results for the state-space model, (3.22) and (3.23), are reported in Table 3.3. All parameters are in quarterly percent units, and means and standard deviations are in fractions. Thus, mean quarterly consumption growth over the period is 0.58% with the standard deviation of the error term about 0.32%, while mean inflation is 0.76% with the disturbance standard deviation about 0.37%. The results indicate that the mean reversion rate of expected consumption is moderate at 0.19. On the other hand, the expected inflation is highly persistent with the mean reversion rate of 0.071. The results also show that the correlations between observable and unobservable factors are essentially zero with statistically insignificant estimates of υ_y and υ_p. Lastly, the correlation between innovations to consumption and innovations to inflation is estimated as -0.60.

Parameter	Estimate	Std. error
$\bar{\mu}_y$	0.578	0.064
κ_y	0.186	0.000
σ_y	0.316	0.044
υ_y	−0.006	0.049
$\hat{\upsilon}_y$	0.201	0.031
$\bar{\mu}_p$	0.758	0.139
κ_p	0.071	0.000
σ_p	0.374	0.013
υ_p	−0.010	0.045
$\hat{\upsilon}_p$	0.234	0.034
$\hat{\rho}$	−0.600	0.000

Table 3.3: Parameter estimates for the system of y_t and p_t. The system is discretized by the Euler approximation and estimated by MLE via the Kalman filter. Data are quarterly, begin in the first quarter of 1952, and end in the second quarter of 2007.

By taking the temporal dependence in the realized consumption growth and inflation rates into account, Wachter (2006) estimates a vector ARMA$(1,1)$ model for consumption growth and inflation and obtains similar results with some differences. Our result of consumption growth 0.58% is similar to 0.55% obtained by Wachter. In contrast, our mean inflation estimate 0.76% is somehow smaller than hers (0.92%). In terms of persistence, our inflation result is comparable with Wachter's results, while she obtains less persistent consumption growth.

There remain only two preference parameters that need to be identified to determine the term structures of interest rates. We estimate these parameters so that the implied time series of interest rates based on (3.11) have the minimum squared errors. The maturities used for this estimation are one, five, and ten years, which roughly correspond to short-, middle-, and long-term of interest rates. The estimation result for the discount rate δ is 0.0010 in quarterly units with a standard error of 0.0004, while the relative risk aversion γ is estimated at 1.205 with a standard error of 0.066. These results are reasonable, suggesting the plausibility of our model.

3.5.3 Model implied interest rates based on estimation

In this subsection, we provide the implied time series of the equilibrium interest rates based on (3.11), given parameter estimates and estimated state variables in the previous subsection. In addition, we examine whether our empirical results imply positive relation between the real activity and equilibrium interest rates.

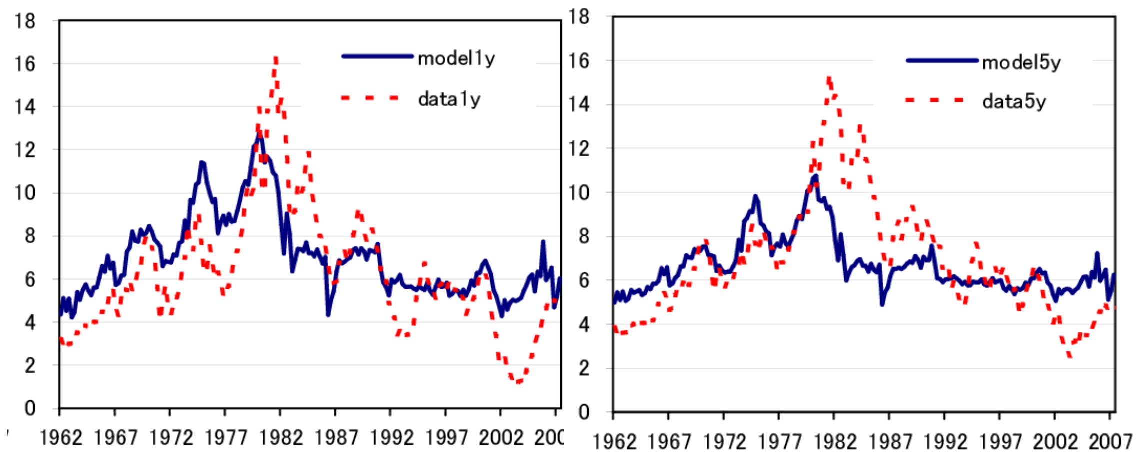

Figure 3.1: Time series of model-implied and actual yield. The bold curve represents model-implied yield and dashed curve corresponds to actual yield data.

Fig. 3.1 illustrates the model-implied time series of one and five-year nominal yields along with actual data. As can be seen, the model-implied yields closely follow the actual data, capturing many of the short- and long-run fluctuations in the actual data. Although the magnitude of fluctuations is somewhat smaller, the entire shape of the graph is similar to that of the actual data. Indeed, the correlations between the theoretical and observed yields is 0.72 for one-year yield and 0.65 for 5-year yield respectively. This is considerably higher than 0.50 for three-month yield or 0.34 for spread on the five-year yield over the three-month yield obtained by Wachter (2006).

Are the interest rates increasing in excess consumption? Our estimation results for parameters implies that $\hat{\rho} < 0$ and $\kappa_y < a_2$ holds in U.S. markets. This corresponds to the first column in Table 3.2. Thus, three of four coefficients in Table 3.2 are positive. Actually, the value of four coefficients are (from the top of Table 3.2): 0.35, 0.1,-0.37,0.14. From this calculation, it is concluded that $m_{y,t}$ is increasing in both of two types of excess consumptions. Unfortunately, the property for $m_{p,t}$ is ambiguous since the sign of coefficients for one of two types of excess consumptions is negative.

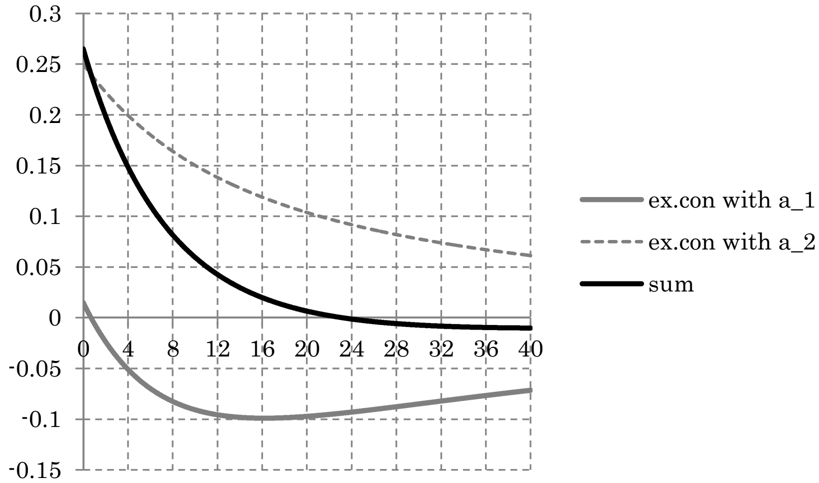

Figure 3.2: The term structure of the partial derivative of spot rate with respect to each excess consumption is given. The horizontal axis is the time to maturity in quarters. The sum of two derivatives is given by the black curve.

To check whether the interest rates are increasing in excess consumption, we calculate the partial derivative of the spot rates with respect to each excess consumption. Figure 3.2 shows this results. By the fact that the coefficient of $m_{p,t}$ with respect to excess consumption with a_1 is negative, the partial derivatives with respect to excess consumption with a_1 are negative for almost all maturities. On the other hand, the partial derivatives with respect to excess consumption with a_2 are positive for all maturities. Let us examine the sum of these derivatives, although it does not express the exact value of exposure of the spot rates to the excess consumption. It is given by the black curve in Figure 3.2. It is found that the sum of derivatives is positive for maturities less than 5 years . That is, our model implies that the spot rates with maturities less than 5 years are increasing in excess consumption.

The remarkable point is that the size of derivatives is close to the regression coefficients in Table 2.1 for those maturities. Thus our model well captures the size of exposure of the spot rates with respect to excess consumption as well as the sign of exposure.

3.6 Conclusion

In this chapter, we propose a new equilibrium model that naturally generates the positive correlation between the nominal interest rates and excess consumption. We focus on the partial observability of economic variables in a pure exchange economy. Departing from the previous studies, we considered the equilibrium model with two unobservable factors. Even with this complexity, we have derived closed form solutions for the nominal equilibrium interest rates. The resulting nominal term structure model turns out to be a two-factor purely Gaussian affine model in which state variables can be expressed as a weighted sum of excess consumptions and price levels under stationary error process assumption.

The model considered here allows us to give a quite different interpretation to the role of excess consumption in determining the interest rates from the consumption habit models. For economic agents engaging in the Bayesian inference, the excess consumption plays a role as an economic indicator helping them to guess the current trend in income growth. Naturally, the economic agents' estimate, hence the equilibrium interest rates, can be increasing in excess consumption under some mild conditions on parameters. This forms a striking contrast to the consumption habit models in which the intertemporal substitution effect induces negative correlation between the excess consumption and interest rates.

Our empirical analysis also supports this view. The estimation results indicate reasonable values for all parameters and, more importantly, the positive correlation between the implied interest rates and excess consumption. As a consequence, the time series of the nominal yield implied by the model captures many of the short- and long-run fluctuations in the actual data with higher correlations than those obtained by Wachter (2006).

3.7 Appendix

3.7.1 Proof of proposition 1

First, it is easy to show

$$\begin{aligned}\ln y_T - \ln y_t &= \left(\frac{1-e^{-\kappa_y(T-t)}}{\kappa_y}\right)(\mu_{y,t}-\bar{\mu}_y) + \left(\bar{\mu}_y - \frac{1}{2}\sigma_y^2\right)(T-t) \\ &\quad + \int_t^T \left(\frac{1-e^{-\kappa_y(T-s)}}{\kappa_y}\right)\upsilon_y + \sigma_y dW_{y,s}\end{aligned}$$

$$
\begin{aligned}
&+\int_t^T \left(\frac{1-e^{-\kappa_y(T-s)}}{\kappa_y}\right)\hat{\upsilon}_y d\widehat{W}_{y,s}, \qquad (3.24)\\
\ln p_T - \ln p_t &= \left(\frac{1-e^{-\kappa_p(T-t)}}{\kappa_p}\right)(\mu_{p,t}-\bar{\mu}_p) + \left(\bar{\mu}_p - \frac{1}{2}\sigma_p^2\right)(T-t)\\
&+\int_t^T \left(\frac{1-e^{-\kappa_p(T-s)}}{\kappa_p}\right)\upsilon_p + \sigma_p dW_{p,s}\\
&+\int_t^T \left(\frac{1-e^{-\kappa_p(T-s)}}{\kappa_p}\right)\hat{\upsilon}_p d\widehat{W}_{p,s}. \qquad (3.25)
\end{aligned}
$$

Then the conditional variance $\mathrm{Var}(\ln p_T + \gamma \ln y_T|\mathcal{F}_t)$ is

$$
\begin{aligned}
\mathrm{Var}(\ln p_T + \gamma \ln y_T|\mathcal{F}_t) &= \gamma^2 \int_t^T \left(\sigma_y + \left(\frac{1-e^{-\kappa_y(s-t)}}{\kappa_y}\right)\upsilon_y\right)^2 + \left(\frac{1-e^{-\kappa_y(s-t)}}{\kappa_y}\right)^2 \hat{\upsilon}_y^2 ds\\
&+2\gamma \int_t^T \left(\frac{1-e^{-\kappa_y(s-t)}}{\kappa_y}\right)\left(\frac{1-e^{-\kappa_p(s-t)}}{\kappa_p}\right)\hat{\upsilon}_p\hat{\upsilon}_y\hat{\rho} ds\\
&+\int_t^T \left(\sigma_p + \left(\frac{1-e^{-\kappa_p(s-t)}}{\kappa_p}\right)\upsilon_p\right)^2 + \left(\frac{1-e^{-\kappa_p(s-t)}}{\kappa_p}\right)^2 \hat{\upsilon}_p^2 ds.
\end{aligned}
\tag{3.26}
$$

By the standard argument of asset pricing theory, the price of nominal bond is given by

$$
\begin{aligned}
P(t,T) &= p_t\, e^{-\delta(T-t)} \frac{E\left[u_c(y_T)\frac{1}{p_T}\middle|\mathcal{F}_t^{y,p}\right]}{u_c(y_t)}\\
&= e^{-\delta(T-t)} E\left[\left(\frac{p_T}{p_t}\right)^{-1}\left(\frac{y_T}{y_t}\right)^{-\gamma}\middle|\mathcal{F}_t^{y,p}\right].
\end{aligned}
$$

By the iteration rule, the bond price is

$$
P(t,T) = e^{-\delta(T-t)} E\left[E\left[\left(\frac{p_T}{p_t}\right)^{-1}\left(\frac{y_T}{y_t}\right)^{-\gamma}\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_t^{y,p}\right]. \tag{3.27}
$$

Since the conditional distribution of $\ln y_T$ and $\ln p_T$ with respect to $\{\mathcal{F}_t\}$ is Gaussian, the inner conditional expectation in (3.27) is

$$
\begin{aligned}
&E\left[\left(\frac{p_T}{p_t}\right)^{-1}\left(\frac{y_T}{y_t}\right)^{-\gamma}\middle|\mathcal{F}_t\right]\\
=\ &\exp\left(-\left(\frac{1-e^{-\kappa_p(T-t)}}{\kappa_p}\right)(\mu_{p,t}-\bar{\mu}_p) - \left(\bar{\mu}_p - \frac{1}{2}\sigma_p^2\right)(T-t)\right)\\
&\times\exp\left(-\gamma\left(\frac{1-e^{-\kappa_y(T-t)}}{\kappa_y}\right)(\mu_{y,t}-\bar{\mu}_y) - \gamma\left(\bar{\mu}_y - \frac{1}{2}\sigma_y^2\right)(T-t)\right)\\
&\times\exp\left(\frac{1}{2}\mathrm{Var}(\ln p_T + \gamma \ln y_T|\mathcal{F}_t)\right).
\end{aligned}
$$

Substituting this equation into (3.27), we obtain

$$\begin{aligned}
P(t,T) &= E\left[\exp\left(-\left(\frac{1-e^{-\kappa_p(T-t)}}{\kappa_p}\right)(\mu_{p,t}-\bar{\mu}_p)-\gamma\left(\frac{1-e^{-\kappa_y(T-t)}}{\kappa_y}\right)(\mu_{y,t}-\bar{\mu}_y)\right)\middle|\mathcal{F}_t^{y,p}\right] \\
&\quad\times\exp\left(\left\{-\delta-\left(\bar{\mu}_p-\frac{1}{2}\sigma_p^2\right)-\gamma\left(\bar{\mu}_y-\frac{1}{2}\sigma_y^2\right)\right\}(T-t)\right) \\
&\quad\times\exp\left(\frac{1}{2}\mathrm{Var}(\ln p_T+\gamma\ln y_T|\mathcal{F}_t)\right).
\end{aligned}$$

Note that the conditional variance is deterministic and can be put outside the conditional expectation with respect to $\{\mathcal{F}_t^{y,p}\}$.

By Proposition 12.6 in Liptser and Shiryaev (1977), $\mu_{y,t}$ is Gaussian under $\{\mathcal{F}_t^{y,p}\}$. Thus, the bond price is

$$\begin{aligned}
P(t,T) &= \exp\left(-\left(\frac{1-e^{-\kappa_p(T-t)}}{\kappa_p}\right)(m_{p,t}-\bar{\mu}_p)-\gamma\left(\frac{1-e^{-\kappa_y(T-t)}}{\kappa_y}\right)(m_{y,t}-\bar{\mu}_y)\right) \\
&\quad\times\exp\left(\frac{1}{2}\left(\frac{1-e^{-\kappa_p(T-t)}}{\kappa_p}\right)^2\phi_{pp,t}+\gamma\left(\frac{1-e^{-\kappa_p(T-t)}}{\kappa_p}\right)\left(\frac{1-e^{-\kappa_y(T-t)}}{\kappa_y}\right)\phi_{py,t}\right. \\
&\qquad\qquad\left.+\frac{1}{2}\gamma^2\left(\frac{1-e^{-\kappa_y(T-t)}}{\kappa_y}\right)^2\phi_{yy,t}\right) \\
&\quad\times\exp\left(\left\{-\delta-\left(\bar{\mu}_p-\frac{1}{2}\sigma_p^2\right)-\gamma\left(\bar{\mu}_y-\frac{1}{2}\sigma_y^2\right)\right\}(T-t)\right) \\
&\quad\times\exp\left(\frac{1}{2}\mathrm{Var}(\ln p_T+\gamma\ln y_T|\mathcal{F}_t)\right). \qquad (3.28)
\end{aligned}$$

In the above equality, we use the definition of $m_{i,t}$ and $\phi_{ij,t}$ $(i,j=y,p)$. Substituting (3.26) into the above equation and differentiating the negative of log price with respect to T, we obtain the instantaneous forward rates as follows,

$$\begin{aligned}
f(t,s) &= \delta+\gamma\left(e^{-\kappa_y(s-t)}m_{y,t}+\left(1-e^{-\kappa_y(s-t)}\right)\bar{\mu}_y-\sigma_y^2/2\right) \\
&\quad+e^{-\kappa_p(s-t)}m_{p,t}+\left(1-e^{-\kappa_p(s-t)}\right)\bar{\mu}_p-\sigma_p^2/2 \\
&\quad-\gamma^2\left(\frac{1-e^{-\kappa_y(s-t)}}{\kappa_y}\right)e^{-\kappa_y(s-t)}\phi_{yy,t} \\
&\quad-\gamma\left(\left(\frac{1-e^{-\kappa_y(s-t)}}{\kappa_y}\right)e^{-\kappa_p(s-t)}+\left(\frac{1-e^{-\kappa_p(s-t)}}{\kappa_p}\right)e^{-\kappa_y(s-t)}\right)\phi_{yp,t} \\
&\quad-\left(\frac{1-e^{-\kappa_p(s-t)}}{\kappa_p}\right)e^{-\kappa_p(s-t)}\phi_{pp,t} \\
&\quad-\frac{1}{2}\gamma^2\left(\left(\sigma_y+\left(\frac{1-e^{-\kappa_y(s-t)}}{\kappa_y}\right)\upsilon_y\right)^2+\left(\frac{1-e^{-\kappa_y(s-t)}}{\kappa_y}\right)^2\hat{\upsilon}_y^2\right)
\end{aligned}$$

$$
\begin{aligned}
&-\gamma\left(\frac{1-e^{-\kappa_y(s-t)}}{\kappa_y}\right)\left(\frac{1-e^{-\kappa_p(s-t)}}{\kappa_p}\right)\hat{\upsilon}_p\hat{\upsilon}_y\hat{\rho} \\
&-\frac{1}{2}\left(\left(\sigma_p+\left(\frac{1-e^{-\kappa_p(s-t)}}{\kappa_p}\right)\upsilon_p\right)^2+\left(\frac{1-e^{-\kappa_p(s-t)}}{\kappa_p}\right)^2\hat{\upsilon}_p^2\right). \qquad (3.29)
\end{aligned}
$$

Under Assumption 1, $\phi_{yy,t}=\bar{\phi}_{yy}, \phi_{pp,t}=\bar{\phi}_{pp}, \phi_{yp,t}=\phi_{py,t}=\bar{\phi}_{yp}=\bar{\phi}_{py}$ where parameters are defined as (3.8), (3.9), and (3.10). Substitution of these equations into (3.29) and tedious calculation yields (3.11). **Q.E.D**.

3.7.2 Proof of proposition 2

Under Assumption 1, the stochastic differential equations for $m_{y,t}$ and $m_{p,t}$ can be expressed as

$$
\begin{aligned}
dm_{y,t} &= \left(c_y-\kappa_y^* m_{y,t}-(\bar{\phi}_{yp}/\sigma_p^2)m_{p,t}\right)dt \\
&\quad +(\kappa_y^*-\kappa_y)d\ln y_t+(\bar{\phi}_{yp}/\sigma_p^2)d\ln p_t, \\
dm_{p,t} &= \left(c_p-(\bar{\phi}_{py}/\sigma_y^2)m_{y,t}-\kappa_p^* m_{p,t}\right)dt \\
&\quad +(\bar{\phi}_{py}/\sigma_y^2)d\ln y_t+(\kappa_p^*-\kappa_p)d\ln p_t,
\end{aligned}
$$

where parameters c_y and c_p are defined as

$$
\begin{aligned}
c_y &= \kappa_y\bar{\mu}_y+\frac{1}{2}\left(\upsilon_y\sigma_y+\bar{\phi}_{yy}+\bar{\phi}_{yp}\right) \\
c_p &= \kappa_p\bar{\mu}_p+\frac{1}{2}\left(\upsilon_p\sigma_p+\bar{\phi}_{pp}+\bar{\phi}_{py}\right).
\end{aligned}
$$

The closed form solution of the system of equations above is given by

$$
\begin{aligned}
m_{y,t} &= \frac{a_1e^{-a_2(t-t_0)}-a_2e^{-a_1(t-t_0)}}{a_1-a_2}m_{y,t_0} \\
&\quad +\frac{\frac{1-e^{-a_2(t-t_0)}}{a_2}-\frac{1-e^{-a_1(t-t_0)}}{a_1}}{a_1-a_2}\left(\kappa_p^*c_y-c_p(\bar{\phi}_{yp}/\sigma_p^2)\right) \\
&\quad +\frac{e^{-a_2(t-t_0)}-e^{-a_1(t-t_0)}}{a_1-a_2}\left(c_y-\kappa_y^*m_{y,t_0}-(\bar{\phi}_{yp}/\sigma_p^2)m_{p,t_0}\right) \\
&\quad +\frac{(a_1-\kappa_y)(\kappa_y^*-a_2)}{a_1-a_2}\int_{t_0}^{t}e^{-a_1(t-s)}d\ln y_s \\
&\quad -\frac{(a_2-\kappa_y)(\kappa_y^*-a_1)}{a_1-a_2}\int_{t_0}^{t}e^{-a_2(t-s)}d\ln y_s \\
&\quad +\frac{(a_1-\kappa_p)(\bar{\phi}_{yp}/\sigma_p^2)}{a_1-a_2}\int_{t_0}^{t}e^{-a_1(t-s)}d\ln p_s
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{(a_2-\kappa_p)(\bar{\phi}_{yp}/\sigma_p^2)}{a_1-a_2}\int_{t_0}^{t} e^{-a_2(t-s)}d\ln p_s, \qquad (3.30)\\
m_{p,t} \;=\; & \frac{a_1 e^{-a_2(t-t_0)}-a_2 e^{-a_1(t-t_0)}}{a_1-a_2} m_{p,t_0}\\
& +\frac{\frac{1-e^{-a_2(t-t_0)}}{a_2}-\frac{1-e^{-a_1(t-t_0)}}{a_1}}{a_1-a_2}\left(\kappa_y^* c_p - c_y(\bar{\phi}_{py}/\sigma_y^2)\right)\\
& +\frac{e^{-a_2(t-t_0)}-e^{-a_1(t-t_0)}}{a_1-a_2}\left(c_p-\kappa_p^* m_{p,t_0}-(\bar{\phi}_{py}/\sigma_y^2)m_{y,t_0}\right)\\
& +\frac{(a_1-\kappa_p)(\kappa_p^*-a_2)}{a_1-a_2}\int_{t_0}^{t} e^{-a_1(t-s)}d\ln p_s\\
& -\frac{(a_2-\kappa_p)(\kappa_p^*-a_1)}{a_1-a_2}\int_{t_0}^{t} e^{-a_2(t-s)}d\ln p_s\\
& +\frac{(a_1-\kappa_y)(\bar{\phi}_{py}/\sigma_y^2)}{a_1-a_2}\int_{t_0}^{t} e^{-a_1(t-s)}d\ln y_s\\
& -\frac{(a_2-\kappa_y)(\bar{\phi}_{py}/\sigma_y^2)}{a_1-a_2}\int_{t_0}^{t} e^{-a_2(t-s)}d\ln y_s, \qquad (3.31)
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= \frac{\kappa_y^*+\kappa_p^*+\sqrt{(\kappa_y^*-\kappa_p^*)^2+4(\bar{\phi}_{yp}/\sigma_p^2)(\bar{\phi}_{py}/\sigma_y^2)}}{2},\\
a_2 &= \frac{\kappa_y^*+\kappa_p^*-\sqrt{(\kappa_y^*-\kappa_p^*)^2+4(\bar{\phi}_{yp}/\sigma_p^2)(\bar{\phi}_{py}/\sigma_y^2)}}{2}.
\end{aligned}
$$

We can change the expression of each integral in (3.30) and (3.31) by integral by parts. For instance,

$$\int_{t_0}^{t} e^{-a_1(t-s)}d\ln y_s = \ln y_t - e^{-a_1(t-t_0)}\ln y_{t_0} - \int_{t_0}^{t} a_1 e^{-a_1(t-s)}\ln y_s ds,$$

It is not difficult to show that a_1 and a_2 are positive. Thus, we can take the limit of $m_{y,t}$ and $m_{p,t}$ by letting t_0 go to $-\infty$ and obtain (3.12) and (3.13). **Q.E.D**.

3.7.3 Proof of proposition 3

If $\hat{\rho}=0$, κ_y^* is given by

$$\kappa_y^* = \left((\kappa_y+\upsilon_y/\sigma_y)^2+(\hat{\upsilon}_y/\sigma_y)^2\right)^{\frac{1}{2}}.$$

Thus, $\kappa_y^*-\kappa_y$ is positive if and only if

$$2\kappa_y\upsilon_y\sigma_y+\upsilon_y^2+\hat{\upsilon}_y^2 \geq 0. \qquad (3.32)$$

Since the correlation coefficient is given by

$$\rho_{y,\mu_y} = \frac{\upsilon_y}{\left(\upsilon_y^2 + \hat{\upsilon}_y^2\right)^{\frac{1}{2}}},$$

substituting this into (3.32) yields (3.16). **Q.E.D**.

3.7.4 Proof of corollary 1

The inequality (3.17) is equivalent to the inequality $-1 \geq -\frac{\left(\upsilon_y^2+\hat{\upsilon}_y^2\right)^{\frac{1}{2}}}{2\kappa_y\sigma_y}$. Combining this inequality with $\rho_{y,\mu_y} \geq -1$ yields (3.16). This concludes the proof. **Q.E.D**.

3.7.5 Derivation of filtering error process

Let us consider a system of matrix linear differential equations

$$\begin{aligned} \frac{d}{dt}U_t &= KU_t + HV_t,\, U_{t_0} = \Phi_{t_0}, \\ \frac{d}{dt}V_t &= GU_t - K^\top V_t,\, V_{t_0} = I, \end{aligned} \tag{3.33}$$

where I is 2×2 identity matrix. It is well-known that the solution of (3.7) is given by $\Phi_t = U_t V_t^{-1}$. Define S_t as

$$S_t = \begin{bmatrix} U_t \\ V_t \end{bmatrix}.$$

Then, the system of equations (3.33) is reduced to

$$\frac{d}{dt}S_t = A\,S_t, \tag{3.34}$$

where the 4×4 matrix A is defined by

$$A = \begin{bmatrix} K & H \\ G & -K^\top \end{bmatrix}.$$

Let us denote the eigen values for A by λ_i $(i = 1, 2, 3, 4)$. Then these values are given by

$$\begin{aligned} \lambda_1 &= \left(\frac{(k_{11}^2 + h_{11}/\sigma_y^2) + (k_{22}^2 + h_{22}/\sigma_p^2) + D^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}, \\ \lambda_2 &= \left(\frac{(k_{11}^2 + h_{11}/\sigma_y^2) + (k_{22}^2 + h_{22}/\sigma_p^2) - D^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned}
\lambda_3 &= -\lambda_1, \\
\lambda_4 &= -\lambda_2, \\
\text{where } D &= \left(\left(k_{11}^2 + h_{11}/\sigma_y^2\right) - \left(k_{22}^2 + h_{22}/\sigma_p^2\right)\right)^2 + \frac{4h_{12}^2}{\sigma_y^2\sigma_p^2}.
\end{aligned}$$

The corresponding eigen vectors x_i $(i = 1, 2, 3, 4)$ are

$$x_i = \begin{bmatrix} \frac{\lambda_i + k_{11}}{g_{11}} \\ \left(\frac{\lambda_i + k_{11}}{h_{12}g_{22}}\right)\left(\frac{\lambda_i^2 - k_{11}^2}{g_{11}} - h_{11}\right) \\ 1 \\ \frac{1}{h_{12}}\left(\frac{\lambda_i^2 - k_{22}^2}{g_{11}} - h_{11}\right) \end{bmatrix}, \quad i = 1, 2, 3, 4.$$

Next, we construct the matrix $[\xi_1 x_1,\ \xi_2 x_2,\ \xi_3 x_3,\ \xi_4 x_4]$ where constants ξ_i $(i = 1, 2, 3, 4)$ satisfy

$$\begin{aligned}
\xi_1\xi_3 &= \frac{(\lambda_2^2 - \kappa_y^2 - h_{11}g_{11})g_{11}}{2(\lambda_2^2 - \lambda_1^2)\lambda_1}, \\
\xi_2\xi_4 &= -\frac{(\lambda_1^2 - \kappa_y^2 - h_{11}g_{11})g_{11}}{2(\lambda_2^2 - \lambda_1^2)\lambda_2}.
\end{aligned}$$

Denote this matrix as

$$R = \begin{bmatrix} Y & Z \\ X & W \end{bmatrix},$$

where W, X, Y, Z are 2×2 submatrices. Constant scalars ξ_i $(i = 1, 2, 3, 4)$ are for the normalization of matrix in the sense that the inverse of R is given by

$$R^{-1} = \begin{bmatrix} W^\top & -Z^\top \\ -X^\top & Y^\top \end{bmatrix}.$$

Denote a diagonal matrix defined by eigen values as

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 \\ 0 & 0 & 0 & -\lambda_2 \end{bmatrix}.$$

Apparently, $A = R\Lambda R^{-1}$ holds and (3.34) can be arranged to

$$\frac{d}{dt}R^{-1}S_t = \Lambda R^{-1}S_t.$$

Since Λ is diagonal, $R^{-1}S_t = e^{\Lambda(t-t_0)}R^{-1}S_{t_0}$ and we obtain

$$S_t = Re^{\Lambda(t-t_0)}R^{-1}S_{t_0}.$$

This yields the next two equations,

$$\begin{aligned}
U_t &= \left(Ye^{\Lambda_1(t-t_0)}W^\top - Ze^{\Lambda_2(t-t_0)}X^\top\right)\Phi_{t_0} \\
&\quad + Ze^{\Lambda_2(t-t_0)}Y^\top - Ye^{\Lambda_1(t-t_0)}Z^\top, \\
V_t &= \left(Xe^{\Lambda_1(t-t_0)}W^\top - We^{\Lambda_2(t-t_0)}X^\top\right)\Phi_{t_0} \\
&\quad + We^{\Lambda_2(t-t_0)}Y^\top - Xe^{\Lambda_1(t-t_0)}Z^\top,
\end{aligned}$$

where Λ_i $(i = 1, 2)$ are submatrices of Λ defined by

$$\Lambda_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},\ \Lambda_2 = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}.$$

Finally, the solution for the matrix Riccati equation is obtained by $\Phi_t = U_t V_t^{-1}$. Especially, the limit, $\overline{\Phi} = \lim_{t\to\infty}$, is given by YX^{-1}, since $e^{\Lambda_2(t-t_0)} \to 0$ as $t \to \infty$. Each element of this limit matrix is obtained as

$$\begin{aligned}
\bar{\phi}_{yy} &= \left(\kappa_y^* + k_{11}\right)\sigma_y^2 = (\kappa_y^* - \kappa_y)\sigma_y^2 - \upsilon_y\sigma_y, \\
\bar{\phi}_{pp} &= \left(\kappa_p^* + k_{22}\right)\sigma_p^2 = (\kappa_p^* - \kappa_p)\sigma_p^2 - \upsilon_p\sigma_p, \\
\bar{\phi}_{yp} &= \bar{\phi}_{py} = \frac{\hat{\upsilon}_y\hat{\upsilon}_p\hat{\rho}}{\lambda_1+\lambda_2}, \\
\text{where}\ \kappa_y^* &= \frac{k_{11}^2 + h_{11}/\sigma_y^2 + \lambda_1\lambda_2}{\lambda_1+\lambda_2},\ \kappa_p^* = \frac{k_{22}^2 + h_{22}/\sigma_p^2 + \lambda_1\lambda_2}{\lambda_1+\lambda_2}.
\end{aligned}$$

3.7.6 Derivation of the Kalman filter

First note that

$$E(w_{t-1}|\mathcal{F}_{t-1}) = E(z_{t-1} - c_z - x_{t-1}|\mathcal{F}_{t-1}) = z_{t-1} - c_z - \hat{x}_{t-1}.$$

From this result and transition equation (3.22), we have

$$x_{t|t-1} = c_x + F\hat{x}_{t-1} + G(z_{t-1} - c_z - \hat{x}_{t-1}).$$

Hence,

$$\begin{aligned} x_t - x_{t|t-1} &= (c_x + Fx_{t-1} + Gw_{t-1} + v_t) - (c_x + F\hat{x}_{t-1} + G(z_{t-1} - c_z - \hat{x}_{t-1})) \\ &= F(x_{t-1} - \hat{x}_{t-1}) - G(z_{t-1} - c_z - w_{t-1} - \hat{x}_{t-1}) + v_t \\ &= F(x_{t-1} - \hat{x}_{t-1}) - G(x_{t-1} - \hat{x}_{t-1}) + v_t \\ &= (F - G)(x_{t-1} - \hat{x}_{t-1}) + v_t \end{aligned}$$

The third equality follows from the observation equation (3.23). Therefore,

$$\begin{aligned} P_{t|t-1} &= E\left[(x_t - x_{t|t-1})(x_t - x_{t|t-1})'\right] \\ &= E\left[\{(F-G)(x_{t-1} - \hat{x}_{t-1}) + v_t\}\{(F-G)(x_{t-1} - \hat{x}_{t-1}) + v_t\}'\right] \\ &= (F-G)E[(x_{t-1} - \hat{x}_{t-1})(x_{t-1} - \hat{x}_{t-1})'](F-G)' + E[v_t v_t'] \\ &= (F-G)\hat{P}_{t-1}(F-G) + Q. \end{aligned}$$

To get the fourth equality, we use the fact that v_t is independent of x_{t-1}. Furthermore, since $\hat{x}_{t-1}$ is a linear function of $z_1, \ldots, z_{t-1}$, it must be independent of v_t. Also, the last equality follows from that F and G are diagonal.

The updating equations can be obtained as follows. By the formula for updating a linear projection we can get[9]

$$\hat{x}_t = x_{t|t-1} + E[(x_t - x_{t|t-1})(z_t - z_{t|t-1})']\left\{E[(z_t - z_{t|t-1})(z_t - z_{t|t-1})']\right\}^{-1}(z_t - z_{t|t-1}). \quad (3.35)$$

Notice that

$$z_{t|t-1} = c_z + x_{t|t-1}, \quad (3.36)$$

and so

$$z_t - z_{t|t-1} = x_t - x_{t|t-1} + w_t.$$

Using this result we can calculate

$$\begin{aligned} E[(z_t - z_{t|t-1})(z_t - z_{t|t-1})'] &= E[\{(x_t - x_{t|t-1}) + w_t\}\{(x_t - x_{t|t-1}) + w_t\}'] \\ &= E[(x_t - x_{t|t-1})(x_t - x_{t|t-1})'] + E[w_t w_t'] \\ &= P_{t|t-1} + R. \end{aligned} \quad (3.37)$$

[9] See, for example, Hamilton (1994, p. 99, equation [4.5.30]).

Here the second equality follows from the fact $E[(x_t - x_{t|t-1})w_t'] = 0$. Similarly,

$$\begin{aligned} E[(x_t - x_{t|t-1})(z_t - z_{t|t-1})'] &= E[(x_t - x_{t|t-1})\{(x_t - x_{t|t-1}) + w_t\}'] \\ &= E[(x_t - x_{t|t-1})(x_t - x_{t|t-1})'] \\ &= P_{t|t-1}. \end{aligned} \tag{3.38}$$

Substituting (3.36), (3.37) and (3.38) into (3.35) gives

$$\hat{x}_t = x_{t|t-1} + P_{t|t-1}(P_{t|t-1} + R)^{-1}(z_t - c_z - x_{t|t-1})$$

The MSE associated with this updated projection, $\hat{P}_t$, can be found from the formula for the MSE of updated linear projection:[10]

$$\begin{aligned} \hat{P}_t &= E[(x_t - \hat{x}_t)(x_t - \hat{x}_t)'] \\ &= E[(x_t - x_{t|t-1})(x_t - x_{t|t-1})'] \\ &\quad - E[(x_t - x_{t|t-1})(z_t - z_{t|t-1})']E[(z_t - z_{t|t-1})(z_t - z_{t|t-1})']E[(z_t - z_{t|t-1})(x_t - x_{t|t-1})'] \\ &= P_{t|t-1} - P_{t|t-1}(P_{t|t-1} + R)^{-1}P_{t|t-1} \end{aligned}$$

[10]See, for example, Hamilton (1994, p. 99, equation [4.5.31]).

[**For Memorandum**]

Chapter 4

Expectation Puzzle without Time-varying Market Price of Risk

4.1 Introduction

In this chapter, we consider a utility function which exhibits decreasing relative risk aversion and explore the implication of this preference to the expectation puzzle. Let us review the expectation puzzle once again. It is illustrated as follows. Let us denote R_i^n as time i spot rate with time to maturity n in months. Consider a simple regression,

$$R_{i+1}^{n-1} - R_i^n = \text{constant} + \phi_n \frac{R_i^n - R_i^1}{n-1} + \text{error term.}$$

The dependent variable is the change in yield in the next month and the dependent variable is the slope of current spot rate curve. The traditional expectation hypothesis which states that the slope of curve contains the information regarding the market expectation of future interest rates implies the regression coefficients are unity for all n.

On the contrary, Campbell and Shiller(1991) documented that this implication has been consistently rejected. The regression coefficient is significantly far from unity and even negative. This situation becomes severe for longer maturities and it is called as "expectation puzzle". Table 2.2 in chapter 2 shows the results reported in Dai and Singleton(2002).

Since the puzzle implies that the slope of current curve and the change in yields in the next month are negatively correlated, it can be considered that the spot rate curve rotates under the mean reversion of short term interest rates. In other words, the expectation puzzle implies that short term interest rates and long term interest rates move in the opposite direction. For instance, when the short term interest rates are relatively high, it is likely that the spot rate curve has negative slope. After we observe the negative slope of the curve, the short term interest rates tend to fall by mean reversion. On the other hand, from the negative correlation with the slope, the long term interest rates rise in average. Thus, when the expectation puzzle occurs, it seems that the rotation of the curve is likely to occur.

There are many researches which successfully explain the expectation puzzle. For instance, Dai and Singleton(2002) and Duffee(2002) explain the puzzle by essentially affine term structure models in which the market price is time-varying. Li and Song(2012) introduce jumps into the affine term structure models and explain the puzzle and the humped shape of volatility curve simultaneously. There are utility based models of term structure which explain the expectation puzzle. Wachter(2006) considers the pure exchange economy with external habit and shows that the puzzle can occur in her model. Buraschi and Jiltsov(2007) show the puzzle can occur by introducing the money in the utility function.

The important point here is that all of the above models explain the puzzle by the time variation of market price of risk. The intuition behind these models is as follows. Suppose that the market price of risk is given as a decreasing function of the short term interest rate. When the short term interest rate is relatively high, it will fall in the next period by mean reversion. On the other hand, the risk premium of long term bonds is small or even negative because the market price of risk is negatively related to the short term interest rate. Thus, the price of long term bonds tends to decline in the next period. Since the short term interest rates fall and long term interest rates rise in the next period, the spot rate curve changes as if it rotates and expectation puzzle occurs.

In this chapter, the following questions are addressed. First, is there an economic story which explains the expectation puzzle other than the story with time-varying market price of risk? Second, what kind of affine term structure model is corresponding to this story? Third, is it possible that an affine term structure model with constant market price of risk explains the expectation puzzle? Finally, what is the difference between the affine term structure model here and the existing affine term structure models? To answer these questions, we introduce a time-additive utility which exhibits decreasing relative risk aversion and investigate the properties of term structure of interest rates in equilibrium. Since the utility function is time-additive, decreasing relative risk aversion implies that the elasticity of intertemporal substitution is increasing. The main results and contributions of the chapter are: we show that time-varying elasticity of intertemporal substitution can explain the rotation of curve even in the absence of time-varying market price of risk. Thus, time-varying elasticity of intertemporal substitution is a possible candidate which explains the expectation puzzle. In fact, if we approximate our pure exchange economy model by an affine term structure model, then it can be shown that this affine term structure model exhibits the negative coefficients in Campbell-Shiller test even in the absence of time-varying market price of risk. Finally, we find a difference between our affine term structure model and existing affine term structure models. In our affine model, there is a new type of state variable which is not found in the past literature.

The chapter is organized as follows. In the next section, a model of pure exchange economy is explained. In section 3, the properties of term structure in equilibrium are investigated. In section 4, the equilibrium term structure is approximated by an affine term structure model and the difference between our affine model and existing affine models is presented. In the final section, the conclusions are given.

4.2 The Model

Consider a pure exchange economy of a single perishable consumption good. The time span of this economy is $[t_0, \tau]$. Let $(\Omega, \mathcal{F}, Q)$ be a complete probability space. The economy is driven by two-dimensional Wiener process $\{W_t : t \in [t_0, \tau]\}$ where $W_t^\top = [W_{1t}, W_{2t}]$. We assume that W_{1t} and W_{2t} are independent. Filtration $\{\mathcal{F}_t : t \in [t_0, \tau]\}$ is chosen to be the $Q-$augmentation of the natural filtration generated by W_t.

The economy is endowed with a flow of the consumption good. The rate of aggregate endowment flow is y_t, $t \in [t_0, \tau]$. It is assumed that y_t follows a stochastic differential equation,

$$\frac{dy_t}{y_t} = \mu_t \, dt + \sigma^\top dW_t, \tag{4.1}$$

where $\sigma^\top = [\sigma_1, \sigma_2]$ is a vector of constants. Without loss of generality, it is assumed that $\sigma \geq 0$. The conditional expectation of instantaneous growth rate of endowment is assumed to follow an Ornstein-Uhlenbeck process,

$$d\mu_t = \kappa(\bar{\mu} - \mu_t)dt + b^\top dW_t, \tag{4.2}$$

where $\bar{\mu}, \kappa$ are positive constants and $b^\top = [b_1, b_2]$ is a vector of constants. We do not restrict the sign of b to allow the correlation between y_t and μ_t.

It is assumed that the representative agent of our economy exists and her/his objective function is expressed as

$$\begin{aligned} &U(\{c_s : s \in [t_0, \tau]\}) = E\left[\int_{t_0}^{\tau} v(c_s, s)ds\right], \\ &\text{where } v(c_s, s) = e^{-\rho s}\left(\alpha c_s + \frac{c_s^{1-\gamma}}{1-\gamma}\right), \quad \alpha \geq 0, \gamma > 0, \end{aligned} \tag{4.3}$$

and $E[\cdot]$ is the expectation operator[1]. The time preference ρ is assumed to be a nonnegative constant[2]. The consumption flow space at each time is assumed to be the set of nonnegative real numbers. Let us denote the first order and the second order derivative of $u(\cdot)$ as $u_c(\cdot)$ and $u_{cc}(\cdot)$ respectively. The coefficient of relative risk aversion is written as

$$RRA(c) = -\frac{v_{cc}(c, s)}{v_c(c, s)}c = \theta(c)\gamma, \tag{4.4}$$

where $\theta(c)$ is defined by

$$\theta(c) = \frac{c^{-\gamma}}{\alpha + c^{-\gamma}}. \tag{4.5}$$

If α is equal to zero, the coefficient of relative risk aversion is constant. Otherwise, $\theta(c)\gamma$ stochastically moves in the open interval $(0, \gamma)$. Since $\theta(c)$ is decreasing in c, the utility function defined by (4.3) exhibits decreasing relative risk aversion in consumption. The parameter γ is the supremum of relative risk aversion, that is, $\sup_{c \in \mathcal{R}^+} RRA(c)$. Note that the

---

[1]This type of utility function was investigated by Ross(1981) which introduces the strong measure of risk aversion.

[2]The parameters are restricted by following two inequalities,

$$\begin{aligned} &\rho > \bar{\mu} - \frac{1}{2}\sigma^\top\sigma + \frac{1}{2}\left(\sigma + \frac{b}{\kappa}\right)^\top\left(\sigma + \frac{b}{\kappa}\right), \\ &\bar{\mu} - \frac{1}{2}\sigma^\top\sigma + \frac{1}{2}(1-\gamma)\left(\sigma + \frac{b}{\kappa}\right)^\top\left(\sigma + \frac{b}{\kappa}\right) > 0. \end{aligned}$$

The first inequality ensures that the limit of the expected utility, $\lim_{\tau\to\infty} U(\{y_s : s \in [t_0, \tau]\})$, is finite. The second inequality is so-called the transversality condition and ensures that the price of discount bond converges to zero as the time to maturity goes to infinity.

reciprocal of $RRA(c)$ is the elasticity of intertemporal substitution, since the utility function is time-additive. Thus, the preference considered here exhibits increasing intertemporal substitution.

It is assumed that all assets traded in the market are in zero net-supply. The price of default-free pure discount bond which matures at $s(\in (t_0, \tau])$ is denoted by $P(t, s)$ $(t \le s)$. We set $P(s, s) = 1$. It is assumed that at least three pure discount bonds of different maturities are traded at any $t \in [t_0, \tau)$. Since the source of the risk in this economy is two-dimensional Wiener process, the market is complete.

Finally, the s-maturity instantaneous forward rate of interest rate and the short rate are denoted as $f(t, s)$ and r_t respectively.

4.3 The Equilibrium Term Structure of Interest Rates

4.3.1 The forward rate of endowment expectation and variance, and the certainty pricing equivalent

Before we derive the instantaneous forward rate of interest rate in equilibrium, we define three variables.

Definition 1 *The forward rate of endowment expectation is defined by*

$$M(t, s) = \frac{\partial}{\partial s} E\left[\ln\left(\frac{y_s}{y_t}\right) \middle| \mathcal{F}_t \right], \ s \ge t. \tag{4.6}$$

And the forward rate of endowment variance is defined by

$$V(t, s) = \frac{\partial}{\partial s} Var\left[\ln\left(\frac{y_s}{y_t}\right) \middle| \mathcal{F}_t \right], \ s \ge t, \tag{4.7}$$

where $Var[\cdot]$ is the variance operator.

Under the assumption on the aggregate endowment process, the forward rate of endowment expectation is given by

$$M(t, s) = e^{-\kappa(s-t)}\mu_t + (1 - e^{-\kappa(s-t)})\bar{\mu} - \frac{1}{2}\sigma^\top \sigma. \tag{4.8}$$

And the forward rate of endowment variance is

$$\begin{aligned} &V(t,s) = \Sigma(t,s)^\top \Sigma(t,s), \\ &\text{where } \Sigma(t,s) = \sigma + \frac{1-e^{-\kappa(s-t)}}{\kappa} b. \end{aligned} \tag{4.9}$$

When $s = t$, the forward rate of endowment expectation is reduced to the conditional expectation of instantaneous growth rate of endowment. The forward rate of endowment variance is reduced to the conditional variance of instantaneous growth rate of endowment respectively.

In equilibrium models, the short rate is equal to the negative of expected rate of change in marginal utility. In our model, it is expressed as[3]

$$r_t = \rho + RRA(y_t)M(t,t) - \frac{1}{2}\left(RRA^2(y_t) - y_t RRA'(y_t)\right)V(t,t). \tag{4.10}$$

The second term of the right hand side is the term of intertemporal substitution effect and the third term is the term of precautionary saving effect[4]. The important point here is that this expression is not specific to our model and can be derived in more general setting[5]. On the other hand, it is difficult to obtain the same expression for the instantaneous forward rate of interest rate in general. As will be clear, our specification of the preference and the stochastic process of endowment allows us to do this. The following variable is important in this procedure.

Definition 2 *The certainty pricing equivalent is defined by,*

$$\hat{y}_{t,s} = v_c^{-1}\left(E\left[v_c(y_s,s)\middle|\mathcal{F}_t\right], s\right), \tag{4.11}$$

where $v_c^{-1}(\cdot, s)$ is the inverse of $v_c(\cdot, s)$ given s.

The certainty pricing equivalent is the certain amount of consumption which gives the same expected marginal utility as the future endowment. Obviously $\hat{y}_{t,s} \leq E\left[y_s|\mathcal{F}_t\right]$ and $\hat{y}_{t,t} = y_t$.

[3]For instance, Wang(1996) derives (4.10) in a model of heterogeneous exchange economy.
[4]This interpretation is usual in the equilibrium models. See Bakshi and Chen(1997) and Wang (1996).
[5]For the formal derivation of (4.10) in the general setting, see Duffie and Zame(1989) and Cox, Ingersoll, Ross(1985a).

It is easy to show

$$\begin{aligned}\hat{y}_{t,s} &= \exp\left(E\left[\ln y_s|\,\mathcal{F}_t\right] - \frac{1}{2}\gamma Var\left(\ln y_s|\,\mathcal{F}_t\right)\right)\\ &= y_t \exp\left(\int_t^s M(t,u) - \frac{1}{2}\gamma V(t,u)\,du\right),\ s \geq t. \qquad (4.12)\end{aligned}$$

Clearly the instantaneous rate of change in certainty pricing equivalent with respect to s is given by $M(t,s) - \frac{1}{2}\gamma V(t,s) > 0$.

4.3.2 The instantaneous forward rate of interest rate in equilibrium

In our economy, the equilibrium price of pure discount bond is given by

$$\begin{aligned}P(t,s) &= \frac{E\left[v_c(y_s,s)|\,\mathcal{F}_t\right]}{v_c(y_t,t)}\\ &= \frac{v_c(\hat{y}_{t,s},s)}{v_c(y_t,t)}. \qquad (4.13)\end{aligned}$$

The last equation holds by the definition of the certainty pricing equivalent. The instantaneous forward rate of interest rate in equilibrium is obtained as follows.

Proposition 4 *In the equilibrium, the instantaneous forward rate of interest rate is determined as*

$$\begin{aligned}f(t,s) &= \rho + RRA(\hat{y}_{t,s})M(t,s)\\ &\quad -\frac{1}{2}\left(RRA^2(\hat{y}_{t,s}) - \hat{y}_{t,s}RRA'(\hat{y}_{t,s})\right)V(t,s) \qquad (4.14)\\ &= \rho + RRA(\hat{y}_{t,s})\left(M(t,s) - \frac{1}{2}\gamma V(t,s)\right),\ s \geq t. \qquad (4.15)\end{aligned}$$

Proof. See appendix.

Note that (4.14) has the same form as (4.10). The only difference is that y_t, $M(t,t)$, and $V(t,t)$ in (4.10) are replaced by $\hat{y}_{t,s}$, $M(t,s)$, and $V(t,s)$ respectively. Thus under the specification of our model, the formula for the interest rate when $s = t$ is naturally extended to the one when $s \geq t$. Then, it is natural to say that the second term of the right hand side of (4.14) is the term of intertemporal substitution effect and the third term is the term of precautionary saving effect.

In (4.15), we can give an interesting interpretation to the instantaneous forward rate of interest rate. In a "certainty economy", the equilibrium interest rate is given by the sum of the time preference and the term of intertemporal substitution effect. The latter term is the product of the reciprocal of elasticity of intertemporal substitution and the instantaneous growth rate of endowment during the time interval $[s, s+dt]$. Equation (4.15) has this form. Note that the instantaneous rate of change in certainty pricing equivalent with respect to s is given by $M(t,s) - \frac{1}{2}\gamma V(t,s)$. Thus, we can transform our pure exchange economy with uncertainty into the pure exchange economy with certainty where the future aggregate endowment at s is known to be $\hat{y}_{t,s}$. In this interpretation, it is natural that $RRA(\hat{y}_{t,s})$ in (4.15) is interpreted as the reciprocal of elasticity of intertemporal substitution rather than the coefficient of relative risk aversion. The risk preference for precautionary savings is imbedded to $\hat{y}_{t,s}$ and $M(t,s) - \frac{1}{2}\gamma V(t,s)$.

In the case of constant relative risk aversion, the forward rate of interest rate has the form,

$$\begin{aligned} f(t,s) &= \rho + \gamma\left(M(t,s) - \frac{1}{2}\gamma V(t,s)\right) \\ &= \rho + \gamma M(t,s) - \frac{1}{2}\gamma^2 V(t,s). \end{aligned} \tag{4.16}$$

Since the forward rate is completely determined by μ_t, the model is a one-factor model in this case. By taking the limit, the short rate is obtained as

$$r_t = \rho + \gamma\left(\mu_t - \frac{1}{2}\sigma^\top\sigma\right) - \frac{1}{2}\gamma^2\sigma^\top\sigma.$$

It is easily verified that the short rate follows an Ornstein-Uhlenbeck process,

$$dr_t = \kappa(\theta - r_t)dt + \gamma b^\top dW_t, \tag{4.17}$$

where $\theta = \rho + \gamma\left(\bar{\mu} - \frac{1}{2}\sigma^\top\sigma\right) + \frac{1}{2}\gamma^2\sigma^\top\sigma$. The vector of market price of risk is given by $\gamma\sigma$. Hence, the equilibrium generates Vasicek model. This is the result which re-states the lemma 2.1 in Goldstein and Zapatero (1996).

Corollary 2 *When the coefficient of relative risk aversion is constant, Vasicek model holds. The process of the short rate is given by* (4.17) *and the market price of risk is given by constant vector* $\gamma\sigma$.

By the argument above, the first γ in (4.16) which is outside the parentheses is better interpreted as the reciprocal of elasticity of intertemporal substitution, whereas the second γ which is in the parentheses should be interpreted as the coefficient of relative risk aversion.

4.3.3 The movements of the forward rate curve

In the economy of decreasing relative risk aversion or increasing intertemporal substitution, the instantaneous forward rate of interest rate depends on both μ_t and y_t. Thus, the equilibrium term structure is exposed to two types of risk: "consumption growth risk" and "consumption level risk". The consumption growth corresponds to μ_t and the consumption level corresponds to y_t. Let us examine the effect of consumption level risk at first.

y_t has an effect on the term structure through the certainty pricing equivalent. By differentiating (4.15) with respect to y_t, we obtain

$$\frac{\partial f(t,s)}{\partial y_t} = RRA'(\hat{y}_{t,s}) \left(M(t,s) - \frac{1}{2}\gamma V(t,s) \right) \frac{\hat{y}_{t,s}}{y_t}. \tag{4.18}$$

Thus we establish the following proposition.

Proposition 5 *When the certainty pricing equivalent is monotonically increasing in s, that is,*

$$\inf_{s \geq t} M(t,s) - \frac{1}{2}\gamma V(t,s) > 0, \tag{4.19}$$

the instantaneous forward rate of interest rate with any maturity $s \geq t$ is decreasing in y_t.

When μ_t satisfies (4.19), the forward rate curve shifts by the change in y_t and so does the spot rate curve. In this sense, y_t corresponds to the "level" factor of the term structure[6].

Remark 1 Even when μ_t is too low to satisfy (4.19), we can maintain the similar statement under a weaker condition. If the parameters satisfy the inequality,

$$\bar{\mu} - \frac{1}{2}\sigma^\top \sigma - \frac{1}{2}\gamma \left(\sigma + \frac{b}{\kappa} \right)^\top \left(\sigma + \frac{b}{\kappa} \right) > 0, \tag{4.20}$$

[6] This property is analogous to the principal components of the term structure in Litterman and Scheinkman(1991).

then $\lim_{s\to\infty} M(t,s) - \frac{1}{2}\gamma V(t,s) > 0$ holds at any t with probability one and there exists a $\mathcal{F}_t-$random variable s_t such that $M(t,s) - \frac{1}{2}\gamma V(t,s) > 0$ for all s satisfying $s > s_t$. Thus, any forward rate of interest rate with the maturity which is longer than s_t is decreasing in y_t.

The effect of μ_t to the term structure is quite different from y_t. The partial derivative of (4.15) respect to μ_t is given by

$$\begin{aligned}
\frac{\partial f(t,s)}{\partial \mu_t} &= RRA(\hat{y}_{t,s})\frac{\partial M(t,s)}{\partial \mu_t} + \frac{\partial f(t,s)}{\partial \hat{y}_{t,s}}\frac{\partial \hat{y}_{t,s}}{\partial \mu_t} \\
&= RRA(\hat{y}_{t,s})e^{-\kappa(s-t)} \\
&\quad + RRA'(\hat{y}_{t,s})\hat{y}_{t,s}\left[M(t,s) - \frac{1}{2}\gamma V(t,s)\right]\left[\frac{1-e^{-\kappa(s-t)}}{\kappa}\right]. \qquad (4.21)
\end{aligned}$$

The first term in the right hand side is the effect of change in the forward rate of endowment expectation (that is $M(t,s)$) and always positive. This is the common property shared with Vasicek model. When μ_t is high, the economy enjoys the economic growth and the interest rates rise.

The second term is the effect of increasing intertemporal substitution. Note that we obtain this second term by differentiating (4.15) where $RRA(\hat{y}_{t,s})$ is better interpreted as the reciprocal of elasticity of intertemporal substitution. It is negative as far as the first bracket is positive. When μ_t rises, the future intertemporal substitution is expected to be in a higher level than current level and the forward rate of interest rate is induced to fall.

Note that there are exponential functions both in the first and second terms. The exponential function in the first term is decreasing in s, whereas $\frac{1-e^{-\kappa(s-t)}}{\kappa}$ in the second term is increasing in s. Thus, the short term forward rates depend heavily on the first term. Indeed, at the short end, the short rate depends only on the first term. On the other hand, the long term forward rates seem to depend heavily on the second term. To obtain the accurate result, however, more rigorous investigation is needed, since $\hat{y}_{t,s}$ and $M(t,s) - \frac{1}{2}\gamma V(t,s)$ vary with s. In fact, we can obtain the following proposition.

Proposition 6 *When* (4.19) *holds, there exists some* s_t^* *such that* $\frac{\partial f(t,s)}{\partial \mu_t} < 0$, $\forall s \geq s_t^*$.

Proof. See appendix.

Remark 2 By the same argument as remark 1, it is easy to show that s_t^* still exists under a weaker condition (4.20) than (4.19).

Thus, when μ_t increases, the long term forward rates fall while the short rate rises. To put differently, the forward rate curve rotates in response to the change in μ_t.

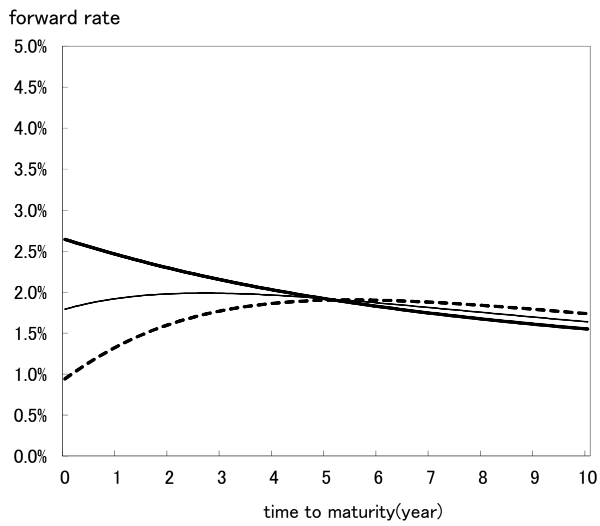

Figure 4.1: Three forward rate curves for $\mu_t = 0,\ 0.01,\ 0.02$ are depicted. The parameters are set at the following: $\rho = 0.01$, $\kappa = 0.3$, $\bar{\mu} = 0.02$, $\sigma^\top = [0, 0]$, $b^\top = [0.005, 0.003]$, $\gamma = 6$, $\alpha = 1$. y_t is set at 0.3. The dashed line corresponds to the case $\mu_t = 0$. The bold line corresponds to the case $\mu_t = 0.02$.

Figure 4.1 shows a numerical example. In this figure, three forward rate curves for $\mu_t = 0,\ 0.01,\ 0.02$ are depicted. The parameters are set at the following values: $\rho = 0.01$, $\kappa = 0.3$, $\bar{\mu} = 0.02$, $\sigma^\top = [0, 0]$, $b^\top = [0.005, 0.003]$, $\gamma = 6$, $\alpha = 1$. y_t is set at 0.3. In this example, $M(t, s) - \frac{1}{2}\gamma V(t, s)$ is positive for all $s \geq t$ and $\hat{y}_{t,s}$ is monotonically increasing in s in each curve. The dashed line corresponds to the case $\mu_t = 0$. The bold line corresponds to the case $\mu_t = 0.02$. The curves cross in the middle range of time to maturity. This means that the forward rate curve rotates by the change in μ_t. Note that the vector of market price of risk is zero vector and constant through time since the vector σ is equal to zero. Thus, in this numerical example, the market price of risk is not time-varying and the rotation of curves is caused by another factor other than time-varying market price of risk.

What is the story behind the rotation of curves in this model? Suppose that μ_t falls. Then the short term interest rate falls, because the effect of intertemporal substitution becomes strong. On the other hand, the fall in μ_t has another effect. Since μ_t follows a mean reverting process and it is persistent unless κ is infinite, the conditional expectation of aggregate consumption in the future is updated to a lower level. This means that the expectation of reciprocal of elasticity of intertemporal substitution in the future, $RRA(\hat{y}_{t,s})$, is updated to be a higher level. The long term interest rates rise by this effect. Even in the case that the market price of risk is constant, this story does work. Therefore, we can say that time-varying elasticity of intertemporal substitution is the main driver in the rotation of curves.

4.4 Approximation by an Affine Term Structure Model

In the previous section, we concluded that the rotation of curves possibly occurs by the change in μ_t even in the absence of time-varying market price of risk. And the main driver of the rotation of curves there is the time-varying elasticity of intertemporal substitution. Thus, it is likely that the time-varying elasticity of intertemporal substitution explains the expectation puzzle. The next questions are: What affine term structure model is corresponding to our pure exchange model? Second, is our affine model with constant market price of risk able to explain the expectation puzzle? Third, if it is so, what is the difference between our affine model and existing affine term structure models? To answer these questions, we formulate an affine term structure model by taking Taylor expansion of the short rate and the vector of market price of risk.

Before doing that, we have to modify the preference of representative agent. So far, we investigated the property of equilibrium interest rates under the assumption that the utility is defined over the consumption level. Under the specification of the endowment process (4.1) and (4.2), the endowment follows a non-stationary process. Thus when we define the utility over the consumption level, the process of equilibrium interest rates is not stationary under decreasing relative risk aversion or increasing elasticity of intertemporal substitution. This property is challenging when we investigate the regression coefficients in the Campbell-Shiller test. For this reason, we introduce consumption habit and modify the preference of representative agent to ensure that the interest rates follow some stationary processes.

Let us denote the consumption habit as z_t and define it by the following equation,

$$z_t = \exp\left(\kappa_c \int_{t_0}^{t} e^{-\kappa_c(t-u)} \ln c_u du\right),$$

where κ_c is a positive constant. Apparently κ_c is the parameter for weights which are put on the past consumption streams when we define the weighted average of past consumption.

Applying Ito's lemma, the process of excess consumption in equilibrium, defined by the ratio of y_t to z_t is given by,

$$d\ln(y_t/z_t) = \left(\mu_t - \frac{1}{2}\sigma^\top\sigma - \kappa_c \ln(y_t/z_t)\right) dt + \sigma^\top dW_t.$$

It is clear that the logarism of excess consumption follows a stationary process, because there is a mean reversion term in the drift of its process.

We assume that the utility function at each time is defined by

$$v(c_t, z_t, t) = e^{-\rho t}\left(\alpha c_t + \frac{c_t^{1-\gamma} z_t^{\gamma}}{1-\gamma}\right).$$

Under this assumption, the coefficient of relative risk aversion is given by,

$$RRA(c_t, z_t) = \frac{(c_t/z_t)^{-\gamma}}{\alpha + (c_t/z_t)^{-\gamma}} \times \gamma.$$

Under the assumptions stated above, it is easy to show that the short rate in equilibrium is given by

$$\rho + RRA(e^{\ln(y_t/z_t)})\left(\mu - \frac{1+\gamma}{2}\sigma^\top\sigma - \kappa_c \ln(y_t/z_t)\right),$$

and the vector of market price of risk is given by $\Lambda = RRA(e^{\ln(y_t/z_t)})\sigma$. Note that the short rate follows a stationary process, since the short rate depends on y_t through the logarism of excess consumption.

To obtain the affine term structure model corresponding to our pure exchange economy model, we take Taylor expansion of the short rate and the vector of market price of risk. Taking expansion around the points where the ratio of consumption to habit is one and μ is $\bar{\mu}$, we can specify the short rate $r(X_t)$ as

$$\begin{aligned} r(X_t) &= \delta_0 + \delta_1 X_{1t} + \delta_2 X_{2t} + \delta_3 X_{3t}, \qquad (4.22)\\ \text{where } \delta_0 &= \rho + \overline{RRA}\left(\bar{\mu} - \frac{1+\gamma}{2}\sigma^\top\sigma\right), \end{aligned}$$

$$
\begin{aligned}
\delta_1 &= \overline{RRA}, \\
\delta_2 &= \delta_3 = -\overline{RRA}\left((\gamma - \overline{RRA})\left(\bar{\mu} - \frac{1+\gamma}{2}\sigma^\top\sigma\right) + \kappa_c\right), \\
\overline{RRA} &= RRA(y,z)|_{y=z},
\end{aligned}
$$

and the vector of state variables $X_t^\top = [X_{1t}, X_{2t}, X_{3t}]$ is defined by

$$
\begin{aligned}
X_{1t} &= \int_{t_0}^{t} e^{-\kappa(t-u)} b^\top dW_u, \\
X_{2t} &= \int_{t_0}^{t} e^{-\kappa_c(t-u)} \sigma^\top dW_u, \\
X_{3t} &= \int_{t_0}^{t} e^{-\kappa_c(t-u)} X_{1u} du.
\end{aligned} \tag{4.23}
$$

The market price of risk is also expanded and obtained as

$$
\begin{aligned}
\Lambda(X_t) &= [\lambda_0 + \lambda_1 X_{1t} + \lambda_2 X_{2t} + \lambda_3 X_{3t}]\,\sigma, \qquad (4.24)\\
\text{where } \lambda_0 &= \overline{RRA}, \\
\lambda_1 &= 0, \\
\lambda_2 &= \lambda_3 = -\overline{RRA}(\gamma - \overline{RRA}).
\end{aligned}
$$

The equivalent martingale measure Q^* is defined by the Radon-Nikodym derivative,

$$
\frac{dQ^*}{dQ} = \exp\left(-\frac{1}{2}\int_{t_0}^{\tau} \Lambda(X_u)^\top \Lambda(X_u) du - \int_{t_0}^{\tau} \Lambda(X_u)^\top dW_u\right). \tag{4.25}
$$

The system of equations (4.22) – (4.25) forms a time-homogeneous affine term structure model. Obviously the short rate follows a Gaussian process since it is linear in X_t. And the vector of market prices of risk is also linear in X_t and time-varying as far as σ is not equal to zero vector. Thus, the model is categorized in the family of purely Gaussian essentially affine models. Obviously, X_{1t} corresponds to μ_t in the model of pure exchange economy. The sum of X_{2t} and X_{3t} corresponds to y_t. Thus the slope of the term structure is mainly determined by X_{1t} while the level is determined by the sum of X_{2t} and X_{3t}.

One point deserves to be mentioned. The process of the first two state variable X_{1t} and X_{2t} is an Ornstein Uhlenbeck process with zero mean. These state variables are typically used when Gaussian affine term structure models are formulated. On the other hand, the third state variable X_{3t} is a weighted sum of past $X_{1u}, u \leq t$. In other words, this state variable accumulates past $X_{1u}s$. This third state variable is not common to the existing

Gaussian affine term structure models and completely new. Thus, our Gaussian affine term structure model is quite different from the existing models in the sense that this third state variable is introduced. This is the answer to the third question which is addressed at the beginning of this section. To this end, we will call this state variable as "accumulation factor".

What is the effect of the presence of the accumulation factor in the affine term structure model? To consider this effect, let us express the instantaneous forward rate of interest rate as the expectation value of future short rate under the forward martingale measure as,

$$
\begin{aligned}
f(t,s) &= E^{F_s}\left[r(X_s) \middle| \mathcal{F}_t \right] \\
&= \delta_0 + \delta_1 E^{F_s}\left[X_{1s} \middle| \mathcal{F}_t \right] + \delta_2 E^{F_s}\left[X_{2s} \middle| \mathcal{F}_t \right] \\
&\quad + \delta_3 E^{F_s}\left[X_{3s} \middle| \mathcal{F}_t \right] \\
&= \delta_0 + \delta_1 E^{F_s}\left[X_{1s} \middle| \mathcal{F}_t \right] + \delta_2 E^{F_s}\left[X_{2s} \middle| \mathcal{F}_t \right] \\
&\quad + \delta_3 \left(e^{-\kappa_c (s-t)} X_{3t} + \int_t^s e^{-\kappa_c (s-u)} E^{F_s}\left[X_{1u} \middle| \mathcal{F}_t \right] du \right),
\end{aligned}
$$

where E^{F_s} denotes expectation operator under the forward martingale measure with respect to the maturity date s. For illustration, let us assume that the market prices of risk are constant. In this case, the volatility vector of Radon-Nikodym derivative $\frac{dQ^{F,s}}{dQ}$ is not random and X_{1u} follows a Markov process also under the forward martingale measure. Then it is obvious that the expectation value $E^{F_s}[X_{1s}|\mathcal{F}_t]$ is increasing in the current value X_{1t}. The coefficient of the expectation in the second term in the right hand side of the last equation is $\delta_1 = \overline{RRA}$ which is positive. Thus, the forward rate of interest rate rises by increase in X_{1t} through this term. This is the effect which is shared with existing affine term structure models.

The fourth term is related to the accumulation factor. The coefficient in this term is δ_3 which is negative under a mild condition $\bar{\mu} > \frac{1+\gamma}{2}\sigma^\top\sigma$. This negativity corresponds to increasing elasticity of intertemporal substitution. Since the expectation of future X_{1u} is integrated in this term, the effect of this term might be larger for longer maturities. Thus, when the maturity is short, the main determinant of the term structure is the second term and X_{1t} has positive relation with the interest rates, while for longer maturities, the main determinant is the fourth term and X_{1t} has negative relation with the interest rates. This mechanism makes the forward rate curve rotate.

By solving the partial differential equation which the bond price satisfies, we obtain the bond price as follows.

Proposition 7 *The bond price in the purely Gaussian essentially affine term structure model formed by the system of equations* (4.22) – (4.25) *is given by*

$$\begin{aligned}
P(X_t, s-t) &= \exp\left(A(s-t) - B_1(s-t)X_{1t}\right. \\
&\qquad \left. - B_2(s-t)X_{2t} - B_3(s-t)X_{3t}\right), && (4.26) \\
B_1(s-t) &= \frac{\delta_1}{\beta-\alpha}\left(e^{-\alpha(s-t)} - e^{-\beta(s-t)}\right) \\
&\quad + \frac{\xi}{\beta-\alpha}\left(\frac{1-e^{-\alpha(s-t)}}{\alpha} - \frac{1-e^{-\beta(s-t)}}{\beta}\right), && (4.27) \\
B_2(s-t) &= B_3(s-t) \\
&= \frac{\kappa\xi}{\alpha\beta} - \delta_1 + \frac{1}{\beta-\alpha}\left(\alpha\delta_1 - \xi\right)\left(\frac{\kappa}{\alpha} - 1\right)e^{-\alpha(s-t)} \\
&\quad - \frac{1}{\beta-\alpha}\left(\beta\delta_1 - \xi\right)\left(\frac{\kappa}{\beta} - 1\right)e^{-\beta(s-t)}, && (4.28) \\
A(s-t) &= -(s-t)\delta_0 + \int_t^s B_1(u-t)\lambda_0\sigma^\top b + \frac{1}{2}B_1(u-t)^2 b^\top b \\
&\quad + B_2(u-t)\lambda_0\sigma^\top\sigma + \frac{1}{2}B_2(u-t)^2\sigma^\top\sigma \, du, && (4.29)
\end{aligned}$$

where the parameters $\alpha, \beta,$ *and* ξ *are defined as*

$$\begin{aligned}
\alpha &= \frac{\kappa + \kappa_c + \lambda_2\sigma^\top\sigma - \sqrt{(\kappa - \kappa_c - \lambda_2\sigma^\top\sigma)^2 - 4\lambda_2\sigma^\top b}}{2}, && (4.30) \\
\beta &= \frac{\kappa + \kappa_c + \lambda_2\sigma^\top\sigma + \sqrt{(\kappa - \kappa_c - \lambda_2\sigma^\top\sigma)^2 - 4\lambda_2\sigma^\top b}}{2}, && (4.31) \\
\xi &= \delta_1(\kappa_c + \lambda_2\sigma^\top\sigma) + \delta_2. && (4.32)
\end{aligned}$$

Proof. See appendix.

Note that there are two types of exponential functions in $B_1(s-t)$. One exponential function is with parameter α and the other is with parameter β. From the definition, α is smaller than β. And these two parameters are positive under some mild conditions on other parameters: $\kappa_c + \lambda_2\sigma^\top\sigma > 0$, $(\kappa_c + \lambda_2\sigma^\top\sigma)\kappa + \lambda_2\sigma^\top b > 0$. These conditions are easily satisfied when σ is close to zero. When α and β are positive, the instantaneous forward rate of interest rate does not explode as the time to maturity goes to infinity.

The third parameter defined in the above proposition ξ is negative when δ_2 is sufficiently small. Note that δ_2 is small means that the effect of increasing intertemporal substitution is strong.

The effect of change in μ_t becomes clear by transforming the pure exchange equilibrium model to the Gaussian affine term structure model. When δ_2 and λ_2 are set to be zero, the model is reduced to Vasicek model. In this case, the exposure $B_1(s-t)$ is equal to $\delta_1 \frac{1-e^{-\kappa(s-t)}}{\kappa}$ and the bond price of any maturity is decreasing in X_{1t}. In general case when δ_2 and λ_2 are not equal to zero, the exposure $B_1(s-t)$ is not always positive. More concretely, $B_1(s-t)$ is positive for small s but negative for large s since $B_1(0) = 0$, $B_1'(0) = \delta_1 > 0$, $\lim_{s\to\infty} B_1(s-t) = \frac{\xi}{\alpha\beta} < 0$ and $B_1(\cdot)$ is continuous. When X_{1t} rises, the short rate rises and the prices of short term bonds become low, whereas the prices of long term bonds become high. The movement of the term structure drastically changes by departure from the model of constant elasticity of intertemporal substitution.

time to maturity(in months)	12	24	48	84	120
our model	-1.679	-1.839	-2.215	-2.978	-4.165
empirical results	-1.425	-1.705	-2.147	-3.096	-4.173

Table 4.1: Regression coefficients in our affine term structure model are given. Paratemeters are set at the following: $\sigma^\top = [0,0], b^\top = [\frac{0.01}{\sqrt{2}}, \frac{0.01}{\sqrt{2}}], \kappa = 0.2, \kappa_c = 0.04, \gamma = 15, \overline{RRA} = 5$. The empirical result reported in Dai and Singleton(2002) is given in the second row for comparison.

The stationary distribution of X_t enables us to calculate the regression coefficients in Campbell-Shiller test (again, i and n are given in months),

$$\phi_n = \frac{Cov\left(R_{i+1}^{n-1} - R_i^n, \frac{R_i^n - R_i^1}{n-1}\right)}{Var\left(\frac{R_i^n - R_i^1}{n-1}\right)}.$$

Table 4.1 shows the regression coefficients calculated at the following parameters: $\sigma^\top = [0,0], b^\top = [\frac{0.01}{\sqrt{2}}, \frac{0.01}{\sqrt{2}}], \kappa = 0.2, \kappa_c = 0.04, \gamma = 15, \overline{RRA} = 5$. For comparison, the empirical result documented in Dai and Singleton (2002) is given in the second row. This table shows that the regression coefficients in our model are close to ones in the empirical results. Note that in this numerical example, the vector of market price of risk is set to be equal zero vector and constant through time. Thus, even in the absence of time-varying market price of risk, an affine term structure model can explain the expectation puzzle.

4.5 Conclusion

In this chapter, we investigated equilibrium interest rates in a pure exchange economy where the utility function of representative agent exhibits decreasing relative risk aversion and increasing intertemporal substitution. It is shown that the forward rate curves rotate by time-varying intertemporal substitution even in the absence of time-varying market price of risk. That is, in our model, the constant market price of risk can be consistent with expectation puzzle. After taking expansion, we show that the affine term structure model approximating our exchange economy exhibits negative coefficients in Campbell-Shiller test. In this affine model, completely new state variable is introduced. It is the "accumulation factor" which integrates the other state variable. Thus, adding accumulation factor makes affine models possible to explain the expectation puzzle under the constant market price of risk.

4.6 Appendix

4.6.1 Proof of proposition 4

By the definition of the instantaneous forward rate of interest rate, we obtain

$$\begin{aligned} f(t,s) &= -\frac{\frac{\partial}{\partial s}E[v_c(y_s,s)|\mathcal{F}_t]}{E[v_c(y_s),s|\mathcal{F}_t]} \\ &= \rho - \frac{\frac{\partial}{\partial s}\left(\alpha + \hat{y}_{t,s}^{-\gamma}\right)}{\alpha + \hat{y}_{t,s}^{-\gamma}}. \end{aligned} \tag{4.33}$$

The partial derivative in the right hand side is calculated as

$$\begin{aligned} \frac{\partial}{\partial s}\left(\alpha + \hat{y}_{t,s}^{-\gamma}\right) &= \frac{\partial \hat{y}_{t,s}^{-\gamma}}{\partial s} \\ &= -\gamma\left(M(t,s) - \frac{1}{2}\gamma V(t,s)\right)\hat{y}_{t,s}^{-\gamma}. \end{aligned} \tag{4.34}$$

Substituting this equation into (4.33), we have

$$\begin{aligned} f(t,s) &= \rho + \left(\frac{\gamma \hat{y}_{t,s}^{-\gamma}}{\alpha + \hat{y}_{t,s}^{-\gamma}}\right) M(t,s) - \frac{1}{2}\gamma\left(\frac{\gamma \hat{y}_{t,s}^{-\gamma}}{\alpha + \hat{y}_{t,s}^{-\gamma}}\right) V(t,s) \\ &= \rho + RRA(\hat{y}_{t,s})M(t,s) - \frac{1}{2}\gamma RRA(\hat{y}_{t,s})V(t,s). \end{aligned} \tag{4.35}$$

The equation (4.15) follows from the last equality. By simple but tedious calculation, we can verify that the following equality holds,

$$\gamma RRA(\hat{y}_{t,s}) = RRA(\hat{y}_{t,s})^2 - \hat{y}_{t,s}RRA'(\hat{y}_{t,s}). \tag{4.36}$$

Substituting this into (4.35), we obtain (4.14). **Q.E.D.**

4.6.2 Proof of proposition 6

Since the first order derivative of relative risk aversion is given by

$$RRA'(\hat{y}_{t.s}) = -RRA(\hat{y}_{t.s})(\gamma - RRA(\hat{y}_{t.s}))\frac{1}{\hat{y}_{t.s}}, \tag{4.37}$$

the right hand side of (4.21) is non-positive for $s > t$ if and only if the following inequality holds:

$$\begin{aligned} &RRA(\hat{y}_{t.s})\frac{e^{-\kappa(s-t)}}{1-e^{-\kappa(s-t)}}\kappa \\ &-\left(M(t,s) - \frac{1}{2}\gamma V(t,s)\right) RRA(\hat{y}_{t.s})(\gamma - RRA(\hat{y}_{t.s})) \le 0. \end{aligned} \tag{4.38}$$

The left hand side of (4.38) is a quadratic function of $RRA(\hat{y}_{t.s})$. When (4.38) holds with equality, then the roots are 0 and

$$\chi(s) = \gamma - \left(\frac{e^{-\kappa(s-t)}}{1 - e^{-\kappa(s-t)}}\right)\left(\frac{\kappa}{M(t,s) - \frac{1}{2}\gamma V(t,s)}\right). \tag{4.39}$$

Therefore, if $RRA(\hat{y}_{t.s}) \in [0, \chi(s)]$, then $\frac{\partial f(t,s)}{\partial \mu_t} \leq 0$. Since the infimum of $M(t,s) - \frac{1}{2}\gamma V(t,s)$ with respect to s is strictly positive, the inequality $\frac{\partial f(t,s)}{\partial \mu_t} \leq 0$ also holds if $RRA(\hat{y}_{t.s}) \in [0, \chi(s)^*]$, where $\chi(s)^*$ is defined by,

$$\chi(s)^* = \gamma - \left(\frac{e^{-\kappa(s-t)}}{1 - e^{-\kappa(s-t)}}\right)\left(\frac{\kappa}{\inf_{u \geq t} M(t,u) - \frac{1}{2}\gamma V(t,u)}\right). \tag{4.40}$$

But $RRA(\hat{y}_{t,s})$ monotonically converges to 0 as $s \to \infty$ since $\inf_{u \geq t} M(t,u) - \frac{1}{2}\gamma V(t,u) > 0$, whereas $\chi(s)^*$ monotonically converges to γ which is an upper bound of $\{RRA(\hat{y}_{t,s}) : s \geq t\}$. Thus there exists s_t^* such that for all $s \geq s_t^*$, $RRA(\hat{y}_{t,s}) \in [0, \chi(s)^*]$ and (4.38) holds. **Q.E.D.**

4.6.3 Proof of proposition 7

Let us define a stochastic process $\{W_t^* : t \geq t_0\}$ as,

$$W_t^* = W_t + \int_{t_0}^{t} \Lambda_u du, \ t \geq t_0.$$

Then, this process is a Wiener process under the measure Q^* defined by (4.25). The risk-neutralized processes of state variables are given by

$$\begin{aligned}
dX_{1t}^* &= \left[-\kappa X_{1t}^* - \lambda_0 \sigma^\top b - \lambda_2 \sigma^\top b (X_{2t}^* + X_{3t}^*)\right] dt + b^\top dW_t^*, \\
dX_{2t}^* &= \left[-\kappa_c X_{2t}^* - \lambda_0 \sigma^\top \sigma - \lambda_2 \sigma^\top \sigma (X_{2t}^* + X_{3t}^*)\right] dt + \sigma^\top dW_t^*, \\
dX_{3t}^* &= \left[-\kappa_c X_{3t}^* + X_{1t}^*\right] dt,
\end{aligned}$$

with the initial values are the same as the original process X_t. The corresponding risk-neutralized short rate is,

$$r_t^* = \delta_0 + \delta_1 X_{1t}^* + \delta_2 X_{2t}^* + \delta_3 X_{3t}^*.$$

Since the bond price function has the exponentially affine form,

$$P(t,s) = \exp\left(A(s-t) - B_1(s-t) X_{1t}^* - B_2(s-t) X_{2t}^* - B_3(s-t) X_{3t}^*\right),$$

it is easy to show that $A(s-t), B_1(s-t), B_2(s-t), B_3(s-t)$ must satisfy the ordinary linear equations,

$$
\begin{aligned}
0 &= \kappa B_1(s-t) - B_3(s-t) + B_1'(s-t) - \delta_1, && (4.41)\\
0 &= \lambda_2\sigma^\top b B_1(s-t) + (\lambda_2\sigma^\top\sigma + \kappa_c)B_2(s-t) + B_2'(s-t) \\
&\quad -\delta_2 && (4.42)\\
0 &= \lambda_2\sigma^\top b B_1(s-t) + \lambda_2\sigma^\top\sigma B_2(s-t) + \kappa_c B_3(s-t) \\
&\quad +B_3'(s-t) - \delta_3 && (4.43)\\
A'(s-t) &= B_1(s-t)\lambda_0\sigma^\top b + \frac{1}{2}B_1(s-t)^2 b^\top b + B_2(s-t)\lambda_0\sigma^\top\sigma \\
&\quad +\frac{1}{2}B_2(s-t)^2\sigma^\top\sigma - \delta_0, && (4.44)
\end{aligned}
$$

with the boundary conditions $A(0) = B_i(0) = 0,\ i = 1,2,3$. In these equations, $B_i'(s-t)$ is the first order derivatives of $B_i(s-t), (i = 1,2,3)$.

At first, we derive $B_2(s-t) = B_3(s-t)$ for all $s \geq t$. From (4.42) and (4.43), the following equation holds,

$$\kappa_c\left(B_2(s-t) - B_3(s-t)\right) = B_2'(s-t) - B_3'(s-t). \tag{4.45}$$

Clearly, the following equation holds,

$$B_2(s-t) - B_3(s-t) = e^{-\kappa_c(s-t)}\left(B_2(0) - B_3(0)\right). \tag{4.46}$$

By the boundary condition, $B_2(s-t) - B_3(s-t) = 0$ holds for any $s \geq t$.

Next, differentiating (4.41), we obtain the following equation,

$$B_3'(s-t) = \kappa B_1'(s-t) + B_1''(s-t), \tag{4.47}$$

where $B_1''(s-t)$ is the second order derivative of $B_1(s-t)$. Substituting this and (4.41) into (4.43), we obtain the following equation,

$$
\begin{aligned}
\delta_1\left(\lambda_2\sigma^\top\sigma + \kappa_c\right) + \delta_3 &= B_1''(s-t)\left(\lambda_2\sigma^\top\sigma + \kappa_c + \kappa\right) + B_1'(s-t) \\
&\quad +B_1(s-t)\left(\lambda_2\sigma^\top b + \kappa(\lambda_2\sigma^\top\sigma + \kappa_c)\right). && (4.48)
\end{aligned}
$$

But this equation, coupled with $B_1(0) = 0$, forms a second order linear ordinary differential equation problem and the solution is given by (4.27).

From (4.47), $B_3(s-t)$ is expressed as

$$B_3(s-t) = \kappa B_1(s-t) + B_1'(s-t) + C, C \in \mathcal{R}. \tag{4.49}$$

Substituting (4.27) into this equation and using the boundary condition $B_3(0) = 0$, we obtain (4.28). **Q.E.D.**

Chapter 5

Term Structure under One-factor Gaussian Endowment Process

5.1 Introduction

It is well known that the continuous compound interest rates follow normal distribution in Vasicek model. Because of this property, this model has many attractable features. For instance, the derivation of rational price of fixed income derivatives is easy when we assume Vasicek model.

On the other hand, Vasicek model has at least two problems. First, the interest rates in this model can be negative with positive probability. When the model is used for pricing fixed income derivative of nominal interest rates, the possibility of negative interest rates seems serious, because it never happens that the nominal interest rate is negative in the real world. When the equilibrium term structure is considered in the framework of pure exchange economy, the nonnegative interest rates can be obtained by assuming the endowment follows a stochastic process other than Gaussian process. For instance, if we assume that the instantaneous expected rate of growth of endowment follows a square root process, then it is easy to show that the equilibrium short rate is always positive under mild condition on parameters. But this assumption means that the instantaneous expected rate of growth of endowment is nonnegative. Apparently it is difficult to justify the assumption that the expected rate of growth of endowment is always positive.

The second problem in Vasicek model is the term structure of volatility is constant through time and it is monotonically decreasing in time to maturity. As is argued in Dai and Singleton (2004), the volatility curve of interest rates is time-varying and humped shaped in the actual fixed income markets[1]. They also argue that one-factor term structure models cannot generate the volatility curve, the shape of which is humped.

In this chapter, we consider a simple pure exchange economy where the endowment follows one-factor mean-reverting process. In this framework, the equilibrium generates Vasicek model when we set the coefficient of relative risk aversion of representative agent constant. The main objective of this chapter is to explore the possibility to overcome the problems of Vasicek model stated above by generalizing the preference of representative agent. As the preference of representative agent, we consider the same utility function as the one in chapter 4.

The main contribution of this chapter is as follows. First, it is shown that under a mild condition on parameters the interest rates are always positive. Thus, the interesting property of the model is that the interest rates never fall to negative values even if the expected rate of growth of endowment is far below zero. The second contribution of this chapter is that it is shown that the term structure of volatility can be humped shaped by allowing the relative risk aversion of representative agent to decrease.

This chapter is organized as follows. In the next section, we describe the model. In section 3, the equilibrium forward rates are derived and the properties of the forward rates are examined. The final section concludes.

5.2 The Model

Consider a pure exchange economy of a single perishable consumption good. The time span of this economy is $[t_0, \tau]$. The economy is endowed with a flow of the consumption good. The rate of endowment flow is y_t at t for $t \in [t_0, \tau]$. The probability space is denoted as $(\Omega, \mathcal{F}, Q)$. The information structure $\{\mathcal{F}_t : t \in [t_0,\ \tau]\}$ is constructed in the usual manner. We set σ-field $\mathcal{F}_t$ is generated by the path of Wiener process $\{W_s : s \in [t_0, t]\}$. We set

[1]This problem is not unique in Vasicek model and many other term structure models have the same problem. There are few models in which the shape of volatility curve is humped.

$\mathcal{F} = \mathcal{F}_{t_0}$.

In this chapter, it is assumed that the endowment follows a stochastic differential equation,

$$d\ln y_t = (g + \kappa(gt - \ln y_t))\, dt + \sigma dW_t, \tag{5.1}$$

where g, κ, and σ are strictly positive constants and $\{W_t : t \in [t_0, \tau]\}$ is one-dimensional Wiener process. The drift term is time-dependent and linear in $\ln y_t$, but the volatility is constant. Thus $\{\ln y_t : t \in [t_0, \tau]\}$ follows an elastic random walk process.

Given $\ln y_t$, $\ln y_s$ $(s \geq t)$ is normally distributed with mean[2],

$$E[\ln y_s | \mathcal{F}_t] \quad = \quad e^{-\kappa(s-t)} \ln y_t + \left(1 - e^{-\kappa(s-t)}\right) gt + g(s-t), \tag{5.2}$$

and variance,

$$Var\left(\ln y_s | \mathcal{F}_t\right) = \frac{1 - e^{-2\kappa(s-t)}}{2\kappa} \sigma^2. \tag{5.3}$$

We assume the same preference for the representative agent of the economy as chapter 3. That is, the objective function of representative agent is the expectation of time-additive utility as follows,

$$U\left(\{c_s : s \in [t_0, \tau]\}\right) = E\left[\int_{t_0}^{\tau} v(c_s, s) ds\right],$$

where $v(c_s, s) = e^{-\rho s}\left(\alpha c_s + \frac{c_s^{1-\gamma}}{1-\gamma}\right)$ and ρ is the time preference and a positive constant. As in the chapter 3, the coefficient of relative risk aversion under this preference is expressed as,

$$RRA(c) = \gamma \left(\frac{c^{-\gamma}}{\alpha + c^{-\gamma}}\right). \tag{5.4}$$

Again, the coefficient of relative risk aversion takes a value in $(0,\ \gamma)$ and is strictly monotone decreasing in consumption.

The time t price of the default-free pure discount bond in zero-net supply which matures at s is denoted by $P(t, s)$ $(t \leq s)$. Without loss of generality, we set $P(s, s) = 1$. We assume that at least two pure discount bonds with different maturities are traded at any $t \in [t_0, \tau]$. Since the source of the risk in this economy is one-dimensional Wiener process, the market is complete in this setting. The instantaneous forward rate and the short rate are denoted as $f(t, s)$ and r_t respectively.

[2] We show this in the appendix.

5.3 Instantaneous Forward Rate of Interest Rate in Equilibrium

Duffie and Zame(1989) analyzed the asset pricing in a pure exchange economy in continuous time and derived the general formula for prices of financial assets. The pure exchange economy in this chapter is a special case of their economy and satisfies the conditions on the utility function and the stochastic process of endowment. Thus we can apply theorem in Duffie and Zame(1989) and the price of the bond maturing at date s is given by,

$$P(t,s) \quad = \quad \frac{E[v_c(y_s,s)|\mathcal{F}_t]}{v_c(y_t,t)} \tag{5.5}$$

where $v_c(\cdot,s)$ is the first-order derivative of $v(\cdot,s)$ with respect to c.

By the definition of instantaneous forward rate of interest rate, we obtain

$$f(t,s) \quad = \quad \rho + \left(\gamma(g+\kappa e^{-\kappa(s-t)}(gt-\ln y_t)) - \frac{1}{2}\gamma^2\sigma^2 e^{-2\kappa(s-t)}\right)\eta(t,s), \tag{5.6}$$

$$\text{where } \eta(t,s) = \frac{E[y_s^{-\gamma}|\mathcal{F}_t]}{\alpha + E[y_s^{-\gamma}|\mathcal{F}_t]},$$

and the short rate is,

$$r_t \quad = \quad \rho + \left(\gamma(g+\kappa(gt-\ln y_t)) - \frac{1}{2}\gamma^2\sigma^2\right)\eta(t,t). \tag{5.7}$$

The long rate, that is the limit of the forward rate is,

$$\lim_{s\to\infty} f(t,s) = \rho. \tag{5.8}$$

Obviously $\eta(t,s)$ fluctuates in the interval $(0,1)$, and when $\alpha \to 0$, $\eta(t,s)$ converges to unity and Vasicek model holds.

5.3.1 Nonnegativity of instantaneous forward rate of interest rates

One remarkable property of this model is that the forward rate has a lower bound and therefore the condition that the forward rate never be negative can be imposed. The following proposition shows this.

Proposition 8 *In equilibrium, the following inequality holds,*

$$f(t,s) \geq \rho - \frac{1}{2}\gamma^2\sigma^2 - \frac{\kappa}{\alpha}e^{\frac{1}{2}\gamma^2\frac{\sigma^2}{2\kappa}}, \ \forall y_t \in \mathcal{R}^+, \ \forall s \in [t,\infty). \tag{5.9}$$

Proof. See appendix.

For sufficiently large ρ, the forward rates of all the maturities are always positive. The spot rate with any maturity is the average of the instantaneous forward rates. Thus, for sufficiently large ρ, the spot rates are all nonnegative with probability one. When α tends to 0, the lower bound explodes. In this case, that is, in the case of Vasicek model, the forward rate of any maturity has no lower bound and takes negative value with positive probability.

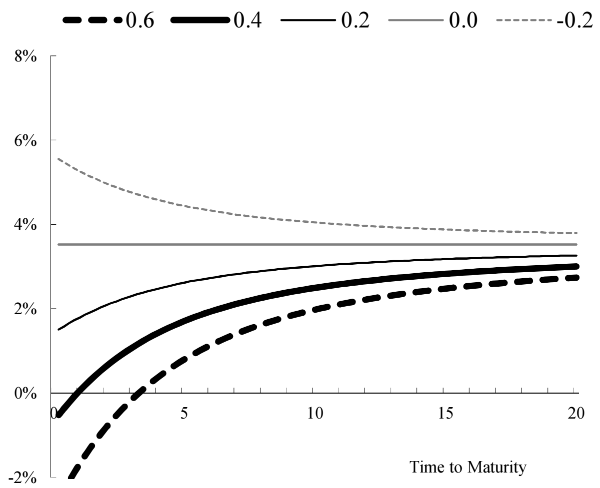

Figure 5.1: The five spot rate curves in Vasicek model are drawn. The level of $\ln y_0$ is chosen to be $\ln y_0 = 0.6, 0.4, 0.2, 0, -0.2$. The parameters are set at the following values: $t = 0$, time preference $\rho = 0.03$, the speed of adjustment $\kappa = 0.4$, the long run average growth rate $g = 0.02$, the volatility of the aggregate consumption growth $\sigma = 0.02$, The coefficient of the relative risk aversion is set at the coefficient relative risk aversion of utility $\alpha c + \frac{c^{1-\gamma}}{1-\gamma}$ when $\gamma = 6, \alpha = 18$ and $\ln c = \ln y_0 = 0.1$.

As a benchmark, Figure 5.1 shows the spot rate curves of Vasicek model. The parameters are set at the following: $t = 0$, time preference $\rho = 0.03$, the speed of adjustment $\kappa = 0.4$, the long run average growth rate $g = 0.02$, the volatility of the aggregate consumption growth $\sigma = 0.02$. Five values of $\ln y_0$ are chosen, $0.6, 0.4, 0.2, 0, -0.2$. The coefficient of the relative risk aversion is set at the coefficient relative risk aversion of utility $\alpha c + \frac{c^{1-\gamma}}{1-\gamma}$ when $\gamma = 6, \alpha = 18$ and $\ln c = \ln y_0 = 0.1$. Note that when $\ln y_0 = 0.6$ and 0.4, the short rate is negative.

Figure 5.2 shows the spot rate curves when the utility is $\alpha c + \frac{c^{(1-\gamma)}}{1-\gamma}$. The level of $\ln y_0$ and the parameters are the same as ones in Figure 5.1. Note that under the specification of parameters given above, the right hand side of (5.9) is positive. Thus the spot rate of any

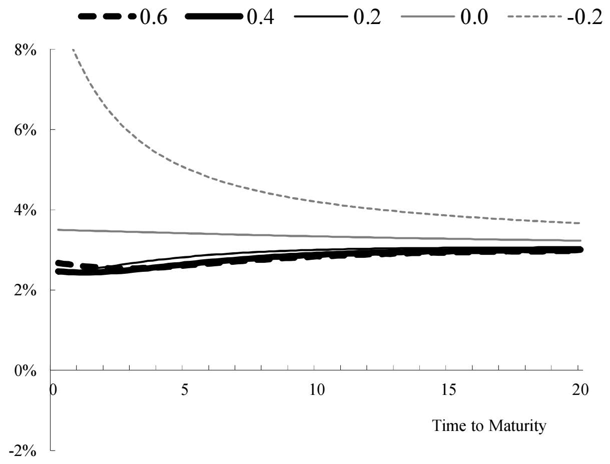

Figure 5.2: The spot rate curve of various $\ln y_0$ when the utility is $\alpha c + \frac{c^{1-\gamma}}{1-\gamma}$ are drawn. The level of $\ln y_0$ and the parameters are the same as ones in Figure fig:vasicek.

maturity is positive. When the logarithm of aggregate consumption is 0.6, the shape of the curve becomes inverted humped. Thus, in contrast to Vasicek model, the short end of spot rate curve does not take a negative value. Note that the inverted humped yield curve never occurs in Vasicek model[3].

The basic reason why the interest rate does not take a negative value is as follows. The intertemporal substitution effect is the product of the expected growth rate of endowment and the reciprocal of elasticity of intertemporal substitution. In the state when the level of aggregate endowment is high, the interest rate is induced to fall, because the instantaneous growth rate of endowment is expected to be low by mean reversion. At the same time, the elasticity of intertemporal substitution becomes high in this state. Note that this change in the elasticity of intertemporal substitution does not occur in Vasicek model. In our model, the total effect is mitigated by time-varying elasticity of intertemporal substitution and the interest rate does not change considerably.

[3]There are models which can generate spot rate curve which is inverted humped. For instance, Constantinides(1991) constructed a model with a specific equivalent martingale measure, which is called "SAINTS (Squared Autoregressive Independent variable Nominal Term Structure) model" and showed that the yield curve can be inverted humped.

5.3.2 Volatility of forward rate of interest rates

In our model, the volatility of the forward rate has an interesting form.

Proposition 9 *The volatility of the forward rate is given by,*

$$\sigma \frac{\partial f(t,s)}{\partial \ln y_t} = -e^{-\kappa(s-t)}\sigma\gamma\left(\eta(t,s)\kappa + (1-\eta(t,s))\left(f(t,s)-\rho\right)\right). \tag{5.10}$$

Proof. See appendix.

When $\alpha = 0$ which is the case of the Vasicek model, the volatility is reduced to $-\gamma\kappa\sigma e^{-\kappa(s-t)}$. On the other hand, when $\alpha \to \infty$, the volatility is proportional to $f(s,t) - \rho$, the difference between the level and the time preference. Thus the volatility is the convex combination of the volatility in Vasicek model and "shifted lognormal volatility ". The weight η_t converges to 0 when $y_t \to \infty$ and converges to unity when $y_t \to 0$. So, when the endowment is large, the volatility is determined mainly by the difference $f(t,s) - \rho$. When the endowment is small, the forward rates fluctuate like Vasicek model.

In modeling the term structure, whether a "lognormal volatility" model can be constructed or not was one of issues. It is known that HJM lognormal model allows riskless arbitrage opportunities[4]. In our framework, the lognormal volatility of forward rates is not supported. Suppose that $\rho = 0$. If we set as $\alpha \to \infty$, the volatility of the forward rate converges to $-\sigma\gamma f(t,s)$. This means that the volatility is proportional to the level of forward rate(lognormal HJM model). But when $\alpha \to \infty$, by (5.6), the forward rate is always zero. Furthermore, when $\alpha \to \infty$, the utility of the representative agent is not well-defined. Thus the lognormal HJM model is not supported also in our model.

Figure 5.3 shows the term structure of volatility that is calculated at the same parameter values as the example in Figure 5.2. It is interesting that "volatility hump" occurs when $\ln y_0 = 0.2, 0.4, 0.6$. The peaks of the curves are at the zone of time to maturity from one-year to three-year. This is consistent with empirical results which report the humped-shaped pattern that peaks around two to three years. As is explained in the previous subsection, the change in interest rate is small when the level of endowment is relatively high, because the intertemporal substitution effect is mitigated by the change in the reciprocal of elasticity

[4]For instance, see Heath, Jarrow, and Morton(1992). See, also Brace, Gatarek, and Musiela(1997).

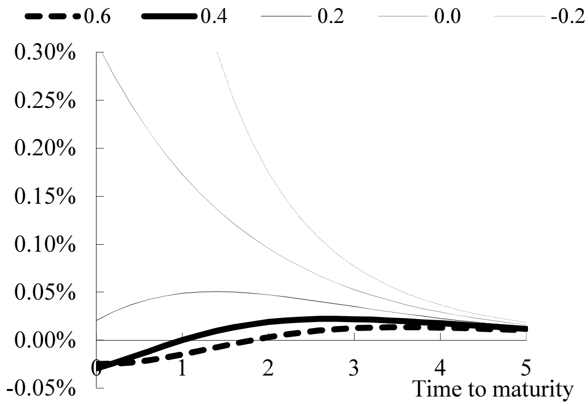

Figure 5.3: The volatility curve of various $\ln y_0$ when the utility is $\alpha c + \frac{c^{1-\gamma}}{1-\gamma}$. The level of $\ln y_0$ and the parameters are the sama as the previous numerical examples.

of intertemporal substitution. This effect is particularly strong for the short end of the term structure and the volatility of the short rate is smaller than the volatility of forward rate with maturity around two to three years.

5.4 Conclusion

In this chapter, a simple pure exchange economy where the endowment follows a one-factor Gaussian process is considered. In this framework, the model generates Vasicek model as a special case when we set the coefficient of relative risk aversion of representative agent constant. We depart from this case and investigate the equilibrium interest rates when the preference of representative agent exhibits decreasing relative risk aversion.

The main conclusions are in two folds. First, it is shown that the interest rates are always nonnegative in equilibrium under the mild condition for parameters. This means that the interest rates never fall to negative values even if the expected rate of growth of endowment is far below zero. This contrasts to the property of the interest rates in Vasicek model.

The second, it is shown that the term structure of volatility is time-varying and the volatility curve has a humped shape. The reason for the humped shape in our model is that the elasticity of intertemporal substitution is time-varying. The change in the elasticity of intertemporal substitution is large for the short end of the curve. Since the effect of intertemporal substitution is the expected instantaneous growth rate of endowment divided by the elasticity of intertemporal substitution, the size of change in intertemporal substitution effect is mitigated and the volatility of the short end is small than the forward rate with two or three years.

5.5 Appendix

5.5.1 Proof of (5.2),(5.3)

Define $\hat{W}_s$ as $(gs - \ln y_s)e^{\kappa(s-t)}$. Ito's lemma gives

$$d\hat{W}_s = -\sigma e^{\kappa(s-t)} dW_s$$

Thus, $\hat{W}_s$ $(s \geq t)$ follows the normal distribution with conditional mean $\hat{W}_t$ and conditional variance $\sigma^2 \frac{1}{2\kappa}\left(e^{2\kappa(s-t)} - 1\right)$. Since $\ln y_s = gs - e^{-\kappa(s-t)}\hat{W}_s$ and $\ln y_s$ is an affine transformation of $\hat{W}_s$, $\ln y_s$ also follows the normal distribution and the conditional mean and variance are given by (5.2),(5.3).

5.5.2 Proof of proposition 8

Since $\frac{E[y_s^{-\gamma}|\mathcal{F}_t]}{\alpha+E[y_s^{-\gamma}|\mathcal{F}_t]} \in (0,1)$, the following inequality holds,

$$\begin{aligned}
f(t,s) &= \rho + \left(g + \gamma\kappa e^{-\kappa(s-t)} gt\right) \frac{E[y_s^{-\gamma}|\mathcal{F}_t]}{\alpha + E[y_s^{-\gamma}|\mathcal{F}_t]} \\
&\quad - \left(\frac{1}{2}\gamma^2\sigma^2 e^{-2\kappa(s-t)}\right) \frac{E[y_s^{-\gamma}|\mathcal{F}_t]}{\alpha + E[y_s^{-\gamma}|\mathcal{F}_t]} \\
&\quad -\gamma\kappa e^{-\kappa(s-t)}(\ln y_t) \frac{E[y_s^{-\gamma}|\mathcal{F}_t]}{\alpha + E[y_s^{-\gamma}|\mathcal{F}_t]} \\
&\geq \rho + (g + \gamma\kappa gt) \inf_{y_t \in \mathcal{R}^+} \left(\frac{E[y_s^{-\gamma}|\mathcal{F}_t]}{\alpha + E[y_s^{-\gamma}|\mathcal{F}_t]}\right) \inf_{s\in[t,\infty)} e^{-\kappa(s-t)} \\
&\quad -\frac{1}{2}\gamma^2\sigma^2 \sup_{y_t \in \mathcal{R}^+} \left(\frac{E[y_s^{-\gamma}|\mathcal{F}_t]}{\alpha + E[y_s^{-\gamma}|\mathcal{F}_t]}\right) \sup_{s\in[t,\infty)} e^{-2\kappa(s-t)} \\
&\quad -\gamma\kappa e^{-\kappa(s-t)}(\ln y_t) \frac{E[y_s^{-\gamma}|\mathcal{F}_t]}{\alpha + E[y_s^{-\gamma}|\mathcal{F}_t]} \\
&\geq \rho - \frac{1}{2}\gamma^2\sigma^2 \\
&\quad -\gamma\kappa e^{-\kappa(s-t)}(\ln y_t) \frac{E[y_s^{-\gamma}|\mathcal{F}_t]}{\alpha + E[y_s^{-\gamma}|\mathcal{F}_t]}. \qquad (5.11)
\end{aligned}$$

Thus when $\ln y_t \leq 0$, the forward rate is larger than $\rho - \frac{1}{2}\gamma^2\sigma^2$. So we consider the case $\ln y_t > 0$, that is $y_t > 1$. In this case, the right hand side of (5.11) satisfies,

$$\begin{aligned}
& \rho - \frac{1}{2}\gamma^2\sigma^2 - \gamma\kappa e^{-\kappa(s-t)}(\ln y_t)\frac{E[y_s^{-\gamma}|y_t]}{\alpha + E[y_s^{-\gamma}|\mathcal{F}_t]} \\
\geq\ & \rho - \frac{1}{2}\gamma^2\sigma^2 - \gamma\kappa e^{-\kappa(s-t)}(\ln y_t)\frac{E[y_s^{-\gamma}|\mathcal{F}_t]}{\alpha} \\
=\ & \rho - \frac{1}{2}\gamma^2\sigma^2 - \frac{\kappa}{\alpha}\frac{\ln y_t}{\left(\frac{y_t^{\gamma e^{-\kappa(s-t)}}}{\gamma e^{-\kappa(s-t)}}\right)}J(s,t),
\end{aligned} \tag{5.12}$$

where the function $J(s,t)$ is defined by,

$$J(s,t) \;=\; \exp\left(-\gamma(1-e^{-\kappa(s-t)})gt - \gamma g(s-t) + \frac{1}{2}\gamma^2\frac{1-e^{-2\kappa(s-t)}}{2\kappa}\sigma^2\right). \tag{5.13}$$

By the way, $\frac{y_t^{\gamma e^{-\kappa(s-t)}}}{\gamma e^{-\kappa(s-t)}} > \ln y_t, \forall y_t \geq 1$ and,

$$\sup_{s\in[t,\infty)} J(s,t) = \exp\left(\frac{1}{2}\gamma^2\frac{\sigma^2}{2\kappa}\right).$$

Thus we obtain the following inequality,

$$f(s,t) \geq \rho - \frac{1}{2}\gamma^2\sigma^2 - \frac{\kappa}{\alpha}\exp\left(\frac{1}{2}\gamma^2\frac{\sigma^2}{2\kappa}\right). \tag{5.14}$$

Q.E.D.

5.5.3 Proof of proposition 9

By Ito's lemma, the volatility of the forward rate is given by $\sigma\frac{\partial f(t,s)}{\partial \ln y_t}$. Simple calculation shows that the derivative $\frac{\partial f(t,s)}{\partial \ln y_t}$ is given by,

$$\begin{aligned}
\frac{\partial f(t,s)}{\partial \ln y_t} \;=\; & -\gamma\kappa e^{-\kappa(s-t)}\frac{E[y_s^{-\gamma}|\mathcal{F}_t]}{\alpha + E[y_s^{-\gamma}|\mathcal{F}_t]} \\
& +\gamma\left(g + \kappa e^{-\kappa(s-t)}(gt - \ln y_t) - \frac{1}{2}\gamma\sigma^2 e^{-\kappa(s-t)}\right) \\
& \times\frac{\partial}{\partial \ln y_t}\left(\frac{E\left[y_s^{-\gamma}|\mathcal{F}_t\right]}{\alpha + E\left[y_s^{-\gamma}|\mathcal{F}_t\right]}\right).
\end{aligned} \tag{5.15}$$

The derivative of the second term in the right hand side is,

$$\begin{aligned}
\frac{\partial}{\partial \ln y_t}\left(\frac{E\left[y_s^{-\gamma}|\mathcal{F}_t\right]}{\alpha + E\left[y_s^{-\gamma}|\mathcal{F}_t\right]}\right) \;&=\; \frac{\partial}{\partial \ln y_t}\left(1 - \frac{\alpha}{\alpha + E\left[y_s^{-\gamma}|\mathcal{F}_t\right]}\right) \\
&=\; \frac{\alpha}{\left(\alpha + E\left[y_s^{-\gamma}|\mathcal{F}_t\right]\right)^2}\frac{\partial E\left[y_s^{-\gamma}|\mathcal{F}_t\right]}{\partial \ln y_t}.
\end{aligned} \tag{5.16}$$

By the normality of $\ln y_s$,

$$\begin{aligned}
\frac{\partial E\left[y_s^{-\gamma}|\mathcal{F}_t\right]}{\partial \ln y_t} &= \frac{\partial}{\partial \ln y_t} \exp\left(-\gamma E\left[\ln y_s|\mathcal{F}_t\right] + \frac{1}{2}\gamma^2 Var_t\left(\ln y_s\right)\right) \\
&= -\gamma e^{-\kappa(s-t)} \exp\left(-\gamma E\left[\ln y_s|\mathcal{F}_t\right] + \frac{1}{2}\gamma^2 Var_t\left(\ln y_s\right)\right) \\
&= -\gamma e^{-\kappa(s-t)} E\left[y_s^{-\gamma-1}|\mathcal{F}_t\right]. \qquad (5.17)
\end{aligned}$$

Substituting (5.17) into (5.16), then into (5.15), and arranging the equation using (5.6), we obtain (5.10). **Q.E.D.**

[**For Memorandum**]

Chapter 6

General Conclusion

In this dissertation, we tried to solve the problems as the empirical challenges stated below:

- the relation between the interest rates and economic activity
- the expectation puzzle
- the humped shape of the term structure of volatility
- the nonnegativity of interest rates.

In chapter 3, we proposed a new equilibrium model that naturally generates the positive correlation between the nominal interest rates and excess consumption. We focused on the partial observability of economic variables in a pure exchange economy and derived closed form solutions for the nominal equilibrium interest rates.

The model considered in this chapter allows us to give a quite different interpretation to the role of excess consumption in determining the interest rates from the consumption habit models. For economic agents engaging in the Bayesian inference, the excess consumption plays a role as an economic indicator helping them to guess the current trend in income growth. Naturally, the economic agents' estimate, hence the equilibrium interest rates, can be increasing in excess consumption under some mild conditions on parameters. This forms a striking contrast to the consumption habit models in which the intertemporal substitution effect induces negative correlation between the excess consumption and interest rates.

Our empirical analysis also supported this view. The estimation results indicate the positive correlation between the implied interest rates and excess consumption. As a consequence, the time series of the nominal yield implied by the model captures many of the short- and long-run fluctuations in the actual data with higher correlations than those obtained by Wachter (2006).

Although the chapter shows some possibilities of partial observability to explain the dynamics of nominal interest rates, there remain some issues to be considered. For instance, the fit gets worse in the longer maturities. More importantly, the model cannot explain the negative relation between the real short rate and the excess consumption. The pair of the facts that the real interest rates have negative and the nominal interest rates have positive relation with the excess consumption should be explained simultaneously. To attain this goal, the model should be revised or another equilibrium model should be considered.

In chapter 4, we investigated equilibrium interest rates in a pure exchange economy where the utility function of representative agent exhibits decreasing relative risk aversion and increasing intertemporal substitution. It is shown that the forward rate curves rotate by time-varying intertemporal substitution even in the absence of time-varying market price of risk. That is, in our model, the constant market price of risk can be consistent with expectation puzzle. After taking expansion, we show that the affine term structure model approximating our exchange economy exhibits negative coefficients in Campbell-Shiller test. In this affine model, completely new state variable which is not found in the literature is introduced. It is the accumulation factor which integrates the other state variable. Thus, adding accumulation factor makes affine models possible to explain the expectation puzzle under the constant market price of risk.

The several important tasks are left. First, the empirical analysis is not conducted. Second, another challenge called LPY(ii) which was documented in Dai and Singleton(2002) is not dealt with in this chapter. The challenge which we deal with in this chapter is only LPY(i). Both challenges should be simultaneously dealt with. Third, risk aversion and intertemporal substitution is not isolated in time-additive utility. Thus, next natural step is to assume a utility function in the class of stochastic differential utilities for the preference of representative agent.

In chpter 5, a simple pure exchange economy where the endowment follows a one-factor Gaussian process is considered. We departed from Vasicek model by generalizing the utility of representative agent and investigated the equilibrium interest rates.

The main conclusions are in two folds. First, it is shown that the interest rates are always nonnegative in equilibrium under the mild condition for parameters. This means that the interest rates never fall to negative values even if the expected rate of growth of endowment is far below zero. This contrasts to the property of the interest rates in Vasicek model.

The second, it is shown that the term structure of volatility is time-varying and the volatility curve has a humped shape. The reason for the humped shape in our model is that the elasticity of intertemporal substitution is time-varying. The change in the elasticity of intertemporal substitution is large for the short end of the curve. Since the effect of intertemporal substitution is the expected instantaneous growth rate of endowment divided by the elasticity of intertemporal substitution, the size of change in intertemporal substitution effect is mitigated and the volatility of the short end is small than the forward rate with two or three years.

There are tasks which remain as future researches. When we argue the non-negativity of interest rates, the interest rates should be nominal. In the framework of this chapter, the inflation risk is not considered at all. In other words, we implicitly assume that the price process is deterministic and we do not make a distinction between nominal and real interest rates. And it is often argued that non-negativity of nominal interest rates holds as a monetary phenomenon. The model in this chapter does not have monetary aspect. Thus, we do not explain the non-negativity of nominal interest rates as a result of monetary phenomenon.

In the past literature, the humped shape of volatility curve is documented in terms of unconditional volatility. In this chapter, the humped shape of volatility curve realizes in some states of economy. This means that we discuss the humped shape of volatility in terms of conditional volatility. Thus the contribution here is that this chapter only provides an economic reasoning by which the humped shape of volatility curve occurs. To explain the humped shape in terms of unconditional volatility, we should further explore the possible extension of equilibrium models.

Overall, although we provided equilibrium models which explain the phenomena we observe in actual fixed income markets, we have to say that the answers obtained are not complete. There still remains tasks as future researches. Nevertheless it is true that the future of equilibrium analysis of interest rates looks bright. Thus we have to keep trying to construct new equilibrium models which explain the phenomena we observe.

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