Stable multigerms, simple multigerms and asymmetric Cantor sets

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Abstract. In this short note, we first show (1) if (n,p) lies inside Mather's nice region then any $\mathcal{A}$-stable multigerm $f:(\mathbb{R}^n,S)\to(\mathbb{R}^p,0)$ and any C^∞ unfolding of f are $\mathcal{A}$-simple, and (2) for any (n,p) there exists a non-negative integer i such that for any integer j $((i\le j))$ there exists an $\mathcal{A}$-stable multigerm $f:(\mathbb{R}^n\times\mathbb{R}^j,S\times\{0\})\to(\mathbb{R}^p\times\mathbb{R}^j,(0,0))$ which is not $\mathcal{A}$-simple. Next, we obtain a characterization of curves among multigerms of corank at most one from the view point of $\mathcal{A}$-stabie multigerms and $\mathcal{A}$-simple multigerms. It turns out that for any (n,p) such that $n<p$ an asymmetric Cantor set is naturally constructed by using upper bounds for multiplicities of $\mathcal{A}$-stable multigerms and upper bounds for multiplicities of $\mathcal{A}$-simple multigerms, and the desired characterization of curves can be obtained by cardinalities of constructed asymmetric Cantor sets.

1. Introduction

For a finite subset $S=\{s_1,\ldots,s_r\}$ $(s_i\neq s_j$ if $i\neq j)$ of $\mathbb{R}^n$ we let $f:(\mathbb{R}^n,S)\to(\mathbb{R}^p,0)$ be a C^∞ map-germ, which is called a *multigerm*. For any i $(1\le i\le r)$ the restriction of f to $(\mathbb{R}^n,s_i)$ is called a *branch of f* and it is denoted by f_i. The integer r is called the *number of branches of f*. Two multigerms $f,\ g:(\mathbb{R}^n,S)\to(\mathbb{R}^p,0)$ are said to be *$\mathcal{A}$-equivalent* if there exist germs of C^∞ diffeomorphisms $\varphi:(\mathbb{R}^n,S)\to(\mathbb{R}^n,S)$ with the condition that $\varphi(s_i)=s_i$ for any i $(1\le i\le r)$ and $\psi:(\mathbb{R}^p,0)\to(\mathbb{R}^p,0)$ such that $f=\psi\circ g\circ\varphi^{-1}$.

A multigerm $f:(\mathbb{R}^n,S)\to(\mathbb{R}^p,0)$ is said to be *$\mathcal{A}$-stable* if for any positive integer d and any C^∞ multigerm $F:(\mathbb{R}^n\times\mathbb{R}^d,S\times\{0\}))\to(\mathbb{R}^p\times\mathbb{R}^d,(0,0))$ of the form $F(x,\lambda)=(f_\lambda(x),\lambda)$ and $f_0=f$, there exist germs of C^∞ diffeomorphisms $H:(\mathbb{R}^n\times\mathbb{R}^d,S\times\{0\}))\to(\mathbb{R}^n\times\mathbb{R}^d,S\times\{0\}))$ with the condition that $H((s_i,0))=(s_i,0)$ for any i $(1\le i\le r)$, $\widetilde{H}:(\mathbb{R}^p\times\mathbb{R}^d,(0,0))\to(\mathbb{R}^p\times\mathbb{R}^d,(0,0))$ and $h:(\mathbb{R}^d,0)\to(\mathbb{R}^d,0)$ such that the following diagram commutes, where $\pi:(\mathbb{R}^p\times\mathbb{R}^d,(0,0))\to(\mathbb{R}^d,0)$ stands for the canonical projection.

$$\begin{array}{ccccc}
(\mathbb{R}^n\times\mathbb{R}^d,(S,0)) & \xrightarrow{F} & (\mathbb{R}^p\times\mathbb{R}^d,(0,0)) & \xrightarrow{\pi} & (\mathbb{R}^d,0)\\
{\scriptstyle H}\big\downarrow & & {\scriptstyle \widetilde{H}}\big\downarrow & & \big\downarrow{\scriptstyle h}\\
(\mathbb{R}^n\times\mathbb{R}^d,(S,0)) & \xrightarrow[(f,\pi)]{} & (\mathbb{R}^p\times\mathbb{R}^d,(0,0)) & \xrightarrow{\pi} & (\mathbb{R}^d,0)
\end{array}$$

A multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ is said to be $\mathcal{A}$-*simple* if there exists a finite number of $\mathcal{A}$-equivalence classes such that for any positive integer d and any C^∞ map $F : U \to V$ where $U \subset \mathbb{R}^n \times \mathbb{R}^d$ is a neighbourhood of $S \times 0$, $V \subset \mathbb{R}^p \times \mathbb{R}^d$ is a neighbourhood of $(0, 0)$, $F(x, \lambda) = (f_\lambda(x), \lambda)$ and $f_0 = f$, there exists a sufficiently small neighbourhood $W \subset U$ of $S \times 0$ such that for any $\{(x_1, \lambda), \cdots, (x_r, \lambda)\} \subset W$ with $F(x_1, \lambda) = \cdots = F(x_r, \lambda)$ the multigerm $f_\lambda : (\mathbb{R}^n, \{x_1, \ldots, x_r\}) \to (\mathbb{R}^p, f_\lambda(x_i))$ lies in one of these finite $\mathcal{A}$-equivalence classes.

THEOREM 1.1. (1) *Suppose that a pair of dimensions (n, p) lies inside the nice region due to Mather. Then any $\mathcal{A}$-stable multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ and any C^∞ unfolding of f are $\mathcal{A}$-simple.*

(2) *For any par of dimensions (n, p) there exists a non-negative integer i such that for any integer j $((i \leq j))$ there exists an $\mathcal{A}$-stable multigerm $f : (\mathbb{R}^n \times \mathbb{R}^j, S \times \{0\}) \to (\mathbb{R}^p \times \mathbb{R}^j, (0, 0))$ which is not $\mathcal{A}$-simple.*

For the definition of Mather's nice region, see [**M6**]. Note that any C^∞ unfolding of an $\mathcal{A}$-stable multigerm is $\mathcal{A}$-stable by Mather's characterization of $\mathcal{A}$-stable multigerms ([**M4**]). Thus, by (1) of Theorem 1.1, the non-negative integer i given in (2) of Theorem 1.1 must satisfy the condition that $(n+i, p+i)$ lies outside Mather's nice region. Topological properties of $\mathcal{A}$-stable map-germs which are $\mathcal{A}$-simple have been well investigated (for instance, see [**D1, D2, D3, D4, D5, DG**]).

Let C_S (resp. C_0) be the set of C^∞ function-germs $(\mathbb{R}^n, S) \to \mathbb{R}$ (resp. $(\mathbb{R}^p, 0) \to \mathbb{R}$). Let m_S (resp. m_0) be the subset of C_S (resp. C_0) consisting of C^∞ function-germs $(\mathbb{R}^n, S) \to (\mathbb{R}, 0)$ (resp. $(\mathbb{R}^p, 0) \to (\mathbb{R}, 0)$). The sets C_S and C_0 have natural $\mathbb{R}$-algebra structures induced by the $\mathbb{R}$-algebra structure of $\mathbb{R}$. For a multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$, let $f^* : C_0 \to C_S$ be the $\mathbb{R}$-algebra homomorphism defined by $f^*(u) = u \circ f$. Put $Q(f) = C_S/f^*(m_0)C_S$. The dimension of $Q(f)$ as a real vector space is called the *multiplicity* of f, and in the case that $n \leq p$ it is finite for an $\mathcal{A}$-stable multigerm and also for an $\mathcal{A}$-simple multigerm. In order to obtain a characterization of curves we construct the natural construction of an asymmetric Cantor set for a given pair of dimensions (n, p) such that $n < p$. For the construction we first recall the known upper bounds for multiplicities. In [**M6, Mn**] Theorem 1.2 of the case that $r = 1$ is proved. However, in [**CTC**] Wall clarifies the meaning of $\gamma(f)$ given in [**M6**] and by using his homomorphism $\bar{t}f : Q(f)^n \to Q(f)^p$ Theorem 1.2 for general r can be proved easily. Thus the proof of it is omitted in this paper.

THEOREM 1.2 ([**M6, Mn**]). *Let $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ $(n \leq p)$ be an $\mathcal{A}$-stable multigerm with corank at most one. Then, the multiplicity of f is restricted in the following way.*

$$\dim_{\mathbb{R}} Q(f) \leq \frac{p + r}{p - n + 1}.$$

THEOREM 1.3 ([**N**]). *Let $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ $(n \leq p, 1 < p)$ be an $\mathcal{A}$-simple multigerm with corank at most one. Then, the multiplicity of f is restricted in the following way.*

$$\dim_{\mathbb{R}} Q(f) \leq \frac{p^2 + (n - 1)r}{n(p - n) + (n - 1)}.$$

Here *corank at most one* for f means that $\max\{n - \mathrm{rank} Jf_i(s_i) \mid 1 \leq i \leq r\} \leq 1$ holds, where $Jf_i(s_i)$ is the Jacobian matrix of the restriction f_i of f at s_i. It is known that Theorem 1.2 gives the best possible bound and in the classification results of $\mathcal{A}$-simple map-germs ([**BG, GH1, GH2, HsK, HnK, KPR, KS, MT, Md, R, WA**]) Theorem 1.3 gives the best possible bound (but, in the case $(n,p,r) = (1,p,1)$ such that $5 < p$, Theorem 1.3 does not give the best possible bound since the effect of fencing curves can not be disregarded as shown in [**A**]). It is known also that every $\mathcal{A}$-stable multigerm with corank at most one is $\mathcal{A}$-simple.

For the number of branches also, there are upper bounds.

THEOREM 1.4. *For any $\mathcal{A}$-stable multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ $(n < p)$ the number of branches of f is restricted in the following way.*

$$r \le \frac{p}{p-n}.$$

THEOREM 1.5 ([**N**]). *For any $\mathcal{A}$-simple multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ $(n < p)$ the number of branches r is restricted in the following way.*

$$r < \frac{p^2}{n(p-n)}.$$

Since for any positive integer r a smooth finite covering with r fibers is $\mathcal{A}$-stable and $\mathcal{A}$-simple, there exists an upper bounds for the number of branches of neither an $\mathcal{A}$-stable multigerm nor an $\mathcal{A}$-simple multigerm in the case that $n = p$.

Now, we construct the natural asymmetric Cantor set for a given pair of dimensions (n,p) such that $n < p$ motivated by Theorems 1.2, 1.3, 1.4 and 1.5. For a given pair of dimensions (n,p) such that $n < p$ we put

$$\begin{aligned} \varphi_{stable,(n,p)}(x) &= \frac{p+x}{p-n+1} \\ \varphi_{simple,(n,p)}(x) &= \frac{p^2+(n-1)x}{n(p-n)+(n-1)}. \end{aligned}$$

Then, note that $\frac{p}{p-n}$ (resp. $\frac{p^2}{n(p-n)}$) is the unique fixed point of the affine function $\varphi_{stable,(n,p)} : \mathbb{R} \to \mathbb{R}$ (resp. $\varphi_{simple,(n,p)} : \mathbb{R} \to \mathbb{R}$). Since for any multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ the multiplicity of f must be greater than or equal to the number of branches, these phenomena suggest that for any i $(1 \le i \le r)$ the branch f_i must be immersive (in other words, $\dim_{\mathbb{R}} Q(f_i) = 1$) if $\frac{p}{p-n} - r < 1$ (resp. $\frac{p^2}{n(p-n)} - r < 1$) for an $\mathcal{A}$-stable multigerm (resp. an $\mathcal{A}$-simple multigerm) f of corank at most one. Furthermore, note that both of $\varphi_{stable,(n,p)}$ and $\varphi_{simple,(n,p)}$ are contractive. Again since for any multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ the multiplicity of f must be greater than or equal to the number of branches, these phenomena suggest that the distribution of multiplicities of branches of f may be uncontrollable.

Let $\mathcal{H}(\mathbb{R})$ be the set of non-empty compact subsets of $\mathbb{R}$. Then, it is known that $\mathcal{H}(\mathbb{R})$ is a complete metric space with respect to the Pompeiu-Hausdorff metric (see [**B, F**]). Define the map $\Phi_{(n,p)} : \mathcal{H}(\mathbb{R}) \to \mathcal{H}(\mathbb{R})$ as

$$\Phi_{(n,p)}(X) = \varphi_{stable,(n,p)}(X) \cup \varphi_{simple,(n,p)}(X).$$

Then, since both of $\varphi_{stable,(n,p)}$ and $\varphi_{simple,(n,p)}$ are contractive, $\Phi_{(n,p)}$ is contractive too (see [**B, F**]). Therefore, by Banach's contraction mapping theorem, we see that there exists the unique fixed point of $\Phi_{(n,p)}$, which is denoted by $\mathcal{C}_{(n,p)}$.

Note that the distribution of $(\dim_{\mathbb{R}} Q(f_1), \ldots, \dim_{\mathbb{R}} Q(f_r))$ for possible $\mathcal{A}$-stable multigerms (resp. $\mathcal{A}$-simple multigerms) $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ of corank at most one is restricted by the coefficient of the linear term $\frac{1}{p-n+1}$ (resp. $\frac{n-1}{n(p-n)+(n-1)}$) and the fixed point $\frac{p}{p-n}$ (resp. $\frac{p^2}{n(p-n)}$) of the affine function $\varphi_{stable,(n,p)}$ (resp. $\varphi_{simple,(n,p)}$). On the other hand, the set $\mathcal{C}_{(n,p)}$ is constructed only by using these four rational numbers $\frac{1}{p-n+1}$, $\frac{n-1}{n(p-n)+(n-1)}$, $\frac{p}{p-n}$ and $\frac{p^2}{n(p-n)}$. Thus, for any given (n,p) such that $n < p$, the set $\mathcal{C}_{(n,p)}$ may be regarded as a visualized clue to investigate both of the distribution of multiplicities of branches of possible $\mathcal{A}$-stable multigerms of corank at most one and the distribution of multiplicities of branches of possible $\mathcal{A}$-simple multigerms of corank at most one simultaneously.

We observe $\mathcal{C}_{(n,p)}$. We see first that $\mathcal{C}_{(n,p)}$ is self-similar by the equality

$$\mathcal{C}_{(n,p)} = \varphi_{stable,(n,p)}(\mathcal{C}_{(n,p)}) \cup \varphi_{simple,(n,p)}(\mathcal{C}_{(n,p)}).$$

Next, let $I_{(n,p)}$ be the closed interval $\left[\frac{p}{p-n}, \frac{p^2}{n(p-n)}\right]$. Then, we see that the intersection $\varphi_{stable,(n,p)}(I_{(n,p)})$ and $\varphi_{simple,(n,p)}(I_{(n,p)})$ is the empty set since we have the following:

$$\frac{1}{p-n+1} + \frac{n-1}{n(p-n)+(n-1)} < 1 < \frac{p^2}{n(p-n)} - \frac{p}{p-n}.$$

Furthermore, for any (n,p) such that $n < p$ each of $\varphi_{stable,(n,p)}$ and $\varphi_{simple,(n,p)}$ is an affine function with one variable and we have

$$\frac{n-1}{n(p-n)+n-1} < \frac{1}{p-n+1}.$$

Thus, it is reasonable to call $\mathcal{C}_{(n,p)}$ the *asymmetric Cantor set relative to* (n,p). The Hausdorff-Besicovitch dimension of the asymmetric Cantor set relative to (n,p) is obtained as the solution of the following equation (for details on Hausdorff-Besicovitch dimensions, see [**B, F**]).

$$\left(\frac{1}{p-n+1}\right)^s + \left(\frac{n-1}{n(p-n)+n-1}\right)^s = 1.$$

We see easily that Hausdorff-Besicovitch dimension of the asymmetric Cantor set is zero if and only if $n = 1$ and it is well-known that the Hausdorff-Besicovitch dimension of a given non-empty compact set is zero if the set is countable (see [**B, F**]). Thus, if $\mathcal{C}_{(n,p)}$ is countable, then we have that $n = 1$. Conversely, if $n = 1$ then $\mathcal{C}_{(n,p)}$ must be countable since $\varphi_{simple,(n,p)}$ is a constant function in this case. Therefore we have the following characterization of curves:

THEOREM 1.6. *Let* (n,p) *be a given pair of dimensions such that* $n < p$. *Then,* $\mathcal{C}_{(n,p)}$ *is a countable set if and only if* $n = 1$.

All results in this paper hold also in complex holomorphic category.

In §2, Theorems 1.1 and 1.4 are proved.

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2. Proofs of Theorems 1.1 and 1.4

For the proofs of Theorems 1.1 and 1.4, we assume that the reader is familiar with Mather theory([**M1, M2, M3, M4, M5, M6**]).

Proof of the assertion (1) of Theorem 1.1. We recall first the following notions given in §7 of [**M5**].

DEFINITION 2.1. (1) Let k be a positive integer and ℓ be a non-negative integer. Put

$$W_\ell = \{z \in J^k(n,p) \mid \mathrm{codim}\mathcal{K}^k(z) \geq \ell\}.$$

Then, W_ℓ is a real closed algebraic set.

(2) The union of irreducible components of W_ℓ whose codimensions are less than ℓ is denoted by W_ℓ^*.

(3) Put $\pi^k(n,p) = \cup_{\ell \geq 0} W_\ell^*$. The set $\pi^k(n,p)$ is also a real closed algebraic set. Let $\sigma^k(n,p)$ be the codimension of $\pi^k(n,p)$. Then, the following holds clearly.

$$\sigma^{k_1}(n,p) \geq \sigma^{k_2}(n,p) \quad (k_1 \leq k_2).$$

(4) Put $\sigma(n,p) = \inf_k \sigma^k(n,p)$.

Mather's nice region can be characterized that the pair of dimensions (n,p) satisfies the condition $n < \sigma(n,p)$. Let $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ be a given $\mathcal{A}$-stable multigerm. Then, the jet extension $j^{p+1}f$ is taransverse to $S \times \mathbb{R}^p \times \mathcal{K}^{p+1}(j^{p+1}f(S))$ and thus for any $s_i \in S$ codimension of $\mathcal{K}^{p+1}(j^{p+1}f(s_i))$ is less than or equal to n. Suppose that $\mathcal{K}^{p+1}(j^{p+1}f(s_i))$ is a subset of $\pi^{p+1}(n,p)$. Then, we have $\sigma(n,p) < \mathrm{codim}\mathcal{K}^{p+1}(j^{p+1}f(s_i)) \leq n$, which contradicts $n < \sigma(n,p)$. Therefore, we have $\mathcal{K}^{p+1}(j^{p+1}f(s_i)) \cap \pi^{p+1}(n,p) = \emptyset$ which means that there are only finitely many $\mathcal{K}^{p+1}$-orbits near $\mathcal{K}^{p+1}(j^{p+1}f(S))$. Since the jet extension $j^{p+1}f$ is taransverse to $\mathcal{K}^{p+1}(j^{p+1}f(S))$, not only f but also any multigerm $g : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ which is near f is $\mathcal{A}$-stable and $\mathcal{A}^{p+1}(j^{p+1}g(S))$ is open in $\mathcal{K}^{p+1}(j^{p+1}g(S))$. Therefore the number of $\mathcal{A}$-orbits which are near $\mathcal{A}(f)$ is finite and thus f is $\mathcal{A}$-simple.

Next we show that any C^∞ unfolding of f is $\mathcal{A}$-simple. Let $F : (\mathbb{R}^n \times \mathbb{R}^d, S \times \{0\}) \to (\mathbb{R}^p \times \mathbb{R}^d, (0,0))$ be a C^∞ unfolding of f. Then, since there are only finitely many $\mathcal{K}^{p+1}$-orbits near $\mathcal{K}^{p+1}j^{p+1}f(S)$, there are only finitely many $\mathcal{K}^{p+d+1}$-orbits near $\mathcal{K}^{p+d+1}(F)$. Since the multigerm F is $\mathcal{A}$-stable, we have that $\mathcal{A}^{p+d+1}(j^{p+d+1}F(S\times\{0\}))$ is in $\mathcal{K}^{p+d+1}(j^{p+d+1}F(S\times\{0\}))$ and $\mathcal{A}^{p+d+1}(j^{p+d+1}G(S\times\{0\}))$ is open in $\mathcal{K}^{p+d+1}(j^{p+d+1}G(S \times \{0\}))$ where $G : (\mathbb{R}^n \times \mathbb{R}^d, S \times \{0\}) \to (\mathbb{R}^p \times \mathbb{R}^d, (0,0))$ is a multigerm near F. Note that G is also $\mathcal{A}$-stable. Therefore, the number of $\mathcal{A}$-orbits which are near $\mathcal{A}(F)$ is finite and thus F is $\mathcal{A}$-simple. □

Note that the above proof works well even in the case $n = \sigma(n,p)$ (that is to say, the case that the pair of dimensions (n,p) lies in the boundary of Mather's nice region) since the equality

$$\sigma(n,p) = \mathrm{codim}\mathcal{K}^{p+1}(j^{p+1}f(s_i))$$

never hold by (2) of Definition 2.1.

Proof of the assertion (2) of Theorem 1.1.

A pair of dimensions (n,p) is inside Mather's nice region if and only if (n,p) satisfies one of the following 5.

(1) $n < \frac{6}{7}p + \frac{8}{7}$ and $p - n \geq 4$,
(2) $n < \frac{6}{7}p + \frac{9}{7}$ and $3 \geq p - n \geq 0$,
(3) $p < 8$ and $p - n = -1$,
(4) $p < 6$ and $p - n = -2$,
(5) $p < 7$ and $p - n \leq -3$.

Therefore, we see that for any pair of dimensions (n,p) there exists a non-negative integer i_1 such that for any integer j_1 $(i_1 \leq j_1)$ the pair of dimensions $(n+j_1, p+j_1)$

lies outside Mather's nice region. Let z be a $(p+i_1+1)$-jet belonging $\pi^{p+i_1+1}(n+i_1, p+i_1)$ and let $f : (\mathbb{R}^n \times \mathbb{R}^{i_1}, S \times \{0\}) \to (\mathbb{R}^p \times \mathbb{R}^{i_1}, (0,0))$ be a representative of z. If f is $\mathcal{A}$-stable, then by putting $i = i_1$ we have the assertion (2) of Theorem 1.1. If f is not $\mathcal{A}$-stable, then by using Mather's construction of $\mathcal{A}$-stable germs we see that there exists a positive integer i_2 and a C^∞ unfolding $F : (\mathbb{R}^n \times \mathbb{R}^{i_1+i_2}, S \times \{0\}) \to (\mathbb{R}^p \times \mathbb{R}^{i_1+i_2}, (0,0))$ of the multigerm f such that F is $\mathcal{A}$-stable. Then, note that $j^{p+i_1+i_2+1}F(S \times \{0\})$ belongs to $\pi^{p+i_1+i_2+1}(n+i_1+i_2, p+i_1+i_2)$ since $j^{p+i_1+1}f(S \times \{0\})$ belongs to $\pi^{p+i_1+1}(n+i_1, p+i_1)$. Therefore, by putting $i = i_1+i_2$ we have the assertion (2) of Theorem 1.1. □

Proof of Theorem 1.4.

Let $\theta_S(f)$ (resp. $\theta_S(n)$) be the C_S-module consisting of germs of C^∞ vector fields along f (resp. along the germ of identity mapping: $(\mathbb{R}^n, S) \to (\mathbb{R}^n, S)$). Let $\theta_0(p)$ be the C_0-module consisting of germs of C^∞ vector fields along the germ of identity mapping: $(\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$. Let $tf : \theta_S(n) \to \theta_S(f)$ (resp. $\omega f : \theta_0(p) \to \theta_S(f)$) be the map defined by $tf(a) = df \circ a$ (resp. $\omega f(b) = b \circ f$). Then, since the given f in Theorem 1.4 is $\mathcal{A}$-stable, we have that

$$\theta_S(f) = tf(\theta_S(n)) + \omega f(\theta_0(p)).$$

Since we have

$$\dim_{\mathbb{R}} \frac{\theta_S(f)}{m_S\theta_S(f)} = pr, \ \dim_{\mathbb{R}} \frac{\theta_S(n)}{m_S\theta_S(n)} = nr \ \text{ and } \ \dim_{\mathbb{R}} \frac{\theta_0(p)}{m_0\theta_0(p)} = p,$$

we have the following desired inequality.

$$pr \leq nr + p.$$

□

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