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SINGULARITIES OF PEDAL CURVES PRODUCED BY SINGULAR DUAL CURVE GERMS IN S^n

Abstract. For an n-dimensional spherical unit speed curve $\mathbf{r}$ and a given point P, we can define naturally the pedal curve of $\mathbf{r}$ relative to the pedal point P. When the dual curve germs are singular, singularity types of pedal curves depend on singularity types of the n-th curvature function germs and the locations of pedal points. In this paper, we investigate sigularity types of pedal curves in such cases.

1. Introduction

Let I be an open interval such that $0 \in I$ and S^n be the n-dimensional unit sphere in $\mathbf{R}^{n+1}$ $(n \geq 2)$. A C^∞ non-singular map $\mathbf{r} : I \to S^n$ is said to be a *spherical unit speed curve* if each of the following $\mathbf{u}_i(s)$ $(1 \leq i \leq n-1)$ is inductively well-defined for any $s \in I$, where initial information are $\mathbf{u}_{-1}(s) \equiv \mathbf{0}$, $\mathbf{u}_0(s) = \mathbf{r}(s)$, $\|\mathbf{u}_0'(s)\| \equiv 1$ and $\kappa_0(s) \equiv 0$.

$$\mathbf{u}_i(s) = \frac{\mathbf{u}_{i-1}'(s) + \kappa_{i-1}(s)\mathbf{u}_{i-2}(s)}{\|\mathbf{u}_{i-1}'(s) + \kappa_{i-1}(s)\mathbf{u}_{i-2}(s)\|} \qquad (1 \leq i \leq n-1),$$

$$\kappa_i(s) = \|\mathbf{u}_{i-1}'(s) + \kappa_{i-1}(s)\mathbf{u}_{i-2}(s)\| \qquad (1 \leq i \leq n-1).$$

The function $\kappa_i : I \to \mathbf{R}$ is called the *i-th curvature function* of $\mathbf{r}$. For a spherical unit speed curve two vectors $\mathbf{u}_i(s)$ and $\mathbf{u}_j(s)$ $(0 \leq i, j \leq n-1$, $i \neq j)$ are perpendicular ([17]). Thus we can define one more vector $\mathbf{u}_n(s)$ uniquely so that $\{\mathbf{u}_0(s), \mathbf{u}_1(s), \ldots, \mathbf{u}_n(s)\}$ is an orthogonal moving frame and $\det(\mathbf{u}_0(s), \ldots, \mathbf{u}_n(s)) = 1$ for any $s \in I$. The map $\mathbf{u}_n : I \to S^n$, which is called the *dual curve* of $\mathbf{r}$ ([1], [21]), defines the *n-th curvature function* in the following way, where the dot in the center is the scalar product.

$$\kappa_n(s) = \mathbf{u}_{n-1}'(s) \cdot \mathbf{u}_n(s).$$

We see that the dual curve $\mathbf{u}_n$ is non-singular at s if and only if $\kappa_n(s) \neq 0$ (see §2).

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For any $i \quad (-1 \le i \le n)$, we put

$$S^i_{\mathbf{u}_i(s)} = (S^n - \{\pm \mathbf{u}_n(s)\}) \cap \langle \mathbf{u}_{-1}(s), \dots, \mathbf{u}_i(s) \rangle_{\mathbf{R}},$$

where $\langle \mathbf{u}_{-1}(s), \dots, \mathbf{u}_i(s) \rangle_{\mathbf{R}}$ means the vector subspace spanned by the vectors $\mathbf{u}_{-1}(s), \dots, \mathbf{u}_i(s)$. Given a spherical unit speed curve $\mathbf{r} : I \to S^n$, choosing a point P of $S^n - \{\pm \mathbf{u}_n(s) \mid s \in I\}$ gives the map which maps $s \in I$ to the unique nearest point in $S^{n-1}_{\mathbf{u}_{n-1}(s)}$ from P. Such a map, which is called the *pedal curve* relative to the *pedal point* P for an n-dimensional unit speed curve $\mathbf{r}$, is denoted by $ped_{\mathbf{r},P}$. Note that since all points in $S^{n-1}_{\mathbf{u}_{n-1}(s)}$ are the nearest points from $\pm \mathbf{u}_n(s)$ the pedal point P for the map-germ $ped_{\mathbf{r},P}$ at s must be outside $\{\pm \mathbf{u}_n(s)\}$.

In [17] we have shown the following.

Theorem 1. ([17]) *Let $\mathbf{r} : I \to S^n$ be an n-dimensional spherical unit speed curve. Suppose that $\kappa_n(0) \neq 0$. Then the following hold.*

1. *The pedal point P is inside $S^n_{\mathbf{u}_n(0)} - S^{n-2}_{\mathbf{u}_{n-2}(0)}$ if and only if the map-germ $ped_{\mathbf{r},P} : (I, 0) \to S^n$ is C^∞ left equivalent to the map-germ given by $s \mapsto (s, 0, \dots, 0)$.*
2. *For any i $(2 \le i \le n)$, the pedal point P is inside $S^{n-i}_{\mathbf{u}_{n-i}(0)} - S^{n-i-1}_{\mathbf{u}_{n-i-1}(0)}$ if and only if the map-germ $ped_{\mathbf{r},P} : (I, 0) \to S^n$ is C^∞ left equivalent to the map-germ given by the following:*

$$s \mapsto (\underbrace{s^i, s^{i+1}, \dots, s^{2i-1}}_{i \text{ elements}}, \underbrace{0, \dots, 0}_{(n-i) \text{ elements}}).$$

Here, two map-germs $f, g : (\mathbf{R}, 0) \to (\mathbf{R}^n, 0)$ are said to be C^∞ *left equivalent* if there exists a germ of C^∞ diffeomorphism $h_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that the identity $g = h_t \circ f$ is satisfied.

The purpose of this paper is to investigate singularities of pedal curves when $\kappa_n(0) = 0$. We say that the n-th curvature function κ_n has an *A_k-type singularity* at 0 $(0 \le k < \infty)$ if $\kappa_n(0) = \kappa_n'(0) = \cdots = \kappa_n^{(k)}(0) = 0$ and $\kappa_n^{(k+1)}(0) \neq 0$.

Theorem 2. *Let $\mathbf{r} : I \to S^n$ be an n-dimensional spherical unit speed curve. Suppose that $P \in S^n_{\mathbf{u}_n(0)} - S^{n-1}_{\mathbf{u}_{n-1}(0)}$. Then the following holds.*

1. *If κ_n has an A_k-type singularity at 0 $(0 \le k \le n-2)$, then the map-germ $ped_{\mathbf{r},P} : (I, 0) \to S^n$ is C^∞ left equivalent to the map-germ given by*

$$s \mapsto (\underbrace{s^{k+2}, s^{k+3}, \dots, s^{2k+3}}_{(k+2) \text{ elements}}, \underbrace{0, \dots, 0}_{(n-k-2) \text{ elements}}).$$

2. *If κ_n has an A_{n-1}-type singularity at 0, then the map-germ $ped_{\mathbf{r},P}$: $(I,0) \to S^n$ is C^∞ right-left equivalent to the map-germ given by*

$$s \mapsto (s^{n+1}, s^{n+2}, \ldots, s^{2n}).$$

Here, two map-germs $f, g : (\mathbf{R}, 0) \to (\mathbf{R}^n, 0)$ are said to be C^∞ *right-left equivalent* if there exist germs of C^∞ diffeomorphisms $h_s : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ and $h_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that the identity $g = h_t \circ f \circ h_s^{-1}$ is satisfied. In the case that $n = 2$ Theorem 2 has been announced in [20]. In the case that $n \geq 3$ it seems to be almost impossible to obtain similar results when κ_n has an A_n-type singularity at 0. We may observe its reason in the following way. It is possible to show that $ped_{\mathbf{r},P}$ is C^∞ right-left equivalent to $\varphi(s) = (s^{n+2}, s^{n+3} + \varphi_2(s), \ldots, s^{2n+1} + \varphi_n(s))$ where $\varphi_j(s) = o(s^{2n+1})$. However, φ is not $\mathcal{A}$-simple since in the case that $n = 3$ fencing curves due to Arnol'd ([2]) have the form of φ and for $n \geq 3$ the local multiplicity of φ is more than $\frac{n^2}{(n-1)}$ which is an upper bound for the local multiplicity of an $\mathcal{A}$-simple map-germ; and the codimension of $T\mathcal{A}(\varphi)$ in $T\mathcal{K}(\varphi)$ is positive (for the restriction on the local multiplicity of an $\mathcal{A}$-simple map-germ, see [18], [19]). Thus, there must exist strong restrictions on higher terms φ_j which can be truncated.

Next, we investigate singularity types of pedal curves when $P \in S^{n-1}_{\mathbf{u}_{n-1}(0)}$. We concentrate on the case that κ_n has an A_0-type singularity at 0. Note that κ_n has an A_0-type singularity at 0 if and only if the function-germ $\kappa_n : (I, 0) \to (\mathbf{R}, 0)$ is non-singular, and the dual curve germ is an ordinary cusp in this case.

Theorem 3. *Let $\mathbf{r} : I \to S^n$ be an n-dimensional spherical unit speed curve. Suppose that κ_n has an A_0-type singularity at 0. Then the following hold.*

1. *The pedal point P is inside $S^n_{\mathbf{u}_n(0)} - S^{n-1}_{\mathbf{u}_{n-1}(0)}$ if and only if the map-germ $ped_{\mathbf{r},P} : (I,0) \to S^n$ is C^∞ left equivalent to the map-germ given by*

$$s \mapsto (s^2, s^3, 0, \ldots, 0).$$

2. *For any i $(1 \leq i \leq n-1)$, the pedal point P is inside $S^{n-i}_{\mathbf{u}_{n-i}(0)} - S^{n-i-1}_{\mathbf{u}_{n-i-1}(0)}$ if and only if the map-germ $ped_{\mathbf{r},P} : (I,0) \to S^n$ is C^∞ right-left equivalent to the map-germ given by*

$$s \mapsto (s^{i+1}, \underbrace{s^{i+3}, s^{i+4}, \ldots, s^{2i+1}}_{(i-1)\ elements}, s^{2i+3}, \underbrace{0, \ldots, 0}_{(n-i-1)\ elements}).$$

3. *The pedal point P is inside $S^0_{\mathbf{u}_0(0)} - S^{-1}_{\mathbf{u}_{-1}(0)}$ if and only if the map-germ*

$ped_{\mathbf{r},P} : (I, 0) \to S^n$ is C^∞ right-left equivalent to the map-germ given by

$$s \mapsto (s^{n+1}, \underbrace{s^{n+3}, s^{n+4}, \ldots, s^{2n+1}}_{(n-1)\ elements}).$$

In the case that $n = 2$ the "only if" parts of Theorem 3 has been announced in [20]. Note that the first assertion of Theorem 2 yields only the "only if" part of the first assertion of Theorem 3. By obtaining a complete list of locations of pedal points inside $S^{n-1}_{\mathbf{u}_{n-1}(0)}$ and singularity types of pedal curves (assertions 2 and 3 of Theorem 3) we can obtain "if" part of the first assertion of Theorem 3.

In §2 we give several preparations to prove Theorems 2 and 3. Theorems 2 and 3 are proved in §3 and §4 respectively.

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2. Preliminaries

We put

$$U(s) = (\mathbf{u}_0(s)^t, \mathbf{u}_1(s)^t, \ldots, \mathbf{u}_n(s)^t),$$

where $\mathbf{u}_i(s)^t$ means the transposed vector of $\mathbf{u}_i(s)$. We further put

$$K(s) = \begin{pmatrix} 0 & \kappa_1(s) & 0 & \cdots & 0 & 0 & 0 \\ -\kappa_1(s) & 0 & \kappa_2(s) & \ddots & 0 & 0 & 0 \\ 0 & -\kappa_2(s) & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & \kappa_{n-1}(s) & 0 \\ 0 & 0 & 0 & \ddots & -\kappa_{n-1}(s) & 0 & \kappa_n(s) \\ 0 & 0 & 0 & \cdots & 0 & -\kappa_n(s) & 0 \end{pmatrix}.$$

Then, the following Serret Frenet type formula holds.

Lemma 2.1. ([17])

$$\frac{d}{ds}U(s)^t = K(s)U(s)^t.$$

By Lemma 2.1 we see that the dual curve $\mathbf{u}_n$ is non-singular at 0 if and only if $\kappa_n(0) \neq 0$. By using Lemma 2.1 again and again we obtain the following:

Lemma 2.2. *Suppose that κ_n has an A_k type singularity at 0 $(k \leq n-1)$. Then, for any i $(0 \leq i \leq n-1)$ properties $\mathbf{u}_i(0)\cdot\mathbf{u}_n^{(\ell)}(0) = 0$ $(0 \leq \ell \leq n-i+k)$ and $\mathbf{u}_i(0) \cdot \mathbf{u}_n^{(n-i+k+1)}(0) \neq 0$ hold.*

Lemma 2.3. ([17]) *The pedal curve of* $\mathbf{r}$ *relative to the pedal point* P *is given by the following expression:*

$$ped_{\mathbf{r},P}(s) = \frac{1}{\sqrt{1-(P\cdot \mathbf{u}_n(s))^2}}(P-(P\cdot \mathbf{u}_n(s))\mathbf{u}_n(s)).$$

Let Ψ_P be the C^∞ map from $S^n - \{\pm P\}$ to S^n given by

$$\Psi_P(\mathbf{x}) = \frac{1}{\sqrt{1-(P\cdot \mathbf{x})^2}}(P-(P\cdot \mathbf{x})\mathbf{x}).$$

We see that the image $\Psi_P(S^n - \{\pm P\})$ is inside the open hemisphere centered at P. Let this open hemisphere be denoted by X_P and set $B_P = \pi(S^n - \{\pm P\})$, where $\pi : S^n \to P^n(\mathbf{R})$ is the canonical projection. Since $\Psi_P(\mathbf{x}) = \Psi_P(-\mathbf{x})$, the map Ψ_P canonically induces the map $\widetilde{\Psi}_P : B_P \to X_P$. Then, Lemma 2.3 shows that $ped_{\mathbf{r},P}$ is factored into three maps in the following way:

$$ped_{\mathbf{r},P}(s) = \widetilde{\Psi}_P \circ \pi \circ \mathbf{u}_n(s).$$

Let $p : B \to \mathbf{R}^n$ be the blow up centered at the origin.

Lemma 2.4. ([17]) *Let* P *be a point of* S^n. *Then, there exist* C^∞ *diffeomorphisms* $h_1 : B_P \to B$ *and* $h_2 : X_P \to \mathbf{R}^n$ *such that the equality* $h_2 \circ \widetilde{\Psi}_P = p \circ h_1$ *holds and the set* $\{[\mathbf{x}] \in B_P \mid \mathbf{x}\cdot P = 0, \mathbf{x} \in S^n\}$ *is mapped to the exceptional set of* p *by* h_1.

Next, we prepare several notions and notations of Mather theory ([11], [12], [13], [14], [15], [16]) which are already common in singularity theory of differentiable mappings. An excellent survey article on Mather theory is [23] which we recommend to the readers.

For any positive integer r let $\mathcal{E}_r$ be the $\mathbf{R}$-algebra of all C^∞ function-germs at the origin of $\mathbf{R}^r$ with usual operations, and let m_r be the unique maximal ideal of $\mathcal{E}_r$.

For any positive integers p, q given a C^∞ map-germ $f : (\mathbf{R}^p, 0) \to (\mathbf{R}^q, 0)$, we let $\theta(f)$ be the $\mathcal{E}_p$-module of vector fields along f. We may identify $\theta(f)$ with $\mathcal{E}_p^q$. For any positive integer r we put $\theta(r) = \theta(id._{\mathbf{R}^r})$, where $id._{\mathbf{R}^r}$ is the identity map-germ of $\mathbf{R}^r$ at the origin. An element of $m_p^\ell\theta(f)$ is a vector field along f such that the Taylor polynomial up to $(\ell-1)$-th degree of it at the origin is zero. The map $f^* : \mathcal{E}_q \to \mathcal{E}_p$ is defined by $f^*(u) = u \circ f$. Two homomorphisms tf (tf is an $\mathcal{E}_p$-homomorphism) and ωf (ωf is an $\mathcal{E}_q$-homomorphism via f^*) are defined in the following way:

$$tf : \theta(p) \to \theta(f), \qquad tf(a) = df \circ a,$$
$$\omega f : \theta(q) \to \theta(f), \qquad \omega f(b) = b \circ f,$$

where df is the differential of f. We put

$$T\mathcal{L}(f)=\omega f(m_q\theta(q)),\quad T\mathcal{C}(f)=f^*m_q\theta(f),$$
$$T\mathcal{A}(f)=tf(m_p\theta(p))+\omega f(m_q\theta(q)),\quad T\mathcal{K}(f)=tf(m_p\theta(p))+f^*m_q\theta(f).$$

The Taylor polynomial up to r-th degree at the origin of f is called *r jet of f at the origin* and is denoted by $j^rf(0)$.

Two map-germs $f,g:(\mathbf{R}^p,0)\to(\mathbf{R}^q,0)$ are said to be *$\mathcal{A}$-equivalent* (resp. $\mathcal{L}$-equivalent) if there exist germs of C^∞ diffeomorphisms $h_s:(\mathbf{R}^p,0)\to(\mathbf{R}^p,0)$ and $h_t:(\mathbf{R}^q,0)\to(\mathbf{R}^q,0)$ (resp. a germ of C^∞ diffeomorphism $h_t:(\mathbf{R}^q,0)\to(\mathbf{R}^q,0)$) such that $g=h_t\circ f\circ h_s^{-1}$ (resp. $g=h_t\circ f$). A C^∞ map-germ $f:(\mathbf{R}^p,0)\to(\mathbf{R}^q,0)$ is said to be *r-$\mathcal{A}$-determined* (resp. *r-$\mathcal{L}$-determined*) if f is $\mathcal{A}$-equivalent (resp. $\mathcal{L}$-equivalent) to any C^∞ map-germ g with $j^rf(0)=j^rg(0)$, and is said to be *finitely $\mathcal{A}$-determined* (resp. *finitely $\mathcal{L}$-determined*) if f is r-$\mathcal{A}$-determined (resp. r-$\mathcal{L}$-determined) by a certain r.

3. Proof of Theorem 2

By composing suitable rotations of S^n to $\mathbf{r}$ if necessary, from the first we may assume that $\mathbf{u}_0(0)=(0,\dots,0,1)$, $\mathbf{u}_1(0)=(0,\dots,0,1,0)$, $\dots$, $\mathbf{u}_{n-1}(0)=(0,1,0,\dots,0)$ and $\mathbf{u}_n(0)=((-1)^\alpha,0,\dots,0)$ where $\alpha=\frac{(n-1)n}{2}$. Suppose that κ_n has an $\mathcal{A}_k$-type singularity at 0 ($0\le k\le n-1$). Then, by Lemma 2.2, we see that the lowest degree of non-zero terms of u_{in} ($1\le i\le n$) is $i+k+1$ for the component function germ u_{in} of the dual curve germ $\mathbf{u}_n=(u_{0n},\dots,u_{nn}):(I,0)\to S^n$.

The assumption that P is a point of $S^n_{\mathbf{u}_n(0)}-S^{n-1}_{\mathbf{u}_{n-1}(0)}$ implies that the scalar product $\mathbf{u}_n(0)\cdot P$ is not zero. Therefore, by Lemma 2.4 the germ of $\widetilde{\Psi}_P:(P^n(\mathbf{R}),\pi\circ\mathbf{u}(0))\to S^n$ is a germ of C^∞ diffeomorphism. It is clear that the canonical projection $\pi:S^n\to P^n(\mathbf{R})$ is a local C^∞ diffeomorphism. Thus, in the case of Theorem 2, the map-germ $ped_{\mathbf{r},P}:(I,0)\to S^n$ is C^∞ left equivalent to the map-germ $(u_{1n},\dots,u_{nn}):(I,0)\to\mathbf{R}^n$ given by

$$s\mapsto(s^{k+2}+\varphi_1(s),s^{k+3}+\varphi_2(s),\dots,s^{k+n+1}+\varphi_n(s)),$$

where $\varphi_j(s)=o(s^{k+n+1})$ ($1\le j\le n$).

Proof of the assertion 1 of Theorem 2. From the arguments above, the map-germ $ped_{\mathbf{r},P}$ is C^∞ left equivalent to $\psi(s)=(s^{k+2}+\psi_1(s),s^{k+3}+\psi_2(s),\dots,s^{k+n+1}+\psi_n(s))$ where $\psi_j(s)=o(s^{k+n+1})$.

Put $f(s)=s^{k+2}$ and apply the Malgrange preparation theorem (for instance, see [6], [7], [9], [23]) to $m_1^{k+2}\mathcal{E}_1$ and f. Then we see that for any function-germ $g\in m_1^{k+2}\mathcal{E}_1$ there exists a certain C^∞ function-germ ψ such that

$$g(s)=\psi(s^{k+2},\dots,s^{2k+3}).$$

Note that $2k+3 \leq k+n+1$ in the case of the assertion 1 of Theorem 2. Thus, for our map-germ $ped_{\mathbf{r},P} : (I,0) \to (S^n, ped_{\mathbf{r},P}(0))$ there exists a germ of C^∞ diffeomorphism $h_t : (S^n, ped_{\mathbf{r},P}(0)) \to (\mathbf{R}^n, 0)$ such that

$$h_t \circ ped_{\mathbf{r},P}(s) = (\underbrace{s^{k+2}, s^{k+3}, \ldots, s^{2k+3}}_{(k+2)\ elements}, \underbrace{0, \ldots, 0}_{(n-k-2)\ elements}). \blacksquare$$

Note that in the case of the assertion 1 of Theorem 2 the following equalities hold:

$$T\mathcal{K}(ped_{\mathbf{r},P}) = T\mathcal{C}(ped_{\mathbf{r},P}) = T\mathcal{A}(ped_{\mathbf{r},P}) = T\mathcal{L}(ped_{\mathbf{r},P}).$$

Proof of the assertion 2 of Theorem 2. It sufficies to show that

$$f(s) = (s^{n+1}, \ldots, s^{2n})$$

is $2n$-$\mathcal{A}$-determined.

Since $n+1$ and $n+2$ are relatively prime, we see that $\gcd(n+1, \ldots, 2n) = 1$, where gcd means the greatest common divisor. Thus, the map $f_{\mathbf{C}}(z) = (z^{n+1}, \ldots, z^{2n})$ $(z \in \mathbf{C})$, which is the complexification of f, is injective. From this and the fact that $f_{\mathbf{C}}$ has an isolated singularity at the origin, by the geometric characterization of finite determinacy due to Mather and Gaffney (see §2 of [23]) we see that f is finitely $\mathcal{L}$-determined. Hence, in order to show that f is $2n$-$\mathcal{A}$-determined it is sufficient to show that

$$m_1^{2n+1}\theta(f+h) \subset T\mathcal{A}(f+h)$$

for any C^∞ map-germ $h : (I,0) \to \mathbf{R}^n$ such that $j^{2n}h(0) = 0$ by Mather's lemma (Corollary 3.2 of [14], see also §4 of [23]).

Let $h : (I,0) \to \mathbf{R}^n$ be a C^∞ map-germ such that $j^{2n}h(0) = 0$. Then, we see easily that the following holds.

$$f^* m_n \mathcal{E}_1 = (f+h)^* m_n \mathcal{E}_1 + f^* m_n^2 \mathcal{E}_1.$$

Thus, by Nakayama's lemma (for instance, see [6], [7], [9], [23]) we see that

$$f^* m_n \mathcal{E}_1 = (f+h)^* m_n \mathcal{E}_1,$$

and therefore both sets are equal to $m_1^{n+1}\mathcal{E}_1$. Consider generators of the following quotient vector space:

$$\frac{(f+h)^* m_n \theta(f+h)}{(f+h)^* m_n^2 \theta(f+h)}.$$

Since we see easily that

$$s^{2n+1}\frac{\partial}{\partial X_\ell} \in T\mathcal{A}(f+h) + (f+h)^* m_n^2 \theta(f+h) \quad (1 \leq \ell \leq n)$$

where $(X_1, \ldots, X_n) \in \mathbf{R}^n$, we have that

$$(f+h)^* m_n \theta(f+h) \subset T\mathcal{A}(f+h) + (f+h)^* m_n^2 \theta(f+h).$$

Apply the Malgrange preparation theorem to $(f+h)^*m_n\theta(f+h)$ and $f+h$. Then, we have the following desired inclusion:

$$m_1^{2n+1}\theta(f+h) \subset m_1^{n+1}\theta(f+h) = (f+h)^*m_n\theta(f+h) \subset T\mathcal{A}(f+h). \ \blacksquare$$

Note that in the case of the assertion 2 of Theorem 2 the following equalities hold but the equality for $T\mathcal{L}(ped_{\mathbf{r},P})$ does not hold:

$$T\mathcal{K}(ped_{\mathbf{r},P}) = T\mathcal{C}(ped_{\mathbf{r},P}) = T\mathcal{A}(ped_{\mathbf{r},P}).$$

4. Proof of Theorem 3

Since $\{S^n_{\mathbf{u}_n(0)} - S^{n-1}_{\mathbf{u}_{n-1}(0)}, S^{n-1}_{\mathbf{u}_{n-1}(0)} - S^{n-2}_{\mathbf{u}_{n-2}(0)}, \ldots, S^0_{\mathbf{u}_0(0)} - S^{-1}_{\mathbf{u}_{-1}(0)}\}$ gives a stratification of $S^n - \{\pm\mathbf{u}_n(0)\}$, the "if" parts of the assertions 1–3 of Theorem 3 follow from the corresponding "only if" parts. Moreover, since the "only if" part of the first assertion of Theorem 3 is contained in the assertion 1 of Theorem 2, we just need to show the "only if" parts of the assertions 2 and 3 of Theorem 3.

By composing suitable rotations of S^n to $\mathbf{r}$ if necessary, we may assume that $\mathbf{u}_0(0) = (0,\ldots,0,1)$, $\mathbf{u}_1(0) = (0,\ldots,0,1,0)$, $\ldots$, $\mathbf{u}_{n-1}(0) = (0,1,0,\ldots,0)$ and $\mathbf{u}_n(0) = ((-1)^\alpha, 0, \ldots, 0)$, where $\alpha = \frac{(n-1)n}{2}$. Since κ_n has an A_0-type singularity at 0, by Lemma 2.2 we see that the lowest degree of non-zero terms of u_{in} $(1 \le i \le n)$ is $i+1$ for the component function-germ u_{in} $(1 \le i \le n)$ of the map-germ $\mathbf{u}_n = (u_{0n}, u_{1n}, \ldots, u_{nn}) : (I,0) \to S^n$. Thus, the map-germ $(u_{1n}, u_{2n}, \ldots, u_{nn}) : (I,0) \to (\mathbf{R}^n, 0)$ has the following form:

$$s \mapsto (s^2 + \varphi_1(s), s^3 + \varphi_2(s), \ldots, s^{n+1} + \varphi_n(s)),$$

where $\varphi_j(s) = o(s^{j+1})$ $(1 \le j \le n)$.

Proof of the "only if" part of the assertion 2 in Theorem 3. In the case of the assertion 2 of Theorem 3, by Lemmas 2.3 and 2.4 the map-germ $ped_{\mathbf{r},P} : (I,0) \to S^n$ is C^∞ left equivalent to

$$s \mapsto (\alpha_1(s), \ldots, \alpha_n(s)),$$

where the function-germ α_j can be written as

$$\alpha_j(s) = \begin{cases} s^{i+j+2} + \psi_j(s) & (1 \le j \le i-1), \\ s^{i+1} + \psi_j(s) & (j = i), \\ s^{i+j+2} + \psi_j(s) & (i+1 \le j \le n), \end{cases}$$

where $\psi_j(s) = o(s^{i+j+2})$ $(1 \le j \le n,\ j \ne i)$ and $\psi_i(s) = o(s^{i+1})$.

Lemma 4.1. (Theorem 3.3 of [8]) *Let $f : (\mathbf{R},0) \to \mathbf{R}$ be a C^∞ function-germ. Suppose that $f(0) = f'(0) = \cdots = f^{(i)}(0) = 0$ and $f^{(i+1)}(0) \ne 0$. Then there exists a germ of C^∞ diffeomorphism $h : (\mathbf{R},0) \to (\mathbf{R},0)$ such* *that $f(h(s)) = \pm s^{i+1}$, where we have $+$ or $-$ according as $f^{(i+1)}(0)$ is > 0 or < 0.*

Note that we can truncate the term of degree $2i+2$ in ψ_j by subtracting α_i^2 since $2i+2 = 2(i+1)$. Thus, by using Lemma 4.1 and coordinate transformations of $\mathbf{R}^n$, we see that the map-germ $ped_{\mathbf{r},P} : (I, 0) \to S^n$ is C^∞ right-left equivalent to the map-germ $s \mapsto (\beta_1(s), \ldots, \beta_n(s))$, where the function-germ β_j can be written as

$$\beta_j(s) = \begin{cases} s^{i+j+2} + \widetilde{\psi}_j(s) & (j \neq i), \\ s^{i+1} & (j = i), \end{cases}$$

where $\widetilde{\psi}_j(s)$ is $o(s^{i+n+2})$. Note that $2i+3 \le i+n+2$ since $i \le n-1$. Thus, in order to finish the proof of the "only if" part of the assertion 2 in Theorem 3, it is enough to show that

$$f(s) = (\underbrace{(s^{i+3}, s^{i+4}, \ldots, s^{2i+1}}_{(i-1)\text{ elements}}, s^{i+1}, s^{2i+3}, \underbrace{0, \ldots, 0}_{(n-i-1)\text{ elements}})$$

is $(2i+3)$-$\mathcal{L}$-determined.

Since $i+1$ and $2i+3$ are relatively prime, we have that

$$\gcd(\underbrace{i+3, i+4, \ldots, 2i+1}_{(i-1)\text{ elements}}, i+1, 2i+3) = 1.$$

Thus, f is finitely $\mathcal{L}$-determined by the geometric characterization of finite determinacy due to Mather and Gaffney. Therefore, in order to show that f is $(2i+3)$-$\mathcal{L}$-determined it is sufficient to show that

$$m_1^{2i+4}\theta(f+h) \subset T\mathcal{L}(f+h)$$

for any C^∞ map-germ $h : (I, 0) \to \mathbf{R}^n$ such that $j^{2i+3}h(0) = 0$ by Mather's lemma.

Let $a_1, \ldots, a_p$ be positive integers such that $\gcd(a_1, \ldots, a_p) = 1$. Then, it is well-known that there exists the smallest integer $K(a_1, \ldots, a_p)$ such that the linear equation $\sum_{j=1}^p a_j x_j = b$ has a non-negative integer solution $(x_1, \ldots, x_p) \in \mathbf{Z}_+^p$ for any integer $b \ge K(a_1, \ldots, a_p)$ (see, for instance, the comment of the problem 1999-8 in [5]. For more details on the number $K(a_1, \ldots, a_p)$, see [2], [3], [4], [5]). From the view point of singularity theory of differentiable mappings the integer $K(a_1, \ldots, a_p)$ is the smallest integer d such that the inclusion $m_1^d\theta(g) \subset T\mathcal{L}(g)$ is satisfied for $g(s) = (s^{a_1}, \ldots, s^{a_p})$.

Suppose that $K(\underbrace{i+3, i+4, \ldots, 2i+1}_{(i-1)\text{ elements}}, i+1, 2i+3) = i+3$. Then, since $2i+4 \ge i+3$, by this supposition we have that

$$m_1^{2i+4}\theta(f+h) \subset T\mathcal{L}(f+h) + m_1^{2(2i+4)}\theta(f+h).$$

Thus, by the Malgrange preparation theorem we have the following desired inclusion:

$$m_1^{2i+4}\theta(f+h) \subset T\mathcal{L}(f+h).$$

Therefore, in order to finish the proof of the "only if" part of the assertion 2 in Theorem 3, it is enough to show the following lemma:

Lemma 4.2. $K(\underbrace{i+3, i+4, \ldots, 2i+1}_{(i-1)\text{ elements}}, i+1, 2i+3) = i+3.$

Proof of Lemma 4.2. In the case that $i = 1$ Lemma 4.2 holds by the Sylvester duality ([22], see also the comment of the problem 1999-8 in [5]). Thus, in the following we assume that $i \geq 2$.

It is clear that there does not exist non-negative integers $k_1, \ldots, k_{i+1}$ such that $i+2 = k_1(i+3) + k_2(i+4) + \cdots + k_{i-1}(2i+1) + k_i(i+1) + k_{i+1}(2i+3)$. Thus, it sufficies to show that $m_1^{i+3}\theta(f) \subset T\mathcal{L}(f)$, which is equivalent to $m_1^{i+3}\theta(f) \subset T\mathcal{L}(f) + m_1^{2(i+3)}\theta(f)$ by an application of the Malgrange preparation theorem similar as in the proof of the assertion 1 of Theorem 2.

We have that $2i+2 = 2(i+1)$, $2i+4 = (i+3)+(i+1)$ and $2i+5$ is $3(i+1)$ if $i = 2$ and $(i+4)+(i+1)$ if $i \geq 3$. Therefore, the inclusion $m_1^{i+3}\theta(f) \subset T\mathcal{L}(f) + m_1^{2(i+3)}\theta(f)$ holds. ■

Proof of the "only if" part of the assertion 3 in Theorem 3. In the case of the assertion 3 of Theorem 3, by Lemmas 2.3 and 2.4 the map-germ $ped_{\mathbf{r},P} : (I,0) \to S^2$ is C^∞ left equivalent to the following:

$$s \mapsto (\underbrace{s^{n+3} + \psi_1(s), s^{n+4} + \psi_2(s), \ldots, s^{2n+1} + \psi_{n-1}(s)}_{(n-1)\text{ elements}}, s^{n+1} + \psi_n(s)),$$

where $\psi_j(s) = o(s^{n+j+2})$ $(1 \leq j \leq n-1)$ and $\psi_n(s) = o(s^{n+1})$. By using Lemma 4.1 and coordinate transformations of $\mathbf{R}^n$, we see that the map-germ $ped_{\mathbf{r},P} : (I,0) \to S^n$ is C^∞ right-left equivalent to the following:

$$s \mapsto (\underbrace{s^{n+3} + \widetilde{\psi}_1(s), s^{n+4} + \widetilde{\psi}_2(s), \ldots, s^{2n+1} + \widetilde{\psi}_{n-1}(s)}_{(n-1)\text{ elements}}, s^{n+1}),$$

where $\widetilde{\psi}_j(s) = o(s^{2n+1})$ $(1 \leq j \leq n-1)$. Thus, in order to finish the proof of the "only if" part of the assertion 3 in Theorem 3, it sufficies to show that

$$f(s) = (\underbrace{s^{n+3}, s^{n+4}, \ldots, s^{2n+1}}_{(n-1)\text{ elements}}, s^{n+1})$$

is $(2n+1)$-$\mathcal{A}$-determined.

Since $n+1$ and $2n+1$ are relatively prime, we have that

$$\gcd(\underbrace{n+3, n+4, \ldots, 2n+1}_{(n-1)\text{ elements}}, n+1) = 1.$$

Thus, f is finitely $\mathcal{L}$-determined by the geometric characterization of finite determinacy due to Mather and Gaffney. Therefore, in order to show that f is $(2n+1)$-$\mathcal{A}$-determined it is sufficient to show that

$$m_1^{2n+2}\theta(f+h) \subset T\mathcal{A}(f+h)$$

for any C^∞ map-germ $h : (I, 0) \to \mathbf{R}^n$ such that $j^{2n+1}h(0) = 0$ by Mather's lemma.

For any C^∞ map-germ $h : (I, 0) \to \mathbf{R}^n$ such that $j^{2n+1}h(0) = 0$, the following holds clearly.

$$f^* m_n \mathcal{E}_1 = (f+h)^* m_n \mathcal{E}_1 + f^* m_n^2 \mathcal{E}_1.$$

Thus, by Nakayama's lemma we see that

$$f^* m_n \mathcal{E}_1 = (f+h)^* m_n \mathcal{E}_1$$

and therefore both sets are equal to $m_1^{n+1}\mathcal{E}_1$.

Suppose that $K(\underbrace{n+3, n+4, \ldots, 2n+1}_{(n-1)\text{ elements}}, n+1) = 2n+4$. Then, by this supposition and the fact that $2n+2 = 2(n+1)$ we have that

$$s^j \frac{\partial}{\partial X_\ell} \in T\mathcal{L}(f+h) + (f+h)^* m_n^3 \theta(f+h) \quad (2n+2 \le j, j \ne 2n+3, 1 \le \ell \le n).$$

Furthermore, for s^{2n+3} we have that

$$s^{2n+3} \frac{\partial}{\partial X_\ell} \in T\mathcal{A}(f+h) + (f+h)^* m_n^3 \theta(f+h) \quad (1 \le \ell \le n).$$

Thus, we have that

$$(f+h)^* m_n^2 \theta(f+h) \subset T\mathcal{A}(f+h) + (f+h)^* m_n^3 \theta(f+h).$$

Hence, by the Malgrange preparation theorem we have the following desired inclusion:

$$m_1^{2n+2}\theta(f+h) = (f+h)^* m_n^2 \theta(f+h) \subset T\mathcal{A}(f+h).$$

Therefore, in order to finish the proof of the "only if" part of the assertion 3 in Theorem 3, it is enough to show the following lemma:

Lemma 4.3. $K(\underbrace{n+3, n+4, \ldots, 2n+1}_{(n-1)\text{ elements}}, n+1) = 2n+4.$

Proof of Lemma 4.3. It is easy to see that there does not exist non-negative integers $k_1, \ldots, k_n$ such that $2n+3 = k_1(n+3) + k_2(n+4) + \cdots + k_{n-1}(2n+1) + k_n(n+1)$. Thus, it sufficies to show that $m_1^{2n+4}\theta(f) \subset T\mathcal{L}(f)$, which is equivalent to $m_1^{2n+4}\theta(f) \subset T\mathcal{L}(f) + m_1^{4n+8}\theta(f)$ by the Malgrange preparation theorem.

We have that $2n+j = (n+j-1)+(n+1)$ $(4 \leq j \leq n+2)$, $2n+(n+3) = 3(n+1)$, $2n+j = (j-1)+(2n+1)$ $(n+4 \leq j \leq 2n+2)$, $2n+(2n+3) = (2n+1)+2(n+1)$, $2n+(2n+4) = 4(n+1)$, $2n+(2n+5) = (n+3)+(2n+1)+(n+1)$, $2n+(2n+6) = (n+3)+3(n+1)$ and $2n+(2n+7) = 2(n+3)+(2n+1)$. Therefore, the inclusion $m_1^{2n+4}\theta(f) \subset T\mathcal{L}(f) + m_1^{4n+8}\theta(f)$ holds. ∎

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