Agreeable Bets with Multiple Priors*

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Abstract

This paper considers a two agent model of trade with multiple priors. First, we characterize the existence of an agreeable *bet* on some event in terms of the set of priors. It is then shown that the existence of an agreeable bet on some event is a strictly stronger condition than the existence of an agreeable *trade*, whereas the two conditions are equivalent in the standard Bayesian framework. Secondly, we show that the two conditions are equivalent when the set of priors is the core of a convex capacity. These results are also related to the no trade theorems under asymmetric information.
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1 Introduction and summary

Imagine two agents, each of whom has a prior distribution over a finite state space Ω. If the two agents do not share priors, there exists an agreeable *trade* between them: by a trade we mean a function $f : \Omega \to \mathbb{R}$, and it is said to be agreeable if the expected value of f for agent 1 is positive and that of $-f$ for agent 2 is positive as well. The converse is also true: if there exists an agreeable trade, the agents do not share priors. A trade f is called a *bet on event* $E \subseteq \Omega$ if f is constant over E and $\Omega \backslash E$; if f is larger on E than on $\Omega \backslash E$, it can be interpreted that agent 1 wins the bet when E occurs, and agent 2 wins when E does not occur. The existence of an agreeable bet on some event is equivalent to the existence of an event for which the prior probabilities of the two agents disagree. Thus if there is no agreeable bet on any event, the priors of the two agents must coincide. To sum up, disagreement of priors, the existence of an agreeable trade, and the existence of an agreeable bet are all equivalent conditions in this framework.

Now suppose that two agents have multiple priors over the state space, and they use the maximin rule à la Gilboa and Schmeidler [9] to evaluate a trade. In this context, a trade is agreeable if the *minimum* expected value of f, where the minimum is taken over the set of priors, is positive for agent 1, and that of $-f$ is positive for agent 2 as well. Billot *et al.* [2] has shown that there exists an agreeable trade (i.e., no trade is not ex ante efficient) if and only if the sets of priors are disjoint (i.e., the agents do not share any priors). Thus, also in the multiple priors framework, the existence of an agreeable trade is equivalent to disagreement of priors. Further characterizations of the existence of an agreeable trade are discussed by Dana [5], Chateauneuf *et al.* [4], and Tallon [16], among others, in general equilibrium models with Choquet expected utility where the sets of priors are given by the core of capacities. But these works do not investigate the relation with the existence of an agreeable bet, which is logically stronger than the other two conditions in principle. Thus it is worth investigating the implications of the existence of an agreeable bet in this context.

The first contribution of this paper is to provide a necessary and sufficient condition for the existence of an agreeable bet in the multiple priors model (Proposition 2 in Section 2). It states that there exists an agreeable bet on an event if and only if the maximum of the probability of the event for one agent is smaller than the minimum of that for the other agent. We show, by an example, that this condition may not be satisfied even if the sets of priors are disjoint. This implies that the existence of an agreeable bet is a strictly stronger condition than the existence of an agreeable trade in the multiple priors model.

We then consider the case where the set of priors of each agent is given as the core of a convex capacity. We show that there exists an agreeable bet on some event if and only if the sets of priors are disjoint (Proposition 4 in Section 3). Therefore, when the set of priors is generated by a convex capacity, the three conditions are equivalent. Mathematically, this result is an application of the separation theorem such that a normal vector is binary. An agreeable trade exists if and only if there exists a hyperplane separating the sets of priors, as is shown in Billot *et al.* [2]. We show that, for an agreeable bet, the normal vector $f \in \mathbb{R}^\Omega$ of the separating hyperplane must be an indicator function, i.e., $f(\omega) \in \{0, 1\}$ for all $\omega \in \Omega$, and that such a special normal vector exists for a convex capacity model.

Although we consider a multiple priors model and *ex ante* agreements, our results help to understand the structure of the so called no trade theorems under asymmetric information in the standard Bayesian framework, i.e., characterizations of an *interim* agreement among agents. We elaborate on this issue in the last part of Section 3: roughly speaking, starting with a posterior, one can construct multiple priors so that an interim agreement with the posterior is translated into an ex ante agreement with the multiple priors, where our results are related.

2 Characterization of agreeable bets on events

Let $P_1, P_2 \subseteq \Delta(\Omega)$ be non-empty closed sets,[1] which will be referred to as the sets of priors for agent 1 and agent 2, respectively. We call a function $f : \Omega \to \mathbb{R}$ a *trade*. A *bet* on event $E \subseteq \Omega$ is a trade f which is constant on both E and $\Omega \backslash E$. Agents evaluate a trade f by the minimum of the expected gain, as axiomatized by Gilboa and Schmeidler [9]. We interpret that $f(\omega)$ is a transfer agent 1 receives from agent 2 when the state is $\omega \in \Omega$. Thus, we say that a trade f is an *agreeable trade* if $\min_{p \in P_1} \sum_{\omega \in \Omega} p(\omega) f(\omega) > 0$ and $\min_{p \in P_2} \sum_{\omega \in \Omega} p(\omega)(-f(\omega)) > 0$. An *agreeable bet* on event E is an agreeable trade which is a bet on E.

Let us begin with characterizing the existence of an agreeable trade.[2]

Lemma 1 *Suppose that P_1 and P_2 are closed and convex. Then, there exists an agreeable trade if and only if $P_1 \cap P_2 = \emptyset$.*

[1] The set $\Delta(\Omega)$ denotes the collection of all probability distributions over Ω.

[2] The result is a special case of the results of Billot *et al.* [2], who considered multiple risk averse agents, and Kajii and Ui [10], who considered multiple risk neutral agents with asymmetric information. See also Billot *et al.* [3] who considered a Choquet expected utility model.

Proof. Since P_1 and P_2 are compact and convex, by the separation theorem, $P_1 \cap P_2 = \emptyset$ if and only if there exists $f : \Omega \to \mathbb{R}$ and $c \in \mathbb{R}$ such that $\min_{p \in P_1} \sum_{\omega \in \Omega} p(\omega) f(\omega) > c > \max_{p \in P_2} \sum_{\omega \in \Omega} p(\omega) f(\omega)$. By letting $g = f - c$, it is clear that the latter statement is equivalent to the condition that there exists $g : \Omega \to \mathbb{R}$ such that $\min_{p \in P_1} \sum_{\omega \in \Omega} p(\omega) g(\omega) > 0$ and $\min_{p \in P_2} \sum_{\omega \in \Omega} p(\omega)(-g(\omega)) > 0$. ∎

Let $\underline{P_i}(E) = \min_{p \in P_i} p(E)$ and $\overline{P_i}(E) = \max_{p \in P_i} p(E)$ be the minimum and maximum probabilities of $E \subseteq \Omega$ for agent $i = 1, 2$. The necessary and sufficient condition for the existence of an agreeable bet on $E \subset \Omega$ is the following.

Proposition 2 *There exists an agreeable bet on $E \subset \Omega$ if and only if*

$$\underline{P_1}(E) > \overline{P_2}(E) \text{ or } \underline{P_2}(E) > \overline{P_1}(E).$$

Proof. Suppose that f is an agreeable bet on E such that $f(\omega) = a$ if $\omega \in E$ and $f(\omega) = b$ otherwise where $a, b \in \mathbb{R}$. Note if $a = b$ then f cannot be an agreeable bet. Thus, either $a - b > 0$ or $a - b < 0$ is true. If $a - b > 0$, we have

$$\min_{p \in P_1} \sum_{\omega \in \Omega} p(\omega) f(\omega) = \underline{P_1}(E)(a - b) + b > 0,$$
$$\min_{p \in P_2} \sum_{\omega \in \Omega} p(\omega)(-f(\omega)) = \overline{P_2}(E)(b - a) - b > 0,$$

and thus

$$\underline{P_1}(E) > -b/(a - b) > \overline{P_2}(E).$$

Similarly, if $a - b < 0$, we have $\underline{P_2}(E) > \overline{P_1}(E)$.

Conversely, suppose that $\underline{P_1}(E) > \overline{P_2}(E)$ or $\underline{P_2}(E) > \overline{P_1}(E)$. If $\underline{P_2}(E) > \overline{P_1}(E)$, let $f : \Omega \to \mathbb{R}$ be such that $f(\omega) = c - 1$ if $\omega \in E$ and $f(\omega) = c$ otherwise where $\underline{P_2}(E) > c > \overline{P_1}(E)$. Then,

$$\min_{p \in P_1} \sum_{\omega \in \Omega} p(\omega) f(\omega) = c - \overline{P_1}(E) > 0,$$
$$\min_{p \in P_2} \sum_{\omega \in \Omega} p(\omega)(-f(\omega)) = \underline{P_2}(E) - c > 0,$$

implying that f is an agreeable bet on E. If $\underline{P_1}(E) > \overline{P_2}(E)$, let $f : \Omega \to \mathbb{R}$ be such that $f(\omega) = 1 - c$ if $\omega \in E$ and $f(\omega) = -c$ otherwise where $\underline{P_1}(E) > c > \overline{P_2}(E)$. A similar calculation shows that f is an agreeable bet. ∎

The condition in the above result is not implied by $P_1 \cap P_2 = \emptyset$, and hence by Lemma 1 the existence of an agreeable bet is a strictly stronger condition than the existence of an agreeable trade, as the following example shows.

Example 1 Let $\Omega = \{1, 2, 3\}$, $P_1 = \{p \in \Delta(\Omega) : (p(1), p(2), p(3)) = t(0.4, 0.4, 0.2) + (1-t)(0.2, 0.2, 0.6),\ 0 \leq t \leq 1\}$, and $P_2 = \{p \in \Delta(\Omega) : (p(1), p(2), p(3)) = (0.3, 0.25, 0.45)\}$. Both P_1 and P_2 are closed convex sets and $P_1 \cap P_2 = \emptyset$. However, for any event $E \subset \Omega$, $\underline{P_1}(E) < \underline{P_2}(E) = \overline{P_2}(E) < \overline{P_1}(E)$.

3 Case of convex capacity

A set function $v : 2^\Omega \to \mathbb{R}$ with $v(\emptyset) = 0$ is called a *capacity* if it is monotone and normalized: that is, $v(E) \geq v(F)$ if $E \supseteq F$ and $v(\Omega) = 1$. A capacity v is said to be *convex* if $v(E) + v(F) \leq v(E \cap F) + v(E \cup F)$ for $E, F \subseteq \Omega$, and it is said to be *concave* if $-v$ is convex. For convex and concave capacities, the following separation theorem due to Frank [8] is known.[3]

Theorem 3 *Let $\mu : 2^\Omega \to \mathbb{R}$ be a convex capacity and $\rho : 2^\Omega \to \mathbb{R}$ be a concave capacity. If $\rho(E) \geq \mu(E)$ for all $E \subseteq \Omega$, then there exists $q : \Omega \to \mathbb{R}$ such that*

$$\rho(E) \geq q(E) \geq \mu(E) \text{ for all } E \subseteq \Omega$$

where $q(E) = \sum_{\omega \in E} q(\omega)$.

The core of a capacity v is defined as:

$$\text{Core}(v) = \{q \in \Delta(\Omega) : q(E) \geq v(E) \text{ for all } E \subseteq \Omega\}$$

where $q(E) = \sum_{\omega \in E} q(\omega)$. The core is a closed convex set. It is known that if v is convex, then $\text{Core}(v) \neq \emptyset$. The following is the main result of this section.

Proposition 4 *Suppose that there exists a convex capacity v_i such that $P_i = \text{Core}(v_i)$ for $i = 1, 2$. Then, there exists an agreeable bet on some event if and only if $P_1 \cap P_2 = \emptyset$.*

The "only if" part is an immediate consequence of Lemma 1. The "if" part follows from Proposition 2 and the following lemma.

[3] This result is also reported in Denneberg [6], and it is a natural extension of the separation theorem for convex functions, taking into account that the Choquet integral of $x \in \mathbb{R}^\Omega$ with respect to a convex capacity is a concave function of $x \in \mathbb{R}^\Omega$. See for instance Murota [13] for a general discussion.

Lemma 5 *Suppose that there exists a convex capacity v_i such that $P_i = \mathrm{Core}(v_i)$ for $i = 1, 2$. If $P_1 \cap P_2 = \emptyset$, then there exists $E \subset \Omega$ such that $\underline{P_1}(E) > \overline{P_2}(E)$.*

Proof. Define $v_2' : 2^\Omega \to \mathbb{R}$ by the rule $v_2'(E) = 1 - v_2(\Omega \backslash E)$ for $E \subseteq \Omega$. It can be readily checked that v_2' is a concave capacity. We shall show that there exists $E \subseteq \Omega$ such that $v_2'(E) < v_1(E)$. Seeking a contradiction, suppose that $v_2'(E) \geq v_1(E)$ for all $E \subseteq \Omega$. By Theorem 3, there exists $q : \Omega \to \mathbb{R}$ such that

$$v_2'(E) \geq q(E) \geq v_1(E) \text{ for all } E \subseteq \Omega.$$

Since $v_2'(\Omega) = v_1(\Omega) = 1$ and $v_2'(\emptyset) = v_1(\emptyset) = 0$, we have $q \in \Delta(\Omega)$. Thus, $q \in \mathrm{Core}(v_1)$. In addition,

$$q(E) = 1 - q(\Omega \backslash E) \geq 1 - v_2'(\Omega \backslash E) = v_2(E) \text{ for all } E \subseteq \Omega.$$

Thus, $q \in \mathrm{Core}(v_2)$, which contradicts to the assumption $\mathrm{Core}(v_1) \cap \mathrm{Core}(v_2) = \emptyset$. Therefore, there exists $E \subset \Omega$ such that $v_2'(E) < v_1(E)$. Note that $v_1(E) = \min_{p \in \mathrm{Core}(v_1)} p(E)$ and $v_2'(E) = 1 - v_2(\Omega \backslash E) = 1 - \min_{p \in \mathrm{Core}(v_2)} p(\Omega \backslash E) = \max_{p \in \mathrm{Core}(v_2)} p(E)$. Thus,

$$\underline{P_1}(E) = \min_{p \in \mathrm{Core}(v_1)} p(E) = v_1(E) > v_2'(E) = \max_{p \in \mathrm{Core}(v_2)} p(E) = \overline{P_2}(E),$$

which completes the proof. ∎

Remark 1 If a capacity v is additive, i.e., it is a probability measure, its core is a singleton $\{v\}$. Hence, Proposition 4 covers the known result for a single prior model. It also applies to the Choquet integral model [15], which is a particular case of the multiple priors model when the capacity is convex.

Remark 2 Lemma 5 can be restated as a separation theorem for the core of convex capacities with a binary normal vector: if $v_i : 2^\Omega \to \mathbb{R}$ is convex for $i = 1, 2$ and $\mathrm{Core}(v_1) \cap \mathrm{Core}(v_2) = \emptyset$, then there exists $f : \Omega \to \{0, 1\}$ such that $\min_{p \in \mathrm{Core}(v_1)} p \cdot f > \max_{p \in \mathrm{Core}(v_2)} p \cdot f$. Hence, it is related to the separation theorem for the core of *discrete* convex set functions shown in Murota [12, Theorem 3.6], which inspired our proof.

Finally, let us relate our results to the so called no trade theorems under asymmetric information in the standard Bayesian framework. It is known that agents do not have a common prior if there exists an interim agreeable bet, i.e., it is common knowledge that a posterior probability of some event for one agent is smaller than that for the other agent. This is a generalized version of Aumann's agreement theorem [1]. But it is also known that the converse does not hold; non-existence of a common prior does not necessarily imply the existence of an interim agreeable bet. The resemblance to our analysis is not a coincidence, since an interim agreeable trade in a single prior model can be characterized in terms of an (ex ante) agreeable trade with multiple priors as follows.

Let $\Pi_i \subseteq 2^\Omega$ be an information partition of agent $i = 1, 2$ with a generic element $\pi_i \in \Pi_i$. Let $q_i(\cdot|\pi_i) \in \Delta(\Omega)$ be a posterior distribution given $\pi_i \in \Pi_i$ for agent i such that $\sum_{\omega\in\pi_i} q_i(\omega|\pi_i) = 1$. A trade $f : \Omega \to \mathbb{R}$ is interim agreeable if $\sum_{\omega\in\pi_1} q_1(\omega|\pi_1)f(\omega) > 0$ for all $\pi_1 \in \Pi_1$ and $\sum_{\omega\in\pi_2} q_2(\omega|\pi_2)(-f(\omega)) > 0$ for all $\pi_2 \in \Pi_2$. Now let $P_i \subseteq \Delta(\Omega)$ be the convex hull of $\{q_i(\cdot|\pi_i)\}_{\pi_i\in\Pi_i}$ for $i = 1, 2$; that is, $P_i := \{p \in \Delta(\Omega) : p = \sum_{\pi_i\in\Pi_i} \lambda(\pi_i)q_i(\cdot|\pi_i)$ where $\lambda \in \Delta(\Pi_i)\}$. Then it can be readily shown that a trade f is interim agreeable if and only if it is (ex ante) agreeable for agents with multiple priors P_1 and P_2. So by Lemma 1, there exists an interim agreeable trade if and only if $P_1 \cap P_2 = \emptyset$. If $p \in P_1 \cap P_2$ exists, it is a (fictitious) common prior in the sense that a conditional probability of p given π_i coincides with $q_i(\cdot|\pi_i)$. This is the essence of the characterization results obtained by Morris [11], Feinberg [7], and Samet [14].

Then Proposition 4 implies that the converse of the generalized agreement theorem holds if the induced set of priors P_i can be expressed as the core of some convex capacity for each agent. In other words, our results suggest a reason why the converse mentioned above may fail: the induced set of priors is not the core of a convex capacity in general. We shall give a simple example below to conclude. A similar example can be constructed for more general cases.

Example 2 Let $\Omega = \{1, 2, 3\}$, $\Pi_i = \{\{1\}, \{2, 3\}\}$, and the prior of player i is $p_i \in \Delta(\Omega)$ such that $(p_i(1), p_i(2), p_i(3)) = (1/2, \alpha/2, (1-\alpha)/2)$. Then $q_i(1|\{1\}) = 1$, $q_i(2|\{2, 3\}) = \alpha$, and $q_i(2|\{2, 3\}) = 1 - \alpha$, and so we have $P_i = \{p \in \Delta(\Omega) : (p(1), p(2), p(3)) = (t, \alpha(1-t), (1-\alpha)(1-t)),\ 0 \le t \le 1\}$. It is straightforward to check that there is no capacity v_i satisfying $P_i = \mathrm{Core}(v_i)$ if and only if $0 < \alpha < 1$.

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