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FUNDAMENTAL GENERA FOR NORMAL SURFACE TRIPLE POINTS BRANCHED OVER ANALYTICALLY IRREDUCIBLE SINGULAR PLANE CURVES

By

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Abstract. In this paper, we study normal surface triple points branched over analytically irreducible singular plane curves. We calculate the fundamental genera of these triple points. Also, we obtain a necessary and sufficient condition for these triple points to be Kodaira singularities.

1. Introduction

In [Oyua], it was proved that for normal surface singularities defined by $z^3 = f(x,y)$, if $f(x,y) \in \mathbb{C}\{x,y\}$ is irreducible and $\mathrm{ord}(f) \geq 2$, then the maximal ideal cycle and the fundamental cycle coincide on the minimal resolution. In this paper, we continue to study this type of singularities. We will compute the fundamental genera and give a necessary and sufficient condition for these singularities to be Kodaira singularities. As we will see later, it makes sense to compute the fundamental genus for a given normal surface singularity since it is useful in the classification of singularities. Also, Kodaira singularities introduced by Karras [Kar80], are an important class of normal surface singularities since the maximal ideal cycle coincides with the fundamental cycle on the minimal resolution while there are many cases where the two cycles do not coincide ([Lau78], [Oyub], [Tom]). Therefore, it is meaningful to consider whether a given

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singularity is Kodaira or not.

Before explaining our results, we prepare some terminology and facts. Let $\varphi : (\tilde{V}, A) \to (V, o)$ be a resolution of a normal surface singularity and $A = \bigcup A_\lambda$ the irreducible decomposition of the exceptional set. We call φ a *good resolution* if the exceptional set A is a simple normal crossing divisor in $\tilde{V}$. Also, φ is called the *minimal resolution* (resp. the *minimal good resolution*) if for any resolution (resp. good resolution) $\varphi' : (\tilde{V}', A') \to (V, o)$ there exists a unique morphism $\pi : \tilde{V}' \to \tilde{V}$ such that $\varphi' = \varphi \circ \pi$. A divisor $D = \sum_\lambda d_\lambda A_\lambda$ $(d_\lambda \in \mathbb{Z})$ on $\tilde{V}$ supported in A is called a *cycle*. We denote d_λ by $\mathrm{cff}_{A_\lambda}(D)$. For a cycle $D = \sum_\lambda d_\lambda A_\lambda$, we define $D_{\mathrm{red}} := \sum_{A_\lambda \subset \mathrm{Supp}(D)} A_\lambda$. Let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_{V,o}$ of V at o. For an element $h \in \mathfrak{m} \setminus \{0\}$, let $(h \circ \varphi)$ be the divisor defined by $h \circ \varphi$ on $\tilde{V}$. The *exceptional part* of $(h \circ \varphi)$ is defined by $(h \circ \varphi)_A := \sum_\lambda v_{A_\lambda}(h \circ \varphi) A_\lambda$, where $v_{A_\lambda}(h \circ \varphi)$ indicates the vanishing order of $h \circ \varphi$ on A_λ. The *maximal ideal cycle* M_A on A is defined by $M_A := \min\{(h \circ \varphi)_A \mid h \in \mathfrak{m} \setminus \{0\}\}$ ([Yau80, Definition 2.11]). The *fundamental cycle* Z_A on A is defined by $Z_A := \min\{D = \sum a_\lambda A_\lambda \mid a_\lambda > 0 \text{ and } D \cdot A_\lambda \leq 0 \text{ for any } \lambda\}$ ([Art66, p.132]). The arithmetic genus $p_a(Z_A) = 1 - \chi(\mathcal{O}_{Z_A})$ of Z_A is called the *fundamental genus* and denoted by $p_f(V, o)$. It can be calculated by the following formula:

$$p_f(V, o) = \frac{Z_A^2 + Z_A \cdot K_{\tilde{V}}}{2} + 1 \tag{1.1}$$

Here, $K_{\tilde{V}}$ is the canonical divisor of $\tilde{V}$ ([Ish14, Definition 7.2.10]). It is well-known that the fundamental genus is independent of a choice of resolution of (V, o) ([Ish14, Proposition 7.2.9]) and useful in the classification of singularities. For example, if $p_f(V, o) = 0$, then (V, o) is a rational singularity ([Ish14, Theorem 7.3.1]), and the definition of minimally elliptic singularities requires $p_f(V, o) = 1$ ([Ish14, Definition 7.6.5]). Also, the case of $p_f(V, o) \geq 2$ has been studied by Tomaru [Tom95] and Konno [Kon12]. Furthermore, for a Kodaira singularity, it is known that the arithmetic genus of the associated pencil equals the fundamental genus ([Kar81]). Kodaira singularities are defined as follows: Let S be a non-singular complex surface and $\Delta \subset \mathbb{C}$ a small open disc around the origin. If $\Phi : S \to \Delta$ is a proper surjective holomorphic map with connected fibers and the generic fiber $S_t := \Phi^{-1}(t)$ $(t \neq 0)$ is a smooth curve of genus g, it is called a *pencil of curves of genus g*.

Definition 1.1 ([Kar80, Definition 2.2])**.** A normal surface singularity (V, o) is called a *Kodaira singularity* if there exists a pencil of curves $\Phi : S \to \Delta$ such that, after a finite number of blowing-ups at finite non-singular points in non-multiple components of the singular fiber S_0, $\sigma : S' \to S$, there is a holomorphic map $\pi : M \to X$ from an open neighborhood M of the proper transform of $\mathrm{Supp}(S_0)$ in S' which defines a resolution of (V, o). Further, if the pencil is of genus g, then we call (X, o) a *Kodaira singularity associated to a pencil of curves of genus* g.

Finally, in order to describe our result, we need to recall the definition of the characteristic exponents of irreducible $f(x, y) \in \mathbb{C}\{x, y\}$ with $\mathrm{ord}(f) \geq 2$. By coordinate changes in x and y, we can always assume that the irreducible curve singularity $(C, o) = \{f(x, y) = 0\}$ has a parametrization $x = t^m$, $y = \sum_{i \geq m} b_i t^i$, which is called the Puiseux expansion of $f(x, y)$ ([BK86, p.385]). Then the characteristic exponents are defined as follows.

Definition 1.2 ([dP00, Definition 5.2.14])**.** We define

$$\begin{aligned} &k_0 := m, \\ &k_j := \min\{i \mid a_i \neq 0, \gcd(i, k_0, \dots, k_{j-1}) < \gcd(k_0, \dots, k_{j-1})\} \quad \text{for } j \geq 1. \end{aligned}$$

We obviously obtain finitely many k_j, say $k_0, k_1, \dots, k_l$. We call them the *characteristic exponents* of $f(x, y)$. Note that $k_0 < k_1 < \cdots < k_l$, $\gcd(k_0, k_1, \dots, k_l) = 1$, and $l \geq 1$.

From now on, let us describe our motivation and background for our research. For a normal surface singularity defined by $z^n = f(x, y)$, where $f(x, y) \in \mathbb{C}\{x, y\}$, there are several results for the fundamental genus and a condition to be a Kodaira singularity. For example, Tomaru proved that if n divides $\mathrm{ord}(f)$ (the order of $f(x, y)$), then it is a Kodaira singularity associated to a pencil of curves of genus (= the fundamental genus) $(n-1)(\mathrm{ord}(f)-2)/2$ ([Tom01, Theorem 4.1]). Also, he proved that if n is sufficiently large, then the singularity is a Kodaira singularity associated to a pencil of curves of genus $(\mu(C) - r(f) + 1)/2$, where $r(f)$ is the number of irreducible components of $f(x, y)$ and $\mu(C)$ is the Milnor number of the curve singularity defined by $f(x, y) = 0$ ([Tom01, Theorem 4.5]). For a Brieskorn-type singularity defined by $x_0^{a_0} + x_1^{a_1} + x_2^{a_2} = 0$ $(2 \leq a_0 \leq a_1 \leq a_2)$, the method for computing the fundamental genus from the exponents a_0, a_1, a_2 was shown by Tomaru [Tom07] for the special case ($\mathrm{lcm}(a_0, a_1) \le a_2$), and later completed by Konno and Nagashima [KN12] for the all cases. Also, Konno and Nagashima obtained a necessary and sufficient conditions for a Brieskorn-type singularity to be a Kodaira singularity. Meng and Okuma [MO14] extended Konno and Nagashima's result to isolated complete intersection singularities of Brieskorn-type.

In the following, we will describe our results of this paper. Let (X, o) be a normal surface singularity defined by $z^3 = f(x, y)$, where $f(x, y) \in \mathbb{C}\{x, y\}$ is irreducible and $\mathrm{ord}(f) > 3$. Our goal is to calculate the fundamental genus $p_f(X, o)$ and to prove a necessary and sufficient condition to be a Kodaira singularity. Since the case of $\gcd(\mathrm{ord}(f), 3) = 3$ can be seen immediately from Tomaru's results, so we assume $\gcd(\mathrm{ord}(f), 3) = 1$. In [Oyua], we showed that the self-intersection numbers M_0^2 and Z_0^2 of the maximal ideal cycle M_0 and the fundamental cycle Z_0 on the minimal resolution were determined by k_0 and k_1. In the process of calculating M_0 and Z_0, we were able to learn more about the resolution of (X, o). Therefore, in this paper, we first compute the fundamental genus $p_f(X, o)$ using that resolution.

Theorem 1.3. *Let $(X, o) \subset \mathbb{C}^3$ be a normal surface singularity defined by $z^3 = f(x, y)$, where $f(x, y) \in \mathbb{C}\{x, y\}$, $\mathrm{ord}(f) > 3$, and $f(x, y)$ is irreducible, and let $k_0, k_1, \ldots, k_l$ be the characteristic exponents of $f(x, y)$.*

1. *If $k_1 > 3k_0$, then $p_f(X, o) = \mathrm{ord}(f) - 1$.*
2. *If $3k_0 > k_1 \ge 3k_0/2$, then $p_f(X, o) = \mathrm{ord}(f) - 2$.*
3. *If $3k_0/2 \ge k_1 > k_0$, then $p_f(X, o) = \mathrm{ord}(f) - 3$.*

Consequently, since we assume $\mathrm{ord}(f) > 3$, we can say that (X, o) is not a rational singularity.

This is proved by Theorems 4.1, 4.2, and 4.3. In Example 4.6 and 4.7, we explicitly calculate the fundamental genera by a different method from that in the proof of main theorems, in order to confirm that our results are correct. Furthermore, we prove the following theorem.

Theorem 1.4. *Let (X, o) be the same as in Theorem 1.3. Then (X, o) is a*

Kodaira singularity if and only if $k_1 > 3k_0$. *If this is the case, it is associated to a pencil of curves of genus* $\mathrm{ord}(f) - 1$.

Our results for normal surface singularities defined by $z^3 = f(x, y)$, where $f(x, y)$ is irreducible are summarized as follows:

		$M_0^2 (= Z_0^2)$	$p_f(X, o)$	Kodaira singularity
$\gcd(\mathrm{ord}(f), 3) = 3$		-3	$\mathrm{ord}(f) - 2$	Yes
	$k_1 > 3k_0$	-1	$\mathrm{ord}(f) - 1$	Yes
$\gcd(\mathrm{ord}(f), 3) = 1$	$3k_0 > k_1 \geq 3k_0/2$	-2	$\mathrm{ord}(f) - 2$	No
	$3k_0/2 > k_1 > k_0$	-3	$\mathrm{ord}(f) - 3$	No

(1.2)

Note that the case of $\gcd(\mathrm{ord}(f), 3) = 3$ comes from [Tom01, Theorem 4.1].

This paper is organized as follows: In Section 2, we recall the construction of the covering resolution of (X, o). In Section 3, we recall the results of [Oyua], and consider the negative components of the fundamental cycle on the covering resolution. In Section 4, we will prove the main theorems.

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2. Resolution of singularities

In this section, we will construct the covering resolution $\phi : (\tilde{X}, \tilde{E}) \to (X, o)$ of a normal surface singularity $(X, o) = \{z^3 = f(x, y)\} \subset \mathbb{C}^3$, where x, y, and z are coordinates of $\mathbb{C}^3$ and $f(x, y) \in \mathbb{C}\{x, y\}$ is irreducible with $\mathrm{ord}(f) \geq 2$ and $\gcd(\mathrm{ord}(f), 3) = 1$. It is obtained by using the triple cyclic covering of the minimal embedded resolution of the irreducible plane curve singularity $(C, o) = \{f(x, y) = 0\} \subset \mathbb{C}^2$. To this purpose, we recall the minimal embedded resolution of irreducible curve singularities in Section 2.1, and we recall the construction of the covering resolution in Section 2.2

2.1 Minimal embedded resolution of irreducible curve singularities

First, we will construct the minimal embedded resolution of (C, o).

Definition 2.1 ([dP00, Definition 5.3.7])**.** Let $\sigma_1 : V_1 \to V_0 := \mathbb{C}^2$ be the blowing-up of $\mathbb{C}^2$ at the origin $Q_0 := o$, the singular point of C. We define $E(1) := \sigma_1^{-1}(Q_0)$ and the strict transform $C(1)$ of C by the closure of $\sigma_1^{-1}(C \setminus \{Q_0\})$. For $i \geq 1$, assuming the composite $\sigma_1 \circ \cdots \circ \sigma_i : V_i \to V_0$ of blowing-ups is obtained, take the blowing-up $\sigma_{i+1} : V_{i+1} \to V_i$ at the unique intersection point Q_i of the strict transform $C(i)$ of C with the exceptional set $E(i) := (\sigma_1 \circ \cdots \sigma_i)^{-1}(Q_0)$ while the following (1) or (2) does not hold.

(1) $C(i)$ is non-singular at Q_i.

(2) $C(i)$ meets only one irreducible component of $E(i)$ transversally at Q_i.

Note that the uniqueness of Q_i follows from the irreducibility of C, and note that Q_i lies on at most two irreducible components of $E(i)$. Thus for some $i = s$, (1) and (2) holds and we obtain a sequence of blowing-ups

$$\mathbb{C}^2 \xleftarrow{\sigma_1} V_1 \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_s} V_s. \tag{2.1}$$

We call $\sigma := \sigma_1 \circ \cdots \circ \sigma_s$ the *minimal embedded resolution* of (C, o). Let $e_i \subset V_i$ be the exceptional curve of σ_i and by the same e_i we denote the strict transform of e_i by $\sigma_{i+1} \circ \cdots \circ \sigma_s$ $(i = 1, 2, \ldots, s-1)$. Therefore, we have $E(s) = \bigcup_{i=1}^{s} e_i$.

Note that the minimal embedded resolution always exists and is determined by the characteristic exponents $k_0, k_1, \ldots, k_l$ of $f(x, y)$. For examples of the minimal embedded resolution, see [dP00, Example 5.3.9] or [Har77, Example 3.9.1]. In this paper, the strict transform of e_i $(i = 1, \ldots, s-1)$ on V_j $(j \geq i)$ is also denoted by e_i.

Let $L_x, L_y \subset \mathbb{C}^2$ be the lines defined by $x = 0$, $y = 0$, respectively. We denote the strict transform of L_x, L_y on V_i by the same L_x, L_y for any i. We put

$$\bar{f}_i := v_{e_i}(f \circ \sigma), \ \bar{x}_i := v_{e_i}(x \circ \sigma), \ \bar{y}_i := v_{e_i}(y \circ \sigma), \tag{2.2}$$

which are the orders of the total transforms of C, L_x, and L_y along e_i. Thus, we have

$$\bar{f}_1 = \mathrm{mult}_{Q_0}(C) = \mathrm{ord}(f),\ \bar{x}_1 = \bar{y}_1 = \bar{h}_1 = 1, \tag{2.3}$$

where $\mathrm{mult}_{Q_0}(C)$ is the multiplicity of C at Q_0. For each $i = 2, \ldots, s$, if the intersection point Q_{i-1} of $C(i-1)$ and $E(i-1)$ lies on e_j for some $j \leq i-2$, then

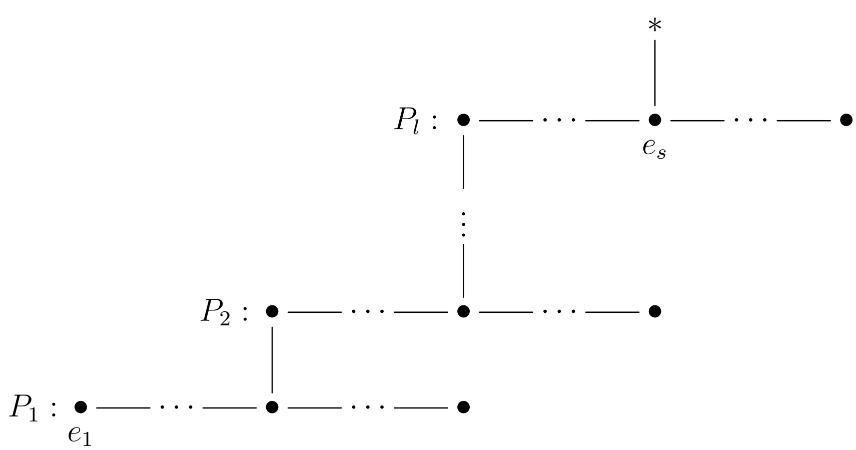
$$\begin{cases} \bar{f}_i = \bar{f}_{i-1} + \bar{f}_j + \mathrm{mult}_{Q_{i-1}}(C(i-1)), \\ \bar{x}_i = \bar{x}_{i-1} + \bar{x}_j + \mathrm{mult}_{Q_{i-1}}(L_x), \\ \bar{y}_i = \bar{y}_{i-1} + \bar{y}_j + \mathrm{mult}_{Q_{i-1}}(L_y), \end{cases} \tag{2.4}$$

otherwise

$$\begin{cases} \bar{f}_i = \bar{f}_{i-1} + \mathrm{mult}_{Q_{i-1}}(C(i-1)), \\ \bar{x}_i = \bar{x}_{i-1} + \mathrm{mult}_{Q_{i-1}}(L_x), \\ \bar{y}_i = \bar{y}_{i-1} + \mathrm{mult}_{Q_{i-1}}(L_y). \end{cases} \tag{2.5}$$

Therefore, we have $\bar{f}_{i+1} \geq \bar{f}_i$, $\bar{x}_{i+1} \geq \bar{x}_i$, and $\bar{y}_{i+1} \geq \bar{y}_i$ for $i = 1, \ldots, s-1$. Also, since C is irreducible, $\bar{x}_i \leq \bar{y}_i$ for any i.

Let Γ_f be the resolution graph of (C, o), i.e., the weighted dual graph of $E(s) \bigcup C(s)$ ([dP00, Definition 5.3.10]). Then Γ_f is a tree consisting of l Puiseux chains $P_1, \ldots, P_l$ as follows:



(2.6)

Here, $\bullet$ corresponds to the irreducible component e_i of $E(s)$ and $*$ corresponds to $C(s)$.

To prove the main theorem, the first Puiseux chain P_1 plays a key role, so we will look at P_1 in detail here. To construct P_1, we perform the Euclidean algorithm for k_0 and k_1 as follows:

$$\begin{aligned} k_1 &= q_1 k_0 + r_1 \quad (q_1 > 0,\ 0 \leq r_1 < k_0) \\ k_0 &= q_2 r_1 + r_2 \quad (q_2 > 0,\ 0 \leq r_2 < k_0) \\ &\ \vdots \\ r_{w-2} &= q_w r_{w-1} \quad (q_w > 0) \end{aligned} \tag{2.7}$$

Note that $w \geq 2$. Then P_1 is determined as follows:

1. If $w \geq 3$, then

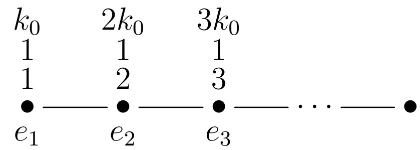
$$P_1 : \underset{e_1}{\bullet} \text{---} \cdots \text{---} \underset{e_{q_1}}{\bullet} \text{---} \underset{e_{q_1+q_2+1}}{\bullet} \text{---} \cdots \text{---} \underset{e_{q_1+q_2+q_3}}{\bullet} \text{---} \cdots \text{---} \underset{e_{q_1+q_2}}{\bullet} \text{---} \cdots \text{---} \underset{e_{q_1+1}}{\bullet} . \tag{2.8}$$

2. If $w = 2$, then

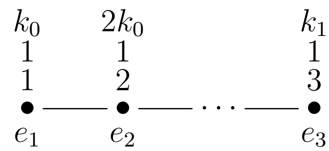
$$P_1 : \underset{e_1}{\bullet} \text{---} \cdots \text{---} \underset{e_{q_1}}{\bullet} \text{---} \underset{e_{q_1+q_2}}{\bullet} \text{---} \cdots \text{---} \underset{e_{q_1+1}}{\bullet} . \tag{2.9}$$

There are 4 possibilities based on how e_1, e_2, and e_3 intersect.

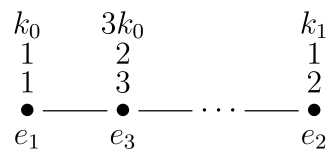
Case 1. $k_1 > 3k_0 \quad (q_1 \geq 3)$



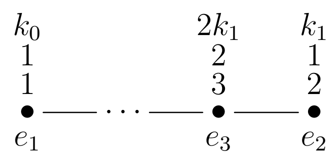
Case 2. $3k_0 > k_1 > 2k_0 \quad (q_1 = 2)$



Case 3. $2k_0 > k_1 \geq 3k_0/2 \quad (q_1 = q_2 = 1)$



Case 4. $3k_0/2 > k_1 > k_0 \quad (q_1 = 1, q_2 \geq 2)$



(2.10)

Here, the three numbers above each vertex are $\bar{f}_i$, $\bar{x}_i$, and $\bar{y}_i$, in that order from top to bottom.

2.2 Covering resolution

For the normal surface singularity (X, o), consider the triple covering map $p : X \to \mathbb{C}^2$ induced from the projection map $\mathbb{C}^3 \to \mathbb{C}^2 : (x, y, z) \mapsto (x, y)$. We put $X' := X \times_{\mathbb{C}^2} V_s$ and $\phi_1 := \mathrm{id} \times \sigma|_{X'}$. Then the surface X' has singularities along $p'^{-1}(E(s))$, which is of the form $z^3 = u^a v^b g(u, v)$ $(a, b \in \mathbb{Z}_{\geq 0})$, where u, v are local coordinates of V_s at an intersection point of irreducible components of $E(s)$ and $g(u, v)$ is a unit. Let $\phi_2 : X'' \to X'$ be the normalization of X' ([dP00]). Then X'' has only cyclic quotient singularities ([Tom01, Lemma 2.5]). Let $\phi_3 : \tilde{X} \to X''$ be the minimal resolution of these cyclic quotient singularities.

We put $\pi = p' \circ \phi_2 \circ \phi_3$. Then we obtain the following diagram:

$$\begin{array}{ccccccc} \mathbb{C}^3 & \xleftarrow{p_1} & \mathbb{C}^3 \times_{\mathbb{C}^2} V_s & & & & \\ \cup & & \cup & & & & \\ (X, o) & \xleftarrow{\phi_1} & X' & \xleftarrow{\phi_2} & X'' & \xleftarrow{\phi_3} & \tilde{X} \\ {\scriptstyle p}\downarrow & & {\scriptstyle p'}\downarrow & & & \swarrow {\scriptstyle \pi} & \\ (\mathbb{C}^2, o) & \xleftarrow{\sigma} & (V_s, E(s)) & & & & \end{array} \tag{2.11}$$

Here, $p_1 : \mathbb{C}^3 \times_{\mathbb{C}^2} V_s \to \mathbb{C}^3$ is the first projection. We put $\phi := \phi_1 \circ \phi_2 \circ \phi_3$.

Let E_i be the strict transform of $p'^{-1}(e_i)_{\text{red}}$ on $\tilde{X}$ by $\phi_2 \circ \phi_3$, and we put $E = \bigcup_{i=1}^{s} E_i$. Note that E_i is not always irreducible. In fact, with reference to Dixon's method ([Dix79, Section 2]), we say the following: Suppose that e_i intersects only e_j (resp. e_i intersects e_j and e_k) in V_s. If $\gcd(\bar{f}_i, \bar{f}_j) = 3$ (resp. $\gcd(\bar{f}_i, \bar{f}_j, \bar{f}_k) = 3$), then E_i splits into three disjoint copies of itself (see below). If E_i is irreducible, we put

$$x_i := v_{E_i}(x \circ \phi),\ y_i := v_{E_i}(y \circ \phi),\ z_i := v_{E_i}(z \circ \phi) \tag{2.12}$$

for $i = 1, 2, \ldots, s$. If E_i is reducible, then $E_i = \bigcup_{j=1}^{3} E_{i,j}$, and $v_{E_{i,1}}(x \circ \phi) = v_{E_{i,2}}(x \circ \phi) = v_{E_{i,3}}(x \circ \phi)$, $v_{E_{i,1}}(y \circ \phi) = v_{E_{i,2}}(y \circ \phi) = v_{E_{i,3}}(y \circ \phi)$, and $v_{E_{i,1}}(z \circ \phi) = v_{E_{i,2}}(z \circ \phi) = v_{E_{i,3}}(z \circ \phi)$, and hence, we simply put them by x_i, y_i, z_i. By [Tom01, Lemma 3.1.] and its proof, we have

$$x_i = \frac{3\bar{x}_i}{\gcd(\bar{f}_i, 3)},\ y_i = \frac{3\bar{y}_i}{\gcd(\bar{f}_i, 3)},\ \text{and}\ z_i = \frac{\bar{f}_i}{\gcd(\bar{f}_i, 3)}. \tag{2.13}$$

Next, we see how to resolve singularities along $p'^{-1}(E(s)) \subset X'$. Suppose that e_i and e_j ($i \neq j$) intersect in V_s. We have $e_i = \{u = 0\}$ and $e_j = \{v = 0\}$ in an open neighborhood U at $e_i \cap e_j$. We put $a = \bar{f}_i$ and $b = \bar{f}_j$. From [Ish14, Section 4.2], we can assume that $p'^{-1}(U)$ is analytically isomorphic to a singularity $X(a, b) = \{z^3 = u^a v^b\}$. If $\gcd(a, b, 3) = 3$, then $X(a, b)$ has 3 connected components, each of them are non-singular. On the other hand, if $\gcd(a, b, 3) = 1$, putting

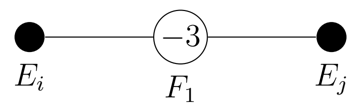
$$\begin{aligned} &r_0 := \gcd(a, b),\ r_1 := \gcd(b, 3),\ r_2 := \gcd(a, 3), \\ &n' := \frac{3}{r_1 r_2},\ a' := \frac{a}{r_0 r_2},\ b' := \frac{b}{r_0 r_1}, \end{aligned} \tag{2.14}$$

we have the following proposition.

Proposition 2.2 ([Tom07, Lemma 4.2]). *The normalization of* $(X(a,b),o)$ *with* $\gcd(a,b,3)=1$ *is a cyclic quotient singularity* $C_{n',\mu}=(\mathbb{C}^2/G_{n',\mu},o)$*, where* μ *is an integer defined by* $a'\mu+b'\equiv 0 \mod n'$*,* $0<\mu<3$*, and* $G_{n',\mu}$ *is the cyclic group generated by* $(e_{n'},e_{n'}^{\mu})\in \mathrm{GL}(2,\mathbb{C})$*, where* $e_{n'}=\exp(2\pi\sqrt{-1}/n')$*.*

Note that the normalization is non-singular if $n'=1$, i.e., either $r_1=3$ or $r_2=3$. Hence, in the following, we assume $n'=3$. It is well-known that the configuration of the exceptional set of the minimal resolution of $C_{3,\mu}$ is given by the Hirzebruch-Jung string of $3/\mu$. The configuration of the minimal resolution of $C_{3,\mu}$ is as follows.

Corollary 2.3. *1. If* $\mu=1$*, i.e.,* $a'\not\equiv b' \mod 3$*, then*



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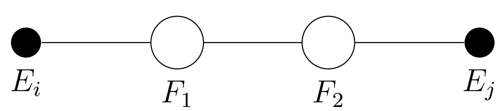
where $\textcircled{$-c$}$ *implies the corresponding exceptional curve is rational and the self-intersection number equals* $-c$*. Furthermore, we have* $v_{F_1}(x\circ\phi)=(x_i+x_j)/3$*,* $v_{F_1}(y\circ\phi)=(y_i+y_j)/3$*, and* $v_{F_1}(z\circ\phi)=(z_i+z_j)/3$*.*

2. If $\mu=2$*, i.e.,* $a'\equiv b' \mod 3$*, then*



,

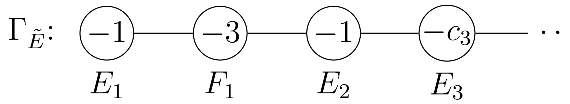
where $\bigcirc$ *implies the corresponding exceptional curve is rational and the self-intersection number equals* -2*. Furthermore, we have* $v_{F_1}(x\circ\phi)=(2x_i+x_j)/3$*,* $v_{F_1}(y\circ\phi)=(2y_i+y_j)/3$*,* $v_{F_1}(z\circ\phi)=(2z_i+z_j)/3$*, and* $v_{F_2}(x\circ\phi)=(x_i+2x_j)/3$*,* $v_{F_2}(y\circ\phi)=(y_i+2y_j)/3$*,* $v_{F_2}(z\circ\phi)=(z_i+2z_j)/3$*.*

Note that the valuations are calculated by the equation $(x\circ\phi)\cdot F_i=(y\circ\phi)\cdot F_i=(z\circ\phi)\cdot F_i=0$. Let F be the union of all exceptional curves obtained by resolving all cyclic quotient singularities on X'' in this way.

We put $\tilde{E}=E\cup F$. Then $\phi:(\tilde{X},\tilde{E})\to(X,o)$ is a good resolution of (X,o) and we call it the *covering resolution* over the minimal embedded resolution.

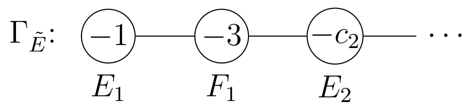
From the above results, we obtain the weighted dual graph $\Gamma_{\tilde{E}}$ of $\tilde{E}$. We put $E_i^2=-c_i$. Then the necessary part of $\Gamma_{\tilde{E}}$ to prove the main theorem is as follows:

Case 1 ($k_1 > 3k_0$).



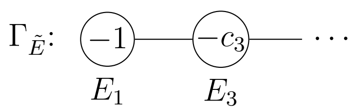
We have $(x_1, x_2, x_3) = (3, 3, 1)$, $(y_1, y_2, y_3) = (3, 6, 3)$, $(z_1, z_2, z_3) = (k_0, 2k_0, k_0)$. Furthermore, by Corollary 2.3, we have $(v_{F_1}(x \circ \phi), v_{F_1}(y \circ \phi), v_{F_1}(z \circ \phi)) = (2, 3, k_0)$.

Case 2 ($3k_0 > k_1 > 2k_0$).



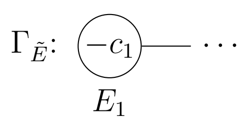
We have $(x_1, x_2) = (3, 3)$, $(y_1, y_2) = (3, 6)$, and $(z_1, z_2) = (k_0, 2k_0)$. Furthermore, we have $(v_{F_1}(x \circ \phi), v_{F_1}(y \circ \phi), v_{F_1}(z \circ \phi)) = (2, 3, k_0)$.

Case 3 ($2k_0 > k_1 \geq 3k_0/2$).



We have $(x_1, x_3) = (3, 2)$, $(y_1, y_3) = (3, 3)$, and $(z_1, z_3) = (k_0, k_0)$.

Case 4 ($2k_0 > k_1 \geq 3k_0/2$).



We have $x_1 = y_1 = 3$ and $z_1 = k_0$.

3. The negative component of the fundamental cycle

In this section, we briefly review the results of [Oyua] and consider the $Z_{\tilde{E}}$-negative components. Let (X, o) be the same as in Theorem 1.3 and $\phi : (\tilde{X}, \tilde{E}) \to (X, o)$ the covering resolution in (2.11). First, we prepare some notations.

Definition and Notation. For an anti-nef cycle $D = \sum d_\lambda \mathcal{E}_\lambda$ on $\tilde{E}$ (i.e., $D \cdot \mathcal{E}_\lambda \leq 0$ for any irreducible component $\mathcal{E}_\lambda$ of $\tilde{E}$), an irreducible component $\mathcal{E}_{\lambda_0}$ is called a *D-negative component* (defined in [TT23]) if $D \cdot \mathcal{E}_{\lambda_0} < 0$. We denote the minimal resolution (resp. the minimal good resolution) by $\phi_0 : (\tilde{X}_0, \tilde{E}_0) \to (X, o)$ (resp. $\phi_0' : (\tilde{X}_0', \tilde{E}_0') \to (X, o)$). Also, we denote the maximal ideal cycle and the fundamental cycle on ϕ_0 (resp. ϕ_0') by M_0 and Z_0 (resp. M_0' and Z_0').

By [Tom01, Proposition 3.3], the maximal ideal cycle $M_{\tilde{E}}$ satisfies $M_{\tilde{E}} = ((\alpha x + \beta y) \circ \phi)_{\tilde{E}}$ for general $\alpha, \beta \in \mathbb{C}$, $M_{\tilde{E}}^2 = -3$, and E_1 is the $M_{\tilde{E}}$-negative component, which satisfies $M_{\tilde{E}} \cdot E_1 = -1$ and $\mathrm{cff}_{E_1}(M_{\tilde{E}}) = 3$. Note that since $(x \circ \phi)_{\tilde{E}} \leq (y \circ \phi)_{\tilde{E}}$, we have $M_{\tilde{E}} = (x \circ \phi)_{\tilde{E}}$. We proved $M_0 = Z_0$ by focusing on the first Puiseux chain P_1 of the resolution graph Γ_f of the irreducible curve singularity defined by $f(x, y) = 0$. In the proof of Theorem 1.3, we will focus on $Z_{\tilde{E}}$-negative components. Hence, in the following, we describe $Z_{\tilde{E}}$-negative components in detail.

Case 1 ($k_1 > 3k_0$). In [Oyua, Theorem 5.1], we proved $M_0^2 = Z_0^2 = -1$. Hence $Z_{\tilde{E}}^2 = Z_0^2 = -1$. Let E_3' be the strict transform of E_3 by ϕ_0. We can see that E_3' is the Z_0-negative component. Therefore, E_3 is the $Z_{\tilde{E}}$-negative component, which satisfies $Z_{\tilde{E}} \cdot E_3 = -1$ and $\mathrm{cff}_{E_3}(Z_{\tilde{E}}) = 1$.

Case 2 ($3k_0 > k_1 > 2k_0$). In [Oyua, Theorem 5.3], we proved $M_0^2 = Z_0^2 = -2$. Hence $Z_{\tilde{E}}^2 = -2$. We see that the strict transform of F_1 by ϕ_0 is the Z_0-negative component. Therefore, F_1 is the $Z_{\tilde{E}}$-negative component, which satisfies $Z_{\tilde{E}} \cdot F_1 = -1$ and $\mathrm{cff}_{F_1}(Z_{\tilde{E}}) = 2$.

Case 3 ($2k_0 > k_1 \geq 3k_0/2$). In [Oyua, Theorem 5.3], we proved $M_0^2 = Z_0^2 = -2$. Hence $Z_{\tilde{E}}^2 = -2$. We see that the strict transform of E_3 by ϕ_0 is the Z_0-negative component. Therefore, E_3 is the $Z_{\tilde{E}}$-negative component, which satisfies $Z_{\tilde{E}} \cdot E_3 = -1$ and $\mathrm{cff}_{E_3}(Z_{\tilde{E}}) = 2$.

Case 4 ($3k_0/2 > k_1 > k_0$). In [Oyua, Theorem 5.5], we proved $Z_{\tilde{E}}^2 = -3$ and E_1 is the $Z_{\tilde{E}}$-negative component, which satisfies $Z_{\tilde{E}} \cdot E_1 = -1$ and $\mathrm{cff}_{E_1}(Z_{\tilde{E}}) = 3$.

4. Fundamental genera and a condition to be a Kodaira singularity

First, we will compute the fundamental genus $p_f(X,o)$ and prove Theorem 1.3.

Theorem 4.1. *Let (X,o) be the same as in Theorem 1.3. If $k_1 > 3k_0$, then $p_f(X,o) = \operatorname{ord}(f) - 1$.*

Proof. Consider the covering resolution $\phi : (\tilde{X}, \tilde{E}) \to (X,o)$ of (X,o) in Section 2. By **Case 1** of Section 3, the $Z_{\tilde{E}}$-negative component is E_3 and $Z_{\tilde{E}} \cdot E_3 = -1$. Hence we have $Z_{\tilde{E}} \cdot K_{\tilde{X}} = \operatorname{cff}_{E_3}(K_{\tilde{X}})(Z_{\tilde{E}} \cdot E_3) = -\operatorname{cff}_{E_3}(K_{\tilde{X}})$, where $K_{\tilde{X}}$ is the canonical divisor on $\tilde{X}$. By (1.1), we have $p_f(X,o) = -(\operatorname{cff}_{E_3}(K_{\tilde{X}}) + 1)/2 + 1$. Therefore we need only to know $\operatorname{cff}_{E_3}(K_{\tilde{X}})$, so, as in the proof of [Tom01, Theorem 4.1], we will compute the vanishing order $v_{E_3}(\omega \circ \phi) = \operatorname{cff}_{E_3}(K_{\tilde{X}})$ of the pull-back of the canonical form $\omega = (dx \wedge dy)/z^2$ onto $\tilde{X}$.

We assume that $E_2 = \{u = 0\}$ and $E_3 = \{v = 0\}$ in an open neighborhood $U \subset \tilde{X}$ of the intersection point of E_2 and E_3, where u, v are local coordinates on U. Since $(x_2, x_3) = (3, 1)$, $(y_2, y_3) = (6, 3)$, and $(z_2, z_3) = (2k_0, k_0)$, we obtain $x \circ \phi = u^3 v$, $y \circ \phi = u^6 v^3$, and $z \circ \phi = u^{2k_0} v^{k_0} h(u,v)$ where $h(u,v)$ is a unit. Hence, we obtain

$$\omega \circ \phi = \frac{3}{u^{4k_0-8} v^{2k_0-3} h(u,v)^2} du \wedge dv \tag{4.1}$$

on $\tilde{X}$. Since the local equation of E_3 is v, we have $v_{E_3}(\omega \circ \phi) = -2k_0 + 3$. Therefore, we obtain $p_f(X,o) = k_0 - 1$. □

Theorem 4.2. *Let (X,o) be the same as in Theorem 1.3. If $3k_0 > k_1 \geq 3k_0/2$, then $p_f(X,o) = \operatorname{ord}(f) - 2$.*

Proof. **Case 2** ($3k_0 > k_1 > 2k_0$). In this case, the $Z_{\tilde{E}}$-negative component is F_1 is and $Z_{\tilde{E}} \cdot F_1 = -1$, so we need to compute $\operatorname{cff}_{F_1}(K_{\tilde{X}})$. We assume that $E_1 = \{u = 0\}$ and $F_1 = \{v = 0\}$ in an open neighborhood $U \subset \tilde{X}$ of the intersection point of E_1 and F_1, where u, v are local coordinates on U. Since $x_1 = 3$, $y_1 = 3$, $z_1 = k_0$, $v_{F_1}(x \circ \phi) = 2$, $v_{F_1}(y \circ \phi) = 3$, and $v_{F_1}(z \circ \phi) = k_0$, we obtain $x \circ \phi = u^3v^2$, $y \circ \phi = u^3v^3$, and $z \circ \phi = u^{k_0}v^{k_0}h(u,v)$. Hence, we obtain

$$\omega \circ \phi = \frac{3}{u^{2k_0-5}v^{2k_0-4}h(u,v)^2} du \wedge dv \tag{4.2}$$

on $\tilde{X}$. Since the local equation of F_1 is v, we have $v_{F_1}(\omega \circ \phi) = -2k_0 + 4$. Therefore, we obtain $p_f(X, o) = k_0 - 2$.

Case 3 ($2k_0 > k_1 \geq 3k_0/2$). Since the $Z_{\tilde{E}}$-negative component is E_3 and $Z_{\tilde{E}} \cdot E_3 = -1$, we will compute $\mathrm{cff}_{E_3}(K_{\tilde{X}})$. We assume that $E_1 = \{u = 0\}$ and $E_3 = \{v = 0\}$ in an open neighborhood $U \subset \tilde{X}$ of the intersection point of E_1 and E_3, where u, v are local coordinates on U. Since $(x_1, x_3) = (3, 2)$, $(y_1, y_3) = (3, 3)$, and $(z_1, z_3) = (k_0, k_0)$, we obtain $x \circ \phi = u^3v^2$, $y \circ \phi = u^3v^3$, and $z \circ \phi = u^{k_0}v^{k_0}h(u,v)$. Hence, we obtain

$$\omega \circ \phi = \frac{3}{u^{2k_0-5}v^{2k_0-4}h(u,v)^2} du \wedge dv \tag{4.3}$$

on $\tilde{X}$. Since the local equation of E_3 is v, we have $v_{E_3}(\omega \circ \phi) = -2k_0 + 4$. Therefore, we obtain $p_f(X, o) = k_0 - 2$. □

Theorem 4.3. *Let (X, o) be the same as in Theorem 1.4. If $3k_0/2 > k_1 > k_0$, then $p_f(X, o) = \mathrm{ord}(f) - 3$.*

Proof. From **Case 4** of Section 3, the $Z_{\tilde{E}}$-negative component is E_1 and $E_1 \cdot Z_{\tilde{E}} = -1$, so we will compute $\mathrm{cff}_{E_1}(K_{\tilde{X}})$.

Let $P \in \tilde{X}$ be the intersection point of E_1 and the strict transform of y-axis and U an open neighborhood of P. We assume that $E_1 = \{v = 0\}$, where u, v are local coordinates on U. Since $x_1 = y_1 = 3$ and $z_1 = k_0$, we obtain $x \circ \phi = v^3$, $y \circ \phi = uv^3$, $z \circ \phi = v^{k_0}h(u,v)$. Hence, we obtain

$$\omega \circ \phi = \frac{-3}{v^{2k_0-5}h(u,v)^2} du \wedge dv \tag{4.4}$$

on $\tilde{X}$. Since the local equation of E_1 is v, we have $v_{E_1}(\omega \circ \phi) = -2k_0 + 5$. Therefore, we obtain $p_f(X, o) = k_0 - 3$. □

Next, we will prove Theorem 1.4. For this purpose, we need the following propositions.

Proposition 4.4 ([Kar80, Proposition 2.7], [Kar81])**.** *Let $\pi : (\tilde{V}, A) \to (V, o)$ be the minimal good resolution of a normal surface singularity, Z_A the fundamental* *cycle on A, and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{V,o}$. Then (V,o) is a Kodaira singularity if and only if $\mathrm{cff}_{A_i}(Z_A)=1$ holds for every Z_A-negative component A_i and there is an element $g\in\mathfrak{m}$ such that the divisor $(g\circ\pi)$ is normal crossing and $Z_A=(g\circ\pi)_A$.*

Proposition 4.5 ([Kar81])**.** *Let (V,o) be a Kodaira singularity. Then the genus of the associated pencil equals the fundamental genus $p_f(V,o)$.*

Since we need to know the coefficients of the Z_0'-negative component for the minimal good resolution, we use the results of Section 3.

Proof of Theorem 1.4. In **Case 1** of Section 3, we saw that $Z_{\tilde{E}}^2=-1$ and E_3 is the $Z_{\tilde{E}}$-negative component, which satisfies $\mathrm{cff}_{E_3}(Z_{\tilde{E}})=1$. Let $\phi':(\tilde{X}',\tilde{E}')\to(X,o)$ be a resolution of (X,o), which is obtained by blowing-down E_1, E_2, and F_1 on the covering resolution $(\tilde{X},\tilde{E})$. In the proof of [Oyua, Theorem 5.1], we proved that $M_{\tilde{E}'}=Z_{\tilde{E}'}=(x\circ\phi')_{\tilde{E}'}$. Hence we can conclude that $M_0'=Z_0'=(x\circ\phi_0')_{\tilde{E}_0'}$ on the minimal good resolution $\phi_0':(\tilde{X}_0',\tilde{E}_0')\to(X,o)$. Let E_3' be the strict transform of E_3 by ϕ_0'. Then E_3' is the Z_0'-negative component and $\mathrm{cff}_{E_3'}(Z_0')=1$. Therefore, (X,o) is a Kodaira singularity by Proposition 4.4. The genus of the associated pencil is $\mathrm{ord}(f)-1$ by Proposition 4.5 and Theorem 4.1.

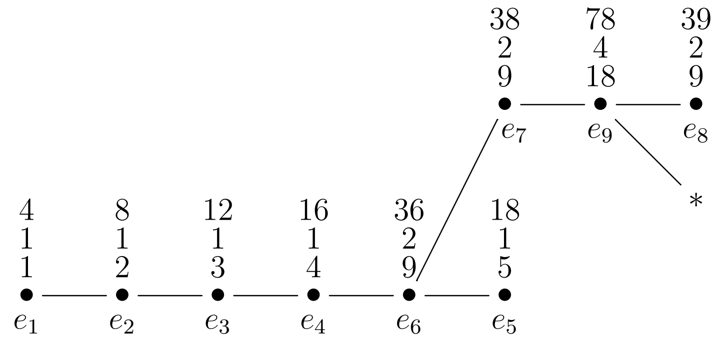
In **Cases 2**, **3**, and **4**, we saw that the coefficient of the $Z_{\tilde{E}}$-negative component is not 1, hence the same holds for the minimal good resolution. Therefore, in these cases, (X,o) is not a Kodaira singularity by Proposition 4.4. ☐

Finally, we compute the fundamental genera for two examples using the adjunction formula and confirm that our results are correct.

Example 4.6. Suppose that $(k_0,k_1,k_2)=(4,18,21)$. If we parameterize $x=t^4$, $y=t^{18}+t^{21}$, then we obtain $f(x,y)=y^4-2y^2x^9-4yx^{15}+x^{18}-x^{21}$ by eliminating t. Note that these characteristic exponents satisfy the condition of **Case 1** of Section 3. Therefore, by Theorem 4.1, we have $p_f(X,o)=3$.

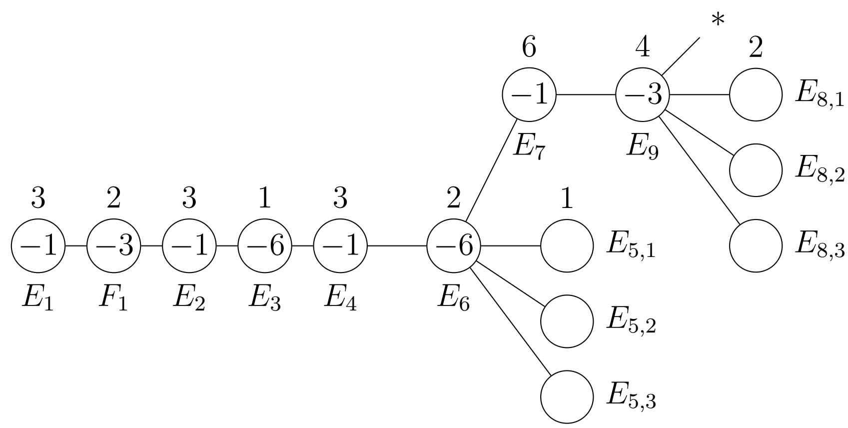
In the following, we compute the minimal good resolution of (X,o). First, the resolution graph Γ_f of the minimal embedded resolution of $(C,o)=\{f(x,y)=0\}$

is as follows:



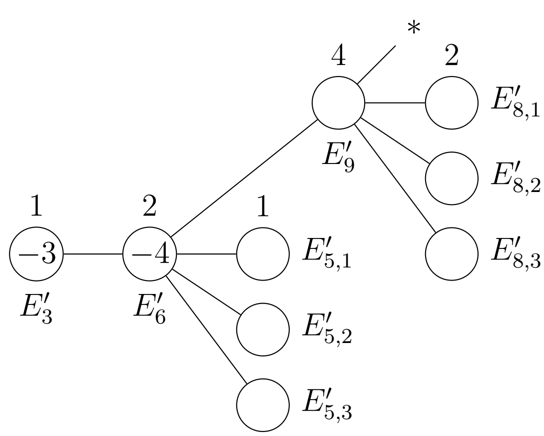
(4.5)

The weighted dual graph $\Gamma_{\tilde{E}}$ of the covering resolution is as follows:



Here, the number above each vertex is the coefficient of the maximal ideal cycle $M_{\tilde{E}}$. We see that E_1 is the $M_{\tilde{E}}$-negative component with $M_{\tilde{E}} \cdot E_1 = -1$, and $M_{\tilde{E}}^2 = -3$.

The weighted dual graph $\Gamma_{\tilde{E}'_0}$ of the minimal good resolution $\phi'_0 : (\tilde{X}'_0, \tilde{E}'_0) \to (X, o)$ is as follows:



Note that this resolution is also the minimal resolution. We see that E_3' is the M_0'-negative component with $M_0' \cdot E_3' = -1$, hence $M_0'^2 = Z_0'^2 = -1$. Let K_0' be the canonical divisor of $\tilde{X}_0'$. Then, by adjunction formula, we obtain

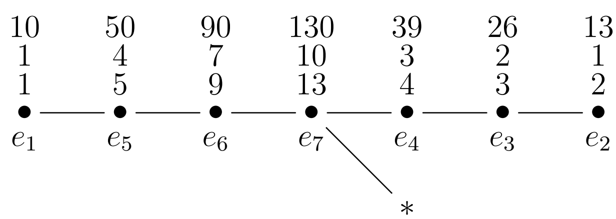
$$K_0' \cdot E_\lambda' = \begin{cases} 1 & (\lambda = 3) \\ 2 & (\lambda = 6) \\ 0 & (\text{otherwise}). \end{cases} \tag{4.6}$$

Therefore, we obtain

$$p_f(X, o) = \frac{-1 + \mathrm{cff}_{E_3'}(Z_0')(K_0' \cdot E_3') + \mathrm{cff}_{E_6'}(Z_0')(K_0' \cdot E_6')}{2} + 1 = 3.$$

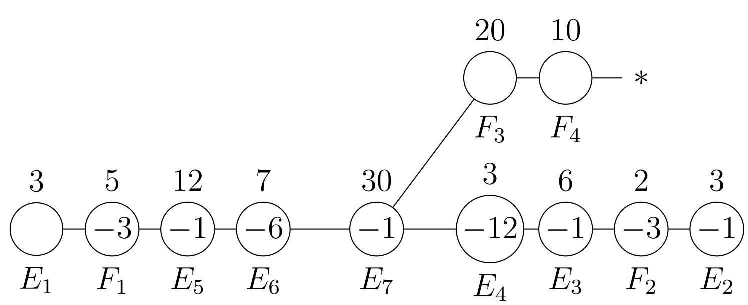
Example 4.7. Suppose that (X, o) is a Brieskorn-type singularity defined by $z^3 = y^{10} - x^{13}$. Since we can parameterize $x = t^{10}$ and $y = t^{13}$, we have $(k_0, k_1) = (10, 13)$. Note that these characteristic exponents satisfy the condition of **Case 4** of Section 3. Therefore, by Theorem 4.1, we have $p_f(X, o) = 7$.

The resolution graph Γ_f of the minimal embedded resolution of $(C, o) = \{f(x, y) = 0\}$ is as follows:

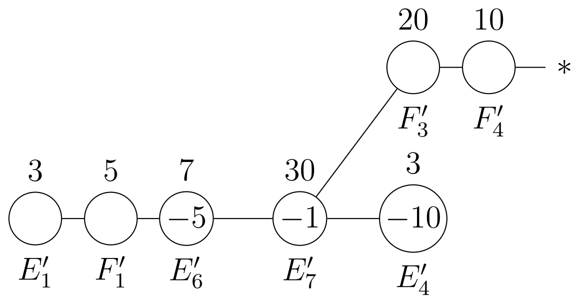


(4.7)

The weighted dual graph $\Gamma_{\tilde{E}}$ of the covering resolution is as follows:



The weighted dual graph $\Gamma_{\tilde{E}_0'}$ of the minimal good resolution is as follows:



We see that E_1' is the M_0'-negative component with $M_0' \cdot E_1' = -1$, hence $M_0'^2 = -3$. We check that $Z_0'^2 = -3$. By adjunction formula, we obtain

$$K_0' \cdot E_i' = \begin{cases} 8 & (i=4) \\ 3 & (i=6) \\ -1 & (i=7) \\ 0 & (\text{otherwise}). \end{cases} \tag{4.8}$$

Also, we have $K_0' \cdot F_j' = 0$ for $j = 1, 2, 3$. Hence we obtain $p_f(X, o) = 7$. Since (X, o) is a Brieskorn-type singularity, we can also check this by using [KN12, Theorem 1.7].

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