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SCALING LIMIT OF THE TIME AVERAGED DISTRIBUTION FOR CONTINUOUS TIME QUANTUM WALK AND SZEGEDY'S WALK ON THE PATH

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Abstract. In this paper, we consider Szegedy's walk, a type of discrete time quantum walk, and corresponding continuous time quantum walk related to the birth and death chain. We show that the scaling limit of time averaged distribution for the continuous time quantum walk induces that of Szegedy's walk if there exists the spectral gap on so-called the corresponding Jacobi matrix .

1. Introduction

Quantum walks, a quantum counterpart of random walks have been extensively developed in various fields during the last two decades. Since quantum walks are very simple models therefore they play fundamental and important roles in both theoretical fields and applications. There are good review articles for these developments such as Kempe [6], Kendon [7], Venegas-Andraca [14, 15], Konno [8], Manouchehri and Wang [9], and Portugal [11].

We investigate the time averaged distribution of a variant of discrete time quantum walk (DTQW) so-called Szegedy's walk [13]. On the path graph, the spectral properties of Szegedy's walk are directly connected to the theory of (finite type) orthogonal polynomials. There are studies of the distribution of Szegedy's walk on the path graph for example [1–3, 5, 10, 12].

In this paper, we focus on scaling limit of the time averaged distributions of both Szegedy's walk and corresponding continuous time quantum walk on the path graph related to the random walk with reflecting walls. According to our main theorem (Theorem 4.1), if there exists the spectral gap, i.e., the limit superior in the size of the path graph tends to infinity of the second largest

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eigenvalue of the Jacobi matrix is less than one (the largest eigenvalue), then the scaling limit of Szegedy's walk is the same as that of corresponding continuous time quantum walk. We should note that existence of the spectral gap of the Jacobi matrix is equivalent to that of the transition matrix of corresponding random walk. A typical example of this case is space homogeneous random walk with $p_j^R = p$ case (the second largest eigenvalue is $2\sqrt{p(1-p)}\cos\pi/n$) treated in [5] except for the symmetric random walk with $p_j^R = 1/2$. Unfortunately we have not been covered with non-spectral gap cases including symmetric random walk and the Ehrenfest model (the second largest eigenvalue is $1-2/n$) treated in [3]. To reveal non-spectral gap case is one of interesting future problems. Also the inverse problem, i.e., the problem that whether the scaling limit of Szegedy's walk causes that of continuous time or not?, can be an interesting future problem.

The rest of this paper is organized as follows. In Sec. 2, we define our setting of discrete time random walk, continuous time quantum walk and discrete time quantum walk on the path graph. Sec. 3 is devoted to show relationships between the time averaged distribution of Szegedy's walk and continuous time quantum walk. In the last section, we state our main theorem (Theorem 4.1) and prove it.

2. Definition of the models

In this paper, we consider the path graph $P_{n+1} = (V(P_{n+1}), E(P_{n+1}))$ with the vertex set $V(P_{n+1}) = \{0, 1, \ldots, n\}$ and the (undirected) edge set $E(P_{n+1}) = \{(j, j+1) : j = 0, 1, \ldots, n-1\}$. On the path graph P_{n+1}, we define a discrete time random walk (DTRW) with reflecting walls as follows:

Let p_j^L be the transition probability of the random walker at the vertex $j \in V(P_{n+1})$ to the left $(j-1 \in V(P_{n+1}))$. Also let $p_j^R = 1 - p_j^L$ be the transition probability of the random walker at the vertex $j \in V(P_{n+1})$ to the right $(j+1 \in V(P_{n+1}))$. For the sake of simplicity, we assume $0 < p_j^L, p_j^R < 1$ except for $j = 0, n$. We put the reflecting walls at the vertex $0 \in V(P_{n+1})$ and the vertex $n \in V(P_{n+1})$, i.e., we set $p_0^R = p_n^L = 1$. We also call this type of DTRW as the birth and death chain.

Let a positive constant C_π be

$$C_\pi := 1 + \sum_{j=1}^{n} \frac{p_0^R \cdot p_1^R \cdots p_{j-1}^R}{p_1^L \cdot p_2^L \cdots p_j^L}$$

then we can define the stationary distribution $\{\pi(0), \pi(1), \ldots, \pi(n)\}$ as

$$\pi(j) = \begin{cases} \frac{1}{C_\pi} & \text{if } j = 0, \\ \frac{1}{C_\pi} \cdot \frac{p_0^R \cdot p_1^R \cdots p_{j-1}^R}{p_1^L \cdot p_2^L \cdots p_j^L} & \text{if } j = 1, 2, \ldots, n. \end{cases}$$

Note that $\pi(j) > 0$ for all $j \in V(P_{n+1})$ and the stationary distribution is satisfied with so-called the detailed balance condition,

$$\pi(j) \cdot p_j^R = p_{j+1}^L \cdot \pi(j+1),$$

for $j = 0, 1, \ldots n-1$.

In order to define a continuous time quantum walk (CTQW) corresponding to the DTRW, we introduce the normalized Laplacian matrix $\mathcal{L}$. Let P be the transition matrix of the DTRW. Also we define diagonal matrices $D_\pi^{1/2} := \mathrm{diag}\left(\sqrt{\pi(0)}, \sqrt{\pi(1)}, \ldots, \sqrt{\pi(n)}\right)$ and $D_\pi^{-1/2} = \left(D_\pi^{1/2}\right)^{-1}$. Note that $D_\pi^{-1/2} = \mathrm{diag}\left(1/\sqrt{\pi(0)}, 1/\sqrt{\pi(1)}, \ldots, 1/\sqrt{\pi(n)}\right)$ by the definition. The normalized Laplacian matrix $\mathcal{L}$ is given by

$$\mathcal{L} := D_\pi^{1/2} \left(I_{n+1} - P\right) D_\pi^{-1/2} = I_{n+1} - D_\pi^{1/2} P D_\pi^{-1/2},$$

where I_{n+1} be the $(n+1) \times (n+1)$ identity matrix. We should remark that the matrix

$$J := D_\pi^{1/2} P D_\pi^{-1/2},$$

is referred as the Jacobi matrix. So we can rewrite $\mathcal{L}$ as $\mathcal{L} = I_{n+1} - J$.

By using the detailed balance condition, we obtain

$$J_{j,k} = J_{k,j} = \begin{cases} \sqrt{p_j^R p_{j+1}^L}, & \text{if } k = j+1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\mathcal{L} = I_{n+1} - J$ is an Hermitian matrix (real symmetric matrix). The CTQW which is discussed in this paper is driven by the time evolution operator (unitary matrix)

$$U_{CTQW}(t) := \exp\left(it\mathcal{L}\right) := \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \mathcal{L}^k,$$

where i is the imaginary unit. Let X_t^C $(t \geq 0)$ be the random variable representing the position of the CTQWer at time t. The distribution of X_t^C is determined by

$$\mathbb{P}\left(X_t^C = k | X_0^C = j\right) := \left|\langle k | U_{CTQW}(t) | j \rangle\right|^2 = \left|\left(U_{CTQW}(t)\right)_{k,j}\right|^2,$$

where $|j\rangle$ is the $(n+1)$-dimensional unit vector (column vector) which j-th component equals 1 and the other components are 0 and $\langle v|$ is the transpose of $|v\rangle$, i.e., $\langle v| = {}^T|v\rangle$.

Hereafter we only consider $X_0^C = 0$, i.e., the CTQWer starts from the left most vertex $0 \in V(P_{n+1})$, cases. The time averaged distribution $\bar{p}_C$ of the CTQW is defined by

$$\bar{p}_C(j) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{P}\left(X_t^C = j | X_0^C = 0\right) dt,$$

for each vertex $j \in V(P_{n+1})$. We define a random variable $\bar{X}_n^C$ as $\mathbb{P}\left(\bar{X}_n^C = j\right) = \bar{p}_C(j)$.

In this paper, we also deal with a type of discrete time quantum walk (DTQW) corresponding to the DTRW so-called Szegedy's walk. The time evolution operator for the DTQW is defined by $U = SC$ with the coin operator C and the shift operator (flip-flop type shift) S. The coin operator C is defined by

$$C = |0\rangle\langle 0| \otimes I_2 + \sum_{j=1}^{n-1} |j\rangle\langle j| \otimes C_j + |n\rangle\langle n| \otimes I_2,$$

where I_2 is the 2×2 identity matrix and $\otimes$ is the tensor product. The local coin operator C_j is defined by

$$C_j = 2|\phi_j\rangle\langle\phi_j| - I_2, \quad |\phi_j\rangle = \sqrt{p_j^L}|L\rangle + \sqrt{p_j^R}|R\rangle,$$

where $|L\rangle = {}^T[1\ 0]$ and $|R\rangle = {}^T[0\ 1]$. The shift operator S is given by

$$S\left(|j\rangle \otimes |L\rangle\right) = |j-1\rangle \otimes |R\rangle, \quad S\left(|j\rangle \otimes |R\rangle\right) = |j+1\rangle \otimes |L\rangle.$$

Let X_t^D $(t = 0, 1, \ldots)$ be the random variable representing the position of the DTQWer at time t. In this paper, we only consider $X_0^D = 0$ cases. The distribution of X_t^D is defined by

$$\begin{aligned} &\mathbb{P}\left(X_t^D = j | X_0^D = 0\right) \\ &:= \left\|\left(\langle j| \otimes I_2\right) U_{DTQW}(t) \left(|0\rangle \otimes |R\rangle\right)\right\|^2 \\ &\ = \left|\left(\langle j| \otimes \langle L|\right) U_{DTQW}(t) \left(|0\rangle \otimes |R\rangle\right)\right|^2 + \left|\left(\langle j| \otimes \langle R|\right) U_{DTQW}(t) \left(|0\rangle \otimes |R\rangle\right)\right|^2 . \end{aligned}$$

We also consider the time averaged distribution $\bar{p}_D$ of the DTQW defined by

$$\bar{p}_D(j) := \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{P}\left(X_t^D = j | X_0^D = 0\right),$$

for each vertex $j \in V(P_{n+1})$. We define a random variable $\bar{X}_n^D$ as $\mathbb{P}\left(\bar{X}_n^D = j\right) = \bar{p}_D(j)$.

3. Relations between $\bar{X}_n^C$ and $\bar{X}_n^D$

Since the Jacobi matrix J is a real symmetric matrix with simple [4] and symmetric [3] eigenvalues, we obtain eigenvalues $1 = \lambda_0 > \lambda_1 > \cdots > \lambda_{n-1} > \lambda_n = -1$ and corresponding eigenvectors $\{|v_\ell\rangle\}_{\ell=0}^n$ as an orthonormal basis of n-dimensional complex vector space $\mathbb{C}^n$. Thus we have the spectral decomposition

$$J = \sum_{\ell=0}^{n} \lambda_\ell |v_\ell\rangle\langle v_\ell|.$$

Noting that $\mathcal{L} = I_{n+1} - J$, the spectral decomposition of $U_{CTQW}(t)$ is given by

$$U_{CTQW}(t) = \sum_{\ell=0}^{n} \exp\left[it\left(1-\lambda_\ell\right)\right] |v_\ell\rangle\langle v_\ell| = e^{it}\sum_{\ell=0}^{n} e^{-it\lambda_\ell}|v_\ell\rangle\langle v_\ell|.$$

Because of simple eigenvalues of the Jacobi matrix J, the time averaged distribution $\bar{p}_C$ is expressed by

$$\bar{p}_C(j) = \sum_{\ell=0}^{n} \left|\langle j|v_\ell\rangle\right|^2 \left|\langle v_\ell|0\rangle\right|^2 = \sum_{\ell=0}^{n} \left|v_\ell(j)\right|^2 \left|v_\ell(0)\right|^2,$$

where $v_\ell(j)$ is the jth component of $|v_\ell\rangle$.

On the other hand, the spectral decomposition of $U_{DTQW}(t)$ is given (see e.g. [3, 5, 12, 13]) by

$$U_{DTQW}(t) = \mu_0|u_0\rangle\langle u_0| + \sum_{\ell=1}^{n-1}\left(\frac{1}{2(1-\lambda_\ell^2)}\sum_{\pm}\mu_{\pm\ell}|u_{\pm\ell}\rangle\langle u_{\pm\ell}|\right) + \mu_n|u_n\rangle\langle u_n|,$$

where

$$\begin{cases} \mu_0 = \lambda_0 = 1, & |u_0\rangle = |\overline{v_0}\rangle, \\ \mu_{\pm\ell} = \exp\left(\pm i\cos^{-1}\lambda_\ell\right), & |u_{\pm\ell}\rangle = |\overline{v_\ell}\rangle - \mu_{\pm\ell}\; S|\overline{v_\ell}\rangle, \\ \mu_n = \lambda_n = -1, & |u_{n-1}\rangle = |\overline{v_{n-1}}\rangle, \end{cases}$$

with

$$|\overline{v_\ell}\rangle = v_\ell(0)|0\rangle\otimes|R\rangle + \sum_{j=1}^{n-1} v_\ell(j)|j\rangle\otimes|\phi_j\rangle + v_\ell(n)|n\rangle\otimes|L\rangle.$$

All the eigenvalues of $U_{DTQW}(t)$ are also simple, the time averaged distribution $\bar{p}_D$ is expressed by

$$\begin{aligned}
&\bar{p}_D(j) \\
&= \left\{ |(\langle j| \otimes \langle L|)\,|u_0\rangle|^2 + |(\langle j| \otimes \langle R|)\,|u_0\rangle|^2 \right\} |\langle u_0|\,(|0\rangle \otimes |R\rangle)|^2 \\
&+ \sum_{\ell=1}^{n-1} \left[\frac{1}{2(1-\lambda_\ell^2)} \sum_{\pm} \left\{ |(\langle j| \otimes \langle L|)\,|u_{\pm\ell}\rangle|^2 + |(\langle j| \otimes \langle R|)\,|u_{\pm\ell}\rangle|^2 \right\} |\langle u_{\pm\ell}|\,(|0\rangle \otimes |R\rangle)|^2 \right] \\
&+ \left\{ |(\langle j| \otimes \langle L|)\,|u_n\rangle|^2 + |(\langle j| \otimes \langle R|)\,|u_n\rangle|^2 \right\} |\langle u_n|\,(|0\rangle \otimes |R\rangle)|^2 .
\end{aligned}$$

More concrete expression of $\bar{p}_D$ in terms of eigenvalues and eigenvectors of the Jacobi matrix J is given as follows (rearrangement of Eq.(10) in [3]):

$$\begin{aligned}
\bar{p}_D(j) = &\frac{1}{2}|v_0(j)|^2 |v_0(0)|^2 + \frac{1}{2}|v_n(j)|^2 |v_n(0)|^2 \\
&+ \frac{1}{2}\sum_{\ell=0}^{n} |v_\ell(j)|^2 |v_\ell(0)|^2 \\
&+ \frac{1}{2}\sum_{\ell=1}^{n-1} \frac{1}{1-\lambda_\ell^2} \left\{ p_{j-1}^R |v_\ell(j-1)|^2 - \lambda_\ell^2 |v_\ell(j)|^2 + p_{j+1}^L |v_\ell(j+1)|^2 \right\} |v_\ell(0)|^2 ,
\end{aligned}$$

with conventions $p_{-1}^R = v_\ell(-1) = p_{n+1}^L = v_\ell(n+1) = 0$.

Now we consider the distribution functions $\bar{F}_n^C(x) := \mathbb{P}\left(\bar{X}_n^C \le x\right) = \sum_{j\le x} \bar{p}_C(j)$ of $\bar{X}_n^C$ and $\bar{F}_n^D(x) := \mathbb{P}\left(\bar{X}_n^D \le x\right) = \sum_{j\le x} \bar{p}_D(j)$ of $\bar{X}_n^D$. For each integer $0 \le k \le n-1$, we have

$$\bar{F}_n^C(k) = \sum_{j=0}^{k} \bar{p}_C(j) = \sum_{j=0}^{k} \left\{ \sum_{\ell=0}^{n} |v_\ell(j)|^2 |v_\ell(0)|^2 \right\}.$$

We also obtain the following expression by using $p_j^L + p_j^R = 1, p_0^R = 1$ and

$p_1^L \left|v_\ell(1)\right|^2 = \lambda_\ell^2 \left|v_\ell(0)\right|^2$:

$$
\begin{aligned}
\bar{F}_n^D(k) &= \sum_{j=0}^{k} \bar{p}_D(j) \\
&= \frac{1}{2}\sum_{j=0}^{k} \left|v_0(j)\right|^2 \left|v_0(0)\right|^2 + \frac{1}{2}\sum_{j=0}^{k} \left|v_n(j)\right|^2 \left|v_n(0)\right|^2 \\
&+ \frac{1}{2}\sum_{j=0}^{k} \left\{ \sum_{\ell=0}^{n} \left|v_\ell(j)\right|^2 \left|v_\ell(0)\right|^2 \right\} + \frac{1}{2}\sum_{j=1}^{k} \left\{ \sum_{\ell=1}^{n-1} \left|v_\ell(j)\right|^2 \left|v_\ell(0)\right|^2 \right\} \\
&+ \frac{1}{2}\sum_{\ell=1}^{n-1} \frac{1}{1-\lambda_\ell^2} \Bigg\{ p_0^R \left|v_\ell(0)\right|^2 - p_1^L \left|v_\ell(1)\right|^2 - p_k^R \left|v_\ell(k)\right|^2 \\
&\qquad\qquad + p_{k+1}^L \left|v_\ell(k+1)\right|^2 \Bigg\} \left|v_\ell(0)\right|^2 \\
&= \sum_{j=0}^{k} \left\{ \sum_{\ell=0}^{n} \left|v_\ell(j)\right|^2 \left|v_\ell(0)\right|^2 \right\} \\
&+ \frac{1}{2}\sum_{\ell=1}^{n-1} \frac{1}{1-\lambda_\ell^2} \left\{ -p_k^R \left|v_\ell(k)\right|^2 + p_{k+1}^L \left|v_\ell(k+1)\right|^2 \right\} \left|v_\ell(0)\right|^2 \\
&= \bar{F}_n^C(k) + \frac{1}{2}\sum_{\ell=1}^{n-1} \frac{1}{1-\lambda_\ell^2} \left\{ -p_k^R \left|v_\ell(k)\right|^2 + p_{k+1}^L \left|v_\ell(k+1)\right|^2 \right\} \left|v_\ell(0)\right|^2 .
\end{aligned}
$$

4. Scaling limit

In this section, we state our main result and prove it.

THEOREM 4.1 *Assume that there exists the spectral gap, i.e.,* $\limsup_{n\to\infty} \lambda_1 < 1 = \lambda_0$*. If* $\frac{\bar{X}_n^C}{n}$ *converges weakly to the random variable* $\bar{X}$ *as* $n \to \infty$ *then* $\frac{\bar{X}_n^D}{n}$ *also converges weakly to the same random variable* $\bar{X}$*.*

Proof of Theorem 4.1

Let $\bar{F}$ be the distribution function of the random variable $\bar{X}$. We assume that

$$
\lim_{n\to\infty} \mathbb{P}\left(\frac{\bar{X}_n^C}{n} \le x \right) = \bar{F}(x) \qquad (1)
$$

for all points x at which $\bar{F}$ is continuous. Hereafter we assume $\bar{F}$ is continuous at x $(0 \le x \le 1)$. Remark that from the definition, Eq. (1) means that

$$\lim_{n\to\infty} \bar{F}_n^C(nx) = \lim_{n\to\infty} \bar{F}_n^C(\lfloor nx \rfloor) = \lim_{n\to\infty} \sum_{j=0}^{\lfloor nx \rfloor} \left\{ \sum_{\ell=0}^{n} |v_\ell(j)|^2 |v_\ell(0)|^2 \right\} = \bar{F}(x), \quad (2)$$

where $\lfloor a \rfloor$ denotes the biggest integer which is not greater than a.

From Eq. (2) and the relation

$$\begin{aligned}
&\mathbb{P}\left(\frac{\bar{X}_n^D}{n} \le x\right) \\
&= \bar{F}_n^D(nx) = \bar{F}_n^D(\lfloor nx \rfloor) \\
&= \bar{F}_n^C(\lfloor nx \rfloor) \\
&\quad + \frac{1}{2}\sum_{\ell=1}^{n-1} \frac{1}{1-\lambda_\ell^2}\left\{ -p_{\lfloor nx \rfloor}^R |v_\ell(\lfloor nx \rfloor)|^2 + p_{\lfloor nx \rfloor+1}^L |v_\ell(\lfloor nx \rfloor + 1)|^2 \right\} |v_\ell(0)|^2,
\end{aligned}$$

if we can prove

$$\lim_{n\to\infty} \sum_{\ell=1}^{n-1} \frac{1}{1-\lambda_\ell^2} |v_\ell(\lfloor nx \rfloor)|^2 |v_\ell(0)|^2 = \lim_{n\to\infty} \sum_{\ell=1}^{n-1} \frac{1}{1-\lambda_\ell^2} |v_\ell(\lfloor nx \rfloor + 1)|^2 |v_\ell(0)|^2 = 0, \quad (3)$$

then we can conclude

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\bar{X}_n^D}{n} \le x\right) = \bar{F}(x),$$

for all points at which $\bar{F}$ is continuous.

From Eq.(2), we obtain

$$0 \le \sum_{j=0}^{\lfloor nx \rfloor} \left\{ \sum_{\ell=1}^{n-1} |v_\ell(j)|^2 |v_\ell(0)|^2 \right\} \le \bar{F}_n^C(\lfloor nx \rfloor) \xrightarrow{n\to\infty} \bar{F}(x).$$

Also we have

$$0 \le \sum_{j=0}^{\lfloor nx \rfloor+1} \left\{ \sum_{\ell=1}^{n-1} |v_\ell(j)|^2 |v_\ell(0)|^2 \right\} \le \bar{F}_n^C\left(\left\lfloor n\left(x + \frac{1}{n}\right)\right\rfloor\right) \xrightarrow{n\to\infty} \bar{F}(x),$$

from continuity of $\bar{F}$ at x. These mean that

$$\lim_{n\to\infty} \sum_{\ell=1}^{n-1} |v_\ell(\lfloor nx \rfloor)|^2 |v_\ell(0)|^2 = \lim_{n\to\infty} \sum_{\ell=1}^{n-1} |v_\ell(\lfloor nx \rfloor + 1)|^2 |v_\ell(0)|^2 = 0. \quad (4)$$

Therefore combining with Eq. (4), we obtain Eq. (3) as follows:

$$
\begin{aligned}
&\limsup_{n\to\infty}\sum_{\ell=1}^{n-1}\frac{1}{1-\lambda_\ell^2}\left|v_\ell(\lfloor nx\rfloor)\right|^2\left|v_\ell(0)\right|^2\\
&\le \limsup_{n\to\infty}\frac{1}{1-\lambda_1^2}\sum_{\ell=1}^{n-1}\left|v_\ell(\lfloor nx\rfloor)\right|^2\left|v_\ell(0)\right|^2\\
&\le \frac{1}{1-\limsup_{n\to\infty}\lambda_1^2}\times\lim_{n\to\infty}\sum_{\ell=1}^{n-1}\left|v_\ell(\lfloor nx\rfloor)\right|^2\left|v_\ell(0)\right|^2\\
&=0,
\end{aligned}
$$

$$
\begin{aligned}
&\limsup_{n\to\infty}\sum_{\ell=1}^{n-1}\frac{1}{1-\lambda_\ell^2}\left|v_\ell(\lfloor nx\rfloor+1)\right|^2\left|v_\ell(0)\right|^2\\
&\le \limsup_{n\to\infty}\frac{1}{1-\lambda_1^2}\sum_{\ell=1}^{n-1}\left|v_\ell(\lfloor nx\rfloor+1)\right|^2\left|v_\ell(0)\right|^2\\
&\le \frac{1}{1-\limsup_{n\to\infty}\lambda_1^2}\times\lim_{n\to\infty}\sum_{\ell=1}^{n-1}\left|v_\ell(\lfloor nx\rfloor+1)\right|^2\left|v_\ell(0)\right|^2\\
&=0.
\end{aligned}
$$

This completes the proof. □

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References

[1] Anahara, Y., Konno, N., Morioka, H., Segawa, E.: Comfortable place for quantum walker on finite path. Quantum Inf. Process. **21**, 242 (2022).

[2] Higuchi, K., Komatsu, T., Konno, N., Morioka, H., Segawa, E.: A discontinuity of the energy of quantum walk in impurities. Symmetry **13**, 1134 (2022).

[3] Ho, C.-L., Ide, Y., Konno, N., Segawa, E., Takumi, K.: A spectral analysis of discrete-time quantum walks related to the birth and death chains. J. Stat. Phys. **171**, 207–219 (2018).

[4] Hora, A., Obata, N.: Quantum Probability and Spectral Analysis of Graphs. Springer (2007).

[5] Ide, Y., Konno, N., Segawa, E.: Time averaged distribution of a discrete-time quantum walk on the path. Quantum Inf. Process. **11** (5), 1207–1218 (2012).

[6] Kempe, J.: Quantum random walks - an introductory overview. Contemporary Physics **44**, 307–327 (2003).

[7] Kendon, V.: Decoherence in quantum walks - a review. Math. Struct. in Comp. Sci. **17**, 1169–1220 (2007).

[8] Konno, N.: Quantum Walks. In: Quantum Potential Theory, Franz, U., and Schürmann, M., Eds., Lecture Notes in Mathematics: Vol. 1954, pp. 309–452, Springer-Verlag, Heidelberg (2008).

[9] Manouchehri, K., Wang, J.: Physical Implementation of Quantum Walks, Springer (2013).

[10] Marquezino, F. L., Portugal, R., Abal, G., Donangelo, R.: Mixing times in quantum walks on the hypercube. Phys. Rev. A **77**, 042312 (2008).

[11] Portugal, R.: Quantum Walks and Search Algorithms, Springer (2013).

[12] Segawa, E.: Localization of quantum walks induced by recurrence properties of random walks. J. Comput. Nanosci. **10**, 1583–1590 (2013).

[13] Szegedy, M.: Quantum speed-up of Markov chain based algorithms. Proc. of the 45th Annual IEEE Symposium on Foundations of Computer Science (FOCS'04), 32–41 (2004).

[14] Venegas-Andraca, S. E.: Quantum Walks for Computer Scientists, Morgan and Claypool (2008).

[15] Venegas-Andraca, S. E.: Quantum walks: a comprehensive review, Quantum Inf. Process. **11**, 1015–1106 (2012).

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