Coloring of planar graphs and its relations to hypergraph coloring

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Preface

This thesis is written on the subject “Coloring of planar graphs and its relations to hypergraph coloring” and it is to be submitted to get the degree of Doctor of Science at Yokohama National University.

I like history and arts along with mathematics. Long time ago, when I was at Shiraz University, I visited many historical places in Persian province, beautiful gardens and architectures, from close distance. At Shiraz University, I also started to practice Kong Fu TOA, an Iranian branch of Kong Fu, and kept doing it for many years to have healthy body and mind. I also interested in wood carving that enables us to make beautiful designs on woods. This skill helped me several times to draw graphs more professionally.

I selected Japan for my Ph.D. period for several reason. This choice originally comes form old time, more than twenty years ago, when I watched a Japanese traditional movie and promised myself to visit this place one day. After many years, time led me to have common researches with some Japanese graph theorists. Meanwhile, I found out there is a book called Asuka and Persia, written by a Japanese author (Noichi Emoto), that explains about a strong historical relation of Iran and Japan.

I found there are something between two cultures look very similar. For example, Omizutori (Japan) and Charshanbe-Suri (Iran), Hanami (Japan) and Sizdabedar (Iran), Kotatsu (Japan) and Korsi (Iran).

These increased my motivation to visit some places of Japan; for example, Nara city. A beautiful city with kind people. In this city, there is a museum that shows historical gifts from several old countries and it was a reason for me how much old people liked to visit here. (I already visited a similar place of Iran, Persepolis, depicted pictures on stone, showing people from many ancient countries coming to Iran with their presents).

In the celebration of Iranian new year (Nowruz), I also succeeded to meet Hisako

Tsunoda, a prominent Japanese artist who has been working in the field of calligraphy for about 35 years and teaches Persian language and literature. From initial time coming to Japan, I also joined to Karate club of YNU and kept learning a new martial art. Many thanks to the members.

During being in Japan, I met many new people. I would like to thank Yaping Mao who was a close friend and provided nice time for me even in difficult times. Thanks to my close friends Ali Talebi-Anaraki, Alireza Soleymanipoor, Park Seyong, Andrea Binotto, Gemechu Yilikal, and Okada Ryutaro. Also, thank Kenta Ozeki for his advice to write this thesis. Thanks to Seiya Negami, Atsuhiro Nakamoto, Yumiko Ohno, Shinya Fujita, Masahiro Sanka, Remiko Lida; in particular for having discussion times on graph theory. Finally, I would like to appreciate my Math Olympiad teacher, Ahmad Peivandi, for his kindness and assistance during a long time. He also made the world of Math for me more fascinating and beautiful.

The author
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Papers underlying the thesis

- M. Hasanvand and K. Ozeki, A Note on 3-Distance Coloring of Planar Graphs, *Submitted*.
- N. Haghparast, M. Hasanvand, and Y. Ohno, The existence of uniform hypergraphs for which the interpolation property of complete coloring fails, *Discrete Math.* **345** (2022) 112722.
- M. Hasanvand, Answering a problem of Matsumoto and Ohno about complete coloring of planar triangulations, *Under prepration.*

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Contents

Introduction

A *graph* consists of finitely many points (called vertices) and finitely many unordered pairs of them (called edges). The sets of vertices and edges of a graph G are usually denoted by $V(G)$ and $E(G)$, respectively. The *degree* $d_G(v)$ of a vertex v refers to the number of edges incident with v. The *maximum degree* $\Delta(G)$ of a graph G is the maximum degree of all vertices v of G. A *k-coloring* of a graph G referes to a function $c : V(G) \to \{1, \ldots, k\}$ such that any two adjacent vertices v and u receive different colors, which means $c(v) \neq c(u)$. The *chromatic number* $\chi(G)$ is the minimum number k such that there is a k-coloring for G.

Graph coloring has much attraction in graph theory. For example, consider coloring a map such that every region colored by one color and any two regions with a common border have different colors. It is natural to ask how many colors are enough for finding such a coloring for an arbitrary map. (Note that we can translate this problem in terms of vertex coloring by replacing each region by a vertex and joining two vertices if corresponding regions have a common border). This problem was given by Francis Guthrie in 1852 and had not been solved for more than 100 years. Finally, Appel and Haken (1976) used a special technique, called discharging method along with a computer assistance to prove this theorem (The Four-Color Theorem).

Theorem 0.1. (Appel and Haken [3]) *If G is a planar graph, then $\chi(G) \leq 4$.*

This theorem has many applications for planar graphs. In addition, coloring of planar graphs has been studied extensively during a long time. In this thesis, we study two types of such colorings, called distance coloring and facial coloring, and we use the second one for hypergraph coloring as well.

Recently, Thomassen (2018) used the four-color theorem to prove that the square of any planar graph has the chromatic number at most 7. This result is also proved by

Hartke, Jahanbekam, Thomas (2016) [13] based on a direct computer assistance. This assertion was originally conjectured by Wegner (1977) [25]. The *square* G^2 of a graph G is the graph obtained from G by connecting any two vertices with distance at most two, and a *subcubic* graph referes to a graph with maximum degree at most three.

Theorem 0.2. ([13, 23]) *If G is a planar subcubic graph, then $\chi(G^2) \leq 7$.*

Let turn our attention to coloring of all graphs. By an induction on the number of vertices, it is easy to show that the chromatic number of a graph is less than or equal to maximum degree plus one. This bound is sharp with respect to the complete graphs.

Observation 0.3. *If G is a graph, then $\chi(G) \leq \Delta(G) + 1$.*

One may ask whether this bound can be pushed down by ignoring some exceptional cases. Brooks characterized those exceptional graphs as the following theorem.

Theorem 0.4. (Brooks [6]) *Let G be a connected graph. Then $\chi(G) \leq \Delta(G)$ if and only if G is neither a complete graph nor an odd cycle.*

For square graphs G^2, it easy to check that $\Delta(G^2) \leq \Delta(G)^2$. Thus we can apply Brooks' Theorem, and it is not difficult to check that $\chi(G^2) \leq \Delta(G)^2$ provided that the square of G is not complete. These exceptional graphs are called *Moore* graphs. Cranston and Rabern (2016) also refined this upper bound a little as the following theorem.

Theorem 0.5. (Cranston and Rabern [8]) *If G is not a Moore graph, then $\chi(G^2) \leq \Delta(G)^2 - 1$.*

Surprisingly, for planar graphs this quadratic bound can be reduced to a linear bound. Moreover, Wegner (1997) proposed the following sharp conjecture.

Conjecture 0.6. ([25]) *If G is a planar graph and $\Delta(G) \geq 8$, then $\chi(G^2) \leq \frac{3}{2}\Delta(G) + 1$.*

This conjecture has been extensively investigated by many authors, see [18, Page 3]. For example, Molloy and Salavatipour (2005) [21] proved this conjecture by replacing the upper bound with $\lceil \frac{5}{3}\Delta(G) \rceil + 25$ provided that $\Delta(G) \geq 241$. The currently best known bound is asymptotically $(\frac{3}{2} + o(1))\Delta(G)$ which was proved by Havet, Heuvel, McDiarmid, Reed (2007) [14] and Amini, Esperet, Heuvel (2013) [2].

Similarly to the square chromatic number, distance coloring of graphs is a natural generalization, which is studied extensively as well. For planar graphs, distance coloring has interesting applications on the frequency assignment problem (radio channel assignment); for example, see [16, 17, 21].

Let start with some definition. The k-th power G^k of a graph G is the graph obtained from G by connecting any two vertices with distance at most k. The chromatic number of G^k is called the *k-distance chromatic number* and is denoted by $\chi_k(G)$. It is not difficult to check that this graph has maximum degree at most

$$\Delta(G^k) \leq \Delta(G)\sum_{i=0}^{k-1}(\Delta(G)-1)^i = \frac{\Delta(G)}{\Delta(G)-2}((\Delta(G)-1)^k - 1) + 1.$$

Let us denote this upper bound by D_k. Fortunately, it was shown that if $k \geq 3$, then there is no graphs whose k-th power would be a complete graph of order $D_k + 1$. Thus by applying Brooks' Theorem, one can easily conclude that $\chi_k(G) \leq D_k$. Surprisingly, this upper bound can be pushed further down as the following result due to Bonamy and Bousquet (2014) [4].

Theorem 0.7. ([4]) *If G is a graph and $k \geq 3$, then $\chi_k(G) \leq D_k - 1$.*

They also conjectured that this upper bound can be replaced by $D_k + 1 - k$ except for finitely many graphs. A weaker version of this conjecture is confirmed by Pierron (2019) [22] by replacing the upper bound with $D_k + 3 - k$.

Conjecture 0.8. ([4]) *For all connected graphs G, except finitely many graphs, $\chi_k(G) \leq D_k + 1 - k$.*

As we have already stated, for planar graphs this upper bound can be pushed down to around square root of it as the following result due to Agnarsson and Halldorsson (2003) [1].

Theorem 0.9. ([1]) *If G is a planar graph and $\Delta(G) \geq 3$, then $\chi_k(G) \leq 2^{k+1}3^{\lceil k/2\rceil}\Delta(G)^{\lfloor k/2\rfloor}$.*

For the case $k = 3$, Theorem 0.9 says that for every planar graph G, we have $\chi_3(G) \leq 144\Delta(G)$. On the other hand, we know there are planar graphs with the maximum degree Δ and 3-distance chromatic number 4Δ minus a constant. For example, consider the complete graph of order 4 and attach $\Delta - 3$ pendant vertices to each vertex. In this thesis, we improve their result to the following version by replacing a better coefficient much closer to the sharp version.

Theorem 0.10. *If G is a planar graph, then $\chi_3(G) \leq (6 + o(1))\Delta(G)$.*

To investigate the sharpness, we will also restrict our attention to subcubic graphs. Conjecture 0.8 suggests the upper bound 19 for all connected subcubic graphs except finitely many. In this thesis, we partially solves Conjecture 0.8, for $k = 3$, for planar graphs with a better bound as follows (and so the exceptional ones must not be planar).

Theorem 0.11. *If G is a subcubic planar graph, then $\chi_3(G) \leq 17$.*

We should note that to prove this result, we use a discharging method without applying a computer assistance. In addition, we conjecture that for this family of planar graphs, this upper bound can be reduced to 12 and show some examples that this conjecture must be sharp.

Let turn our attention to the second type of coloring. For coloring problem, we usually try to have different colors on both ends of each edge. There is another concept of coloring that is necessary to have at least one pair of colors at both ends of all edges. More precisely, a *complete k-coloring* refers to a function $c : V(G) \to \{1, \ldots, k\}$ such that any two adjacent vertices have different colors and for any two colors $i, j \in \{1, \ldots, k\}$, there is one edge uv such that $c(v) = i$ and $c(u) = j$. The *achromatic number* of a graph G is the maximum number k such that there is a complete k-coloring for G and is denoted by $\psi(G)$. Note that the gap of the chromatic number and the achromatic number can be arbitrary large enough; for example, for paths P of size $\binom{k}{2}$ we have $\chi(G) \leq 2 \leq k \leq \psi(G)$, where k is an odd integer with $k \geq 3$. An interesting property of this kind of coloring is that if we have a complete $(k+1)$-coloring of G, we can find a complete k-coloring provided that $k \geq \chi(G)$. This property is called *the interpolation property* of complete coloring and was discovered by Harary, Hedetniemi, and Prins (1967) [12].

Theorem 0.12. ([12]) *Every graph G admits a complete t-coloring for every t with $\chi(G) \leq t \leq \psi(G)$.*

Graphs can be generalized to hypergraphs by removing size limitation of edges. In fact, a hypergraph H consists of a set $V(H)$ of vertices and a set $E(H)$ of subsets of $V(H)$. Every element of $E(H)$ is called a *hyperedge*. A hypergraph is called *m-uniform*, if all hyperedges have the same size m. Note that graphs defined here are 2-uniform hypagraphs. A *complete k-coloring* of an m-uniform hypergraph H refers to a function $c : V(H) \to \{1, \ldots, k\}$ such that any two vertices in the same hyperedge have different colors and for any subsets of colors $\{1, \ldots, k\}$ with size m, there is at least one hyperedge such that those colors appear on its vertices. The parameters $\chi(H)$ and $\psi(H)$ can be similarly defined for hypergraphs.

Recently, Edwards and Rzążewski (2020) [10] investigated complete coloring of graphs and hypergraphs (they also defined another extention of complete coloring). In addition, they showed that some basic properties of complete colorings of graphs do not carry over to the case of hypergraphs. In particular, the interpolation property fails for hypergraphs accoring to the following theorem.

Theorem 0.13. ([10]) *Let k be a positive integer with $k \geq 9$. There exists a k-uniform hypergraph H which has a complete $\chi(H)$-coloring, and a complete $\psi(H)$-coloring, but no complete t-coloring for some t with $\chi(H) < t < \psi(H)$.*

Moreover, they formulated the following two problems for generalizing Theorem 0.13 to 3-uniform hypergraphs, and for studying a weaker version of the interpolation property of complete colorings of hypergraphs.

Problem 0.14. *(Edwards and Rzążewski (2020) [10]) Does there exist a 3-uniform hypergraph not satisfying the interpolation property?*

Problem 0.15. *(Edwards and Rzążewski (2020) [10]) Does there exist a uniform hypergraph H with $\psi(H) \geq \chi(H) + 2$ such that H has a complete $\chi(H)$-coloring and a complete $\psi(H)$-coloring, but no complete t-coloring for any t with $\chi(H) < t < \psi(H)$?*

In this thesis, we generalize Theorem 0.13 to all integers k with $k \geq 3$ by modifying some parts of their proof. In Section 2.3, we answer Problem 0.15 positively by giving several kinds of 3-uniform hypergraphs, which consequently shows that the answer of Problem 0.14 is positive. In particular, we formulate the following stronger assertion.

Theorem 0.16. *There exists a* 3*-uniform hypergraph* H *with* $\psi(H) \geq \chi(H)+3$ *such that* H *has a complete* $\chi(H)$*-coloring and a complete* $\psi(H)$*-coloring, but no complete* t*-coloring for any* t *satisfying* $\chi(H) < t < \psi(H)$.

There is another concept that enables us to make hypergraphs from planar graphs. A *face hypergraph* refers to a hypergraph obtained from a planar graph G whose vertices are the same as G and there is a one-to-one correspondence between the faces of G and hyperedges of H such that each hyperedge of H consists of all vertices of its corresponding face. This concept was introduced by Kündgen and Ramamurthi [19]. In this thesis, we deal with planar triangulations, planar graphs whose faces are triangular, and our face hypergraphs are obtained from planar triangulations.

Recently, Matsumoto and Ohno (2020) [20] investigated complete colorings for a special family of face hypergraphs using terms of facial complete colorings of planar triangulations. They posed the following problem in their paper to study the interpolation property of those hypergraphs. (For convenience, we write Problem 0.17 in terms of hypergraphs which is equivalent to Problem 5 in [20].) They also remarked that the answer is positive, if one replaces the weaker condition $4 \leq t < \psi(H)$. Recall that by the Four-Color Theorem, every planar triangulation is 4-colorable [3].

Problem 0.17. (Matsumoto and Ohno (2020) [20]) *Does there exist a* 3*-uniform face hypergraph* H*, obtained from a planar triangulation, such that* H *has a complete* $\psi(H)$*-coloring, but no complete* t*-coloring for some* t *satisfying* $\chi(H) \leq 4 < t < \psi(H)$*?*

Moreover, they put forward the following conjecture to suggest a family of hypergraphs satisfying the interpolation property.

Conjecture 0.18. (Matsumoto and Ohno (2020) [20]) *Let* H *be a* 3*-uniform face hypergraph obtained from a planar triangulation. If* H *is* 3*-colorable, then it admits a complete* t*-coloring for every* t *with* $\chi(H) \leq t \leq \psi(H)$.

In this thesis, we disprove Conjecture 0.18 by a particular hypergraph of order 12, which consequently shows that the answer of Problem 0.17 is positive. It is known that a planar triangulation is 3-colorable if and only if the degree of every vertex is even [24].

Theorem 0.19. *There exists a* 3*-uniform* 3*-colorable face hypergraph of order* 12*, obtained from a planar triangulation, having a complete* 6*-coloring but with no complete* 5*-coloring.*

They also gave a sufficient condition for such a 3-colorable face hypergraph to admit a complete k-coloring as the following theorm. Here, we denote by $\alpha'(H)$ is the maximum number of vertex disjoint hyperedges of H.

Theorem 0.20. (Matsumoto and Ohno (2020) [20]) *Let H be a 3-uniform face hypergraph obtained from a planar triangulation. If H is 3-colorable and $\alpha'(H) \geq 4\binom{k}{3}$, then H admits a complete k-coloring.*

In addition, they characterized such hypergraphs with the achromatic number exactly 3, and posed the following related problem for the existence of a complete 4-coloring. Moreover, they showed that the lower bound cannot be replaced by $\frac{1}{4}(|V(H)|+1)$ by giving an infinite family of examples.

Problem 0.21. (Matsumoto and Ohno (2020) [20]) *Does there exist a real number m with $3 \leq m < 4$ such that if H is a 3-uniform face hypergraph obtained from a planar triangulation with the chromatic number 4 satisfying $\alpha'(H) \geq \frac{1}{m}|V(H)|$, then H has a complete 4-coloring.*

In this thesis, we answer their problem negatively, as the following theorem, by showing that the lower bound cannot also be replaced by $\frac{1}{3}|V(H)|$ that is the largest bound which can be examined.

Theorem 0.22. *There are infinitely many 3-uniform face hypergraphs H obtained from planar triangulations with the chromatic number 4 satisfying $\alpha'(H) \geq \frac{1}{3}|V(H)|$ while H has no complete 4-coloring.*

Chapter 1

3-Distance Coloring of Planar Graphs

In 2018, Thomassen showed that every subcubic planar graph has the 2-distance chromatic number at most 7, which was originally conjectured by Wegner (1977). In this chapter, we consider 3-distance coloring of this family of graphs, and prove that every subcubic planar graph has the 3-distance chromatic number at most 17 and conjecture that this number can be pushed down to 12. In addition, we show that every planar graph G with maximum degree at most Δ has the 3-distance chromatic number at most $(6 + o(1))\Delta$.

1.1 Introduction

In this chapter, all graphs are considered to be simple (without loops and multiple edges). The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The degree $d_G(v)$ of a vertex v refers to the number of edges incident with v. A (proper) coloring of a graph G refers to a function $c : V(G) \to \mathbb{Z}$ such that any two adjacent vertices receive different colors. The minimum number of needed colors is called the *chromatic number* $\chi(G)$ of G. For a positive integer k, the *k-th power* G^k of G is the graph obtained from G by connecting any two vertices with distance at most k. We write $\chi_k(G)$ for the chromatic number of G^k, which is said to be the *k-distance chromatic number* of G. A graph is called *planar*, if it can be embedded on the plane. A graph embedded on the plane is a *plane graph*. Two faces of a plane graph are said to be *adjacent*, if they share a common edge. For a face f, we denote by $d(f)$ the length of its boundary walk. A face f is said to be k^+-face (resp. k^--face) if $d(f) \geq k$ (resp. $d(f) \leq k$). A graph G is called *essentially k-edge-connected*, if $d_G(X) \geq k$, for every vertex set X satisfying $2 \leq |X| \leq |V(G)| - 2$, where $d_G(X)$ denotes the number of edges with exactly one end in X. For a bipartite graph G with bipartition (X, Y), the *bipartite complement* of G refers to the graph obtained from the complete bipartite graph with the bipartition (X, Y) by deleting the edges of G. For a family of graphs $\mathcal{A}$, we say that G has an *$\mathcal{A}$-minor*, if there are vertex-disjoint connected subgraphs whose contraction results in a simple graph in $\mathcal{A}$ after deleting some edges and some isolated vertices. A graph is called *subcubic*, if it has maximum degree at most three.

It is known that every planar graph is 4-colorable [3]. For the 2-distance chromatic number of planar graphs, we cannot impose a fixed upper bound, because $\chi_2(G) \geq \Delta(G)+1$. In 1977, Wegner [25] proved that $\chi_2(G) \leq 8$ for subcubic planar graphs G. He also conjectured that the upper bound can be reduced to 7. Recently, this conjecture has been independently confirmed by Thomassen (2018) [23] and Hartke, Jahanbekam, Thomas (2016) [13]. We should notice that the latter used a computer assistance.

Theorem 1.1. *For all subcubic planar graphs G, we have $\chi_2(G) \leq 7$.*

For the 2-distance chromatic number of all planar graphs, Wegner [25] proposed the following sharp conjecture which has been studied by many authors, see [18, Page 3]. The currently best known bound is $(\frac{3}{2} + o(1))\Delta(G)$, see [2, 14].

Conjecture 1.2. ([25]) *If G is a planar graph and $\Delta(G) \geq 8$, then $\chi_2(G) \leq \frac{3}{2}\Delta(G) + 1$.*

Cranston and Kim (2008) [7] already showed that for all subcubic graphs G, except the Petersen graph, $\chi_2(G) \leq 8$. For 3-distance coloring, Bonamy and Bousquet (2014) [4] showed that $\chi_3(G) \leq 20$ for all subcubic graphs G. Moreover, they conjectured that the upper bound can be improved by one, except for finitely many graphs.

Conjecture 1.3. ([4]) *For all subcubic connected graphs G, except finitely many, we have $\chi_3(G) \leq 19$.*

In Section 1.2, we study 3-distance coloring of subcubic planar graphs by proving the following theorem. In addition, we conjecture that this upper bound can be reduced to 12, which is sharp if it is true.

Theorem 1.4. *For all subcubic planar graphs G, we have $\chi_3(G) \leq 17$.*

For the 3-distance chromatic number of all planar graphs, the best known upper bound is obtained by Agnarsson and Halldórsson (2003) [1, Corollary 3.6] who showed that if G is a planar graph, then $\chi_3(G)$ is less than $144\Delta(G)$. In Section 1.3, we improve their result by proving the following theorem.

Theorem 1.5. *For all planar graphs G, we have $\chi_3(G) \leq (6 + o(1))\Delta(G)$.*

This result is much closer to the best known lower bound in the sense that there are planar graphs G with 3-distance chromatic number $4\Delta(G)$ minus a constant. Note also that for the 4-distance chromatic number, the upper bound cannot be linear, because $\chi_4(G) \geq \Delta(G)^2 + 1$ when G has a tree containing a vertex v such that itself and all its neighbours have degree $\Delta(G)$.

1.2 Subcubic planar graphs

In this section, we are going to give an upper bound on the 3-distance chromatic number of subcubic planar graphs. Before doing so, let us state the following assertion that is useful to find coloring of the whole graph from colorings of some subgraphs of the graph.

Proposition 1.6. *Let G be a graph, and let X, Y be the vertex subsets with $X \cap Y = \emptyset$ and $X \cup Y = V(G)$. Suppose that the induced subgraphs $G[X]$ and $G[Y]$ have k-colorings $c : X \to A$ and $c' : Y \to B$, where A and B are two sets of colors with size k, respectively. Let P be the bipartite graph with bipartition (A, B) such that*

$$ab \in E(P),$$

if and only if there are two vertices $v \in X$ and $u \in Y$ such that $uv \in E(G)$, $c(v) = a$, and $c'(u) = b$. Then G itself has a k-coloring if the bipartite complement of P admits a perfect matching.

Proof. Let M be a perfect matching of the bipartite complement of P. We shall extend coloring of $G[X]$ to all vertices of G by permuting colors of vertices in Y. More precisely, for every $v \in X$, we define $c''(v) = c(v)$, and for every $u \in Y$, we define $c''(u) = a$, where

a is the unique vertex adjacent to $b = c'(u)$ in M. Note that for any two adjacent vertices $v \in X$ and $u \in Y$ satisfying $c(v) = a$ and $c'(u) = b$, we must have $ab \notin E(M)$ and so $c''(v) = a \neq c''(u)$. Therefore, it is easy to see that c'' is a k-coloring G. $\square$

We need also the following well-known lemma in our proof.

Lemma 1.7. (Hall [11]) *Let G be a bipartite graph with bipartition (A, B). Then G admits a matching covering all vertices of A if and only if for every $S \subseteq A$, $|N_G(S)| \geq |S|$.*

For planar graphs, there is a simple formula relating order, size, and the number of faces. It was first discovered by Euler (1752).

Theorem 1.8. (Euler's Formula) *If G is a connected graph embedded on the plane, then*

$$|V(G)| - |E(G)| + |F(G)| = 2.$$

One the other hand, it is obvious that summation of degree faces and summation of degree vertices are the same. By a combination of these two equalities, one can easily derive the following corollary about cubic planar graphs which be used in our proof. Roughly speaking, average degree faces is around six and also there must be some faces with degree at most five.

Corollary 1.9. *If G is a connected cubic graph embedded on the plane, then* $\sum_{f \in F(G)}(d(f) - 6) = -12$.

Proof. Since G is cubic, we have $|E(G)| = \frac{3}{2}|V(G)|$. On the other hand, $3|V(G)| = \sum_{v \in V(G)} d_G(v) = \sum_{f \in F(G)} d(f)$. Thus $|V(G)| - |E(G)| + |F(G)| = -\frac{1}{2}|V(G)| + |F(G)| = -\frac{1}{6}\sum_{f \in F(G)} d(f) + \sum_{f \in F(G)} 1$. Thus the assertion can be proved by Theorem 1.8. $\square$

Now, we are ready to prove the main result of this chapter.

Proof. The proof is based on discharging method. Let G be a counterexample with the minimum $|V(G)|$. It is easy to see that G is connected. Let $H = G^3$. We are going to prove the following claims.

Claim 1. The graph G is cubic and triangle-free.

Suppose that G has a vertex x with degree at most two. By the minimality, if $d_G(x) = 1$ then the third power of the graph $G - x$ must have a 17-coloring. If $d_G(x) = 2$, then the third power of the graph $G-x+e$ must have a 17-coloring, where e is an edge joining two neighbours of x in G. Since $d_H(x) \leq 14$, we can extend it to a 17-coloring of H, which is a contradiction.

Now, suppose that G has a triangle xyz. We first contract this triangle into a new vertex v to obtain a new subcubic planar graph G'. By the minimality, the third power of the graph G' must have a 17-coloring, so we can extend it to a 17-coloring of H, because the degrees of the vertices x, y, and z are at most 15 in H, which is again a contradiction. □

Claim 2. The graph G is essentially 4-edge-connected.

Suppose, to the contrary, that G has a vertex set X with $2 \leq |X| \leq |V(G)| - 2$ such that $d_G(X) \leq 3$. Let G_x and G_y be the subcubic planar graphs obtained from G by contracting Y and X, respectively, where $Y = V(G) \setminus X$. Note that we delete multiple edges (if necessary). We denote by x and y two vertices corresponding to X and Y in the new graphs. Note that $x \in V(G_y)$ and $y \in V(G_x)$. By the minimality of G, the third power of the contacted graphs G_x and G_y have 17-colorings $c : V(G_x) \to A$ and $c' : V(G_y) \to B$, where A and B are two sets of colors with size 17, respectively.

Let P be a bipartite graph with bipartition (A, B) such that $ab \in E(P)$ if and only if there are two adjacent vertices $v \in X$ and $u \in Y$ in H satisfying $c(v) = a$ and $c'(u) = b$. It is not difficult to check that P has maximum degree at most 9. In addition, for all but possibly six colors in A are adjacent to at most 6 colors of B in the graph P. Those exceptional colors must appear on the vertices of X_1 or appear on at least two vertices of X_2, where X_i denotes the set of all vertices in X with distance exactly i from y in G_x. Note that $|X_1| \leq 3$ and $|X_2| \leq 2|X_1|$.

Let P' be the bipartite complement of P. If the graph P' has a perfect matching, then it follows from Proposition 1.6 that G has a 17-coloring, a contradiciton. Thus by Lemma 1.7, there is a subset S of A satisfying $|N_{P'}(S)| < |S|$. Since P' has minimum degree at least 8, we must have $8 < |S|$. In addition, since all but possibly at most six vertices in S must have degree at least 11, we must have $11 < |S|$. Since a vertex in $B \setminus N_{P'}(S)$ has degree at least 8, we must also have $|A \setminus S| \geq 8$ while $|A \setminus S| + |S| < 17$. This is a contradiction, as desired. □

Now we fix an embedding of G into the plane, and consider G as a plane graph. We have two more claims related to small faces of G.

Claim 3. There is no 4-face adjacent to a 6^--face, and there is no 4-face connected to a 5^--face by an edge.

Suppose that G has a 4-face $v_1 \dots v_4$. By Claim 2, every vertex v_i in the 4-face has a neighbor that is not on the 4-face, and let v_{i+4} be such a neighbor. We may assume that this face has a common edge with a 6^--face or it is connected to a 5^--face by an edge as in Figure 1.1. Let $V = \{v_1, \dots, v_8\}$. By the minimality of G, the graph $(G - \{v_1, \dots, v_4\})^3$ has a 17-coloring. We erase colors on $v_5, \dots, v_8$. For each $v \in V$, we write $L(v)$ to be the set of colors which do not appear in the neighbours of v in H.

In the case when the 4-face has a common edge with a 6^--face, the list sizes of the vertices v_5, v_6, v_7, v_8, v_1, v_2, v_3, and v_4 are at least 4, 3, 3, 4, 8, 7, 7, and 8, respectively. It is easy to check that we can extend the coloring to all vertices of H using the colors from these lists by a greedy algorithm with respect to the order mentioned above.

In the case when the 4-face is connected to a 5^--face by an edge, the list sizes of the vertices v_6, v_8, v_5, v_7, v_1, v_2, v_3, and v_4 are at least 3, 5, 3, 3, 7, 7, 7, and 7, respectively. We may assume that the equalities hold. If there is a path Q of size at most three connecting v_6 and v_8 in G, then Q together with the path v_6, v_2, v_1, v_4, v_8 or v_6, v_2, v_3, v_4, v_8 bounds a disk on the plane, but we see that this contradicts Claim 2 and the first case. Thus, one can conclude that there is no path of size at most three connecting v_6 and v_8 in G and hence $v_6v_8 \notin E(H)$. Thus, we can use the same color to v_6 and v_8. Similarly, we obtain $v_5v_7 \notin E(H)$ and we can use the same color to v_5 and v_7. Since $|L(v_6)| + |L(v_8)| > |L(v_4)|$, there is a color $b \in L(v_6) \cup L(v_8)$ such that $b \in L(v_6) \cap L(v_8)$ or $b \notin L(v_4)$. It is easy to check that, first giving the color b to both v_6 and v_8 in the case $b \in L(v_6) \cap L(v_8)$, and to one of v_6 and v_8 otherwise, we can extend the coloring to all vertices of H using the colors from these lists by a greedy algorithm with respect the order mentioned above. □

Claim 4. There are no two adjacent 5-faces, and there is no 5-face adjacent to two adjacent 6^--faces.

Suppose that G has a 5-face $v_1 \dots v_5$. By Claim 2, every vertex v_i in the 5-face has a neighbor that is not on the 5-face, and let v_{i+5} be such a neighbor. Assume that this face has a common edge with a 5-face or a common edge with two adjacent 6^--faces, as Figure 1.2. Let $V = \{v_1, \dots, v_{10}\}$. By the minimality of G, the graph $(G - \{v_1, \dots, v_5\})^3$

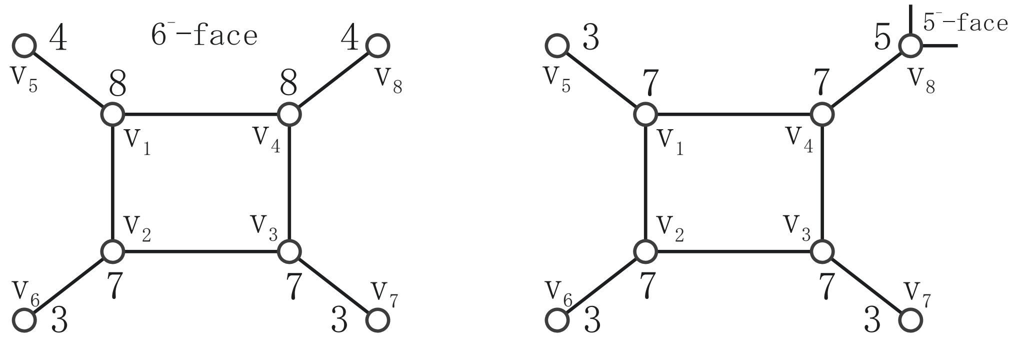

Figure 1.1: A 4-face adjacent to a 6^{-}-face, and a 4-face connected to 5^{-}-face by an edge.

has a 17-coloring. We erase colors on $v_6, \ldots, v_{10}$. For each $v \in V$, we write $L(v)$ to be the set of colors which do not appear in the neighbours of v in H. Lower bounds of the list sizes are illustrated in Figure 1.2; we may assume that the equality holds. Let C be the 5-cycle $C = v_6v_9v_7v_{10}v_8$. By the same reason as in the proof of Claim 3, we see that for any two vertices $v, u \in V(C)$ with $vu \in E(C)$, there exists no path of size at most three connecting v and u in G. Thus, the induced subgraph $H[V]$ must be a complete graph minus the edges of C.

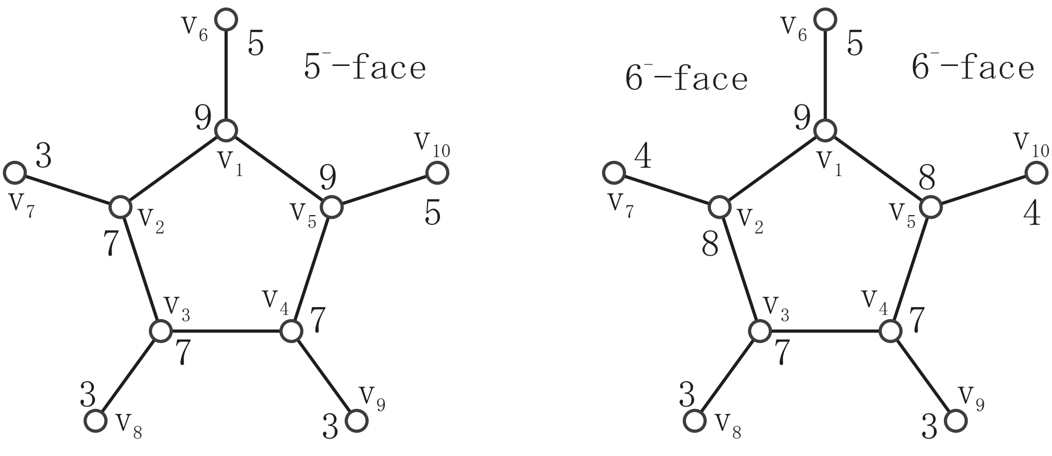

Figure 1.2: A 5-face adjacent to a 5-face, and a 5-face adjacent to two 6^{-}-faces.

Suppose that there is a color $b \in L(u) \cap L(u')$ such that $u, u' \in V(C)$, $u \neq u'$, and $uu' \in E(C)$. Then, we can first color both of vertices u and u' by the same color b and next start our greedy algorithm for the remaining vertices with respect to the order v_9, v_8, v_7, v_{10}, v_6, v_3, v_4, v_2, v_5, v_1. We may therefore assume that every color appears in at most two lists of the vertices in $V(C)$. Let $L(C)$ to be the union of $L(u)$ over all $u \in V(C)$.

Since

$$9.5 = \frac{1}{2}\sum_{i=6}^{10} |L(u_i)| \leq |L(C)|,$$

there are at least 10 colors in $L(C)$.

Now we construct the bipartite graph P such that one bipartition is V, the other bipartition is the set of colors, and a vertex $u \in V$ is connected by an edge to the color i if and only if $i \in L(u)$. If there is a matching M of P that covers all vertices in V, then coloring each vertex u in V with the color that is matched by M to u, all vertices in V receive distinct colors in the list, and we obtain a 17-coloring of H. Thus, we may assume that P does not have such a matching. By Lemma 1.7, there is a subset S of V satisfying $|N_P(S)| < |S|$. If S contains all vertices in C, then it follows from the observation in the previous paragraph that $|N_P(S)| \geq |N_P(V(C))| \geq 10 \geq |S|$, a contradiction. Thus, we may assume that $V(C) \setminus S \neq \emptyset$, and hence $|S| \leq 9$.

First, we consider the case when the 5-face has a common edge with a 5-face. Since $|N_P(u)| = |L(u)| \geq 3$ for each $u \in V$, we see that $3 < |S|$. In addition, since there are only three vertices u in V with $|N_P(u)| = |L(u)| = 3$ and all other vertices u' satisfy $|N_P(u')| = |L(u')| \geq 5$, we see that $5 < |S|$. Since there are only five vertices u in V with $|N_P(u)| = |L(u)| \leq 5$ and all other vertices u' satisfy $|N_P(u')| = |L(u')| \geq 7$, we see that $7 < |S|$. Since $V(C) \setminus S \neq \emptyset$, S must contain at least one of v_1 and v_5, which shows $9 \leq |N_P(S)| < |S| \leq 9$, a contradiction.

Secondly, we consider the other case. Since $|N_P(u)| = |L(u)| \geq 3$ for each $u \in V$, we see that $3 < |S|$. Since there are only two vertices u in V with $|N_P(u)| = |L(u)| = 3$ and all other vertices u' satisfy $|N_P(u')| = |L(u')| \geq 4$, we see that $4 < |S|$. Since $V(C) \setminus S \neq \emptyset$, S must contain at least one vertex in $V \setminus V(C)$, which shows $7 \leq |N_P(S)| < |S|$. This implies that S contains at least one of v_1, v_2 and v_5, and hence $8 < |S|$. Then, S contains v_1, v_2 and v_5, and hence $9 \leq |N_P(S)| < |S| \leq 9$, a contradiction. This completes the proof of Claim 4. □

By Corollary 1.9, we have

$$\sum_{f \in F(G)} (d(f) - 6) = -12,$$

where $F(G)$ is the set of all faces. Let us consider the following discharging procedure: Every 7^+-face will send charge 1/3 to every adjacent 5-face and also will send charge 1/2 to every adjacent 4-face. We are going to show that the charge of every face would be nonnegative which is a contradiction. By Claim 3, every 4-face is adjacent to at least four 7^+-faces and so it receives at least charge $4 \times 1/2 = 2$. By Claim 4, every 5-face is adjacent to at least three 7^+-faces and so it receives at least charge $3 \times 1/3 = 1$. Let f be a 7^+-face and let c_i be the number of i-faces adjacent to it for $i \in \{4, 5\}$. Recall that the face f will send charge $c_4/2 + c_5/3$ to other faces. By Claims 3 and 4, one can conclude that $d(f) \geq 3c_4 + 2c_5$. Thus, if $d(f) \geq 8$, then $d(f) - 6 \geq \frac{1}{6}d(f) \geq c_4/2 + c_5/3$. Even in the case $d(f) = 7$, since either $c_4 = 2$ and $c_5 = 0$, or $c_4 = 1$ and $c_5 \leq 1$, or $c_4 = 0$ and $c_5 \leq 3$, we therefore have $d(f) - 6 \geq c_4/2 + c_5/3$. This confirms our claim and completes the proof. □

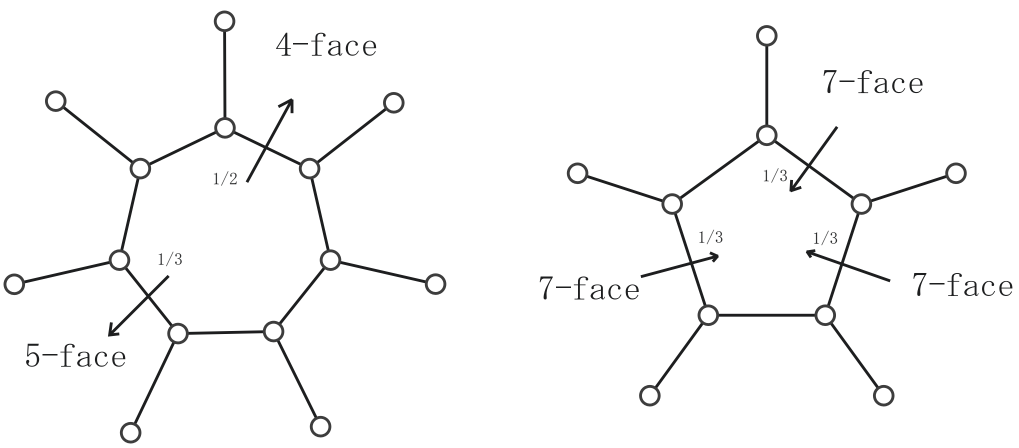

Figure 1.3: Examples of charging and discharging procedures.

It would be an interesting problem to determine the sharp upper bound in Theorem 1.4. We believe the following conjecture would be true. To see the sharpness, we can consider the graph obtained from the Cartesian product of the 5-cycle and the path of size 2 by subdividing two edges which are contained in a common 4-cycle but no 5-cycles. Indeed, the third power of this graph is the complete graph of order 12.

Conjecture 1.10. *If G is a planar subcubic graph, then $\chi_3(G) \leq 12$.*

In our proof, we used discharging method along with some simple configurations. We believe by considering larger configurations, it would be possible to push the upper bound further down. So we leave it for the interested readers.

1.3 Graphs with bounded maximum degree

It would be an interesting problem to establish upper bounds on the 3-distance chromatic number of planar graphs with higher maximum degree. For the square chromatic number, Wegner (1977) [25] conjectured that if G is a planar graph with $\Delta(G) \geq 8$, then $\chi_2(G) \leq \frac{3}{2}(\Delta(G)) + 1$. In the following, theorem, we establish an upper bound $\chi_3(G)$, which is linearly depending on $\chi_2(G)$ for graphs on surfaces.

Lemma 1.11. *For every integer g, there is a positive integer c_g such that if G is a graph embedded on a surface with genus g, then $\chi_{2k+1}(G) \leq c_g \chi_{2k}(G)$. In particular, if G is planar, then*

$$\chi_{2k+1}(G) \leq 4\chi_{2k}(G).$$

In addition, if every $\mathcal{A}$-minor free graph has chromatic number at most $c_{\mathcal{A}}$ for a family $\mathcal{A}$ of graphs, then for every $\mathcal{A}$-minor free graph G we have $\chi_{2k+1}(G) \leq c_{\mathcal{A}}\chi_{2k}(G)$.

Proof. It is known that there is a positive integer c_g such that every graph which can be embedded on a surface with genus g is c_g-colorable [15]. Note that $c_0 = 4$ by the Four-Color Theorem [3]. Let $c : V(G) \to \mathbb{Z}$ be a coloring of the graph G^{2k} using $\chi_{2k}(G)$ colors. For every color i, we let X_i be the set of all vertices having the same color i, where $1 \leq i \leq \chi_{2k}(G)$. Let G_0 be the graph obtained from G by connecting any two vertices with distance exactly $2k+1$. We claim that $G_0[X_i]$ is embeddable on the same surface of G. Let P be a path of size $2k+1$ with end vertices v and u in X_i. Assume that that P has a common middle vertex x with another path P' of size $2k+1$ with end vertices v' and u' in X_i. We may assume that x is closer to u in P and also is closer to u' on P'. These imply that the distance of u and v is at most $2k$, and so they must have different colors whenever $u \neq u'$. Therefore, the edges corresponding to P in $G_0[X_i]$ can be drawn on the surface without crossing other edges. (Note that the arguments stated above also show that if G is $\mathcal{A}$-minor free, then $G_0[X_i]$ should be $\mathcal{A}$-minor free). Thus every graph $G_0[X_i]$ is c_g-colorable and so the graph G^{2k+1} must be $c_g\chi_{2k}(G)$-colorable. Hence the proof is completed. □

Havet, van den Heuvel, McDiarmid and Reed (2007) [14] and Amini, Esperet, and Heuvel (2013) [2] showed that the square chromatic number of every planar graph G is at most $\left(\frac{3}{2} + o(1)\right)\Delta(G)$. By applying this result together with Theorem 1.11, one can conclude Theorem 1.5.

Lemma 1.12. *(Restated) For all planar graphs G, we have $\chi_3(G) \leq (6 + o(1))\Delta(G)$.*

Note that there are planar graphs with the maximum degree Δ and 3-distance chromatic number 4Δ minus a constant. For example, consider the complete graph of order 4 and attach $\Delta - 3$ pendant vertices to each vertex.

Chapter 2

Complete Coloring of Hypergraphs

In 1967 Harary, Hedetniemi, and Prins showed that every graph G admits a complete t-coloring for every t with $\chi(G) \leq t \leq \psi(G)$, where $\chi(G)$ denotes the chromatic number of G and $\psi(G)$ denotes the achromatic number of G which is the maximum number r for which G admits a complete r-coloring. Recently, Edwards and Rzążewski (2020) showed that this result fails for hypergraphs by proving that for every integer k with $k \geq 9$, there exists a k-uniform hypergraph H with a complete $\chi(H)$-coloring and a complete $\psi(H)$-coloring, but no complete t-coloring for some t with $\chi(H) < t < \psi(H)$. They also asked whether there would exist such an example for 3-uniform hypergraphs and posed another problem to strengthen their result. In this chapter, we generalize their result to all cases k with $k \geq 3$ and settle their problems by giving several examples of 3-uniform hypergraphs. In particular, we disprove a recent conjecture due to Matsumoto and Ohno (2020) who suggested a special family of 3-uniform hypergraph to satisfy the desired interpolation property.

2.1 Introduction

In this chapter, all hypergraphs are considered simple. Let H be a hypergraph. The vertex set and the hyperedge set of H are denoted by $V(H)$ and $E(H)$, respectively. A vertex subset of $V(H)$ is said to be *independent*, if there is no hyperedge of H including two different vertices of it. A hypergraph is said to be *k-uniform*, if its hyperedges all have the same size k. We say that a vertex set S *covers* a hyperedge e, if S includes at least one vertex of e. A *face hypergraph* refers to a hypergraph obtained from a graph G embedded in some surface whose vertices are the same vertices of G and there is a one-to-one correspondence between the faces of G and hyperedges of H such that each hyperedge of H consists of all vertices of its corresponding face. This concept was introduced by Kündgen and Ramamurthi [19]. The minimum number of colors needed to color the vertices of H such that any two vertices lying in the same hyperedge have different colors (proper property) is denoted by $\chi(H)$. A *complete t-coloring* of a k-uniform hypergraph H is a coloring of its vertices, using t colors, such that any two vertices lying in the same hyperedge have different colors, and also every set of k different colors appears in at least one hyperedge. Note that a uniform hypergraph may have not a complete coloring, see [9]. For a hypergraph H, we denote by $\psi(H)$ the largest integer t such that H has a complete t-coloring, if such t exists. Otherwise, we define $\psi(H) = 0$. The numbers $\chi(H)$ and $\psi(H)$ are called the *chromatic number* and the *achromatic number* of H, respectively. It was proved in [10, 20] that a given uniform hypergraph H may have not a complete $\chi(H)$-coloring even if it admits a complete coloring. We say that a hypergraph H satisfies *interpolation property*, if it admits a complete t-coloring for every integer t with $\chi(H) \leq s < t \leq \psi(H)$, provided that H has a complete s-coloring.

In 1967 Harary, Hedetniemi, and Prins studied the interpolation property for complete colorings of graphs and established the following result.

Theorem 2.1. ([12]) *Every graph G admits a complete t-coloring for every t with $\chi(G) \leq t \leq \psi(G)$.*

Recently, Edwards and Rzążewski (2020) showed that Theorem 2.1 cannot be extended to k-uniform hypergraphs for all integers k with $k \geq 9$.

Theorem 2.2. ([10]) *Let k be a positive integer with $k \geq 9$. There exists a k-uniform hypergraph H which has a complete $\chi(H)$-coloring, and a complete $\psi(H)$-coloring, but no complete coloring for some t with $\chi(H) < t < \psi(H)$.*

In addition, they formulated the following two problems for generalizing Theorem 2.2 to 3-uniform hypergraphs, and for studying a weaker version of the interpolation property of complete colorings of hypergraphs.

Problem 2.3. (Edwards and Rzążewski (2020) [10]) *Does there exist a 3-uniform hypergraph not satisfying the interpolation property?*

Problem 2.4. (Edwards and Rzążewski (2020) [10]) *Does there exist a uniform hypergraph H with $\psi(H) \geq \chi(H) + 2$ such that H has a complete $\chi(H)$-coloring and a complete $\psi(H)$-coloring, but no complete t-coloring for any t satisfying $\chi(H) < t < \psi(H)$?*

In this chapter, we generalize Theorem 2.2 to all cases k with $k \geq 3$ by modifying some parts of their proof. In Section 2.3, we answer Problem 2.4 positively by giving several kinds of 3-uniform hypergraphs, which consequently shows that the answer of Problem 2.3 is positive. In particular, we formulate the following stronger assertion.

Theorem 2.5. *There exists a 3-uniform hypergraph H with $\psi(H) \geq \chi(H) + 3$ such that H has a complete $\chi(H)$-coloring and a complete $\psi(H)$-coloring, but no complete t-coloring for any t satisfying $\chi(H) < t < \psi(H)$.*

Recently, Matsumoto and Ohno (2020) [20] investigated complete colorings for a special family of face hypergraphs using terms of facial complete colorings of planar triangulations. They posed the following problem in their paper to study the interpolation property of those hypergraphs. (For convenience, we write Problem 2.6 in terms of hypergraphs which is equivalent to Problem 5 in [20].) They also remarked that the answer is positive, if one replaces the weaker condition $4 \leq t < \psi(H)$. It is known that every planar triangulation is 4-colorable [3].

Problem 2.6. (Matsumoto and Ohno (2020) [20]) *Does there exist a 3-uniform face hypergraph H, obtained from a planar triangulation, such that H has a complete $\psi(H)$-coloring, but no complete t-coloring for some t satisfying $\chi(H) \leq 4 < t < \psi(H)$?*

Moreover, they put forward the following conjecture to suggest a family of hypergraphs satisfying the interpolation property. In Section 2.4, we disprove Conjecture 2.7 by a particular hypergraph of order 12, which consequently shows that the answer of Problem 2.6 is positive. It is known that a planar triangulation is 3-colorable if and only if the degree of every vertex is even [24].

Conjecture 2.7. (Matsumoto and Ohno (2020) [20]) *Let H be a 3-uniform face hypergraph obtained from a planar triangulation. If H is 3-colorable, then it admits a complete t-coloring for every t with $\chi(H) \leq t \leq \psi(H)$.*

2.2 The existence of uniform hypergraphs for which the interpolation property fails

The following theorem makes a stronger version for Theorem 2.2.

Theorem 2.8. *Let k be a positive integer with $k \geq 3$. There exists a k-uniform hypergraph H which has a complete $\chi(H)$-coloring and a complete $\psi(H)$-coloring, but no complete t-coloring for some t with $\chi(H) < t < \psi(H)$.*

Proof. We may assume that $k \geq 4$, as the assertion holds for $k = 3$ by Theorem 2.9. Let r be a sufficiently large integer compared to k. Define H to be the k-uniform hypergraph with $V(H) = \{v_{i,j} : 1 \leq i \leq k,\ 1 \leq j \leq r\}$ and $E(H) = E_1 \cup E_2$ such that

$$E_1 = \{\{v_{i,p_i} : 1 \leq i \leq k\} : (p_1, \ldots, p_k) \in \mathcal{A} \text{ and } f(p_1, \ldots, p_k) \leq 1\}, \text{ and}$$

$$E_2 = \{\{v_{i,p_i} : 1 \leq i \leq k\} : (p_1, \ldots, p_k) \in \mathcal{A} \text{ and } p_1 < \cdots < p_k\},$$

where $\mathcal{A}$ denotes the set of all sequences $(p_1, \ldots, p_k)$ such that all p_i are distinct and $1 \leq p_i \leq r$ and $f(p_1, \ldots, p_k) = |\{(i,j) : |p_i - p_j| = 1 \text{ and } 1 \leq i < j \leq k\}|$. We call the i-th part of H as the set of all vertices $v_{i,j}$ with $1 \leq j \leq r$, and call the j-th position of H as the set of all vertices $v_{i,j}$ with $1 \leq i \leq k$. According to this construction, one can prove the following three assertions:

(a1) There is no hyperedge including two vertices of the same position.

(a2) There is no hyperedge including two vertices of the same part.

(a3) For any two vertices in different parts and different positions, there is a hyperedge including them.

We prove only the last assertion as the other ones are obvious. Let $v_{i,j}$ and $v_{i',j'}$ be two arbitrary vertices of H in different parts and different positions so that $i \neq i'$ and $j \neq j'$. Since r is large enough, there is an integer s with $1 \leq s \leq r$ such that $\{j, j'\} \cap \{s, \ldots, s + 2k + 2\} = \emptyset$. Consider the sequence $(p_1, \ldots, p_k)$ satisfying $p_i = j$, $p_{i'} = j'$, and $p_t = s + 2t$ for every $t \in \{1, \ldots, k\} \setminus \{i, i'\}$. Obviously, this sequence is in $\mathcal{A}$ and $f(p_1, \ldots, p_k) \leq 1$. Thus the hyperedge corresponding to this sequence must be in E_1. Note that this hyperedge includes both of $v_{i,j}$ and $v_{i',j'}$. Hence the claim holds.

To show that this hypergraph has a complete k-coloring, we take color set $\{c_1, c_2, \ldots, c_k\}$ and for each i with $1 \le i \le k$, we color all vertices in the i-th part with the color c_i. By (a_2) this is a proper coloring and each hyperedge contains all k colors. For complete r-coloring, we take a color set $\{c_1, c_2, \ldots, c_r\}$ and for each j with $1 \le j \le r$, we color all vertices in the j-th position with the color c_j. According to (a_1), it is a proper coloring. In addition, if $\{c_{p_1}, c_{p_2}, \ldots, c_{p_k}\}$ is a k-subset of $\{c_1, c_2, \ldots, c_r\}$ with $p_1 < \cdots < p_k$, then the hyperedge $\{v_{1,p_1}, v_{2,p_2}, \ldots, v_{k,p_k}\}$ of E_2 contains this color set. Therefore, $\chi(H) = k$ and $\psi(H) \ge r$.

Now, we show that H has no complete t-coloring for every integer t with $\frac{k-2}{k-1}r + k + 1 \le t < r$. Suppose, to the contrary, that H has a complete t-coloring using colors $c_1, \ldots, c_t$. Define X to be the set of colors appearing in at least two parts and define Y to be the set of colors appearing in only one part. We are going to prove the following two assertions:

(b1) Each color of X appears in only one position and all vertices of this position are colored only by this color.

(b2) Each part has only one color from Y so that $|Y| = k$ and $|X| = t - k$.

Consider a color $x \in X$. If $x \in X$ occurred in more than one position, then by the definition of X, there must be two vertices having the same color x with different parts and different positions. Thus by (a_3) there is a hyperedge including both of them. This shows that the coloring is not proper, a contradiction. Thus all occurrences of x are in the same position. Now, since $|X| < r$, there is one position whose colors are not in X. In other words, there are k vertices with different parts whose colors are in Y. On the other hand, each part contains at most one color of Y; otherwise, if two colors of Y are in the same part, then by (a2) there is no hyperedge including them which is impossible. Therefore, $|Y| = k$ and $|X| = t - k$. Consequently, we can define y_i to be the unique color in Y appearing in the i-th part, where $1 \le i \le k$. Assume that the color $x \in X$ appears in the j-th position. We are going to show that all vertices of this position are colored by this color. If we consider a given arbitrary vertex $v_{i,j}$ of this position, then there is a hyperedge of H containing all colors of the set $\{y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_k\}$. Let $(p_1, \ldots, p_k) \in \mathcal{A}$ be the sequence corresponding to this hyperedge. Obviously, the color of v_{t,p_t} must be y_t for every $t \in \{1, \ldots, k\}$ with $t \ne i$. Thus the color x must appear in the

i-th part, and so the vertex $v_{i,j}$ must be colored with x. Therefore, all of vertices of the j-th position are colored with the color x. Hence the assertions hold.

Obviously, there are $r - |X|$ positions not colored by the colors of X. Since $r - |X| \leq r/(k-1) - 1$, we can conclude that there are $k-1$ consecutive positions $\{s, s+1, \ldots, s+k-2\}$ of H colored only with colors of X. Define Z to be the set of all those $k-1$ colors along with the color y_2. By the assumption, there is a hyperedge $e \in E(H)$ including all colors of Z. Let $(p_1, \ldots, p_k) \in \mathcal{A}$ be the sequence corresponding to this hyperedge. Obviously, by $(b2)$, the vertex v_{2,p_2} must be colored by y_2. We know that $\{p_1, \ldots, p_k\} \setminus \{p_2\} = \{s, s+1, \ldots, s+k-2\}$. Since $k \geq 4$, there must be three integers $a, b, c \in \{1, \ldots, k\}$ such that $\{p_a, p_b, p_c\} = \{s, s+1, s+2\}$. Thus $f(p_1, \ldots, p_k) \geq 2$ and so $e \notin E_1$. Moreover, according to the situation of the position containing the color y_2, we have either $\max\{p_1, p_3\} < p_2$ or $p_2 < \min\{p_1, p_3\}$ and so $e \notin E_2$. This is a contradiction. Hence the theorem is proved. □

2.3 Solution to Problem 2.4 using 3-uniform hypergraphs

In this section, we are going to answer Problem 2.4 by giving several examples of 3-uniform hypergraphs without the interpolation property. These examples will be introduced by their incidence graphs; the *incidence graph* of a hypergraph H is a bipartite graph G with $V(G) = V(H) \cup E(H)$ in which a vertex $v \in V(H)$ and a hyperedge $e \in E(H)$ are adjacent in G if and only if $v \in e$. In what follows, we represent the vertices of $V(H)$ and $E(H)$ in the figure of incidence graph by black and white vertices, respectively.

2.3.1 A hypergraph of order 9

A positive answer to Problem 2.4 is given in the following theorem.

Theorem 2.9. *There exists a 3-uniform hypergraph H of order 9 with $\psi(H) \geq \chi(H) + 2$ such that H has a complete $\chi(H)$-coloring and a complete $\psi(H)$-coloring, but no complete t-coloring for any t satisfying $\chi(H) < t < \psi(H)$.*

Proof. Let H be the 3-uniform hypergraph of order 9 whose incidence graph is shown in

Figure 2.1 such that its vertices are colored black. If H has a complete k-coloring for $k \geq 6$, then it has at least twenty hyperedges. However, H has exactly ten hyperedges and hence $\psi(H) \leq 5$. In fact, H has a complete 3-coloring and a complete 5-coloring (see Figures 2.1 and 2.2, respectively). Therefore, $\chi(H) = 3$ and $\psi(H) = 5$.

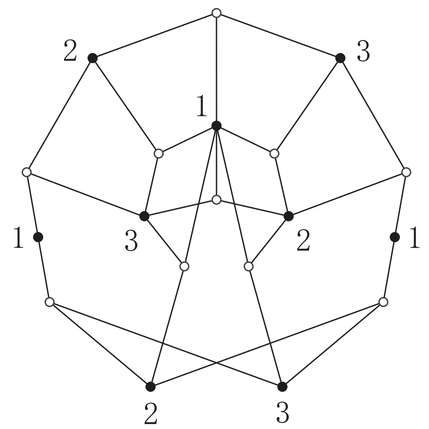

Figure 2.1: A complete 3-coloring of H

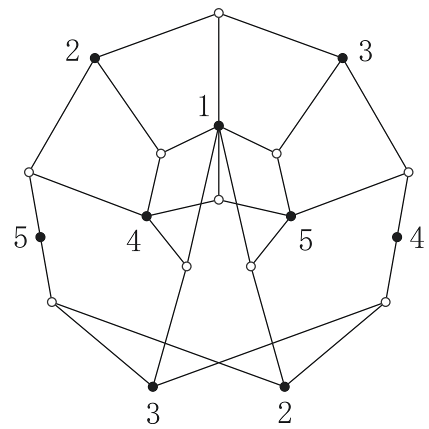

Figure 2.2: A complete 5-coloring of H

Next, we show that H has no complete 4-coloring. Suppose, to the contrary, that H has a complete 4-coloring using colors $c_1, \ldots, c_4$. Since H has nine vertices, there exists at least one color appearing on at least three vertices of H, say color c_1. Note that those vertices with the same color form an independent set. It is easy to check that there are exactly three independent sets of H with size three (which are shown as vertices numbered by $1, 2$ and 3 in Figure 2.1). Since the vertices of every such vertex set cover all hyperedges of H, the triad $\{c_2, c_3, c_4\}$ does not appear on any hyperedge of H. Hence H has no complete 4-coloring and so it is a desired hypergraph. □

2.3.2 A 3-regular 3-uniform hypergraph of order 15

Another positive answer to Problem 2.4 is given in the next theorem.

Theorem 2.10. *There exists a* 3*-uniform* 3*-regular hypergraph* H *of order* 15 *with* $\psi(H) \geq \chi(H) + 2$ *such that* H *has a complete* $\chi(H)$*-coloring and a complete* $\psi(H)$*-coloring, but no complete* t*-coloring for any* t *satisfying* $\chi(H) < t < \psi(H)$*.*

Proof. Let H be the hypergraph with the vertex set $\{v_{i,j} : 1 \leq i \leq 3, 1 \leq j \leq 5\}$ consisting of those hyperedges e_{ij} with $1 \leq i \leq 3$ and $1 \leq j \leq 5$ in which

$$e_{ij} = \{v_{i,j+1}\} \cup \{v_{t,j} : 1 \leq t \leq 3, t \neq i\},$$

where $v_{i,6} = v_{i,1}$. The incidence graph of this hypergraph is shown in Figure 2.3 such that its vertices are colored black. Obviously, H is 3-uniform and 3-regular. If H has a complete k-coloring for $k \geq 6$, then H must have at least twenty hyperedges. However, H has exactly fifteen hyperedges and hence $\psi(H) \leq 5$. In fact, H has a complete 3-coloring and a complete 5-coloring (see Figures 2.3 and 2.4, respectively). Therefore, $\chi(H) = 3$ and $\psi(H) = 5$.

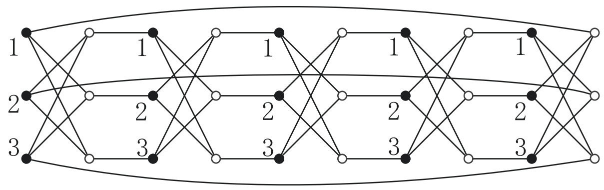

Figure 2.3: A complete 3-coloring of H

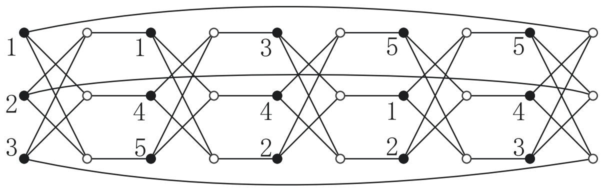

Figure 2.4: A complete 5-coloring of H

Suppose, to the contrary, that H has a complete 4-coloring using colors $c_1, \ldots, c_4$. Since $|V(H)| = 15$, there exists a color appearing on at least four vertices, say c_1. Define $X_i = \{v_{i,j} : 1 \leq j \leq 5\}$ for each i with $1 \leq i \leq 3$. According to the construction of H, it is not difficult to check that every independent set of size four must be a subset of X_1, X_2, or X_3. Hence the color c_1 only appears on vertices of a set X_t where $1 \leq t \leq 3$. If c_1 appears on five vertices, then it must appear on all vertices of X_t. In this case, the triad $\{c_2, c_3, c_4\}$ does not appear, because all hyperedges of H are covered by the vertices of X_t. Therefore, each color appears on at most four vertices. Since H has 15 vertices, every color must appear on four vertices, except one color which appears on three vertices. We may assume that for each $i \in \{1, 2, 3\}$, the color c_i appears on exactly four vertices of X_i. Then the remaining three vertices are colored by c_4 so that each X_i includes exactly one of them. Let us define $Y_j = \{v_{i,j} : 1 \leq i \leq 3\}$ for each j with $1 \leq j \leq 5$. It is easy to check that if a vertex in X_i and a vertex in $X_{i'}$ are colored by the same color provided that $i \neq i'$, both of them cannot be in the set $Y_j \cup Y_{j+1}$ for all $j \in \{1, \ldots, 5\}$; where $Y_6 = Y_1$. Now, since three vertices are colored by c_4 and each X_i includes exactly one of them, we derive a contradiction. Therefore, H has no complete 4-coloring and it is a desired hypergraph. □

2.3.3 Solution to a stronger version of Problem 2.4

Our aim in this subsection is to present a 3-uniform 3-colorable hypergraph having a complete 6-coloring but no complete t-coloring for each $t \in \{4, 5\}$. This hypergraph is made from the complete 3-uniform hypergraph H of order 6 with size $\binom{6}{3}$ by splitting every vertex into two vertices. In fact, we used an innovative computer search for finding how to split the vertices satisfactorily. A similar method was already used to construct the hypergraph considered in the proof of Theorem 2.9.

Theorem 2.11. *There exists a* 3*-uniform hypergraph* H *of order* 12 *with* $\psi(H) \geq \chi(H)+3$ *such that* H *has a complete* $\chi(H)$*-coloring and a complete* $\psi(H)$*-coloring, but no complete* t*-coloring for any* t *satisfying* $\chi(H) < t < \psi(H)$*.*

Proof. Let H be the 3-uniform hypergraph whose incidence graph is shown in Figure 2.5 such that its vertices are colored black. If H has a complete k-coloring for $k \geq 7$, then it has at least thirty-five hyperedges. However, H has exactly twenty hyperedges and hence $\psi(H) \leq 6$. In fact, H has a complete 3-coloring and a complete 6-coloring (see Figures 2.5 and 2.6, respectively). Therefore, $\chi(H) = 3$ and $\psi(H) = 6$.

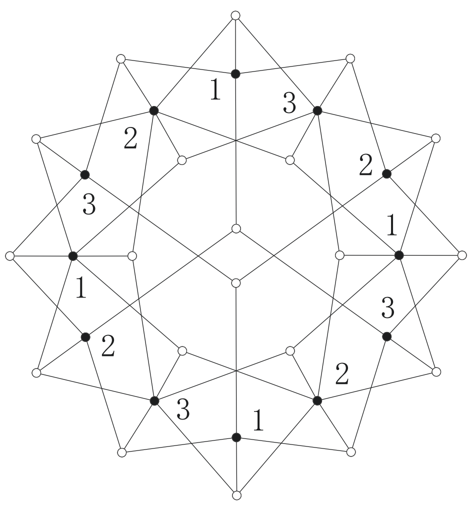

Figure 2.5: A complete 3-coloring of H.

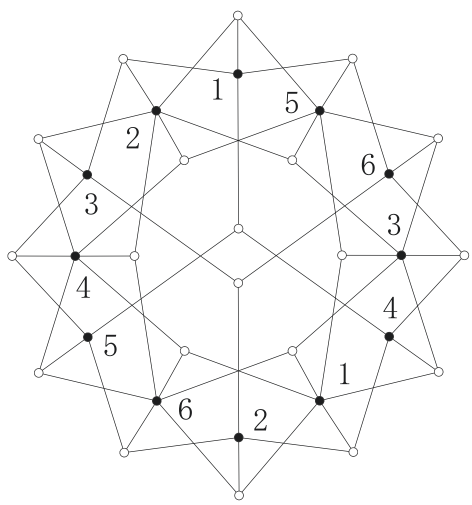

Figure 2.6: A complete 6-coloring of H.

Next, we show that H has neither a complete 4-coloring nor a complete 5-coloring. According to the construction of H, it is not hard to check that there are exactly three independent sets X_1, X_2 and X_3 of size four (which are shown as vertices numbered by $1, 2$ and 3 in Figure 2.5, respectively). Moreover, every independent set of size three must be a subset of X_1, X_2, or X_3. Suppose, to the contrary, that H has a complete 4-coloring using colors $c_1, \ldots, c_4$. First, we assume that there exists a color appearing on at least four vertices of H, say color c_1. Since H has no independent sets of size five, the color c_1 must appear on all four vertices of a set X_i, where $i \in \{1, 2, 3\}$. Since these four vertices cover all hyperedges of H, the triad $\{c_2, c_3, c_4\}$ does not appear on any hyperedge of H, a contradiction. Now, since H has 12 vertices, we may assume that every color appears on exactly three vertices of H. On the other hand, H has at most three disjoint independent sets of size three, a contradiction. Therefore, H has no complete 4-coloring.

Suppose, to the contrary, that H has a complete 5-coloring using colors $c_1, \ldots, c_5$. As we have observed above, no color can appear on at least four vertices. Since H has 12 vertices, there must be a color appearing on exactly three vertices of H, say c_1. Call the set of all vertices having the color c_1 by S. Since the size of S is three, it must be a subset of X_1, X_2, or X_3, say X_1. We may assume that the unique vertex in $X_1 \setminus S$ is colored by c_2. Since X_1 covers all hyperedges of H, the triad $\{c_3, c_4, c_5\}$ does not appear on any hyperedge of H, a contradiction. Therefore, H has no complete 5-coloring and it is a desired hypergraph. □

As we have shown, the answer of Problem 2.4 is positive even by replacing the lower bound of $\chi(H) + 3$. The natural question arises whether the answer would be positive even by replacing greater lower bounds.

Problem 2.12. *For any integer t_0 with $t_0 \geq 4$, does there exist a uniform hypergraph H with $\psi(H) \geq \chi(H) + t_0$ such that H has a complete $\chi(H)$-coloring and a complete $\psi(H)$-coloring, but no complete t-coloring for any t satisfying $\chi(H) < t < \psi(H)$?*

2.4 An exceptional example for Conjecture 2.7

A counterexample of Conjecture 2.7 is given in the following theorem which answers both of Problems 2.3 and 2.6 as well. This hypergraph was first found by writing a C++ code for checking complete colorings of hypergraphs and by applying it on the specified outputs of *plantri* program due to Brinkmann and McKay [5]. Note that this face hypergraph is unique by searching among all 3-colorable planar triangulations on up to 23 vertices.

Theorem 2.13. *There exists a* 3*-uniform* 3*-colorable face hypergraph of order* 12*, obtained from a planar triangulation, having a complete* 6*-coloring but with no complete* 5*-coloring.*

Proof. Let H be the 3-uniform face hypergraph obtained from the planar triangulation shown in Figure 2.7. If H has a complete k-coloring for $k \geq 7$, then H has at least thirty-five hyperedges. However, H has exactly twenty hyperedges and hence $\psi(H) \leq 6$. In fact, H has a complete 3-coloring and a complete 6-coloring (see Figures 2.7 and 2.8, respectively). Therefore, $\chi(H) = 3$ and $\psi(H) = 6$.

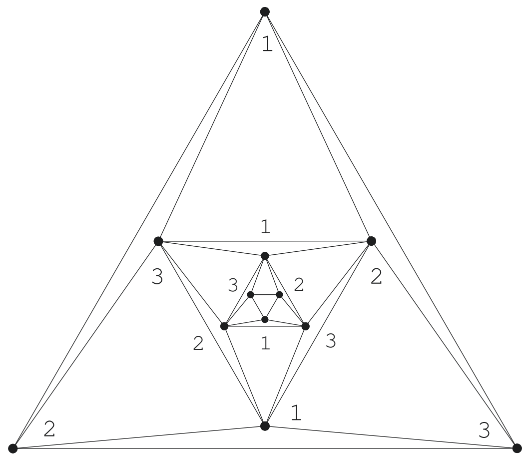

Figure 2.7: A complete 3-coloring of H

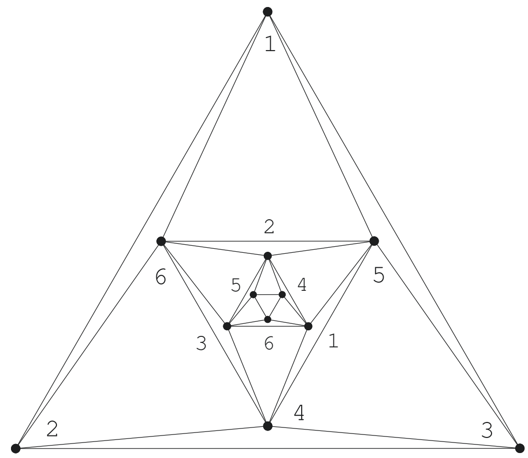

Figure 2.8: A complete 6-coloring of H

Suppose, to the contrary, that H has a complete 5-coloring using colors $c_1, \ldots, c_5$. We call the vertices of H by $v_1, \ldots, v_6$ and $w_1, \ldots, w_6$ which are shown in Figure 2.9. We may assume that w_1, w_2, and w_3 are colored by c_1, c_2, and c_3, respectively. We may also assume that each of the colors c_4 and c_5 appears on at least one of w_4, w_5, and w_6; otherwise, it is enough to change the colors of them to make this property along with maintaining the property of complete 5-coloring. According to the features of the hypergraph H, we can also assume that w_4, w_5, and w_6 are colored by c_4, c_5, and c_2, respectively. It is not difficult to check that for a given arbitrary proper coloring of the octahedron, every pair of colors is contained in at most two kinds of triads appeared on faces of the octahedron. Thus the octahedron $v_1 v_2 \cdots v_6$ minus the face $v_4 v_5 v_6$ has at most two kinds of colored faces including both of c_3 and c_4. Since there exist three remaining triads containing c_3 and c_4, one can conclude that the color c_4 must appear on either v_4 or v_6. Similarly, with respect to the colors c_1 and c_5 on this octahedron, one can also conclude that the color c_5 must appear on either v_4 or v_5.

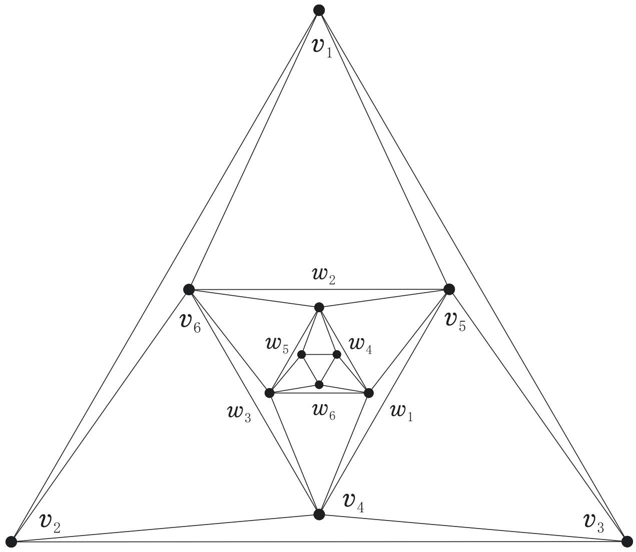

Figure 2.9: A vertex labelling of H

To complete the proof, we shall consider three cases.

Case A: The vertex v_4 is colored by c_2.
In this case, the vertices v_5 and v_6 must be colored by c_5 and c_4, respectively. Since at least one face is colored by the triad $\{c_1, c_4, c_5\}$, the color c_1 must also appear on the vertex v_1. Consequently, it is easy to see that the triad $\{c_3, c_4, c_5\}$ cannot appear, which is a contradiction.

Case B: The vertex v_4 is colored by c_4.
In this case, the vertex v_5 must be colored by c_5 and so the vertex v_6 must be colored by c_1. Since at least one face is colored by the triad $\{c_3, c_4, c_5\}$, the color c_3 must also appear on the vertex v_3. Consequently, it is easy to see that the triad $\{c_1, c_3, c_5\}$ cannot appear which is again a contradiction.

Case C: The vertex v_4 is colored by c_5.
The proof of this case is similar to Case B (by exchanging the colors c_4 and c_5 and exchanging the colors c_1 and c_3, and using the symmetry of H).

Hence the proof is completed. □

We would like to know whether the 3-uniform face hypergraph stated in the proof of Theorem 2.13 is a unique exceptional face hypergraph not satisfying the interpolation property. For this purpose, we pose the following problem which is a revised version of

Problem 2.6.

Problem 2.14. *Let H_0 be the 3-uniform face hypergraph stated in the proof of Theorem 2.13. Does there exist a 3-uniform face hypergraph H with $H \neq H_0$, obtained from a planar triangulation, not satisfying the interpolation property?*

2.5 Answering a problem of Matsumoto and Ohno about complete coloring of planar triangulations

Recently, Matsumoto and Ohno [20] showed that if a 3-uniform face hypergraph obtained from a planar triangulation is 3-colorable, then its achromatic number can be large enough, provided that its matching number would be lare enoguh. In addition, they characterized such hypergraphs with the achromatic number exactly 3. Here, we denote by $\alpha'(H)$ is the maximum number of vertex disjoint hyperedges of H.

Theorem 2.15. ([20]) *Let H be a 3-uniform face hypergraph obtained from a planar triangulation. If H is 3-colorable and $\alpha'(H) \geq 4\binom{k}{3}$, then H admits a complete k-coloring.*

The condition on the chromatic number is necessary, because they constructed an infinite family of such 3-uniform face hypergraphs H with the chromatic number 4 and without complete 4-coloring while $\alpha'(H) \geq \frac{1}{4}(|V(H)|+1)$. Motivated by this observation, they posed the following problem in their paper.

Problem 2.16. (see Problem 4 [20]) *Does there exist a real number m with $3 \leq m < 4$ such that if H is a 3-uniform face hypergraph obtained from a planar triangulation with the chromatic number 4 satisfying $\alpha'(H) \geq \frac{1}{m}|V(H)|$, then H has a complete 4-coloring.*

In the following theorem, we answer to Problem 2.16 negatively by an infinite family of planar triangulations. It remains to decide whether by inserting a condition on minimum degree or maximum degree the result holds or not.

Theorem 2.17. *There are infinitely many 3-uniform face hypergraphs H obtained from planar triangulations with the chromatic number 4 satisfying $\alpha'(H) \geq \frac{1}{3}|V(H)|$ while H has no complete 4-coloring, where $\alpha'(H)$ is the maximum number of vertex disjoint hyperedges.*

Proof. Let n be an integer divisible by 3 with $n \geq 9$. Let k be an odd positive integer. First, consider a cycle C of order $k+4$ with vertices $c_0, c_1, \ldots, c_{k+3}$. Next, add two vertices u_0 and u to this cycle and join them to all those vertices. Finally, for every i with $1 \leq i \leq k$, insert two vertices x_i and y_i into the face $c_{i-1}c_iu_0$ and insert six edges x_ic_{i-1}, x_ic_i, x_iu_0, y_ic_{i-1}, y_ic_i, and y_ix_i into this graph. Call the resulting graph G_k. Obviously, G_k is a triangulation on the plane (the graph G_1 is depicted in Figure 2.10). It is easy to check that $c_1x_1y_1, \ldots, c_kx_ky_k$, $u_0c_0c_{k+3}$, and $uc_{k+1}c_{k+2}$ are $|V(G)|/3$ vertex disjoint faces of G in which $|V(G)| = 3k+6$. Let H be the 3-uniform face hypergraph obtained from G. Suppose, to the contrary, that H has a complete 4-coloring (with respect to the colors 1, 2, 3, and 4). We may assume that four vertices u_0, c_0, c_1, x_1 are colored by $1, 2, 3$, and 4 respectively. First assume that u is colored by 4. According to the construction of H, all vertices of the cycle C must be colored by 2 and 3. Since C has odd order, we derive a contradiction. Now, assume that u is colored by 1. In this case, for every i with $1 \leq i \leq k$, the vertices c_{i-1}, c_i, x_i must be colored by $2, 3, 4$. Therefore, all vertices $y_1, \ldots, y_k$ must have the same color 1. This implies that every hyperedge of H (every face of G) contains a vertex with the color 1. Thus the triple $\{1, 2, 3\}$ cannot appear. Hence we again derive a contradiction, as desired. $\square$

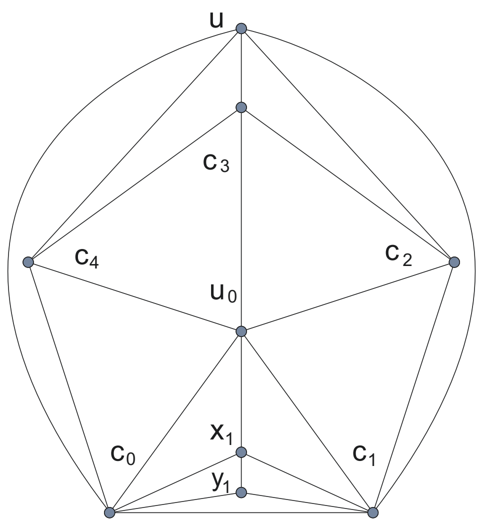

Figure 2.10: A 4-chromatic triangulation of order 9 having three vertex disjoint faces but with no proper facial 3-complete 4-coloring.

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