Yokohama Mathematical
Journal Vol. 68, 2022

NOTE ON GRAPH COVERINGS WITH VOLTAGE ASSIGNMENTS

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(Received October 1, 2022; Revised January 13, 2023)

Abstract. It has been known that any covering space of a suitable topological space can be covered by a regular covering of it which admits a group action as its covering transformation. We shall establish a theorem which describes the relation between a given graph covering and a regular covering which cover it in a combinatorial way with the two notions of ordinary and permutation voltage assignments.

Introduction

In 1986, the first author [15] proposed the conjecture that a connected graph G can be covered by a planar graph if and only if G can be embedded on the projective plane. The sufficiency is clear since such a planar graph can be obtained from G embedded on the projective plane by pulling it back via the covering projection from the sphere to the projective plane. This conjecture has been called "*Planar Cover Conjecture*" and known as one of famous open problems in topological graph theory and there have been published a lot of papers [1–3, 5–25, 27, 28, 31] related to it.

Negami has proved in [15] that if G can be obtained as the quotient of the planar graph by a suitable group action, then G can be embedded on the projective plane, as stated in the conjecture. Such a covering admitting a group action is said to be *regular*. One might wonder if the existence of a planar covering of a given graph guarantees that of its planar regular covering; if it were true, then the conjecture would follow. Actually we can show that any covering can be covered by a regular covering, by a method in algebraic topology, which will be reviewed in Section 1, but its planarity cannot be guaranteed in general.

However, Negami [26] has developed a theory recently to give a new proof scheme for the conjecture, focusing on the possible regular coverings which cover

2020 Mathematics Subject Classification: 05C10, 57M10
Key words and phrases: topological graph theory, graph coverings, voltage assignments

a given planar covering. Our purpose in this paper is to provide the tools to enable us to deal with his theory in a concrete way.

There has been known a combinatorial way to construct a covering of a given graph using two types of "voltage assignments", developed in [4,29]. An *ordinary voltage assignment* $\alpha : E(G) \to A$ assigns an element $\alpha(e)$ of a finite group A to each edge e of a given graph G and derives a regular covering G^α of G where A acts as its covering transformation group. On the other hand, a *permutation voltage assignment* $\rho : E(G) \to S_{\{1,\dots,n\}}$ assigns a permutation ρ_e over the n numbers $\{1,\dots,n\}$ to each edge e of G and derives a general covering G_ρ, where S_N denotes the group consisting of all permutations over a set N in general. (See Section 2 for their detailed definitions.)

They look very similar but work differently; G^α gives only regular coverings while G_ρ may not be regular and can control all finite coverings of G. We would like to understand the relation between these two notions and to know how we construct a regular covering which covers a given covering of a connected graph in particular. The following theorems are our answer:

THEOREM 1. *Let G be a connected graph and let G_ρ be any n-fold covering of G derived by a permutation voltage assignment $\rho : E(G) \to S_{\{1,\dots,n\}}$. Then there exists uniquely a regular covering G^α of G such that any regular covering $\tilde{G}$ of G which covers G_ρ covers G^α.*

We call the regular covering G^α in the theorem the *canonical regular covering* of G over G_ρ. Note that this is determined by the pair of G and G_ρ. (See the diagram below.) Theorem 2 gives the concrete form of G^α:

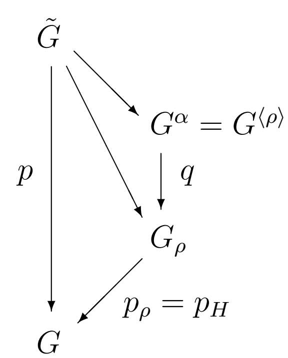

Figure 1 The canonical regular covering

THEOREM 2. *If G_ρ is connected and ρ is normalized along a spanning tree of G, then the ordinary voltage assignment $\alpha : E(G) \to \langle\rho\rangle$ with $\alpha(e) = \rho_e$ for each edge $e \in E(G)$ derives the canonical regular covering G^α of G over G_ρ, where $\langle\rho\rangle$ stands for the subgroup in $S_{\{1,\dots,n\}}$ generated by $\{\rho_e : e \in E(G)\}$.*

Let $G^{\langle\rho\rangle}$ denote the canonical regular covering of G over G_ρ given in the above theorem, that is, $G^\alpha = G^{\langle\rho\rangle}$. Notice that the notation $G^{\langle\rho\rangle}$ itself has a meaning for any permutation voltage assignment ρ in general. However, $G^{\langle\rho\rangle}$ may be disconnected when ρ is not normalized. We shall define the notion of being *normalized* and discuss the connectedness of graph coverings in Section 3.

If the reader knows the theory of covering spaces in algebraic topology, then he may know a way to prove the existence of the canonical regular covering of a given connected graph. We shall explain it briefly in Section 1. However, it is important that Theorem 2 gives a combinatorial way to give its canonical regular covering, beyond theoretical understanding. In Section 4, we shall describe the structure of $G^{\langle\rho\rangle}$ in detail, which will work as the proof of our two theorems.

As an application of our method, we shall discuss "abelian coverings" in Section 5, which will show the relation between abelian coverings and non-abelian coverings. Also we shall illustrate an concrete example of a planar covering of $K_{3,3}$, related to Planar Cover Conjecture, from the point of view of our theory in Section 6.

This note has been written mainly for researchers working in graph theory and our terminology is quite standard in that field. However, we assume the reader's knowledge of basic topology and group theory. Section 3 will show an important argument to join the theories in topology and in graph theory.

1. Reviewing covering spaces

We shall describe briefly the well-developed general theory of covering spaces in algebraic topology [30] in this section. We assume that any topological space discussed here has no pathological structure, as graphs, surfaces and manifolds. The notation has been adapted to our use in graph theory. We shall refer to the total described below as "the classification of covering spaces".

Let $\tilde{X}$ and X be two topological spaces, which are assumed to be arcwise connected. If there exists a local homeomorphism $p : \tilde{X} \to X$, that is, a surjective continuous map which induces a homeomorphism between suitable open neighborhoods of corresponding points in $\tilde{X}$ and X, then $\tilde{X}$ or the pair $(\tilde{X}, p)$ is called a *covering space* of X and p its *covering projection* or simply a projection. If the projection p is an n-to-1 map for a finite number $n > 0$, then $\tilde{X}$ is called an *n-fold covering* of X.

The *fundamental group* $\pi_1(X)$ of a topological space X is defined as a group consisting of closed curves based at a fixed point in X up to continuous deformation (or up to *homotopy*) and the product of two elements in $\pi_1(X)$ corresponds to the closed curve going along the first one and next along the second one. It has been known that any covering space $\tilde{X}$ of X is associated with a subgroup H in $\pi_1(X)$ unique up to conjugation and that its projection $p : \tilde{X} \to X$ induces an isomorphism $p_\# : \pi_1(\tilde{X}) \to H$. We denote such a covering space by $\tilde{X}_H$ and its projection by $p_H : \tilde{X}_H \to X$.

It is clear that an arc α starting at a base point in X can be lifted to an arc in $\tilde{X}_H$, say $\tilde{\alpha}$, by pulling it back naturally via p_H. It depends on the homotopy type of α whether or not the terminus of $\tilde{\alpha}$ coincides with its origin, even if α is a closed curve. The existence of the isomorphism between $\pi_1(\tilde{X}_H)$ and H implies that the subgroup H consists of the homotopy types of closed curves in X that can be lifted to a closed curve in $\tilde{X}_H$.

Let $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}_H$ be the base point of $\pi_1(X)$ and that of $\pi_1(\tilde{X}_H)$, respectively, with $p_H(\tilde{x}_0) = x_0$. Take any point $\tilde{x}$ in $\tilde{X}_H$ which projects to x_0 and consider two arcs $\tilde{\alpha}$ and $\tilde{\beta}$ starting at $\tilde{x_0}$ and terminating at $\tilde{x}$ in $\tilde{X}_H$. Then $\tilde{\alpha} \cdot \tilde{\beta}^{-1}$ becomes a closed curve based at $\tilde{x}_0$ and hence it can be regarded as an element $\tilde{h}$ in $\pi_1(\tilde{X}_H)$. That is, $\tilde{\alpha}\tilde{\beta}^{-1} = \tilde{h}$ and $\tilde{\alpha} = \tilde{h}\tilde{\beta}$. Since p_H induces an injective homorphism from $\pi_1(\tilde{X}_H)$ to $\pi_1(X)$, we may regard this equality as $\alpha = h\beta$, omitting their "tildes" to get the corresponding elements in $\pi_1(X)$.

This means that α belongs to the right coset $H\beta$ of H in $\pi_1(X)$ and hence there is a bijection between the pre-images of x_0 in $\tilde{X}_H$ and the right cosets Hg of H in $\pi_1(X)$. We can conclude the same fact also for any point $x \neq x_0$ in X, modifying the above argument slightly. Therefore, the number of pre-images of x is equal to $|p_H^{-1}(x)| = (\pi_1(X) : H)$. This value, say n, does not depend on a point $x \in X$ and is often called the *covering index* of p_H and $\tilde{X}_H$ is called an *n-fold* covering of X.

Suppose that H is a normal subgroup in $\pi_1(X)$ in particular. Then the quotient $\pi_1(X)/H$ becomes a group and acts on the set of pre-images $p_H^{-1}(x)$ for each point $x \in X$ naturally. Thus, the quotient group $\pi_1(X)/H$ acts on the whole $\tilde{X}_H$ as the *covering transformation group*. That is, all points in $\tilde{X}_H$ equivalent under this group action project to the same point in X. In particular, the covering space $\tilde{X}_{\{1\}}$ corresponding to the trivial subgroup $\{1\}$ in $\pi_1(X)$ is called the *universal covering space* of X and $\pi_1(X)$ itself acts on $\tilde{X}_{\{1\}}$, where "1" stands for the identity element in $\pi_1(X)$.

Let $\tilde{X}_N$ be the covering of X associated with another subgroup N in $\pi_1(X)$ which is contained in H, denoted by $N < H$. Then $\tilde{X}_N$ corresponds to the right coset decomposition of N in $\pi_1(X)$ and we can define a covering projection $q : \tilde{X}_N \to \tilde{X}_H$ so that q maps a point in $\tilde{X}_N$ corresponding to a right coset Ng to a point corresponding to Hg, which is well-defined since $N < H$, and that $p_N = p_H \circ q$. In this case, we say that the covering $p_N : \tilde{X}_N \to X$ *factors through* p_H and simply that $\tilde{X}_N$ *covers* $\tilde{X}_H$. Since $\{1\}$ is contained in any subgroup H, the universal covering $p_{\{1\}} : \tilde{X}_{\{1\}} \to X$ covers any covering of X.

NOTE: Although a graph is a combinatorial object in graph theory, we usually regard it as a 1-dimensional topological space which can be expressed as a simplicial complex consisting only of vertices and edges. Thus, the classification of covering spaces works also for the coverings of any connected graph G. Note that $\pi_1(G)$ is isomorphic to the "free group".

The *free group* F_r of rank r is the group consisting of all words over r alphabets and their inverses. The product of two words first joins them and cancels consecutive inverse pairs as well as possible. The rank of $\pi_1(G)$ of a connected graph G as a free group coincides with the cycle rank of G as a graph, which is equal to $|E(G)|-|V(G)|+1$. Therefore, if G is a tree, then it admits no covering except itself since $\pi_1(G)$ is trivial. On the other hand, if G is not a tree, that is, if G contains at least one cycle, then $\pi_1(G)$ contains infinitely many elements and its universal covering becomes an infinite tree.

2. graph coverings with voltage assignments

In this section, we shall introduce two combinatorial ways to construct the covering spaces of a given graph, which has been developed in [4, 29]. First, we should translate the notion of covering spaces into one described in terms of graph theory, as follows.

Let $\tilde{G}$ and G be two connected simple graphs and let $p : \tilde{G} \to G$ be a covering projection between them. We assume implicitly that p maps each vertex in $\tilde{G}$ to a vertex in G. Recall that p must induce a homeomorphism between suitable open neighborhoods of two corresponding vertices $\tilde{v}$ and $v = p(\tilde{v})$ in the topological sense. An enough small open neighborhoods of v in G consists of the vertex v and the segments of edges incident to v. This means that the same number of edges are incident to both $\tilde{v}$ and v and that p induces a bijective correspondence between those edges.

Since two graphs are assumed to be simple, such a correspondence induces a bijection between their neighbors. So we define a *covering* $p : \tilde{G} \to G$ as a surjective map $p : V(\tilde{G}) \to V(G)$ such that it induces a bijection between $N(\tilde{v})$ and $N(p(\tilde{v}))$ for each vertex $\tilde{v} \in V(\tilde{v})$, where $N(v)$ stands for the neighborhood of a vertex v in general, that is, $N(v)$ is the set of vertices adjacent to v.

First we shall show a combinatorial way to construct a regular covering of a given connected graph G. Let A be a finite group and let $\alpha : E(G) \to A$ be an assignment of elements of A to edges. The value $\alpha(e)$ is called a *voltage* assigned to an edge e. The assignment α is usually called a *voltage assignment*. However, we shall refer to it as an *ordinary voltage assignment* to distinguish it from what will be defined later.

Precisely speaking, each edge $e = uv$ should have a direction $u \to v$ to define the voltage assignment α and it must satisfies the condition that $\alpha(vu) = \alpha(uv)^{-1}$ for the edge vu having the opposite direction. However, we shall describe all for undirected graphs, assuming implicitly defined edge directions, to avoid the usage of noisy notations.

Let u be any vertex of G and prepare the set $U = \{u \times \tau : \tau \in A\}$ where the symbolical notation $u \times \tau$ denotes a vertex prepared virtually for each element $\tau \in A$. Let v be another vertex of G to which u is joined by an edge $e = uv$ and let $V = \{v \times \tau : \tau \in A\}$ be the set of vertices defined for v as well as for u. Put $V(G^\alpha)$ as the union of those sets so defined and join each vertex $u \times \tau$ to $v \times (\alpha(e) \cdot \tau)$ for each eage $e = uv \in E(G)$ and each element $\tau \in A$.

The resulting graph G^α over $V(G^\alpha)$ is called the *covering* of G *derived by* the voltage assignment α. It is easy to see that G^α is a covering space of G in the topological sense and that A acts on G^α; the action carries any vertex $u \times \tau$ to $u \times (\tau \cdot \alpha)$ for each $\alpha \in A$. Thus, G^α is a regular covering of G having the covering transformation group A. Its projection $p^\alpha : G^\alpha \to G$ projects each vertex $u \times \tau$ to u for any vertex $u \in V(G)$.

Now we shall show another combinatorial construction of a graph covering. Let $S_{\{1,\dots,n\}}$ denote the group consisting of all permutations over $\{1, \dots, n\}$, which is isomorphic to the symmetric group S_n of degree n. Let $\rho : E(G) \to S_{\{1,\dots,n\}}$ be an assignment of permutations to edges; $\rho(e) = \rho_e \in S_{\{1,\dots,n\}}$. As well as the ordinary voltage assignment α, we assume that $\rho(vu) = \rho(uv)^{-1}$ for each edge $e = uv$ and call ρ a *permutation voltage assignment* of G.

Put $U = \{u_1, \dots, u_n\}$ as the set of n copies of any vertex $u \in V(G)$ and let $V = \{v_1, \dots, v_n\}$ be the set similarly defined for another vertex $v \in V(G)$. Add the edges $u_i v_{\rho_e(i)}$ for $i = 1, \dots, n$ if there exists an edge $e = uv$ between u and v in G. The resulting graph over $V(G) \times \{1, \dots, n\}$ is denoted by G_ρ and is called the *covering* of G *derived by* ρ. Its projection $p_\rho : G_\rho \to G$ maps each u_i to u.

It is easy to see that G_ρ is actually an n-fold covering of G and that any n-fold covering of G, regular or not, can be obtained as G_ρ with a permutation voltage $\rho : E(G) \to S_{\{1,\dots,n\}}$ suitably defined.

NOTE: Any covering G_ρ constructed combinatorially can be regarded as a covering space of G in the topological sense. However, G_ρ may be disconnected for some permutation voltage assignment ρ even if G is connected. In such a case, the classification of covering spaces does not work well. Hereafter, we shall assume implicitly that both G and G^ρ are connected. Under such an assumption, G_ρ corresponds to a suitable subgroup H in $\pi_1(G)$ and $\tilde{G}_H = G_\rho$.

What is H? To answer this question, Negami [21] has discussed the relation between two ways, topological and combinatorial, to construct graph coverings and conclude the following.

Let $\rho : E(G) \to S_{\{1,\ldots,n\}}$ be a permutation voltage assignment for a connected graph G and let $W = e_1 \cdots e_k$ be any closed walk in G based at a fixed vertex v_0, presented as a sequence of edges. Then the homotopy type of W as a closed curve can be regarded as an element in $\pi_1(G)$. Define the *voltage* $\rho(W)$ for W by $\rho(W) = \rho(e_1) \cdots \rho(e_k)$. This induces a well-defined group homomorphism $\rho : \pi_1(G) \to S_{\{1,\ldots,n\}}$. Put $H = \rho^{-1}(S_{\{2,\ldots,n\}})$, where $S_{\{2,\ldots,n\}}$ is the subgroup in $S_{\{1,\ldots,n\}}$ consisting of all permutations fixing 1. This H is the answer.

It is well-known that the cycle space of G is generated by the cycles $C_1, \ldots, C_r$ corresponding to the edges $e_1^*, \ldots, e_r^*$ of a cotree T^* in G, that is, those not belonging to a spanning tree T in G. Then we can redefine a permutation voltage assignment $\rho^* : E(G) \to S_{\{1,\ldots,n\}}$ by $\rho^*(e_i^*) = \rho(C_i)$ and $\rho^*(e) = \mathrm{id}$ for each edge e lying on T, where id stand for the identity permutation. Since the induced homomorphism $\rho^* : \pi_1(G) \to S_{\{1,\ldots,n\}}$ coincides with ρ, this permutation voltage ρ^* derives the same covering as $\tilde{G}_H = G_\rho$.

Similarly, the covering G^α of G derived by an ordinary voltage assignment $\alpha : E(G) \to A$ must be the regular covering $\tilde{G}_N$ of G associated with some normal subgroup N in $\pi_1(G)$. Since $\pi_1(G)/N$ acts on $\tilde{G}_N$ as its covering transformation group, the group A must be isomorphic to $\pi_1(G)/N$.

Let $a_1, \ldots, a_{|A|}$ denote all the elements of A, that is, $A = \{a_1, \ldots, a_{|A|}\}$. Then $\alpha(e)$ induces a permutation over this A. Define a permutation voltage assignment $\rho_\alpha : E(G) \to S_{\{1,\ldots,|A|\}}$ as one that assigns a permutation over the subscripts $\{1, \ldots, |A|\}$ induced by $\alpha(e)$ to each edge $e \in E(G)$. Then G_{ρ_α} is a covering of G equivalent to G^α and we can recognize the normal subgroup N for $\tilde{G}_N = G^\alpha = G_{\rho_\alpha}$ in the same way as mentioned above for G_ρ.

3. Connectedness of coverings

As we have mentioned in the previous note, we should guarantee the connectedness of coverings of a connected graph G when we apply the classification of covering spaces to them. Here we shall discuss a criterion for a covering of a connected graph to be connected.

Let G be a connected graph and let T be any spanning tree of G. A permutation voltage $\rho : E(G) \to S_{\{1,\ldots,n\}}$ is said to be *normalized* along T if $\rho(e) = \mathrm{id}$ for each edge e lying on T. That is, the only edges $e_1^*, \ldots, e_r^*$ of the cotree T^* corresponding to T receive non-trivial permutations in $S_{\{1,\ldots,n\}}$. If ρ is not normalized, then we can obtain another permutation voltage assignment ρ^* in the same way as described in the previous section and this ρ^* is normalized along T. We call it the *normalization* of ρ along T.

A subgroup H in $S_{\{1,\dots,n\}}$ is said to be *transitive* over $\{1,\dots,n\}$ if there exists an element $\tau \in H$ with $\tau(x) = y$ for any pair of numbers $x, y \in \{1,\dots,n\}$.

Lemma 3. *Let $\rho : E(G) \to S_{\{1,\dots,n\}}$ be a permutation voltage assignment of a connected graph G normalized along a spanning tree T of G. Then the covering G_ρ derived by ρ is connected if and only if $\langle\rho\rangle$ is transitive over $\{1,\dots,n\}$.*

Proof. Let $u \in V(G)$ be any vertex of G and let $u_1,\dots,u_n$ be the pre-images of u via the projection $p_\rho : G_\rho \to G$. First suppose that G_ρ is connected. Then there exists a path Q from u_i to u_j in G_ρ. Let $\tau_{i,j} \in S_{\{1,\dots,n\}}$ denote the product of voltages assigned to edges along the path Q. It is clear that $\tau_{i,j}$ belongs to $\langle\rho\rangle$ and $\tau_{i,j}(i) = j$ since Q starts at u_i and terminates at u_j. This implies that $\langle\rho\rangle$ is transitive over $\{1,\dots,n\}$. The necessity follows.

Conversely suppose that $\langle\rho\rangle$ is transitive over $\{1,\dots,n\}$. Let $T_1,\dots,T_n$ be the mutually disjoint n trees each of which projects to the spanning tree T along which ρ is normalized. We may assume that u_i lies on T_i and take another vertex v_k lying on T_i that projects to v. Then there is a path $\tilde{e}_1 \cdots \tilde{e}_\ell$ along T_i joining u_i to v_k, where $\tilde{e}_1,\dots,\tilde{e}_\ell$ are the edges lying along the path in this order.

Since ρ is normalized along T, the voltage assigned to each edge $p_\rho(\tilde{e}_j)$ of T is the identity permutation, $\rho(p_\rho(\tilde{e}_j)) = \mathrm{id}$, and hence $\tilde{e}_j$ joins two vertices in G_ρ which has the same index. This implies that $k = i$ and that all vertices lying on T_i has the same index i as well as u_i. Thus, we assume that u_i lies on T_i for any vertex u of G and any index i.

Since $\langle\rho\rangle$ is transitive, there is a sequence $e_1,\dots,e_\ell$ of edges in G, for any pair of numbers $i, j \in \{1,\dots,n\}$, such that the composition $\rho_{e_\ell} \cdots \rho_{e_1} \in \langle\rho\rangle$ maps i to j. If $e_1 = uv$ and if $\rho_{e_1}(i) = i_1$, then there is an edge $u_i v_{i_1}$ in G_ρ. Start at u_i and move to v_{i_1} along the edge $u_i v_{i_1}$. If $e_2 = u'v'$ and if $\rho_{e_2}(i_1) = i_2$, then we can go from v_{i_1} to u'_{i_1} along T_{i_1} and there is an edge $u'_{i_1} v'_{i_2}$. Move to v'_{i_2}. Continue the same process as above for edges $e_1,\dots,e_\ell$ in order. Then we find a path from u_i to u_j in G_ρ, which joins two trees T_i and T_j, and hence G_ρ is connected. ∎

Notice that the above argument for the necessity works even if ρ was not assumed to be normalized. To understand why the assumption of being normalized is needed, consider the following easy example.

Let G be the complete graph over three vertices u, v and x_0. Then the two edges x_0u and x_0v induce a spanning tree of G, say T. Define $\rho : E(G) \to S_{\{1,2,3\}}$ by $\rho(x_0u) = (12)$, $\rho(uv) = (123)$ and $\rho(x_0v) = \mathrm{id}$. Then ρ is not normalized along T and is not also along any other spanning tree of G. It is easy to see that the covering G_ρ consists of a triangle and a hexagon, and hence it is not connected. However, we have $\langle\rho\rangle = \langle(12),(123)\rangle$ and it is transitive over $\{1,2,3\}$. This shows that the sufficiency of the lemma does not hold without the assumption of ρ being normalized.

Now consider the normalization ρ^* of ρ along T, which assigns (12)(123) to uv and id to the two edges x_0u and x_0v of T. Since $(12)(123) = (13)$, we have $\langle\rho^*\rangle = \langle(13)\rangle$ and $\langle\rho^*\rangle$ is not transitive over $\{1, 2, 3\}$. The covering G_{ρ^*} is not connected. This coincides with the fact stated in the lemma.

We shall give a similar criterion for a regular covering of a connected graph G to be connected. An ordinary voltage assignment $\alpha : E(G) \to A$ is said to be *normalized* along a spanning tree T of G if the voltages $\alpha(e)$ on all edges e lying along T is the identity element of A; $\alpha(e) = \text{id}$. We denote the subgroup in A generated by all voltages $\alpha(e)$ by $\langle\alpha\rangle = \langle\alpha(e) : e \in E(G)\rangle$.

Lemma 4. *Let A be a finite group and let $\alpha : E(G) \to A$ be an ordinary voltage assignment of a connected graph G normalized along a spanning tree T of G. Then the covering G^α derived by α is connected if and only if $A = \langle\alpha\rangle$.*

Proof. First suppose that G^α is connected. Then there exists a path Q in G^α which joins $u \times \text{id}$ to $u \times \tau$, for any element $\tau \in A$. Then the product of voltages $\alpha(e)$ along Q must be equal to τ. This implies that τ belongs to $\langle\alpha\rangle$ and hence $A \subset \langle\alpha\rangle$. Since the later is a subset of A, we have $A = \langle\alpha\rangle$.

Now suppose that $A = \langle\alpha\rangle$. It suffices to show that $u \times \text{id}$ can be connected to $v \times \tau$ by a path in G^α for any two vertices $u, v \in V(G)$ and any element $\tau \in A$. Then τ can be expressed as the products of voltages $\alpha(e_1), \dots, \alpha(e_\ell)$ and we can find a path Q from $u \times \text{id}$ to $u \times \tau$, using the copies $T_1, \dots, T_{|A|}$ of the spanning tree T of G by the way similar to that in the proof of Lemma 3. Extend Q to join $u \times \text{id}$ to $v \times \tau$, adding the path between $u \times \tau$ and $v \times \tau$ along the tree T_i which contains both of them; the added path preserves "τ" since α is normalized along T. The sufficiency follows. ■

The previous example of a graph G over three vertices u, v and x_0 works to illustrate the above lemma. However, the voltage assignment must be regarded as an ordinary one $\alpha : E(G) \to S_{\{1,2,3\}}$, that is, $\alpha(x_0u) = (12)$, $\alpha(uv) = (123)$ and $\alpha(x_0v) = \text{id}$. Then the regular covering G^α is 6-fold since $S_{\{1,2,3\}}$ acts on it. We have $\langle\alpha\rangle = \langle(12), (123)\rangle = S_{\{1,2,3\}}$, but this covering is not connected and has three components each of which is a cycle of length 6. Also the regular covering G^{α^*} derived by the normalization α^* of α along T is disconnected and $\langle\alpha^*\rangle = \langle(13)\rangle \neq S_{\{1,2,3\}}$. Thus, the assumption of G^α being normalized is needed actually, as well as in Lemma 3.

Remark that a covering of a connected graph is assumed to be connected implicitly, as stated in the second paragraph of Section 2, unless stated otherwise and that the notations G_ρ and G^α with voltage assignments are just methods to exhibit coverings of a graph.

4. Canonical regular coverings

Let G be a connected graph and let $\rho : E(G) \to S_{\{1,\dots,n\}}$ be a permutation voltage assignment. The covering G_ρ derived by ρ may be disconnected in general, but we suppose that G_ρ is connected and that ρ is normalized along a spanning tree T of G, as in Theorem 2. The following arguments consisting of six parts will give a proof of Theorems 1 and 2, describing the desired regular covering G^α in details:

• *Construction*

Here, we shall discuss the regular covering G^α under a slightly wider situation. Let A be a subgroup in $S_{\{1,\dots,n\}}$ and suppose that A contains $\langle\rho\rangle$. Put $A_i = \{\tau \in A : \tau(1) = i\}$ for each $i = 1, \dots, n$. Then A decomposes into n mutually disjoint subsets $A_1, \dots, A_n$ and A_1 becomes a subgroup of A in particular. Define an ordinary voltage assignment $\alpha : E(G) \to A$ by $\alpha(e) = \rho_e$. Then the covering G^α of G derived by α becomes a regular covering of G with its covering transformation group A.

Under this construction, the set of pre-images of each vertex $u \in V(G)$ in G^α decomposes into n mutually disjoint sets $U_1, \dots, U_n$ each of which, say U_i, corresponds to A_i. Denote the corresponding element in U_i by $u \times \tau$ symbolically for each $\tau \in A_i$. That is, we have $U_i = \{u \times \tau : \tau \in A_i\}$. Suppose that there is an edge $e = uv$ in G and define V_i's similarly for another vertex $v \in V(G)$. Since $\alpha(e) = \rho_e$, any vertex $u \times \tau \in U_i$ must be joined to the vertex $v \times (\rho_e \cdot \tau) \in V_j$ for some j by an edge in G^α. Here, we have $\tau(1) = i$ and hence $\rho_e \cdot \tau(1) = \rho_e(i)$. This implies that $j = \rho_e(i)$ and that such edges form a matching between the vertices in U_i and those in $V_{\rho_e(i)}$, corresponding to the edge uv in G. ■

• *Connectedness*

We should confirm that the coverings that we deal with in this section are all connected, to use the classification of covering spaces. Both G and G_ρ are connected by the assumption itself in the theorem. On the other hand, the regular covering G^α constructed above may be disconnected in general. However, Since ρ is normalized, if $A = \langle\rho\rangle$, then G^α must be connected by Lemma 4. Hereafter, we assume that G^α is connected, setting $G^\alpha = G^{\langle\rho\rangle}$.

Recall that G_ρ itself is assumed to be connected. This guarantees that $\langle\rho\rangle$ is transitive over $\{1, \dots, n\}$ by Lemma 3. Thus, there exists an element τ in A such that $\tau(1) = i$ for any $i = 1, \dots, n$, and hence all U_i's are non-empty. ■

• *Projections*

Let $p^\alpha : G^\alpha \to G$ be the projection of the regular covering G^α and discuss how p^α factors through the projection $p_\rho : G_\rho \to G$ of G_ρ. Since the pre-images of each vertex $u \in V(G)$ in G_ρ correspond to the numbers $1, \ldots, n$ by definition, we denote them by $u_1, \ldots, u_n$ and let $v_1, \ldots, v_n$ be the pre-images of another vertex $v \in V(G)$. Suppose that these two vertices are joined by an edge $e = uv$ in G. Then each u_i is joined to $v_{\rho_e(i)}$ by an edge in G_ρ.

Define a map $q : V(G^\alpha) \to V(G_\rho)$ by $q(u \times \tau) = u_i$ for $u \times \tau \in U_i$; recall that $\tau(1) = i$ by the definition of U_i. Each edge in G^α can be expressed as $(u \times \tau,\, v \times (\rho_e \cdot \tau))$. Since $\rho_e \cdot \tau(1) = \rho_e(i)$, q carries this edge to the edge $u_i v_{\rho_e(i)}$ in G_ρ, which exists actually as seen above. Therefore, the map q so defined works as a covering projection from G^α onto $q(G^\alpha)$. Since G_ρ is connected, the image $q(G^\alpha)$ must coincide with the whole G_ρ. We have $p_\rho \circ q(u \times \tau) = p_\rho(u_i) = u$ if $u \times \tau \in U_i$. This implies that $p^\alpha = p_\rho \circ q$, as we want. ■

NOTE: The assumption of G_ρ being connected is very important in the previous argument. If G_ρ is not connected, the projection q may not be surjective. For example, consider a graph over four vertices u, v, w and x_0 having five edges x_0u, x_0v, x_0w, uv and vw, as G. Then G has a spanning tree T induced by three edges x_0u, x_0v and x_0w. Assign (1) to uv, (23) to vw and the identity permutation $\mathrm{id} = (1)$ to the other edges.

This permutation voltage assignment $\rho : E(G) \to S_{\{1,2,3\}}$ is normalized along T. Since $\langle \rho \rangle = \langle (1), (23) \rangle = \langle (23) \rangle$ is not transitive over $\{1, 2, 3\}$, the covering G_ρ is not connected by Lemma 3. Actually, G_ρ has two components, one of which consists of the vertices numbered by 1 and the other vertices form another component. On the other hand, $G^{\langle \rho \rangle}$ is a 2-fold covering of G since $|\langle \rho \rangle| = 2$ and it is connected. Thus, there exists no sujection from $G^{\langle \rho \rangle}$ to G_ρ in this case. However, our assumption excludes such a case.

If we dare to follow the previous argument for the above example, then we would have $A_1 = A$, $A_2 = \emptyset$ and $A_3 = \emptyset$ since the voltage (23) on vw does not move 1, and the projection q maps the whole G^α to the first component of G_ρ described above; q is not surjective at all. If we use $A_i^{(2)} = \{\tau \in A : \tau(2) = i\}$ instead of A_i, then we have $A_1^{(2)} = \emptyset$ and q will map G^α isomorphically to the second component of G_ρ.

Since we have confirmed the connectedness of both G_ρ and G^α, we can apply the classification of covering spaces to these coverings. That is, G_ρ and G^α are associated with subgroups H and N in $\pi_1(G)$ and N is normal in particular. Thus, we have $G_\rho = \tilde{G}_H$ and $G^\alpha = \tilde{G}_N$. We shall use the notations $p_H : G_\rho \to G$ and $p_N : G^\alpha \to G$ below, instead of p_ρ and p^α.

• *Group actions*

As we have already seen, the covering $p_N : G^\alpha \to G$ is regular and the group $A = \langle \rho \rangle$ acts on G^α as its covering transformation group. On the other hand, the subgroup H in $\pi_1(G)$ is isomorphic to $\pi_1(\tilde{G}_H)$ as shown in Section 1 and $H \cap N$ is a normal subgroup in H. This implies that the covering $q : G^\alpha \to G_\rho$ is a regular covering of $G_\rho = \tilde{G}_H$ associated with the normal subgroup $H \cap N$ and the group isomorphic to $H/(H \cap N)$ acts on $G_\alpha = \tilde{G}_N$ as its covering transformation group theoretically. We would like to know what group acts on G^α as the covering transformation group of $q : G^\alpha \to G_\rho$ concretely.

The answer is $A_1 = \{\tau \in A : \tau(1) = 1\}$, which is a subgroup in A. If $\tau \in A_1$ and $\tau' \in A_i$, then $\tau' \cdot \tau(1) = \tau'(1) = i$ and hence the group action by A_1 leaves each A_i invariant. This implies that $u \times \tau \in A_i$ are all equivalent under the group action by A_1 and are projected to the same vertex u_i by the projection $q : G^\alpha \to G_\rho$. Note that A_i does not become a group at all for $i \neq 1$. ■

• *Minimality*

Now we shall show that the covering index of G^α is the smallest among those regular coverings of G that cover G_ρ. This is a preparation to prove that G^α is actually the minimal one in a stronger sense discussed in the next part.

Let B be any finite group and let $\beta : E(G) \to B$ be an ordinary voltage assignment of G. This derives the regular covering $p : G^\beta \to G$ with its covering transformation group B. Suppose that this projection p factors through $p_H : G_\rho \to G$, that is, there exists a covering projection $q : G^\beta \to G_\rho$ with $p_H \circ q = p$. This implies that $p_\#(\pi_1(G^\beta))$ is a normal subgroup in H and hence $q_\#(\pi_1(G^\beta))$ also is normal in $\pi_1(G_\rho)$. Thus, G^β can be regarded as a regular covering of G_ρ, and its covering transformation group, say B_1, becomes a subgroup in B.

Let u be any vertex of G. Then each pre-image of u in G^β can be presented as $u_i \times \tau$ for some $\tau \in B_1$ if it projects to u_i in G_ρ. Put $U_i = \{u_i \times \tau : \tau \in B_1\}$. Then $p^{-1}(u)$ decomposes into the n mutually disjoint sets $U_1, \dots, U_n$ and $q(U_i) = \{u_i\}$. Conversely, put $B_i = \{\tau \in B : u \times \tau \in U_i\}$; recall that each vertex in $p^{-1}(u)$ has an expression $u \times \tau$ for some $\tau \in B$, according to the definition of G^β.

Since B_1 leaves each U_i invariant, there exists an element $\tau_1 \in B_1$ with $b\,\tau_1 = b'$ for any two elements b, $b' \in B_i$. This means that b and b' are contained in a common coset of B_1 in B and that $b_i B_1 = B_i$ for some element $b_i \in B$; $b_1 = \mathrm{id}$ in particular. Thus, $B = B_1 \cup \cdots \cup B_n$ gives a left coset decomposition of B_1 in B.

Take an edge $e = uv$ of G and define the n mutually disjoint sets $V_1, \dots, V_n$ with $p^{-1}(v) = V_1 \cup \cdots \cup V_n$ for the vertex v, as well as $U_1, \dots, U_n$ for u. Then we have $q(V_i) = \{v_i\}$ and the action by B_1 leaves each V_i invariant. By the definition of G^β, a voltage $\beta(e)$ is assigned to this edge e and an edge in G^β joins $u \times \tau$ to $v \times (\beta(e) \cdot \tau)$.

Suppose that $u \times \tau \in U_i$ and $v \times (\beta(e) \cdot \tau) \in V_j$. Then we have $u \times \tau = u_i \times \tau'$ and $v \times (\beta(e) \cdot \tau) = v_j \times \tau''$ for some τ', $\tau'' \in B_1$. Since any edge between U_i

and V_j projects to the edge u_iv_j, the number j is determined, not depending on the choice of $u \times \tau \in U_i$, and must be equal to $\rho_e(i)$. Thus we can define a correspondence Φ between the ordinary voltages for G^β and the permutation voltages for G_ρ by $\Phi(\beta(e)) = \rho_e$ for each edge $e \in E(G)$.

Let $\langle\beta\rangle$ denote the subgroup in B generated by the elements given as $\beta(e)$ for all edges e of G and extend Φ naturally to a correspondence from $\langle\beta\rangle$ to $\langle\rho\rangle$. The product of any element b in $\langle\beta\rangle$ from the left side preserves the coset decomposition $B_1 \cup \cdots \cup B_n$ of B and shuffles these indexes, which corresponds to a permutation over $\{1, \ldots, n\}$ given as the product of ρ_e's.

This means that the extended Φ becomes a group homomorphism between $\langle\beta\rangle$ and $\langle\rho\rangle$. Since Φ is surjective, we have $|\langle\beta\rangle| \geq |\langle\rho\rangle|$ and hence $|B| \geq |A|$. This implies that the regular covering G^α of G with $\alpha(e) = \rho_e$ $(e \in E(G))$ has the smallest covering index among all regular coverings of G that cover G_ρ. ■

• *Canonicity*

We have just shown that $G^{\langle\rho\rangle} = G^\alpha$ is the minimum regular covering of G that covers G_ρ. This covering G^α is associated with a normal subgroup N in $\pi_1(G)$ and N is contained in the subgroup H in $\pi_1(G)$ that G_ρ is associated with. Let $p_M : \tilde{G}_M \to G$ be another regular covering of G whose projection p_M factors through $p_H : G_\rho \to G$ and which is associated with a normal subgroup M in $\pi_1(G)$. Then M must be contained in H as well as N is.

Consider the set $NM = \{n \cdot m : n \in N, m \in M\}$. It is easy to see that NM is contained in H and is a normal subgroup in $\pi_1(G)$ since so are both N and M. Under this situation, if $M \not\subset N$, then NM would be a normal subgroup in H which contains N as its proper subset; $N \subsetneq NM$. This implies that $\tilde{G}_N$ would cover $\tilde{G}_{NM}$ and $|V(\tilde{G}_{NM})| < |V(\tilde{G}_N)|$. This contradicts the minimality of $\tilde{G}_N$. Therefore, N must contain M and hence $\tilde{G}_M$ covers $\tilde{G}_N$. That is, the projection $p_M : \tilde{G}_M \to G$ factors through $p_N : \tilde{G}_N = G^{\langle\rho\rangle} \to G$. ■

It is not so difficult to show group-theoretically that any subgroup H of finite index in a group Γ contains a normal subgroup in Γ which also has finite index. Notice that the normal subgroup N corresponding to $G^{\langle\rho\rangle}$ must coincides with such a normal subgroup maximal in H, by the minimality of $G^{\langle\rho\rangle}$.

Corollary 5. *The n-fold covering G_ρ of a connected graph G derived by a permutation voltage ρ is regular if and only if $|\langle\rho\rangle| = n$.*

Proof. Since $G^{\langle\rho\rangle}$ is the minimum regular covering of G which covers G_ρ by Theorem 2, it is clear that G_ρ is regular if and only if $G_\rho = G^{\langle\rho\rangle}$. In this case, $G^{\langle\rho\rangle}$ itself must be an n-fold covering and hence we have $|\langle\rho\rangle| = n$. ■

NOTE: A graph G is said to be *bipartite* if it has a vertex coloring with two colors, say black and white, such that any two vertices having the same color are not adjacent in G and it is well-known that G is bipartite if and only if G contains no odd cycle. A covering $\tilde{G}$ of G is called a *bipartite covering* of G if $\tilde{G}$ is bipartite. It has been known in [14] that there exists the *canonical bipartite covering* $b : B(G) \to G$ of G such that any bipartite covering of G covers $B(G)$, as well as the canonical regular covering $G^{\langle\rho\rangle}$.

Unfortunately, "the canonical planar covering" of G does not exist in general even if G has a planar covering. If Planar Cover Conjecture is true, then any nonplanar graph G having a planar covering must have a 2-fold planar covering. Thus, if G had the canonical planar covering, this 2-fold planar covering of G would be canonical. However we can construct easily a nonplanar graph which has two inequivalent 2-fold planar coverings, according to the result in [13]; they cannot cover each other and hence they are not canonical.

5. Abelian coverings

A covering of G is said to be *abelian* if it is a regular covering and if its covering transformation group is an abelian group. We may be able to discuss many things within the category of abelian coverings since abelian groups have the well-known strictures. We shall show here an application of our arguments in the previous section, which joins abelian coverings and coverings whose regularity is uncertain.

The following lemma gives an essential fact on abelian coverings, but it will lose its raison d'etre after we established Theorem 7, which will present the exact fact on abelian coverings:

Lemma 6. *A covering G_ρ of a connected graph G with a permutation voltage assignment ρ is covered by an abelian covering of G if and only if the canonical regular covering $G^{\langle\rho\rangle}$ of G over G_ρ is abelian.*

Proof. The sufficiency is clear since $G^{\langle\rho\rangle}$ itself covers G_ρ. Put $\tilde{G}_H = G^{\langle\rho\rangle}$ for the subgroup H in $\pi_1(G)$ that $G^{\langle\rho\rangle}$ is associated with, to use the topological arguments developed in Section 1.

Suppose that an abelian covering $\tilde{G}_N$ of G covers G_ρ. Then there is a covering projection $q : \tilde{G}_N \to \tilde{G}_H$ with $p_N = p_H \circ q$ by the canonicity of $\tilde{G}_H = G^{\langle\rho\rangle}$. This implies that $N < H$. Since $\tilde{G}_H$ is a regular covering of G, H also is a normal subgroup in $\pi_1(G)$ and the quotient $\pi_1(G)/H$ acts on $\tilde{G}_H$ as well as $\pi_1(G)/N$ does on $\tilde{G}_N$. Then we have a surjective homomorphism $\bar{q} : \pi_1(G)/N \to \pi_1(G)/H$ such that $\bar{q}(Ng) = Hg$ for each element $g \in \pi_1(G)$. Since $\pi_1(G)/N$ is abelian, so is $\pi_1(G)/H$ and hence $\tilde{G}_H$ also is an abelian covering of G. Therefore, the necessity follows. ■

As we have just discussed in the previous proof, it is easy to see that if an abelian covering covers a regular covering, then the latter becomes ableian. We shall not assume the regularity of a given covering in the following theorem:

Theorem 7. *Let G be a connected graph and let $\tilde{G}$ be a covering of G. Then there exists no abelian covering of G which covers $\tilde{G}$ if $\tilde{G}$ is not abelian.*

Proof. Suppose that there exists an ableian covering of G which covers a covering G_ρ of G. Then $G^{\langle\rho\rangle}$ is an abelian covering of G and $\langle\rho\rangle$ is an abelian group by Lemma 6. First we would like to determine the order of this abelian group $\langle\rho\rangle$.

Since G_ρ is assumed to be connected, $\langle\rho\rangle$ is transitive over $\{1,\dots,n\}$ by Lemma 3 and hence there exists an element $\tau_{i,j}$ in $\langle\rho\rangle$ such that $\tau_{i,j}(i) = j$. Consider $\tau_{1,j}$ in particular and take any element $\tau \in \langle\rho\rangle$ with $\tau(1) = j$, as well as $\tau_{1,j}$. Choose any number $h \in \{1,\dots,n\}$ and confirm what is $\tau(h)$.

$$\tau(h) = \tau \cdot \tau_{j,h}(j) = \tau \cdot \tau_{j,h} \cdot \tau_{1,j}(1)$$

Since $\langle\rho\rangle$ is abelian, we may rearrange their order and hence:

$$\tau(h) = \tau_{1,j} \cdot \tau_{j,h} \cdot \tau(1) = \tau_{1,j} \cdot \tau_{j,h}(j) = \tau_{1,j}(h)$$

This means that $\tau = \tau_{1,j}$ as permutations over $\{1,\dots,n\}$ and $\tau_{1,j}$ exists uniquely.

The n permutations $\tau_{1,1},\dots,\tau_{1,n}$ are all distinct and any element in $\langle\rho\rangle$ must be identical to one of them, depending on which number it maps 1 to. Therefore, we have $\{\tau_{1,1},\dots,\tau_{1,n}\} = \langle\rho\rangle$ as sets, and hence $|\langle\rho\rangle| = n$. By Corollary 5, the n-fold covering G_ρ must be identical to $G^{\langle\rho\rangle}$ and hence G_ρ itself is an abelian covering of G. Conversely, if G_ρ is not abelian, then there does not exist any abelian covering of G which covers G_ρ. ■

Notice that the abelian covering in the statement of Theorem 7 is "of G", but it is not "of $\tilde{G}$". We can construct an abelian covering of $\tilde{G}$ freely as $\tilde{G}^\alpha$, using an ordinary voltage assignment $\alpha : E(\tilde{G}) \to A$ for any abelian group A. However, if $\tilde{G}$ is not an abelian covering of G, then this covering $\tilde{G}^\alpha$ is not an ableian covering of G although $\tilde{G}^\alpha$ covers G.

6. Examples

The first author has discussed an example of an irregular 4-fold covering of $K_{3,3}$ and its regular 8-fold covering in [26]. The former is a planar covering while the latter is not planar but it can be embedded on the torus. If an m-fold covering of a graph covers its n-fold covering, then n divides m. This implies that the 8-fold regular covering of $K_{3,3}$ becomes the minimum regular covering which covers the irregular 4-fold covering of $K_{3,3}$ and hence the 8-fold one must be the canonical regular covering of the 4-fold one. Thus, the 8-fold regular covering of $K_{3,3}$ can be obtained as $G^{\langle\rho\rangle}$ by the way described in our arguments.

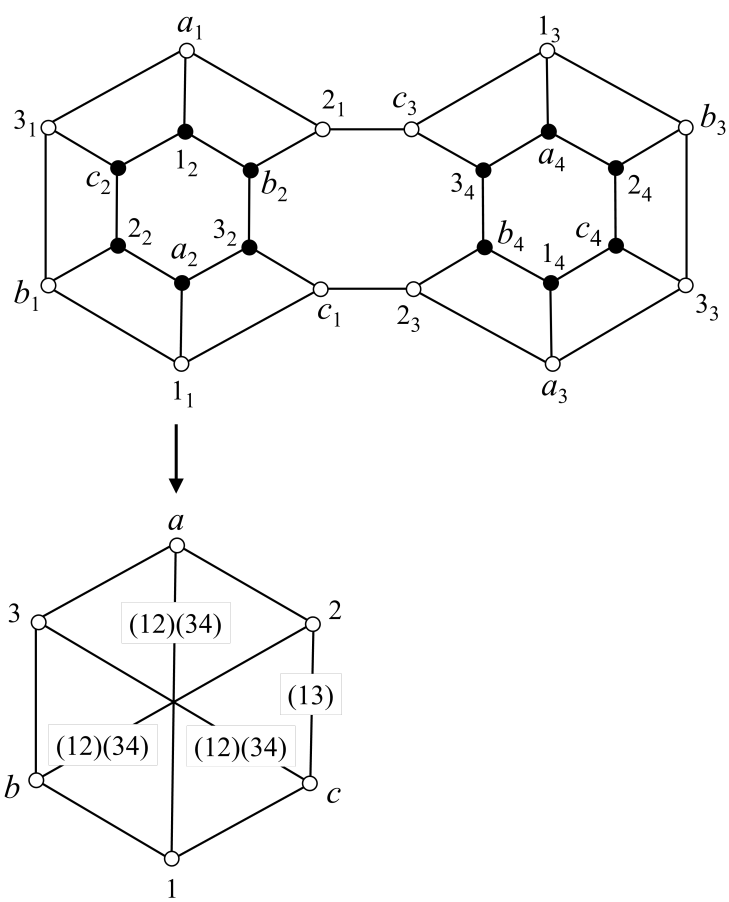

Figure 2 The 4-fold covering of $K_{3,3}$ with permutation voltages

Figure 2 presents the irregular 4-fold covering of $K_{3,3}$ and its permutation voltage assignment ρ. The labels on edges in the lower graph $K_{3,3}$ indicate the permutations over $\{1, 2, 3, 4\}$ assigned to the edges. The edges with no label receive the identity permutation id. Notice that the derived covering of $K_{3,3}$ with this permutation voltage ρ is not regular actually since two vertices black and white projecting to the same vertex cannot be transfered by any group action.

Put $\sigma = (12)(34)$ and $\tau = (13)$; the former is assigned to three diagonals $1a$, $2b$ and $3c$ of the hexagon and the latter only to the edge $2c$ placed at the right end. Then we have $\sigma\tau = (1234)$ and hence $\sigma^2 = \tau^2 = (\sigma\tau)^4 = \mathrm{id}$.

$$\begin{aligned}\langle\rho\rangle &= \langle\, \sigma, \tau \mid \sigma^2 = \tau^2 = (\sigma\tau)^4 = \mathrm{id}\,\rangle \\ &= \{\, \mathrm{id}, (13), (24), (12)(34), (13)(24), (14)(23), (1234), (1432)\,\} \\ &\cong D_4\end{aligned}$$

This means that the group $\langle\rho\rangle$ generated by σ and τ is isomorphic to the dihedral group D_4 of order 8 and that D_4 acts on the 8-fold regular covering $K_{3,3}^{\langle\rho\rangle}$, which coincides with one given in [26]. Since D_4 is not abelian, the 4-fold covering of $K_{3,3}$ given in Figure 2 admits no abelian covering of $K_{3,3}$ which covers it.

Unfortunately, we have determined the concrete form of $\langle\rho\rangle$ by hand. If there is a rapid algorithm to determine the full set of elements generated by given generators, it will help us to analyze planar coverings of nonplanar graphs according to the theory given in [26].

Acknowledgment. The contents of this paper has been presented partially in an annual conference [24] entitled "Discrete Mathematics and its Applications 2013". The authors apologize to all participants for the delay in writing it down as a paper.

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