Around Frankl Conjecture

TAKASHI NISHIMURA† AND SHIN TAKAHASHI‡

To the memory of Professor Etsuo Yoshinaga

INTRODUCTION

Let X be the finite set $\{1, 2, \ldots, n\}$ and let $\mathcal{F}$ be a system of subsets of X.

In 1979, P. Frankl proposed the following conjecture(conjecture 1.7 in [**1**], page 32). In this paper, we shall give several results to this conjecture.

CONJECTURE. *Suppose that $\mathcal{F}$ satisfies the following two conditions:*

$$\begin{aligned} &(1) \qquad \#\mathcal{F} \geq 2, \\ &(2) \qquad F, F' \in \mathcal{F} \Rightarrow F \cap F' \in \mathcal{F}. \end{aligned}$$

Then, there exists an element i of X such that the following inequality satisfies.

$$\#\{F \in \mathcal{F} | i \in F\} \leq \frac{1}{2} \#\mathcal{F}$$

Here, $\#\mathcal{F}$ means the number of elements of $\mathcal{F}$.

We say $\mathcal{F}$ *is closed with respect to intersection operator* if $\mathcal{F}$ satisfies the condition (2) of the above conjecture.

DEFINITION: We say an element I of $\mathcal{F}$ is *maximal in $\mathcal{F}$* if I satisfies the following two conditions:

$$\begin{aligned} &(1) \qquad I \neq X, \\ &(2) \qquad \#I \geq \#F \qquad \text{for any } F \text{ of } \mathcal{F} - \{X\}. \end{aligned}$$

Any $\mathcal{F}$ has a maximal element of $\mathcal{F}$. Although maximal elements are not unique in general, it is clear that the number of elements of a maximal element is unique for any $\mathcal{F}$.

†Department of Mathematics, Faculty of Education, Yokohama National University, Yokohama 240, Japan

‡41-7, Miwachou, Machida, Tokyo 195, Japan

THEOREM 1. *Let $\mathcal{F}$ be a system of subsets of X which satisfies two conditions of Frankl conjecture. Let I be a maximal element of $\mathcal{F}$.*

(1-A) Suppose that $\#I$ is equal to $n-1$ or $n-2$. Then Frankl conjecture is true.

(1-B) Suppose that $\#I$ is less than or equal to $\frac{n}{2}$. Then Frankl conjecture is true.

We can prove the following theorems 2 and 3 similarly as theorem 1-B.

THEOREM 2. *Let $\mathcal{F}$ be a system of subsets of X which satisfies two conditions of Frankl conjecture. Suppose that the following inequality holds.*

$$\#\mathcal{F} > 2^n - 2^{n-[\frac{n-1}{2}]-1}$$

Then Frankl conjecture is true.

DEFINITION: We say a system $\mathcal{F}$ of subsets of X is a *weakly abstract complex of parity type* if for any $F \in \mathcal{F}$, any $G \subset F$ with $\#G \equiv \#F \pmod 2$ must be an element of $\mathcal{F}$.

It is clear that any abstract complex is a weakly abstract complex of parity type.

THEOREM 3. *Frankl conjecture is true for any weakly abstract complex of parity type which satisfies two conditions of Frankl conjecture.*

Proofs of theorems 1, 2 and 3 are given in §§1, 2 and 3 respectively. Lastly, in §4 an approach from other viewpoint and conjectures are given .

§1. PROOF OF THEOREM 1

PROOF OF THEOREM 1-A: First, we prove theorem 1-A in the case $\#I = n-1$. In this case, without loss of generality, we may assume that $I = \{2, 3, \ldots, n\}$. We set

$$\mathcal{F}_1 = \{F \in \mathcal{F} | 1 \in F\}$$

and

$$\widetilde{\mathcal{F}_1} = \{F \in \mathcal{F} | F \subset I\}.$$

Then, we see

$$\mathcal{F} = \mathcal{F}_1 \cup \widetilde{\mathcal{F}_1}, \quad \mathcal{F}_1 \cap \widetilde{\mathcal{F}_1} = \phi.$$

Furtermore, we see easily that the map

$$\mathcal{F}_1 \to \widetilde{\mathcal{F}_1}$$

defined by

$$F \mapsto F \cap I$$

is injective. Thus, we have

$$\#\mathcal{F}_1 \leq \#\widetilde{\mathcal{F}_1}.$$

Therefore, we have

$$\begin{aligned}\#\mathcal{F}_1 &= \frac{1}{2}(\#\mathcal{F}_1 + \#\mathcal{F}_1) \\ &\leq \frac{1}{2}(\#\mathcal{F}_1 + \#\widetilde{\mathcal{F}_1}) \\ &= \frac{1}{2}\#\mathcal{F}.\end{aligned}$$

Next, we prove theorem 1-A in the case $\#I = n-2$. In this case, without loss of generality, we may assume that $I = \{3, 4, \ldots, n\}$. We set

$$\begin{aligned}\mathcal{F}_{1,2} &= \{F \in \mathcal{F} | \{1,2\} \subset F\}, \\ \widetilde{\mathcal{F}_{1,2}} &= \{F \in \mathcal{F} | F \subset I\}, \\ \mathcal{F}_1' &= \{F \in \mathcal{F} | F \cap \{1,2\} = \{1\}\}\end{aligned}$$

and

$$\mathcal{F}_2' = \{F \in \mathcal{F} | F \cap \{1,2\} = \{2\}\}.$$

Then, these four sets are mutually disjoint and

$$\mathcal{F} = \mathcal{F}_{1,2} \cup \widetilde{\mathcal{F}_{1,2}} \cup \mathcal{F}_1' \cup \mathcal{F}_2'.$$

Furthermore, we see easily that the map

$$\mathcal{F}_{1,2} \to \widetilde{\mathcal{F}_{1,2}}$$

defined by

$$F \mapsto F \cap I$$

is injective. Thus, we have

$$\#\mathcal{F}_{1,2} \leq \#\widetilde{\mathcal{F}_{1,2}}.$$

We define two numbers i_1, i_2 as

$$\#\mathcal{F}_{i_1}' \leq \#\mathcal{F}_{i_2}' \qquad \text{and} \qquad \{i_1, i_2\} = \{1, 2\}.$$

Then, we have

$$\begin{aligned}&\#\{F \in \mathcal{F} | i_1 \in F\} \\ &= \#\mathcal{F}_{i_1}' + \#\mathcal{F}_{1,2} \\ &= \frac{1}{2}(\#\mathcal{F}_{i_1}' + \#\mathcal{F}_{i_1}' + \#\mathcal{F}_{1,2} + \#\mathcal{F}_{1,2}) \\ &\leq \frac{1}{2}(\#\mathcal{F}_{i_1}' + \#\mathcal{F}_{i_2}' + \#\mathcal{F}_{1,2} + \#\widetilde{\mathcal{F}_{1,2}}) \\ &= \frac{1}{2}\#\mathcal{F}.\end{aligned}$$

∎

PROOF OF THEOREM 1-B: We set

$$\mathcal{F}_i = \{F \in \mathcal{F} | i \in F\}$$

for each i of X. Then, counting elements of elements of $\mathcal{F}$ by two different ways yields the following equality:

$$\sum_{F \in \mathcal{F}} \#F = \sum_{i \in X} \#\mathcal{F}_i. \tag{1-1}$$

Suppose that Frankl conjecture is false in this case. Then, we have

$$\sum_{i \in X} \#\mathcal{F}_i > \frac{n}{2} \#\mathcal{F}. \tag{1-2}$$

On the other hand, by the assumption of theorem 1-B, we have

$$\sum_{F \in \mathcal{F}} \#F \leq \frac{n}{2} \#\mathcal{F}. \tag{1-3}$$

(1-2) and (1-3) contradict (1-1). Therefore, Frankl conjecture is true in this case. ∎

§2. PROOF OF THEOREM 2

LEMMA 2-1. *Let $\mathcal{F}$ be a system of subsets of X which satisfies two conditions of Frankl conjecture. We suppose that $\#\mathcal{F}$ is greater than $2^n - 2^{n-k-1}$ for a certain non-negative integer k. Then the set $\mathcal{G}_k = \{G \in 2^X | \#G \leq k\}$ is included in $\mathcal{F}$.*

PROOF OF LEMMA 2-1: Suppose that there exists an element G of $\mathcal{G}_k$ which does not belong to $\mathcal{F}$. We would like to deduce a contradiction from this assumption. We set

$$\mathcal{F}^c = \{X - F | F \in \mathcal{F}\}.$$

Then, since $\mathcal{F}$ is closed with respect to intersection operator, $\mathcal{F}^c$ is closed with respect to union operator. Namely, the following holds.

$$F, F' \in \mathcal{F}^c \Rightarrow F \cup F' \in \mathcal{F}^c$$

Furthermore, we see $(X - G)$ does not belong to $\mathcal{F}^c$ since we assumed G does not belong to $\mathcal{F}$. From these properties of $\mathcal{F}^c$, we see that $\mathcal{F}^c$ satisfies the following property.

Property 2-2. *For any subset H of $X - G$, H does not belong to $\mathcal{F}^c$ or $(X - G) - H$ does not belong to $\mathcal{F}^c$.*

Therefore, we have the following inequalities:

$$\#\mathcal{F}^c \leq 2^n - 2^{\#(X-G)-1} \leq 2^n - 2^{n-k-1}. \tag{2-3}$$

However, (2-3) contradicts our assumption:

$$\#\mathcal{F}^c = \#\mathcal{F} > 2^n - 2^{n-k-1}.$$

Therefore, we see

$$\mathcal{G}_k \subset \mathcal{F}.$$

∎

We set

$$\mathcal{F}_i = \{F \in \mathcal{F} | i \in F\}$$

for any $i \in X$.

Lemma 2-4. *Let $\mathcal{F}$ be a system of subsets of X which satisfies two conditions of Frankl conjecture. Suppose that $\#\mathcal{F}$ is greater than $2^n - 2^{n-[\frac{n-1}{2}]-1}$. Then, we have*

$$\sum_{i\in X} \#\mathcal{F}_i \leq \frac{n}{2}\#\mathcal{F}.$$

Proof of lemma 2-4: We prove lemma 2-4 by induction with respect to $\#\mathcal{F}$.

(I) $\#\mathcal{F} = 2^n$.

In this case, $\mathcal{F}$ must be the set 2^X. Since $\#\mathcal{F}_i = 2^{n-1}$ for any $i \in X$, we have

$$\sum_{i\in X} \#\mathcal{F}_i = n2^{n-1} = \frac{n}{2}2^n \leq \frac{n}{2}\#\mathcal{F}.$$

(II) Suppose that lemma 2-4 holds in the case $\#\mathcal{F}' > \ell$. Under this assumption, we show lemma 2-4 holds also in the case $\#\mathcal{F} = \ell$.

Let C be an element of $2^X - \mathcal{F}$ which satisfies the following condition:

$$\#C \leq \#D \qquad \text{for any } D \text{ of } 2^X - \mathcal{F}.$$

Then, we see that the set $\mathcal{F} \cup \{C\}$ is closed with respect to intersection operator. By the induction hypothesis, we have

$$\sum_{i \in X} \#\mathcal{F}_i + \#C \leq \frac{n}{2}(\ell + 1). \tag{2-5}$$

Furthermore, by lemma 2-1 we have

$$\#C \geq [\frac{n-1}{2}] + 1. \tag{2-6}$$

(2-5) and (2-6) yield

$$\begin{aligned}\sum_{i \in X} \#\mathcal{F}_i &\leq \frac{n}{2}\ell + \frac{n}{2} - \#C \\ &\leq \frac{n}{2}\ell + \frac{n}{2} - [\frac{n-1}{2}] - 1 \\ &\leq \frac{n}{2}\ell = \frac{n}{2}\#\mathcal{F}.\end{aligned}$$

∎

PROOF OF THEOREM 2: Suppose that Frankl conjecture is false in this case. Then,

$$\#\mathcal{F}_i > \frac{1}{2}\#\mathcal{F}$$

for any $i \in X$. Thus, we have

$$\sum_{i \in X} \#\mathcal{F}_i > \frac{n}{2}\#\mathcal{F}.$$

This contradicts lemma 2-4. Therefore, Frankl conjecture is true in this case. ∎

§3. PROOF OF THEOREM 3

LEMMA 3-1. *Let $\mathcal{F}$ be a weakly abstract complex of parity type. Suppose that $n \geq 2$. Then, the following inequality holds:*

$$\sum_{F \in \mathcal{F}} \#F \leq \frac{n}{2}\#\mathcal{F}. \tag{3-2}$$

PROOF OF LEMMA 3-1: We prove lemma 3-1 by induction with respect to $n(= \#X)$.

(I) $n = 2$.

In this case, $\mathcal{F}$ must be one of the following:

$$\{\phi\},\ \{\{1\}\},\ \{\{2\}\},\ \{\phi,\{1\}\},\ \{\phi,\{2\}\},$$
$$\{\{1\},\{2\}\},\ \{\phi,\{1,2\}\},\ \{\phi,\{1\},\{2\}\},\ \{\phi,\{1\},\{1,2\}\},$$
$$\{\phi,\{2\},\{1,2\}\},\ \{\phi,\{1\},\{2\},\{1,2\}\}.$$

In each case, the inequality (3-2) holds.

(II) Suppose that lemma 3-1 holds in the case $n = k$. Under this assumption, we show lemma 3-1 holds also in the case $n = k+1$.

Let $\mathcal{F}$ be a system of subsets of $X = \{1, 2, \ldots, k+1\}$, which is a weakly abstract complex of parity type. For any i of X we set

$$\begin{aligned}\mathcal{F}_i &= \{F \in \mathcal{F} \mid i \in F\},\\ \mathcal{F}_i(*) &= \{F \in \mathcal{F} \mid i \notin F\},\\ \mathcal{F}_i(i) &= \{F - \{i\} \mid i \in F \in \mathcal{F}\}.\end{aligned}$$

Then, for any i of X we have

$$\begin{aligned}\sum_{F\in\mathcal{F}} \#F &= \sum_{F\in\mathcal{F}_i} \#F + \sum_{F\in\mathcal{F}_i(*)} \#F\\ &= \sum_{F\in\mathcal{F}_i(i)} \#F + \sum_{F\in\mathcal{F}_i(*)} \#F + \#\mathcal{F}_i\\ &\leq \frac{k}{2}(\#\mathcal{F}_i(i) + \#\mathcal{F}_i(*)) + \#\mathcal{F}_i \qquad \text{(by the induction hypothesis)}\\ &= \frac{k}{2}\#\mathcal{F} + \#\mathcal{F}_i.\end{aligned}$$

Thus, we see

$$\begin{aligned}(k+1)(\sum_{F\in\mathcal{F}} \#F) &\leq \sum_{i\in X}(\frac{k}{2}\#\mathcal{F} + \#\mathcal{F}_i)\\ &= \frac{k(k+1)}{2}\#\mathcal{F} + \sum_{i\in X}\#\mathcal{F}_i.\end{aligned} \tag{3-3}$$

From (3-3) and the equality $\sum_{F\in\mathcal{F}} \#F = \sum_{i\in X} \#\mathcal{F}_i$, the concluding inequality

$$\sum_{F\in\mathcal{F}} \#F \leq \frac{(k+1)}{2}\#\mathcal{F}$$

holds. ∎

PROOF OF THEOREM 3: Lemma 3-1 and the equality $\sum_{F\in\mathcal{F}} \#F = \sum_{i\in X} \#\mathcal{F}_i$ prove theorem 3. ∎

§4. APPENDICES

It seems to be quite difficult to generalize our theorem 1. On the other hand, we estimate that our assumptions of theorem 2 and 3 are probably too limited. At present, we do not know what type approach is hopeful for the complete answer to Frankl conjecture. Perhaps, the following proposition might be useful.

PROPOSITION 4-1. *Suppose that Frankl conjecture is true for any $\mathcal{F}$ which satisfies the following three conditions:*

$$\begin{aligned} &(1) \quad \#\mathcal{F} \geq 2, \\ &(2) \quad F, F' \in \mathcal{F} \Rightarrow F \cap F' \in \mathcal{F}, \\ &(3) \quad \phi \in \mathcal{F}. \end{aligned}$$

Then Frankl conjecture is true.

PROOF OF PROPOSITION 4-1: Let $\mathcal{G}$ be a system of subsets of $X = \{1, 2, \ldots, n\}$ which satisfies

$$\begin{aligned} &(1) \quad \#\mathcal{G} \geq 2, \\ &(2) \quad G,\ G' \in \mathcal{G} \Rightarrow G \cap G' \in \mathcal{G}, \\ &(3) \quad \phi \notin \mathcal{G}. \end{aligned}$$

It is sufficient to prove Frankl conjecture for the given $\mathcal{G}$. Let J be a minimal element of $\mathcal{G}$, namely, J satisfies

$$\#J \leq \#G$$

for any G of $\mathcal{G}$. Then, $\#J \neq 0$ since we have assumed $\phi \notin \mathcal{G}$. Since J is minimal and $\mathcal{G}$ is closed with respect to intersection operator, we see

$$J \subset G$$

for any G of $\mathcal{G}$.

We set

$$\mathcal{G}_{\hat{J}} = \{G - J \mid G \in \mathcal{G}\}.$$

Then, we see $\#\mathcal{G}_{\hat{J}} \geq 2$, $\phi \in \mathcal{G}_{\hat{J}}$ and $\mathcal{G}_{\hat{J}}$ is closed with respect to intersection operator. Therefore, from the assumption of proposition 4-1, there exists an element i of X such that the inequality

$$\#\{G \in \mathcal{G}_{\hat{J}} \mid i \in G\} \leq \frac{1}{2} \#\mathcal{G}_{\hat{J}}$$

holds. From the construction of $\mathcal{G}_{\hat{J}}$, the inequality

$$\#\{G \in \mathcal{G} \mid i \in G\} \leq \frac{1}{2} \#\mathcal{G}$$

also holds for the given $\mathcal{G}$ and the same element i of X. ∎

Lastly, we propose two conjectures relating Frankl conjecture. Conjecture 4-2 is stronger than Frankl conjecture.

CONJECTURE 4-2. *Let $\mathcal{F}$ be a system of subsets of $X = \{1, 2, \ldots, n\}$ which satisfies*

$$\begin{aligned} &(1) \qquad X \notin \mathcal{F}, \\ &(2) \qquad F, F' \in \mathcal{F} \Rightarrow F \cap F' \in \mathcal{F}. \end{aligned}$$

Then, there exists an element i of X such that the following sharp inequality satisfies.

$$\#\{F \in \mathcal{F} | i \in F\} < \frac{1}{2}\#\mathcal{F}.$$

Concerning this conjecture, slight modifications of the proof of theorem 1 show that the following theorem 1' holds.

THEOREM 1'. *Let $\mathcal{F}$ be a system of subsets of X which satisfies two conditions of conjecture 4-2. Let I be a maximal element of $\mathcal{F}$.*

(1'-A) Suppose that $\#I$ is equal to $n-1$ or $n-2$. Then conjecture 4-2 is true.

(1'-B) Suppose that $\#I$ is less than $\frac{n}{2}$. Then conjecture 4-2 is true.

Let $\mathcal{F}$ be a system of subsets of $X = \{1, 2, \ldots, n\}$. We define *the abstract complex induced by $\mathcal{F}$* (denoted by $\mathcal{F}_*$) as

$$\mathcal{F}_* = \{G \mid G \subset F \in \mathcal{F}\}.$$

CONJECTURE 4-4. *For any $\mathcal{F} \subset 2^X$, the following inequality holds.*

$$\Big(\sum_{F \in \mathcal{F}} \#F\Big)/\#\mathcal{F} \geq \Big(\sum_{F \in \mathcal{F}_*} \#F\Big)/\#\mathcal{F}_*$$

REFERENCES

[1] Akiyama, J. and Frankl, P., "Gendai Kumiawaseron (in Japanese)," first edition, Kyouritsu shuppan, Tokyo, 1987.