

RECOGNIZING RIGHT-LEFT EQUIVALENCE LOCALLY

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The present paper is a survey of the author's recent results on recognizing C^r right-left equivalence of C^∞ map-germs ($0 \leq r \leq \infty$).

We say two C^∞ map-germs are C^r right-left equivalent if they coincide under germs of appropriate C^r co-ordinate systems of the source space and the target space, where a C^0 co-ordinate system means a co-ordinate system given by a homeomorphism. We often encounter the situations where we would like to decide whether or not given two map-germs are C^r right-left equivalent. In the case that one of them is of full rank (resp. linear), the implicit function theorem (resp. the rank theorem) answers our purpose (possibly except for $r = 0$). However, how can we decide in general case? By using a simple systematic method explained in Section 4, we can obtain many results to the problem. In Section 1, we give a series of criteria for C^r right-left equivalence of C^∞ map-germs ($1 \leq r \leq \infty$). In Section 2, infinitesimal refinements of criteria for C^∞ right-left equivalence of C^∞ map-germs are given. Next, we consider C^0 right-left equivalence. In Section 3, we give a series of criteria for C^0 right-left equivalence of \mathcal{K} -equivalent map-germs. All of the results are derived from one simple idea, which is the key of our systematic method and explained exhaustively in Section 4. In Section 5 we give several applications of our results, which show how useful our method is.

The results for $r = \infty$ are all valid both in the real analytic category and in the complex analytic category as well.

1. Criteria for C^r right-left equivalence ($1 \leq r \leq \infty$). For a given C^∞ map-germ $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$, any C^r map-germ $\Phi : (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^p, 0)$ such that $\Phi(x, 0) = f(x)$ is called a C^r deformation-germ of f . A C^r deformation-germ $\Phi : (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^p, 0)$ of $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ is said to be C^r -trivial if

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there exist germs of C^r diffeomorphisms $h : (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^n \times \mathbf{R}^k, (0, 0))$ and $H : (\mathbf{R}^p \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^p \times \mathbf{R}^k, (0, 0))$ such that the following diagram (*) commutes, where $\pi : (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^k, 0)$, $\pi' : (\mathbf{R}^p \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^k, 0)$, are canonical projections:

$$(*) \quad \begin{array}{ccccc} (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) & \xrightarrow{(\Phi, \pi)} & (\mathbf{R}^p \times \mathbf{R}^k, (0, 0)) & \xrightarrow{\pi'} & (\mathbf{R}^k, 0) \\ \downarrow h & & \downarrow H & & \parallel \\ (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) & \xrightarrow{(f, \pi)} & (\mathbf{R}^p \times \mathbf{R}^k, (0, 0)) & \xrightarrow{\pi'} & (\mathbf{R}^k, 0) \end{array}$$

For given two C^∞ map-germs $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$, we consider the following conditions (i_r), (ii_r), (iii_r) and (iv_r).

(i_r) The map-germ f is C^r right-left equivalent to g .

(ii_r) There exist a germ of C^r diffeomorphism $s : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and a C^r map-germ $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), M(0))$ such that the following (a) and (b) are satisfied:

(a) $f(x) = M(x)g(s(x))$,

(b) the C^r map-germ $F : (\mathbf{R}^n \times \mathbf{R}^p, (0, 0)) \rightarrow (\mathbf{R}^p, 0)$ given by

$$F(x, \lambda) = f(x) - M(x)\lambda$$

is a C^r -trivial deformation-germ of f .

(iii_r) There exist a germ of C^r diffeomorphism $s : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and a C^r map-germ $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), M(0))$ such that (a), (b) of condition (ii_r) and the following (c) are satisfied:

(c) The C^r map-germ $G : (\mathbf{R}^n \times \mathbf{R}^p, (0, 0)) \rightarrow (\mathbf{R}^p, 0)$ given by

$$G(x, \lambda) = g(x) - M(s^{-1}(x))^{-1}\lambda$$

is a C^r -trivial deformation-germ of g .

(iv_r) There exist a germ of C^r diffeomorphism $s : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and a C^r map-germ $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), M(0))$ such that (a), (b) of condition (ii_r) and the following (d) are satisfied:

(d) The germ $(H(\{0\} \times \mathbf{R}^p), 0)$ is transverse to the germ $(\{0\} \times \mathbf{R}^p, 0)$, where H is the germ of C^r diffeomorphism of $(\mathbf{R}^p \times \mathbf{R}^p, 0)$ given in the above commutative diagram (*) with k, Φ replaced by p, F .

First, we consider rank zero cases.

THEOREM 1.1 ([15]). *Let $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be C^∞ map-germs with rank zero. Then condition (ii_r) implies condition (i_r) for $1 \leq r \leq \infty$.*

Next, we consider positive rank cases.

EXAMPLE 1.1. Let $f, g : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ be given by

$$f(x, y) = (x, y^3 + xy),$$

$$g(x, y) = (x, y^3)$$

and $M : (\mathbf{R}^2, 0) \rightarrow (GL(2, \mathbf{R}), E_2)$ be given by

$$M(x, y) = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix},$$

where E_2 is the unit 2 by 2 matrix. Then $f(x, y) = M(x, y)g(x, y)$.

It is well known that any C^∞ deformation-germ of the map-germ f is C^∞ -trivial. Thus, (ii_∞) is satisfied. However, for any $1 \leq r \leq \infty$ condition (i_r) does not hold (in fact, f and g are even not topologically right-left equivalent).

This example shows that condition (ii_r) does not necessarily imply condition (i_r) in positive rank cases. Nevertheless, the following holds under no assumptions.

THEOREM 1.2 ([15]). *Condition (iii_r) implies condition (i_r) for $1 \leq r \leq \infty$.*

Although Theorem 1.2 is interesting in itself, we prefer the C^r triviality of the linearly parametrized deformation-germ of *only one* of f or g to those of both of f and g . Thus, we are led to condition (iv_r) .

THEOREM 1.3 ([15]). *Condition (iv_r) implies condition (i_r) for $1 \leq r \leq \infty$.*

In the case $r = \infty$, we have

THEOREM 1.4 ([15]). *For any C^∞ map germs $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$, the following hold:*

- (1) $(i_\infty) \Leftrightarrow (iii_\infty) \Leftrightarrow (iv_\infty)$.
- (2) $(i_\infty) \Leftrightarrow (ii_\infty) \Leftrightarrow (iii_\infty) \Leftrightarrow (iv_\infty)$ if the rank of f is zero.

Therefore, we may answer the C^∞ recognition problem completely by using our conditions in principle.

2. Infinitesimal refinements of criteria for C^∞ right-left equivalence. First, we review infinitesimal notations briefly. For details on them, see [9], [14], [15], [21].

Let \mathcal{E}_n (resp. m_n) denote the set of C^∞ function-germs $(\mathbf{R}^n, 0) \rightarrow \mathbf{R}$ (resp. $(\mathbf{R}^n, 0) \rightarrow (\mathbf{R}, 0)$). The set \mathcal{E}_n has a natural \mathbf{R} -algebra structure and the set m_n is the unique maximal ideal in \mathcal{E}_n . For any positive integer ℓ , m_n^ℓ means the ℓ -times product of m_n . For $\ell = 0$, m_n^0 is \mathcal{E}_n .

For a C^∞ map-germ $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$, let $\theta(f)$ denote the \mathcal{E}_n -module consisting of germs of C^∞ vector fields $\zeta : (\mathbf{R}^n, 0) \rightarrow T(\mathbf{R}^p)$ such that $\pi_p \circ \zeta = f$, where $\pi_p : T(\mathbf{R}^p) \rightarrow \mathbf{R}^p$ denotes the canonical projection. By using the standard identification of $T(\mathbf{R}^p)$ with $\mathbf{R}^p \times \mathbf{R}^p$, $\theta(f)$ may be identified with the free \mathcal{E}_n -module with p -generators. When the given f is the identity map-germ $(\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$, $\theta(f)$ may be denoted by $\theta(n)$.

For a C^∞ map-germ $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$, let $Tf : (T\mathbf{R}^n, \pi_n^{-1}(0)) \rightarrow (T\mathbf{R}^p, \pi_p^{-1}(0))$ denote the tangent map-germ of f , where $\pi_n : T\mathbf{R}^n \rightarrow \mathbf{R}^n$, $\pi_p : T\mathbf{R}^p \rightarrow \mathbf{R}^p$ are canonical projections. We define

$$tf : \theta(n) \rightarrow \theta(f), \quad wf : \theta(p) \rightarrow \theta(f)$$

by $tf(\xi) = Tf \circ \xi$, $wf(\eta) = \eta \circ f$. By using tf and wf , we define

$$\begin{aligned} T\mathcal{A}(f) &= tf(m_n\theta(n)) + wf(m_p\theta(p)) \quad \text{and} \\ T_e\mathcal{A}(f) &= tf(\theta(n)) + wf(\theta(p)). \end{aligned}$$

THEOREM 2.1. *Let $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be C^∞ map-germs with rank zero. Suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), M(0))$ such that $f(x) = M(x)g(s(x))$. Suppose furthermore that there exists an integer k ($k \geq 0$) such that*

- (a) *each entry of $M - M(0)$ belongs to m_n^{k+1} and*
- (b) *$m_n^k\theta(f) \subset T_e\mathcal{A}(f)$.*

Then f and g are C^∞ right-left equivalent.

Although there are no proofs of Theorem 2.1 in a series of author's papers [13]–[18], by the proof of Theorem 2.2 below it is clear that conditions (a) and (b) of Theorem 2.1 imply the C^∞ -triviality of $f(x) - M(x)\lambda$. Thus, Theorem 2.1 follows from Theorem 1.1.

THEOREM 2.2 ([14]). *Let $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be C^∞ map-germs. Suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), M(0))$ such that $f(x) = M(x)g(s(x))$. Suppose furthermore that there exists an integer k ($k \geq 0$) such that*

- (a) *each entry of $M - M(0)$ belongs to m_n^{k+1} ,*
- (b) *$m_n^k\theta(f) \subset T_e\mathcal{A}(f)$ and*
- (c) *$m_n^k\theta(g) \subset T_e\mathcal{A}(g)$.*

Then f and g are C^∞ right-left equivalent.

Theorem 2.2 was stated (but not proved) first by A. A. du Plessis ([19], page 128). Conditions (a) and (b) (resp. (a) and (c)) of Theorem 2.2 imply the C^∞ -triviality of $f(x) - M(x)\lambda$ (resp. $g(x) - M(s^{-1}(x))^{-1}\lambda$). Thus, Theorem 2.2 follows from Theorem 1.2 (for details, see [14]).

THEOREM 2.3 ([14]). *Let $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be C^∞ map-germs. Suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), M(0))$ such that $f(x) = M(x)g(s(x))$. Suppose furthermore that there exists a positive integer k such that*

- (a) *each entry of $M - M(0)$ belongs to m_n^k and*
- (b) *$m_n^k\theta(f) \subset T\mathcal{A}(f)$.*

Then f and g are C^∞ right-left equivalent.

Conditions (a) and (b) imply condition (iv $_\infty$). Thus Theorem 2.3 follows from Theorem 1.3 (for details, see [14]). Although we can deduce infinitesimal results from Theorem 1.3 in a different way, Theorem 2.3 is the most standard infinitesimal refinement of Theorem 1.3 and quite useful as shown in the following examples and §5.3.

EXAMPLE 2.1 (taken from [12]). This example is almost the same as Example 1.1 in [14]. Let $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$ be given by

$$f(x, y) = (x, xy + y^3, xy^2 + y^{10}).$$

Thanks to D. Mond, the following have been known as information on f ([12], Theorem 4.2.2:7).

- (2.1.1) The map-germ f is 10-determined with respect to C^∞ right-left equivalence.
- (2.1.2) The map-germ f is not 9-determined with respect to C^∞ right-left equivalence.
- (2.1.3) $m_2^8\theta(f)$ is contained in $TA(f)$.

Let $N(x, y)$ be a 3 by 3 matrix with entries belonging to m_2^8 . Then, by Theorem 2.3, $g = f + Nf$ is C^∞ right-left equivalent to f . In fact, $f = (E_3 + N)^{-1}g$ and $M = (E_3 + N)^{-1}$ satisfies $M(0) = E_3$ and each entry of $M - M(0)$ belongs to m_2^8 .

EXAMPLE 2.2 (taken from [6]). Let $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$ be given by

$$f(x, y) = (x, y^3 + x^2y, y^4 + xy^2).$$

Thanks to T. Gaffney and A. A. du Plessis, the following has been known as one of information on f ([6], Example (3.6)):

$$(2.2.1) \quad m_2^3\theta(f) \subset TA(f).$$

Let $N(x, y)$ be a 3 by 3 matrix with entries belonging to m_2^3 . Then, by Theorem 2.3, $g = f + Nf$ is C^∞ right-left equivalent to f . In fact, $f = (E_3 + N)^{-1}g$ and $M = (E_3 + N)^{-1}$ satisfies $M(0) = E_3$ and each entry of $M - M(0)$ belongs to m_2^3 . Combining this result with direct co-ordinate manipulations yields the same M -determinacy result as in Example (3.6) of [6]. Note that the only information which we require is (2.2.1).

3. Criteria for C^0 right-left equivalence. A C^∞ deformation-germ $\Phi : (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^p, 0)$ of f is said to be *Thom trivial* (resp. *transversely Thom trivial*) if there exist C-regular stratifications in the sense of Bekka ([2]), \mathcal{S} of $\mathbf{R}^n \times \mathbf{R}^k$, \mathcal{T} of $\mathbf{R}^p \times \mathbf{R}^k$ and $\{\mathbf{R}^k\}$ of \mathbf{R}^k such that the following (T1) and (T2) (resp. (T1), (T2) and (T3)) hold:

(T1) The map-germ

$$(\Phi, \pi) : (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^p \times \mathbf{R}^k, (0, 0))$$

is a Thom map-germ with respect to \mathcal{S} and \mathcal{T} .

(T2) The map-germ

$$\pi' : (\mathbf{R}^p \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^k, 0)$$

is a stratified map-germ (or equivalently in this case, a Thom map-germ) with respect to \mathcal{T} and $\{\mathbf{R}^k\}$.

(T3) The stratum T_0 of \mathcal{T} , which contains the origin $(0, 0)$ of $\mathbf{R}^p \times \mathbf{R}^k$, is transverse to $\{0\} \times \mathbf{R}^k (\subset \mathbf{R}^p \times \mathbf{R}^k)$.

Here $\pi : (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^k, 0)$, $\pi' : (\mathbf{R}^p \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^k, 0)$ are canonical projections. For the definition of a C-regular stratification, see [2]. We remark that the notion of a C-regular stratification is an extended one of a Whitney stratification and it

is known that every C -regular stratification admits a controlled tube system ([2]). For the definitions of a stratified map-germ and a Thom map-germ, see [2], [7], [11].

By Thom's second isotopy lemma ([2], [7], [11]), we see that for any Thom trivial deformation-germ $\Phi : (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^p, 0)$ of f , there exist germs of homeomorphisms $h : (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^n \times \mathbf{R}^k, (0, 0))$ and $H : (\mathbf{R}^p \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^p \times \mathbf{R}^k, (0, 0))$ such that the diagram (*) in Section 1 commutes, where $\pi : (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^k, 0)$, $\pi' : (\mathbf{R}^p \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^k, 0)$ are canonical projections.

For given two C^∞ map-germs $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$, we consider the following four conditions.

(i) The map-germ f is topologically equivalent to g .

(ii) There exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), M(0))$ such that the following (a) and (b) are satisfied:

$$(a) f(x) = M(x)g(s(x)),$$

(b) the C^∞ map-germ $F : (\mathbf{R}^n \times \mathbf{R}^p, (0, 0)) \rightarrow (\mathbf{R}^p, 0)$ given by

$$F(x, \lambda) = f(x) - M(x)\lambda$$

is a Thom trivial deformation-germ of f .

(iii) There exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), M(0))$ such that (a), (b) of condition (ii) and the following (c) are satisfied:

(c) The C^∞ map-germ $G : (\mathbf{R}^n \times \mathbf{R}^p, (0, 0)) \rightarrow (\mathbf{R}^p, 0)$ given by

$$G(x, \lambda) = g(x) - M(s^{-1}(x))^{-1}\lambda$$

is a Thom trivial deformation-germ of g .

(iv) There exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), M(0))$ such that (a) of condition (ii) and the following (d) are satisfied:

(d) The C^∞ map-germ $F : (\mathbf{R}^n \times \mathbf{R}^p, (0, 0)) \rightarrow (\mathbf{R}^p, 0)$ given by

$$F(x, \lambda) = f(x) - M(x)\lambda$$

is a transversely Thom trivial deformation-germ of f .

THEOREM 3.1 ([16]). *Let $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be C^∞ map-germs with rank zero. Then condition (ii) implies condition (i).*

THEOREM 3.2 ([16]). *Condition (iii) implies condition (i).*

THEOREM 3.3 ([18]). *Condition (iv) implies condition (i).*

Note that Example 1.1

$$f(x, y) = (x, y^3 + xy)$$

$$g(x, y) = (x, y^3)$$

again shows that condition (ii) does not necessarily imply condition (i) in positive rank case.

4. **The simple systematic method.** Let $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be a C^∞ map-germ and $\Phi : (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^p, 0)$ be a C^r deformation-germ of f , where $0 \leq r \leq \infty$. Suppose that there exists a C^r \mathcal{A} -morphism from Φ to f . By the definition of a C^r \mathcal{A} -morphism, there exist C^r map-germs $h : (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^n \times \mathbf{R}^k, (0, 0))$, $H : (\mathbf{R}^p \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^p \times \mathbf{R}^k, (0, 0))$ and $\varphi : (\mathbf{R}^k, 0) \rightarrow (\mathbf{R}^k, 0)$ such that the following (4.1) and (4.2) hold (for the definition of C^r \mathcal{A} -morphism, see [17]).

(4.1) For any representatives \tilde{h} of h and \tilde{H} of H , there exist neighborhoods U of the origin in \mathbf{R}^n , V of the origin in \mathbf{R}^k and W of the origin in \mathbf{R}^p such that the restrictions $\tilde{h}|_{U \times \{\lambda\}}$ and $\tilde{H}|_{W \times \{\lambda\}}$ are C^r diffeomorphisms for any $\lambda \in V$.

(4.2) The following diagram commutes, where $\pi : (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^k, 0)$, $\pi' : (\mathbf{R}^p \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^k, 0)$, are canonical projections:

$$\begin{array}{ccccc} (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) & \xrightarrow{(\Phi, \pi)} & (\mathbf{R}^p \times \mathbf{R}^k, (0, 0)) & \xrightarrow{\pi'} & (\mathbf{R}^k, 0) \\ \downarrow h & & \downarrow H & & \downarrow \varphi \\ (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) & \xrightarrow{(f, \pi)} & (\mathbf{R}^p \times \mathbf{R}^k, (0, 0)) & \xrightarrow{\pi'} & (\mathbf{R}^k, 0). \end{array}$$

By (4.2), we may write

$$h(x, \lambda) = (h_1(x, \lambda), \varphi(\lambda)) \quad \text{and} \quad H(y, \lambda) = (H_1(y, \lambda), \varphi(\lambda)).$$

Let $\varphi'_H : (\mathbf{R}^k, 0) \rightarrow (\mathbf{R}^p, 0)$ be the C^r map-germ given by

$$(4.3) \quad \varphi'_H(\lambda) = H_1(0, \lambda).$$

The map-germ (4.3) is the key idea in our study.

We put also $h' : (\mathbf{R}^n \times \mathbf{R}^k, (0, 0)) \rightarrow (\mathbf{R}^n \times \mathbf{R}^p, (0, 0))$ as

$$h'(x, \lambda) = (h_1(x, \lambda), \varphi'_H(\lambda))$$

and $H' : (\mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^p, (0, 0, 0)) \rightarrow (\mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^p, (0, 0, 0))$ as

$$H'(x, \lambda, y) = (h'(x, \lambda), H_1(y, \lambda) - \varphi'_H(\lambda)).$$

Then we can show that $\{h', H', \varphi'_H\}$ is a C^r \mathcal{K} -morphism from Φ to F , where F is the graph deformation-germ of f given by $F(x, y) = f(x) - y$. (For details, see [17]. In [17] the argument is pursued only for C^∞ deformation-germs. However, C^r deformation-germs can be treated by the exactly parallel argument.)

Next, returning to the situations in Sections 1–3, we let $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be C^∞ map-germs. We suppose that there exist a C^r map-germ $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), M(0))$ and a germ of C^r diffeomorphism $s : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ such that $f(x) = M(x)g(s(x))$. We concentrate on considering the following C^r deformation-germ of f :

$$(4.4) \quad f(x) - M(x)\lambda.$$

This deformation-germ is linear with respect to parameter variables. Remark that the parameter space of (4.4) is p -dimensional. Thus, if there exists a C^r \mathcal{A} -morphism $\{h, H, \varphi\}$ from Φ to f , then the map-germ (4.3) is a map-germ between the same dimensional spaces. We suppose furthermore that the deformation-germ (4.4) is C^r -trivial. Then, there exists

a C^r \mathcal{A} -morphism $\{h, H, \varphi\}$ from Φ to f . Thus, from the above argument we see that there exists a C^r \mathcal{K} -morphism $\{h', H', \varphi'_H\}$ from (4.4) to the graph deformation-germ F of f . In particular, we have the following equality (for details, see [15], [16]):

$$f(h_1(x, g(s(x)))) = H_1(0, g(s(x))).$$

Finally, we can show the following.

LEMMA 4.1 ([15], [16]). *If the map-germ (4.3) is a germ of C^r diffeomorphism, then the endomorphism-germ of $(\mathbf{R}^n, 0)$ given by*

$$x \mapsto h_1(x, g(s(x)))$$

is also a germ of C^r diffeomorphism.

Thus, we see that for $0 \leq r \leq \infty$ in order to obtain C^r right-left equivalence of f and g , it suffices to find that the map-germ (4.3) is a germ of C^r diffeomorphism.

5. Several applications

5.1. C^r right-left equivalence of C^r -stable map-germs ($1 \leq r \leq \infty$)

DEFINITION 5.1. A C^∞ map-germ $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ is said to be C^r -stable if every C^∞ deformation-germ of f is C^r -trivial ($0 \leq r \leq \infty$).

There are several apparently different definitions of C^r -stability. For the relation between them, see [20]. Our definition of C^r -stability is called P - C^r -stability in [20].

From the argument in Section 4, we have the following (for the definition of C^r \mathcal{K} -versality, see [17]).

THEOREM 5.1 ([17]). *For any C^r -stable map-germ f , its graph deformation-germ is C^r \mathcal{K} -versal for $0 \leq r \leq \infty$.*

For $1 \leq r < \infty$, the uniqueness of C^r \mathcal{K} -versal deformation-germ of a given map-germ may be proved easily by a slight modification of Martinet's proof of the uniqueness of C^∞ \mathcal{K} -versality (pp. 155–156 of [1], pp. 21–22 of [8]), because in order to prove the uniqueness we need only one implication, *the C^r \mathcal{K} -versality implies the infinitesimal C^{r-1} \mathcal{K} -versality*, which is clear. Thus, by using Martinet's argument (p. 158 of [1], p. 28 of [8]), we see that Theorem 5.1 yields a C^r generalization of Mather's classification theorem ($1 \leq r \leq \infty$) without any difficulty. Note that Theorem 1.2 also yields the same generalization of Mather's classification theorem as a trivial corollary.

THEOREM 5.2 ([15], [17]). *Let $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be C^r -stable map-germs ($1 \leq r \leq \infty$). Suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), M(0))$ such that $f(x) = M(x)g(s(x))$. Then f and g are C^r right-left equivalent.*

5.2. C^0 right-left equivalence of C^0 -stable map-germs

DEFINITION 5.2. A C^∞ map germ $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ is said to be *MT-stable* if the jet extension of it is multi-transverse to the Thom-Mather canonical stratification of the jet space.

Concerning C^0 right-left equivalence of MT-stable map-germs, there is a well-known theorem due to M. Fukuda and T. Fukuda.

THEOREM 5.3 ([3]). *Let $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be MT-stable map-germs. Suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), M(0))$ such that $f(x) = M(x)g(s(x))$. Then, they are C^0 right-left equivalent.*

DEFINITION 5.3. A C^∞ map-germ $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ is said to be *Thom stable* if every C^∞ deformation-germ of f is Thom trivial.

As a consequence of the definition of MT-stability, every C^∞ deformation-germ of an MT-stable map-germ is Thom trivial (see [7], [11]). Thus, every MT-stable map-germ is Thom stable. By Thom's second isotopy lemma ([2], [7], [11]), every Thom stable map-germ is C^0 -stable in the sense of Definition 5.1. As a trivial corollary of Theorem 3.2, we obtain a generalization of Theorem 5.3.

THEOREM 5.4 ([16]). *Let $f, g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be Thom stable map-germs. Suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), M(0))$ such that*

$$f(x) = M(x)g(s(x)).$$

Then, they are C^0 right-left equivalent.

5.3. An estimate of the order of C^∞ determinacy. As an application of Theorem 2.3, we show the following.

THEOREM 5.5. *Let $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be a C^∞ map-germ. Suppose that there exist positive integers k, ℓ such that*

- (a) $m_n^k \theta(f) \subset TA(f)$ and
- (b) $m_n^\ell \theta(f) \subset TK(f)$.

Then, f is $(k + \ell - 1)$ -determined with respect to C^∞ right-left equivalence.

The set $TK(f)$ is defined by

$$TK(f) = tf(m_n \theta(n)) + f^* m_p \theta(f),$$

where $f^*(u) = u \circ f$. This set is the tangent space of the orbit through f by the action of the group \mathcal{K} which was introduced by Mather in [9]. For details on the group \mathcal{K} , $TK(f)$ and the definition of determinacy, see [9], [21]. Theorem 5.5 is a similar estimate to the well-known estimate due to Gaffney ([4]). Theorem 5.5 has been stated already in [13] without the proof. Several applications of Theorem 5.5 to divergent diagrams have been obtained in [13].

Proof. Let ${}_{k-1}\mathcal{K}$ be the set of all pairs of (s, M) , where s is a germ of C^∞ diffeomorphism $(\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), E_p)$ is a C^∞ map-germ with each entry of $M - M(0)$ belonging to m_n^k . Here, E_p is the p by p unit matrix. The set ${}_{k-1}\mathcal{K}$ is a group by the operation $(s_1, M_1) * (s_2, M_2) = (s_1 \circ s_2, M_1 M_2)$, where $s_1 \circ s_2$ is the composition of s_1 and s_2 and $M_1 M_2$ is the product of matrices of M_1 and M_2 . The group ${}_{k-1}\mathcal{K}$ is a subgroup of the group \mathcal{K} and the tangent space of the orbit through f by the action of the group ${}_{k-1}\mathcal{K}$ is

$$(5.3.1) \quad T_{{}_{k-1}\mathcal{K}}(f) = tf(m_n \theta(n)) + f^* m_p m_n^k \theta(f).$$

Condition (b) of Theorem 5.5 implies

$$(5.3.2) \quad m_n^{k+\ell} \theta(f) \subset tf(m_n^{k+1} \theta(n)) + f^* m_p m_n^k \theta(f) \subset T_{{}_{k-1}\mathcal{K}}(f).$$

By (5.3.2), we see that f is $(k + \ell)$ -determined with respect to the group ${}_{k-1}\mathcal{K}$.

Let $g : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be a C^∞ map-germ with $j^{k+\ell-1} f(0) = j^{k+\ell-1} g(0)$. Then, since $k > 0$, (5.3.2) implies

$$(5.3.3) \quad m_n^{k+\ell} \theta(g) \subset tg(m_n^{k+1} \theta(n)) + g^* m_p m_n^k \theta(g) + m_n^{k+\ell+1} \theta(g).$$

By Mather's lemma (Lemma 3.1 of [10]), (5.3.3) implies that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ and a C^∞ map-germ $M : (\mathbf{R}^n, 0) \rightarrow (GL(p, \mathbf{R}), E_p)$ such that

$$(5.3.4) \quad f(x) = M(x)g(s(x)) \quad \text{and}$$

$$(5.3.5) \quad \text{each entry of } M - M(0) \text{ belongs to } m_n^k.$$

By Theorem 2.3, (5.3.4), (5.3.5) and condition (a) of Theorem 5.5 imply that f and g are C^∞ right-left equivalent. ■

Remark 5.1. Note that in the proof of Theorem 5.5 we use Mather's lemma only for orbits by ${}_{k-1}\mathcal{K}$ group action, whose tangent spaces are much simpler than $TA(g)$.

Since $TA(f) \subset TK(f)$, Theorem 5.5 yields the following well-known estimate due to du Plessis and Wall as a trivial corollary.

THEOREM 5.6 ([19], [21]). *Let $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$ be a C^∞ map-germ. Suppose that there exists a positive integer k such that*

$$m_n^k \theta(f) \subset TA(f).$$

Then, f is $(2k - 1)$ -determined with respect to C^∞ right-left equivalence.

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