Differentiable odd functions

By

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0. Introduction

In 1942, H.Whitney showed that the differentiable even function $f(x)$ defined in a neighborhood of the origin in R was written as $g(x^2)$ and the odd function $f(x)$ was written as $xg(x^2)$ ([1]). In this note we will try to extend this result in the case of a gereral differentiable function $f(x_1, \cdots, x_n)$ defined in a neighborhood of the origin in R^n . We obtained the following result.

Theorem

If $f(x_1, \cdots, x_n)$ is a differentiable odd function for all variables $x_1, \cdots, x_n$ which means

$$f(x_1, \cdots, x_n) = -f(x_1, \cdots, x_{i-1}, -x_i, x_{i+1}, \cdots, x_n)$$

for all x_i, there exists a differentiable function $g(x_1, \cdots, x_n)$ such that

$$f(x_1, \cdots, x_n) = x_1 \cdots x_n g(x_1^2, \cdots, x_n^2).$$

About similar result for an even function, T.H.Bröcker deals with it in his text ([2]) as an Exercise.

This note is organized as follows. In § 1, we introduce the tool, called Mather-Malgrange preparation theorem, which is used in the proof of our result. In § 2, we shall prove our result.

1. Preliminaries

Let $\mathcal{E}_n$ be an R-algebra consist of smooth function germs at $0 \varepsilon R^n$ and let $\mathfrak{m}_n$ be a maximal ideal of $\mathcal{E}_n$ i.e.

$$\mathfrak{m}_n = \{ \phi \ \varepsilon \ \mathcal{E}_n \mid \phi(0) = 0 \}.$$

Let $\mathfrak{m}_n^k$ be a k-th product ideal and let $\mathfrak{m}_n^\infty = \bigcap_{k=1}^{\infty} \mathfrak{m}_n^k$. Let $f:(R^n, 0) \to (R^p, 0)$

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be a smooth map germ and let $f^*: \mathcal{E}_p \to \mathcal{E}_n$ be a homomorphism induced by f.
For $\phi \in \mathcal{E}_n$ we use the following notations

$$\hat{\phi} = \phi + m_n^\infty \in \mathcal{E}_n / m_n^\infty,$$

$$\bar{\bar{\phi}} = \phi + f^* m_p \mathcal{E}_n + m_n^\infty \in \mathcal{E}_n / (f^* m_p \mathcal{E}_n + m_n^\infty).$$

Mather-Malgrange preparation theorem

The following two conditions for $\phi_1, \cdots, \phi_k \in \mathcal{E}_n$ are equivalent:

(1) $\phi_1, \cdots, \phi_k$ generate $\mathcal{E}_n$ as $\mathcal{E}_p$-module via f^*,

(2) $\bar{\bar{\phi}}_1, \cdots, \bar{\bar{\phi}}_k$ generate $\mathcal{E}_n / (f^* m_p \mathcal{E}_n + m_n^\infty)$ as R-module.

2. Proof of Theorem

Let $\phi: (R^n, 0) \to (R^n, 0)$ be the map germ given by

$$\phi(x_1, \cdots, x_n) = (x_1^2, \cdots, x_n^2).$$

Since

$$\mathcal{E}_n / (\phi^* m_n \mathcal{E}_n + m_n^\infty) = \mathcal{E}_n / (\langle x_1^2, \cdots, x_n^2 \rangle \varepsilon_n + m_n^\infty)$$

$$= \langle \overline{\overline{x_1^{r_1} \cdots\cdots x_n^{r_n}}} \mid r_i = 0 \text{ or } 1, 1 \leqq i \leqq n \rangle R,$$

we have

$$\mathcal{E}_n = \langle x_1^{r_1} \cdots\cdots x_n^{r_n} \mid r_i = 0 \text{ or } 1, 1 \leqq i \leqq n \rangle \phi^* \mathcal{E}_n$$

from Mather-Malgrange preparation theorem.
Therefore for $f \in \mathcal{E}_n$, there exist $h_{r_1} \in \mathcal{E}_n$ such that

(1) $f(x_1, \cdots, x_n) = \sum_{r_1, \dots, r_n} h_{r_1, \cdots, r_n}(x_1^2, \cdots, x_n^2)\, x_1^{r_1} \cdots x_n^{r_n}.$

Because $f(x_1, x_2, \cdots, x_n) = -f(-x_1, x_2, \cdots, x_n)$, we obtain

(2) $f(x_1, \cdots, x_n) = \sum_{r_2, \dots, r_n} h_{1, r_2, \cdots, r_n}(x_1^2, \cdots, x_n^2)\, x_1 x_2^{r_2} \cdots x_n^{r_n}$

$$- \sum_{r_2, \dots, r_n} h_{0, r_2, \cdots, r_n}(x_1^2, \cdots, x_n^2)\, x_2^{r_2} \cdots x_n^{r_n}.$$

From (1) and (2), we obtain

(3) $f(x_1, \cdots, x_n) = \sum_{r_2, \dots, r_n} h_{1, r_2, \cdots, r_n}(x_1^2, \cdots, x_n^2)\, x_1 x_2^{r_2} \cdots x_n^{r_n}.$

Similarly, since $f(x_1, \cdots, x_n) = f(x_1, -x_2, x_3, \cdots, x_n)$, we obtain

(4) $f(x_1, \cdots, x_n)$

$$= \sum_{r_3, \dots, r_n} h_{1,1,r_3, \cdots, r_n}(x_1^2, \cdots, x_n^2)\, x_1 x_2 x_3^{r_3} \cdots x_n^{r_n}$$

$$- \sum_{r_3, \dots, r_n} h_{1,0,r_3, \cdots, r_n}(x_1^2, \cdots, x_n^2)\, x_1 x_3^{r_3} \cdots x_n^{r_n}.$$

From (3), (4), we obtain

$$f(x_1,\cdots,x_n) = \sum_{r_3,\ldots,r_n} h_{1,1,r_3,\cdots,r_n}(x_1^2,\cdots,x_n^2)\, x_1x_2x_3^{r_3}\cdots x_n^{r_n}.$$

Similarly, using $f(x_1,\cdots,x_n) = -f(x_1,\cdots,x_{i-1},-x_i,x_{i+1},\cdots,x_n)$ for $3 \leqq i \leqq n$, finally we obtain

$$f(x_i,\cdots,x_n) = x_1x_2\cdots x_n h_{1,1,\cdots,1}(x_1^2,x_2^2,\cdots,x_n^2)$$

hence let $g = h_{1,1,\cdots,1}$. This completes the proof.

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