Pluri-Genera δ_m of Normal Surface Singularities with C^*-Action

By

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Abstract. We shall give a classification of normal surface singularities with C^*-action with respect to the behavior of some pluri-genera δ_m for m. Moreover, as its application we obtain a few relations between the deformation theory and δ_m-genera of such singularities.

1. Introduction and Statements of results. Let (X, x) be a normal surface singularity. Let U be a stein neighborhood of x in X and K the canonical line bundle of U-$\{x\}$ and $\mathcal{O}(mK)$ the invertible sheaf of sections of m-th power of K. For an element ω of $\Gamma(U-\{x\}, \mathcal{O}(mK))$, by using local coodinates $\{U_\alpha, (z_\alpha^1, z_\alpha^2)\}_\alpha$ of $U-\{x\}$, we write ω as $\omega=\psi_\alpha(z_\alpha)(dz_\alpha^1\wedge dz_\alpha^2)^m$ and define a continuous (2, 2)-form $(\omega\wedge\bar{\omega})^{1/m}$ as follows:

$$(\omega\wedge\bar{\omega})^{1/m}|U_\alpha:=|\psi_\alpha(z_\alpha)|^{2/m}\left(\frac{1}{2\pi}\right)^2 dz_\alpha^1\wedge d\bar{z}_\alpha^1\wedge dz_\alpha^2\wedge d\bar{z}_\alpha^2\,.$$

Let $L^{2/m}(U-\{x\})$ be the vector subspace of $\Gamma(U-\{x\}, \mathcal{O}(mK))$ consisting of all elements which satisfy $\int_{V-\{x\}}(\omega\wedge\bar{\omega})^{1/m}<\infty$ for some neighborhood V of x in U. Then, K. Watanabe [17] defined following pluri-genera of (X, x).

DEFINITION 1.

$$\delta_m=\dim_C\Gamma(U-\{x\}, \mathcal{O}(mK))/L^{2/m}(U-\{x\}) \qquad (m\geqq 1)\,.$$

This integer is determined independently by the choice of the Stein neighborhood U. On the other hand, in [1] M. Artin defined the geometric genus p_g of normal surface singularities. Here we note that if $m=1$, $\delta_1=p_g$ by H. B. Laufer [6].

DEFINITION 2.

$$\delta:=\limsup \delta_m/m^2\,.$$

In [17], Watanabe shows that $\delta<\infty$.

Let $\pi: Z\to X$ be the minimal good resolution of normal surface singularity (X, x). When (X, x) has a (good) C^*-action, P. Orlik-P. Wagreich [9] have

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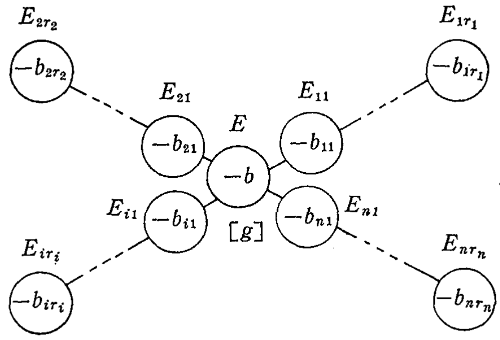
proved that the weighted dual graph can be written as follows:



$\cdots(1)$

where $-b$, $-b_{ij}$ are selfintersection numbers and g is genus of the center E in the graph, all curves E_{ij} in branchs are $\boldsymbol{P}^1$. We set $d_i/e_i := b_{j1} - \frac{1}{|b_{j2}} - \cdots - \frac{1}{|b_{jr_i}}$ (continuous fractional number), with $e_i < d_i$, and e_i and d_i are relatively prime.

Now we state our results in the following.

THEOREM 1. *Normal $\boldsymbol{C}^*$-surface singularitieis are classified as follows by behaviors of δ_m-genera for m:*

δ	δ_m	structure
>0	When $m \longrightarrow \infty$, δ_m diverges with second order	(i) $g \geqq 2$ (ii) $g=1$ and $n \geqq 1$ (iii) $g=0$ and $\sum_{i=1}^{n} \frac{d_i - 1}{d_i} > 2$
$=0$	(I) $\delta_m = 1$ for any $m \geqq 1$	$g=1$ and $n=0$ (i. e, simple elliptic singularities)
	(II) $\delta_m = \begin{cases} 0 & \text{if } m \not\equiv 0 \pmod{L} \\ 1 & \text{if } m \equiv 0 \pmod{L} \end{cases}$	$g=0$ and $\sum_{i=1}^{n} \frac{d_i - 1}{d_i} = 2$
	(III) $\delta_m = 0$ for any $m \geqq 1$	$g=0$ and $\sum_{i=1}^{n} \frac{d_i - 1}{d_i} < 2$ or cyclic quotient singularities

where we set $L := \text{l.c.m.}(d_1, \cdots, d_n)$. And furthermore if $\delta > 0$, then we have

$$\delta = \frac{1}{2} \frac{(2g - 2 + \sum_{i=1}^{n} (d_i - 1/d_i)^2}{b - \sum_{i=1}^{n} e_i/d_i}.$$

REMARK. We may assume that $b - \sum_{i=1}^{n} e_i/d_i$ is always positive (cf, [12] p. 185).

COROLLARY 1. *Let (X, x) be a normal $\boldsymbol{C}^*$-surface sigularity with data as above* (1), *then following three conditions are equivalent*;

i) $\delta_m=0$ *for any* $m\geqq 1$,

ii) $g=0$ *and* $\sum_{i=1}^{n}\frac{d_i-1}{d_i}<2$, *or cyclic quotient singularities,*

iii) *quotient singularities.*

The deformation theory of normal C^*-surface singularities has been studied by H. Pinkham [13]. From his results and Theorem 1, we obtain a few relations betweem δ and the deformation theory.

COROLLARY 2. *Let* (X, x) *be a normal* C^**-surface singularity, then we have*

i) *If* $8\delta<b-\sum_i e_i/d_i$, *then* $T_X^1(\nu)=0$ *for* $\nu>0$,

ii) *If* $4\delta<b-\sum_i e_i/d_i$, *then any deformations of* Z *to which extends blows down to deformations of* X.

Furthermore we consider the case that X is an affine cone of a projective curve.

THEOREM 2. *Let* Y *be an embedded non-singular curve in* $\boldsymbol{P}^n$ *by a holomorphic line bundle* L *with* $b:=C_1(L)\geqq 2g+1$, g *genus of* Y. *Then, for* $(C_Y, \{0\})$, *following three conditions are equivalent*;

i) $T^1_{C_Y}(\nu)>0$ *for any* $\nu>0$, ii) $8\delta<b$, iii) $4g-4<b$.

We would like here to express our gratitude to Dr. Kimio Watanabe for many suggestions for this paper.

2. Classification of normal C*-surface singularities with respect to the behavior of δ_m. The conditions which determine the analytic types of normal C^*-surface singularities were clearly described by H. Pinkham in [12, Th. 2-1] by using the results of P. Orlik-P. Wagreich [9]. And he showed how to obtain the geometric genus $p_g=\delta_1$ from its conditions (cf [12, Th. 5-1]). Recently K. Watanage-S. Ohyanagi proved the extended formula of Pinkham's.

For any integer $m\geqq 1$ and $k\geqq 0$, let $D_m^{(k)}$ be the divisor on E:

$$D_m^{(k)}:=kD-\sum_{i=1}^{n}\left[\frac{ke_i+m(d_i-1)}{d_i}\right]\cdot P_i$$

where D is an associated divisor to the conormal sheaf of E in (1) and $P_i:=E_{i1}\cap E$, and for $a\in\boldsymbol{R}$, $[a]$ is the greatest integer less than, or equal to a.

THEOREM [17, Th. 2.21].

$$\delta_m=\sum_{k\geqq 0}\dim_C H^1(E, \mathcal{O}_E((1-m)K_E+D_m^{(k)}))\,.$$

K. Watanabe has given some formula of δ_m for other singularities of several types ([17]).

PROOF OF THOREM 1. By Riemann-Roch theorem on E [4], we rewrite the formula of Theorem P-W-O as follows;

$$\delta_m = \sum_{k\in\Lambda_m^1} \{C_1(mK_E - D_m^{(k)}) - g + 1\}$$
$$- \sum_{k\in\Lambda_m^2} \{\dim_C H^0(E, \mathcal{O}((1-m)K_E + D_m^{(k)}) + C_1(mK_E - D_m^{(k)}) - g + 1\} \tag{1}$$

where we set $\Lambda_m^1 := \{k\in \boldsymbol{Z}^+;\ C_1(mK_E - D_m^{(k)}) > 2g-2\}$, $\Lambda_m^2 := \{k\in\boldsymbol{Z}^+;\ 0\leqq C_1(mK_E - D_m^{(k)}) \leqq 2g-2\}$ and $\boldsymbol{Z}^+$ the set of non-negative integers. Now we put

$$\alpha_{mi}^{(k)} := \frac{ke_i + m(d_i-1)}{d_i} - \left[\frac{ke_i + m(d_i-1)}{d_i}\right],\ \alpha_m^{(k)} := \sum_{i=1}^n \alpha_m^{(k)},$$

then $0\leqq \alpha_{mi}^{(k)} < 1$ and $0\leqq \alpha_m^{(k)} < n$. And if we put $D := 2g-2+\sum_{i=1}^n \frac{d_i-1}{d_i}$, then we have

$$C_1(mK_E - D_m^{(k)}) = mD - k(b - \sum_i e_i/d_i) - \alpha_m^{(k)} \tag{2}$$

So, by easy computations we can see

$$\Lambda_m^1 = \left\{k\in\boldsymbol{Z}^+;\ \frac{mD - \alpha_m^{(k)} - (2g-2)}{b - \sum_i e_i/d_i} > k\right\}.$$

$$\Lambda_m^2 = \left\{k\in\boldsymbol{Z}^+;\ \frac{mD - \alpha_m^{(k)} - (2g-2)}{b - \sum_i e_i/d_i} \leqq k \leqq \frac{mD - \alpha_m^{(k)}}{b - \sum_i e_i/d_i}\right\}.$$

Therefore $\sharp\Lambda_m^2 \leqq \frac{2g-2}{b-\sum_i e_i/d_i} + 1$, and moreover when $m\to\infty$, $\dim_C H^0(E, \mathcal{O}((1-m)K_E + D_m^{(k)}))$ diverges at most first order. So that the second term of (2) diverges at most first order. Now we use the notation $\sim$ in the sense that both terms are equal up to the parts which diverges at most first order. Hence we consider three cases; $D>0$, $D=0$, $D<0$. Let $D>0$, then

$$\delta_m \sim \sum_{k\in\Lambda_m^1} \{C_1(mK_E - D_m^{(k)}) - g + 1\}$$
$$\sim \sum_{k\in\Lambda_m^1} \{C_1(mK_E - D_m^{(k)})\} = \sum_{k\in\Lambda_m^1} \{mD - k(b - \sum_{i=1}^n e_i/d_i) - \alpha_m^{(k)}\}$$
$$\sim \sum_{k\in\Lambda_m^1} \left\{mD - k(b - \sum_{i=1}^n e_i/d_i)\right\} \sim \frac{D^2}{2(b - \sum_{i=1}^n e_i/d_i)} m^2. \tag{3}$$

So that if $D>0$, then

$$\delta = \frac{D^2}{2(b - \sum_{i=1}^n e_i/d_i)} > 0.$$

And the singularities satisfying $D>0$ are classified in next three cases; i) $g\geqq 2$, ii) $g=1$ and $n>0$, iii) $g=0$ and $\sum_i \frac{d_i-1}{d_i} > 2$.

Next we consider the case: $D=0$. They are classified two cases; i) $g=1$ and $n=0$ (i.e, simple elliptic singularities [14]), ii) $g=0$ and $\sum_i \frac{d_i-1}{d_i} = 2$. For case i) we can easily see by Theorem *P-W-O* that $\delta_m=1$ for any $m\geqq 1$. For cace ii) we have

$$C_1(mK_E-D_m^{(k)})=-k(b-\sum_i e_i/d_i)-\alpha_m^{(k)}\leqq 0\,.$$

Since $b-\sum_i e_i/d_i>0$, if $k>0$, then $C_1(mK_E-D_m^{(k)})<0$. So we may only consider the case: $k=0$. Since $C_1(mK_E-D_m^{(k)})=-\alpha_m^{(0)}\leqq 0$, we have $\delta_m=-\alpha_m^{(0)}+1$. Hence we have

$$\delta_m=\begin{cases} 0 \text{ if } m(d_i-1)\not\equiv 0 \bmod d_i \text{ for all } i\,, \\ 1 \text{ if } m(d_i-1)\equiv 0 \bmod d_i \text{ for all } i\,, \end{cases} \tag{4}$$

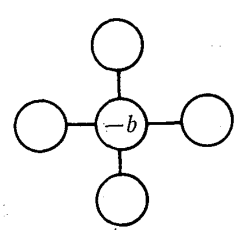
Moreover since d_i and d_i-1 are relatively prime, and $\alpha_{mi}^{(0)}=\dfrac{m(d_i-1)}{d_i}-\left[\dfrac{m(d_i-1)}{d_i}\right]$, we obtain

$$\delta_m=\begin{cases} 0 \text{ if } m\not\equiv 0 \bmod \text{l.c.m.}(d_1,\cdots,d_n)\,, \\ 1 \text{ if } m\equiv 0 \bmod \text{l.c.m.}(d_1,\cdots,d_n)\,, \end{cases} \tag{5}$$

Finally we consider the remainder case: $D<0$. Since $C_1(mK_E-D_m^{(k)})=mD-k(b-\sum_{i=1}^n e_i/d_i)-\alpha_m^{(k)}<0$ for any $m\geqq 1$, we have that $\delta_m=0$ for any $m\geqq 1$. Therefore we have the desired classification saying in Theorem 1.

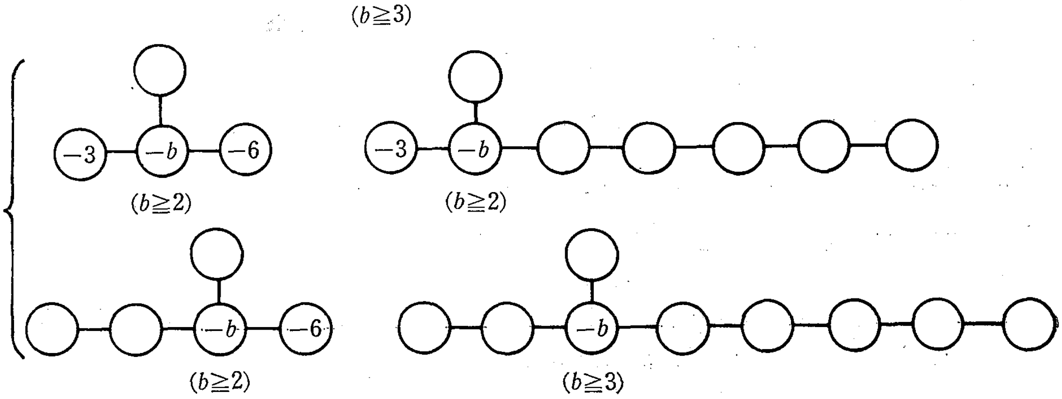
REMARK. We consider the singularities belonging to the type II) of the case $\delta=0$. The combinations $(d_1,\cdots,d_n)$ of positive integers which satisfies $\sum_{i=1}^n \dfrac{d_i-1}{d_i}=2$ are only following four types: (2, 2, 2, 2), (2, 3, 6), (2, 4, 4), (2, 3, 3). Therefore the weighted dual graphs of these singularities are exhausted by the following list:

(2, 2, 2, 2)

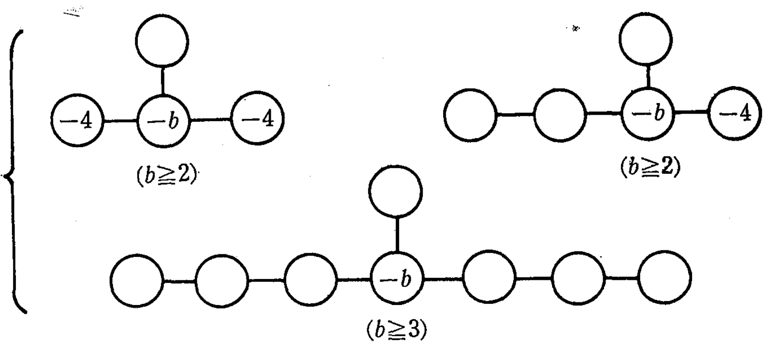


$(b\geqq 3)$

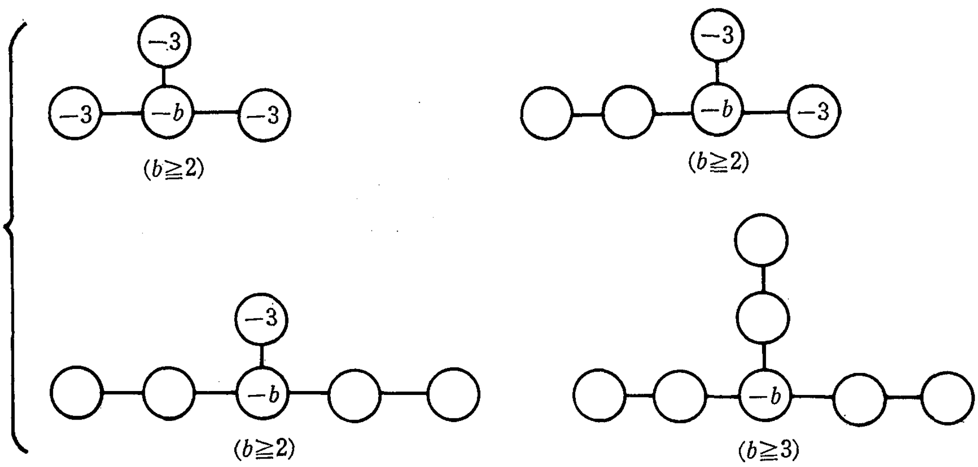
(2, 3, 6)



(2, 4, 4)



(3, 3, 3)



where we adopted the convention that ○=(−2). Moreover we note that above singularities are rational, but not Gorenstein. So it seems that the equations of above singularities are fairly complicated. For (2, 2, 2, 2)-type, when $b=3$, H. Pinkham computed its equation ([12]).

If a finite group G acts on $\boldsymbol{C}^2$ and no elements $g\in G$ fixes a line $\boldsymbol{C}^1$, then the quotient space $\boldsymbol{C}^2/G$ is a normal analytic space ([3]). The singularity which holomorphically isomorphic to a singularity on $\boldsymbol{C}^2/G$ is called the quotient singularity. In [2], E. Brieskorn proved that the analytic type of the quotient singularity is determined only by the graph of the minimal resolution and classified the dual graphs of quotient singularities.

PROOF OF COROLLARY. 1. The equivalence i) $\Leftrightarrow$ ii) follows from Theorem 1, and it is easy to see that iii) $\Leftrightarrow$ ii) from Definition 1 and the definition of the quotient singularity. So it suffices to show that ii) $\Rightarrow$ iii). The combinations $(d_1, \cdots, d_n)$ of positive integers which satisfies $\sum_{i=1}^{n}\frac{d_i-1}{d_i}<2$ are following four types: (2, 2, n) $n\geqq 2$, (2, 3, 3), (2, 3, 4), (2, 3, 5). So by the results of E. Brieskorn [2], these singularities are quotient singularities.

REMARK. The equivalence i) $\Leftrightarrow$ iii) was already proved by K. Watanabe, without the assumption "with $\boldsymbol{C}^*$-action".

3. Deformations and δ of normal C^*-surface singularities. For definitions of the deformation of singularities and $T_X^1(\nu)$, we do not describe them in here. We refer [11], [13], [15] as good references for these articles. In this section we study a few relations of deformations and δ of normal C^*-surface singularities as applications of Theorem 1.

Let (X, x) be a normal C^*-surface singularity with a data of (1) in 1. Then H. Pinkham [13] proved following two theorems.

THEOREM P-1. *If* $b>4g-4+2n+\sum_{i=1}^{n}(e_i-2)/d_i$, *then* $T_X^1(\nu)=0$ *for any* $\nu>0$ (i.e., negative grading).

THEOREM P-2. *If* $b>2g-2+n+\sum_{i=1}^{n}(e_i-1)/d_i$, *then any deformation of* Z *to which extends belows down to a deformation of* X.

Now we prove Corollary 2.

PROOF OF COROLLARY 2. i), if $\delta>0$, since $\delta=\dfrac{1}{2}\dfrac{(2g-2+\sum_i(d_i-1/d_i))^2}{b-\sum_i e_i/d_i}$, we obtain the equivalence:

$$8\delta<b-\sum_i e_i/d_i \Leftrightarrow b>4g-4+\sum_i (e_i-1)/d_i\,.$$

If $\delta=0$, then by Theorem 1 we can see that such singularities are all satisfying the inequality in Theorem P-1.

Proof of ii) is similary done as i) from Theorem P-2.

Let Y be a non-singular curve with genus g and embedded in $\boldsymbol{P}^n$ by a holomorphic line bundle L. D. Mumfold [7] proved that if $b=C_1(L)\geqq 2g+1$, then the affine cone C_Y is normal. And in [8], he proved that if $b>4g-4$, then $T_{C_Y}^1(\nu)=0$ for all $\nu>0$. H. Pinkham [11] gave the elementary proof for the latter results, and generalized the results to Theorem P-1 for normal C^*-surface singularities. In the following, we prove the converse of Mumfold's result by using Pinkham's technique.

PROPOSITION. *Let* $b\geqq 2g+1$. *If* $T_{C_Y}^1(\nu)=0$ *for all* $\nu>0$, *then we have* $b>4g-4$.

PROOF. We may only consider the case: $g\geqq 2$. Let Θ_Y be the tangent sheaf of Y and N_Y the normal sheaf of the embedding $Y \hookrightarrow \boldsymbol{P}^n$. Then we have following sheaf exact sequence:

$$0 \longrightarrow \mathcal{O}_Y(\nu) \longrightarrow \mathcal{O}_Y(\nu+1)^{n+1} \longrightarrow \Theta_{P^n}|_Y\otimes\mathcal{O}_Y(\nu) \longrightarrow 0$$

$$0 \longrightarrow \Theta_Y(\nu) \longrightarrow \Theta_{P^n}|_Y\otimes\mathcal{O}_Y(\nu) \longrightarrow N_Y(\nu) \longrightarrow 0\,,$$

where we use the notation $\mathcal{F}(\nu)$ for the tensor product of a locally free sheaf $\mathcal{F}$ with the sheaf of sections of the ν-th power $\nu\cdot L$ of L. By the definition of $T_{C_Y}^1(\nu)$, we have an exact sequence ([11], p. 38):

$$H^0(Y, \mathcal{O}_Y(\nu+1))^{n+1} \longrightarrow H^0(Y, N_Y(\nu)) \longrightarrow T_{C_Y}^1(\nu) \longrightarrow 0\,.$$

From above sequences we obtain the following commutative diagram:

$$\begin{tikzcd}
 & & & & & H^1(Y,\, \Theta_{Pn}|_Y \otimes \mathcal{O}_Y(\nu)) \\
H^0(Y,\, \mathcal{O}_Y(\nu)) \arrow[dr] & & & & H^1(Y,\, \Theta_Y(\nu)) \arrow[ur] & \\
 & H^0(Y,\, \mathcal{O}_Y(\nu+1))^{n+1} \arrow[rr, "\alpha_\nu"] \arrow[dr, "\beta_\nu"'] & & H^0(Y,\, N_Y(\nu)) \arrow[r] \arrow[ur] & T^1_{C_Y}(\nu) \arrow[r] & 0 \\
 & & H^0(Y,\, \Theta_{Pn}|_Y \otimes \mathcal{O}_Y(\nu)) \arrow[ur, "\gamma_\nu"'] & & &
\end{tikzcd}$$

So that it suffices to show that if $b \leqq 4g-4$, then there is $\nu > 0$ such that γ_ν is not surjective. From sheaf isomorphism:

$$\Theta_{Pn}|_Y \cong (\mathcal{O}_Y(\nu+1))^{n+1},$$

we have

$$\begin{aligned} H^1(Y,\, \Theta_{Pn}|_Y \otimes \mathcal{O}_Y(\nu)) &\cong (H^1(Y,\, \mathcal{O}_Y(\nu+1))^{n+1} \\ &\cong (H^0(Y,\, \mathcal{O}_Y(K_Y-(\nu+1)\cdot L))^{n+1} \end{aligned}$$

Since $g \geqq 2$ and $b \geqq 2g+1$, then

$$C_1(K_Y-(\nu+1)\cdot L) = 2(g-1)-(\nu+1)b < 0 \qquad \text{for any} \quad \nu \geqq 0.$$

Hence we have that $H^1(Y,\, \Theta_{Pn}|_Y \otimes \mathcal{O}_Y(\nu)) = 0$ for any $\nu \geqq 0$. Therefore it suffices to show that if $b \leqq 4g-4$, then there is $\nu > 0$ such that $H^1(Y,\, \Theta_Y(\nu))$ is not zero. But from eqivalence: $b \leqq 4g-4 \Leftrightarrow C_1(2K_Y-L) \geqq 0$, and isomorphism: $H^1(Y,\, \Theta_Y(1)) \cong H^0(Y,\, \mathcal{O}(2K_Y-L))$, we have the desired result.

PROOF OF THEOREM 2. The equivalence ii) $\Leftrightarrow$ iii) follows from Theorem 1, and iii) $\Rightarrow$ i) from Theorem P-1. Moreover i) $\Rightarrow$ iii) follows from the above proposition.

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