Functional Laws of the Iterated Logarithm for Sums of Functions of Random Variables

By

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1. Introduction.

Recently, Philipp [15] established functional laws of the iterated logarithm for the empirical distribution functions of some weakly dependent random variables. Philipp's method relies on the exponential bounds which are obtained by the martingale approximation of a sequence of bounded random variables (Propositions 3.3.1 and 4.2.1 in Philipp [15]).

Yoshihara [19] proved that the analogous exponential bound is obtained from the convergence rate to normality and showed that the above Philipp's result remains valid under less restrictive conditions in the mixing case.

In this paper, using Philipp's technique and Yoshihara's one, well shall prove the functional law of the iterated logarithm for

$$(1.1)\qquad f_N(t)=(2N\log\log N)^{-1/2}\sum_{k=1}^{N}\{H_{Nk}(t,\xi_k)-EH_{Nk}(t,\xi_k)\},\quad N\geqq 3,$$

under some conditions, (Theorem 2). In Section 5, among others, we show some examples which are new, (Theorems 3-5).

2. Assumptions and the main results.

Let $\{\xi_i\}$ be a (not necessarily strictly stationary) e-dimensional vector-valued sequence of weakly dependent random variables. Let

$$(2.1)\qquad \{H_{nk}(t,y),\ k=1,2,\cdots;\ n=1,2,\cdots\}$$

be a sequence of Borel measurable functions defined on $T\times R^e$, where T is a finite closed interval or an infinite one.

In what follows, we always assume that

$$(2.2)\qquad EH^2_{nk}(t,\xi_k)<\infty\qquad (k=1,2,\cdots;\ n=1,2,\cdots)$$

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for all $t \in T$ and for any integer n ($\leqq 1$), put

$$\eta_{nk}(t)=H_{nk}(t, \xi_k)-EH_{nk}(t, \xi_k), \tag{2.3}$$

$$S_n(t)=\sum_{k=1}^{n} \eta_{nk}(t), \tag{2.4}$$

and

$$s_n^2(t)=E|S_n(t)|^2. \tag{2.5}$$

Next, let $D(T)$ be the space of functions f on T that are right continuous and have finite left-hand limits. Let $\{C_j\}$ be a monotone increasing sequence of finite closed intervals such that $T=\bigcup_{j=1}^{\infty} C_j$. The mapping $p_j: D(T)\to R^1$ defined as

$$p_j(x)=\sup_{t\in C_j}|x(t)| \tag{2.6}$$

is a seminorm. We shall endow $D(T)$ with the topology defined by the metric d which constructed in the following way:

$$d(x, y)=\sum_{j=1}^{\infty} 2^{-j}\{p_j(x-y)/(1+p_j(x-y))\}. \tag{2.7}$$

For x, $x_n \in D(T)$, let

$$x^{(j)}=x|_{C_j}; \qquad x_n^{(j)}=x_n|_{C_j}. \tag{2.8}$$

We shall consider the following assumptions:

ASSUMPTION I. For each j ($j=1, 2, \cdots$), there exist a positive number τ and a closed interval $[c_j, d_j](\subset C_j)$ such that for each n the interval $[c_j, d_j]$ can be devided into disjoint intervals $I_{j,i}^{(n)}$ ($i=1, 2, \cdots$) for which

$$|I_{j,i}^{(n)}|\geqq n^{-\tau}, \qquad \bigcup_{i=1}^{\infty} I_{j,i}^{(n)}=[c_j, d_j], \tag{2.9}$$

where $|I|$ denote the length of the interval I, and

$$|H_{nk}(t, y)-H_{nk}(s, y)|\leqq M_0|t-s| \tag{2.10}$$

provided that s and t belong to the same interval $I_{j,i}^{(n)}$.

ASSUMPTION II. (IIA). There exist a nonnegative function v on $T\times T$ and positive number γ, α_1, α_2, A_1 and A_2 such that for each j and N (fixed) and for any s, t ($s\neq t$) in C_j and m and n ($0\leqq m<m+n\leqq N$) the following inequalities hold:

$$v^2(s, t)\leqq M_{1,j}^2|t-s|^{\gamma} \tag{2.11}$$

$$\left|n^{-1}E\left(\sum_{j=m+1}^{m+n}(\eta_{Nj}(t)-\eta_{Nj}(s))\right)^2-v^2(s, t)\right|\leqq A_1|t-s|^{\gamma}n^{-\alpha_1}, \tag{2.12}$$

$$\Delta(s, t)=\sup_z\left|P\left(v^{-1}(s, t)\left(\sum_{j=m+1}^{m+n}(\eta_{Nj}(t)-\eta_{Nj}(s))\right)<\sqrt{n}\,z\right)-\Phi(z)\right| \tag{2.12}$$

$$\leqq A_2 n^{-\alpha_2} \quad \text{for } |s-t|\geqq N^{-1} \text{ and } v(s, t)>0.$$

where M_{1j} is some constant and

(2.14) $$\Phi(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-u^2/2}du\,.$$

(IIB) for each $t\in T$,

(2.15) $$s_n^2(t)=n\sigma^2(t)(1+o(1)) \qquad \text{as } n\to\infty\,,$$

if $\sigma^2(t)>0$.

(IIC) Let j be an arbitrary positive integer. For arbitrary m, $t_i\in C_j$ ($i=1, 2, \cdots, m$) and $\beta_i\in R^1$ ($i=1, \cdots, m$) such that

(2.16) $$v^2=\liminf_{n\to\infty} n^{-1}E(\sum_{k=1}^{n}\sum_{i=1}^{m}\beta_i\eta_{n,k}(t_i))^2>0$$

the inequality

(2.17) $$\limsup_{n\to\infty}\frac{|\sum_{k=1}^{n}(\sum_{i=1}^{m}\beta_i\eta_{nk}(t_i))|}{v(2n\log\log n)^{1/2}}\leqq 1 \qquad \text{a.s.}$$

holds.

Let $X=\{X(t): t\in T\}$ be a separable real-valued, sample continuous Gaussian process with mean zero and continuous covariance $R(s, t)$ satisfying

(2.18) $$E(X(t)-X(s))^2\leqq g(|t-s|)\,, \qquad t,\ s\in C_j$$

for any j (fixed) where g is a continuous nondecreasing function such that $g(|u|)\leqq A|u|^{\alpha_4}$ for some $\alpha_4>0$.

Now, let

(2.19) $$f_N(t)=\frac{S_N(t)}{(2N\log\log N)^{1/2}}\,, \qquad t\in T\,.$$

Then, we have the following theorem.

THEOREM 1. *Let* $\{\xi_i, -\infty<i<\infty\}$ *be a strictly stationary sequence of random variables. Let* $X=\{X(t), t\in T\}$ *be a Gaussian process defined above. Suppose that Assumptions* (I), (IIA) *and* (IIB) *are satisfied. Suppose that*

(2.20) $$N^{-1/2}S_N(\cdot)\xrightarrow{D} X\,.$$

Furthermore, suppose that there is a ρ $(0<\rho<\gamma)$ *such that*

(2.21) $$(1-\rho)/2<\delta=\min(\alpha_1, \alpha_2)\,.$$

Then, for each j $(\geqq 1)$ *and* ε (>0), *there is with probability one a random index* $N_0=N_0(\varepsilon)$ *such that*

(2.22) $$|f_N(t)-f_N(s)|\leqq c|t-s|^{(\gamma-\rho)/2}+\varepsilon$$

for all pairs (s, t) $(s, t\in C_j)$ *and all* $N\geqq N_0$, *where* $f_N(t)$ $(t\in T)$ *is the function defined by* (2.19), *and* c *is an absolute constant.*

Next, let $H(R)$ be the reproducing kernel (r.k.) Hilbert space with r.k. $R(s, t)$

and $\|\cdot\|_H$ the norm of $H(R)$. The following theorem is a general functional law of the iterated logarithm which can be applied to many satistical problems.

THEOREM 2. *Suppose that in addition to the hypotheses of Theorem* 1 *Assumption* (IIC) *is satisfied and the covariance function* $R(s, t)$ *is positive definite Then, the sequence* $\{f_N(t), N\geqq 3\}$ *is with probability one relatively compact in* $D(T)$ *and the set of limit points of the sequence coincides with the set*

$$K=\{h\in H(R)\,|\,\|h\|_H\leqq 1\}. \tag{2.23}$$

(*cf. Philipp* [15],*Berkes and Philipp* [3] *and Yoshihara* [19]).

REMARK. Using Theorem 2, we can prove new results. As examples, we show Theorems 3-5 in Section 5.

3. Proofs.

The following lemma is prove by the same method used in the proof of Lemma 2 in Yoshihara [19].

LEMMA 3.1. *Let j and N be fixed. If the hypotheses of Theorem* 1 *are satisfied, then*

$$\begin{aligned} &P(|\sum_{k=H+1}^{H+Q}\eta_{Nk}(t)-\sum_{k=H+1}^{H+Q}\eta_{Nk}(s)|\geqq 3Al^{(\gamma-\rho)/2}(2Q\log\log Q)^{1/2}) \\ &\leqq c\{\exp(-M_{1j}^{-2}l^{-\rho}A^2\log\log Q)+A^{-2}Q^{-\delta}l^{\delta}\} \end{aligned} \tag{3.1}$$

uniformly for all pairs (s, t) $(s, t\in C_j)$, all H and all Q, where A and c are some positive constants and $l=|t-s|\geqq N^{-1}$.

LEMMA 3.2. *Suppose that Assumption* (I) *is satisfied. Let m and p are arbitrary positive integers. If s and $s+mp\in C_j$, then*

$$\begin{aligned} &\sup_{s\leqq t\leqq s+mp}|\sum_{k=H+1}^{H+Q}\eta_{Nk}(t)-\sum_{k=H+1}^{H+Q}\eta_{Nk}(s)| \\ &\leqq 3\max_{1\leqq i\leqq m}|\sum_{k=H+1}^{H+Q}(\eta_{Nk}(s+ip)-\eta_{Nk}(s+(i-1)p)|+M_0pQ \end{aligned} \tag{3.2}$$

uniformly in $H\geqq 0$, where Q is an arbitrary positive integer.

PROOF. We shall only consider the case $H=0$. The proofs of other cases are analogous, If both s and t lie in an interval $I_{j,i}^{(N)}$, then by (2.10)

$$|\sum_{k=1}^{Q}\eta_{Nk}(t)-\sum_{k=1}^{Q}\eta_{Nk}(s)|\leqq\sum_{k=0}^{Q}|\eta_{Nk}(t)-\eta_{Nk}(s)|\leqq M_0pQ. \tag{3.3}$$

If $s\in I_{j,i}^{(N)}$ and $t\in I_{j,i+1}^{(N)}$ ($I_{j,i}^{(N)}$ and $I_{j,i+1}^{(N)}$ being adjacent intervals), then $s+p\in I_{j,i+1}^{(N)}$ and so by (2.10)

$$
\begin{aligned}
&|\sum_{k=0}^{Q}(\eta_{Nk}(t)-\eta_{Nk}(s))| \\
(3.4)\qquad &\leqq |\sum_{k=0}^{Q}(\eta_{Nk}(s+p)-\eta_{Nk}(s))|+|\sum_{k=0}^{Q}(\eta_{Nk}(s+p)-\eta_{Nk}(t))| \\
&\leqq |\sum_{k=0}^{Q}(\eta_{Nk}(s+p)-\eta_{Nk}(s))|+M_0 pQ\,.
\end{aligned}
$$

Thus, we have (3.2) from (3.4) and the proof is completed.

(a) Now, we proceed to prove Theorem 1. To prove Theorem 1, it is enough to show that the conclusion of Theorem 1 holds for each C_j ($j=1, 2, \cdots$). Since C_j is finite and closed for each $j(\geqq 1)$, so if we can show that the conclusion holds for the interval [0, 1], then the conclusions in the general cases are easily obtained by the completely analogous method to the above special case. Hence, we shall consider the case where the interval [0, 1]. We use Philipp's method in [15].

For integers P and Q ($\geqq 1$), let

$$
(3.5)\qquad Z(P, Q, t_1, t_2)=|\sum_{k=P+1}^{P+Q}(\eta_{Nk}(t_2)-\eta_{Nk}(t_1))| \qquad (0\leqq t_1<t_2\leqq 1)
$$

Let N be sufficiently large. Put $n=[\log N/\log 2]$ and $m=[(\log N)^{1/2}]$ where $[s]$ denotes the largest integer p such that $p\leqq s$. We write N, t_1 and t_2 as follows:

$$
\begin{aligned}
&N=2^n+\sum_{j=1}^{n}\varepsilon_j 2^{j-1}=2^n+\sum_{j=d}^{n}\varepsilon_j 2^{j-1}+\theta_0 2^d \\
(3.6)\qquad & \\
&t_i=a_i 2^{-m}+\sum_{k=m+1}^{d} b_{ik}2^{-k}+\theta_i 2^{-d} \qquad (i=1, 2)
\end{aligned}
$$

where $\varepsilon_j=0, 1$, $b_{i,k}=0, 1$ and $0\leqq\theta_i<1$ ($i=0, 1, 2$), and $d=[n/2]$. We note that from Lemma 3.2 (with $m=1$)

$$
(3.7)\qquad Z(P, Q, h2^{-d}, (h+\theta)2^{-d})\leqq Z(P, Q, h2^{-d}, (h+1)2^{-d})+M_0 Q 2^{-d}
$$

and

$$
\begin{aligned}
&Z(P, Q, s, t)\leqq Z(P, Q, a_1 2^{-m}, a_2 2^{-m}) \\
(3.8)\qquad &\quad +\sum_{j=1}^{2}\sum_{i=m+1}^{d} Z(P, Q, a_{j,i}2^{-i}, (a_{j,i}+1)2^{-i}) \\
&\quad +\sum_{j=1}^{2} Z(P, Q, a_{j,d+1}2^{-d}, (a_{j,d+1}+1)2^{-d})+2M_0 Q 2^{-d}\,.
\end{aligned}
$$

Put

$$
(3.9)\qquad \chi(k)=(2k\log\log k)^{1/2} \qquad (k\geqq 3)\,.
$$

Let A be a positive number such that $AM_{1j}^2/M_0\geqq 2$. We define the following events:

$$E_n(a_1, a_2)=\{Z(0, 2^n, a_1 2^{-m}, a_2 2^{-m})\geqq A((a_2-a_1)2^{-m})^{(\gamma-\rho)/2}\chi(2^n)\}$$

$$E_n=\bigcup_{0\leqq a_1, a_2<2^m} E_n(a_1, a_2)$$

$$F_n(k, b)=\{Z(0, 2^n, b2^{-k}, (b+1)2^{-k})\geqq A2^{-k(\gamma-\rho)/2}\chi(2^n)\}$$

$$F_n=\bigcup_{m<k\leqq d}\ \bigcup_{0\leqq b<2^k} F_n(k, b)$$

$$G_n(b_1, b_2, j, h)=\{Z(2^n+h2^j, 2^{j-1}, b_1 2^{-m}, b_2 2^{-m}) \geqq A((b_2-b_1)2^{-m})^{(\gamma-\rho)/2}(n-j)^{-2}\chi(2^n)\} \tag{3.10}$$

$$G_n=\bigcup_{0\leqq b_1, b_2<2^m}\ \bigcup_{d\leqq j\leqq n}\ \bigcup_{0\leqq h<2^{n-j}} G_n(b_1, b_2, j, h)$$

$$H_n(k, b, j, h)=\{Z(2^n+h2^j, 2^{j-1}, b2^{-k}, (b+1)2^{-k}) \geqq A2^{-(\gamma-\rho)k/2}(n-j)^{-2}\chi(2^n)\}$$

$$H_n=\bigcup_{d\leqq j\leqq n}\ \bigcup_{m<k\leqq j/2}\ \bigcup_{0\leqq b<2^k}\ \bigcup_{0\leqq h<2^{n-j}} H_n(k, b, j, h)$$

LEMMA 3.3. *Suppose that the conditions of Theorem* 1 *are satisfied. Then, with probability one only a finite number of events* E_n, F_n, G_n *and* H_n *occur.*

The proof of this lemma is completely analogous to the proof of Lemma 3.3.8 in Philipp [15] and so is omitted.

PROOF OF THEOREM 1. The proof is easily obtained from Lemma 3.3 (see, the proof of Theorem 3.1 in Philipp [15]).

(b) Next, we shall consider Theorem 2. For the sequence $\{f_N\}$, defined by (2.19), let

$$f_n^{(j)}=f_n|_{C_j} \qquad (j=1, 2, \cdots) \tag{3.11}$$

where $\{C_j\}$ is the sequence of the closed intervals defined in Assumption (I). Let $H_j(R)$ be the r.k. Hilbert space with r.k. $R(t, s)$ restricted to $C_j\times C_j$ and $\|\cdot\|_{H_j}$ the norm of $H_j(R)$.

LEMMA 3.4. *Suppose that the hypotheses of Theorem* 2 *are satisfied. If for each* j *and for almost all* ω *the set of limit points of* $\{f_N^{(j)}, N\geqq 3\}$ *concides with the set*

$$K_j=\{h\in H_j(R)\,|\,\|h\|_{H_j}\leqq 1\}, \tag{3.12}$$

then the conclusion of Theorem 2 *holds.*

PROOF. The proof is easily obtained by the same method as the one used in the proofs of Lemma 3.4 and 3.5 in Mangano [9].

PROOF OF THEOREM 2. By Lemma 3.4, it suffices to prove that the conclusion of Theorem 2 holds for each j. Hence, as in the proof of Theorem 1, we need only to show the case $C_j=[0, 1]$. But the proof in the case is obtained by the completely same method as the one used in the proof of Theorem 3.2 in Philipp [15] and so is omitted.

4. Modification.

In this section, we shall consider the case where $H_{Nj}(t, x)$ $(0\leqq t\leqq 1)$ $(j=1, \cdots, N)$ do not satisfy (2.10), but

(4.1) $$G_{Nj}(t, x)=H_{Nj}\left(\frac{[tN]}{N}, x\right) \quad (0\leqq t\leqq 1) \quad (j=1, \cdots, N)$$

satisfy (2.10).

For a sequence $\{\xi_j\}$ of random variables, let

(4.1) $$\zeta_{Nj}(t)=G_{Nj}(t, \xi_j)-EG_{Nj}(t, \xi_j) \qquad (0\leqq t\leqq 1).$$

Further, put

(4.3) $$X_N(t)=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\eta_{Nj}(t) \qquad (0\leqq t\leqq 1)$$

where $\eta_{Nj}(t)$'s are the ones defined by (2.3) and

(4.4) $$Y_N(t)=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\zeta_{Nj}(t) \qquad (0\leqq t\leqq 1).$$

Then, it is obvious that

(4.5) $$Y_N(t)=X_N\left(\frac{[tN]}{N}\right) \qquad (0\leqq t\leqq 1).$$

PROPOSITION. *Suppose that for some* α (>0), δ (>0) *and* K $(\geqq 0)$

(4.6) $$E|X_N(t)-X_N(s)|^{\alpha}\leqq\frac{K}{N^{1+\delta}}|t-s|$$

if $|t-s|\leqq 1/N$. *Then, for any* $\varepsilon>0$

(4.7) $$P\{\sup_{0\leq t\leq 1}|X_N(t)-Y_N(t)|>\varepsilon\ i.o.\}=0.$$

PROOF. Firstly, we note that

$$P(\sup_{0\leq t\leq 1}|X_N(t)-Y_N(t)|>\varepsilon)$$
$$\leqq\sum_{k=0}^{N-1}P\left(\sup_{0\leq t<1/N}\left|X_N\left(t+\frac{k}{N}\right)-X_N\left(\frac{k}{N}\right)\right|>\varepsilon\right).$$

So, by the method of the proof of Theorem 12.3 in Billingsley [4] and (4.6)

$$P(\sup_{0\leq t\leq 1}|X_N(t)-Y_N(t)|>\varepsilon)\leqq\sum_{k=0}^{N-1}\frac{K}{\varepsilon^{\alpha}N^{1+\delta}}\frac{1}{N}=\frac{K}{\varepsilon^{\alpha}N^{1+\delta}}.$$

Hence, by the Borel-Cantelli lemma, we have the desired conclusion.

By Proposition, we can easily prove the analogous result to Theorem 2 if we use $\{Y_N(t), 0\leqq t\leqq 1\}$ instead of $\{X_N(t), 0\leqq t\leqq 1\}$.

5. Examples.

(I) Strassen's version of the log log law. Let

$$H_{Nk}(t,\ x)=\frac{x}{\sigma^2}I\Big(t-\frac{k}{N}\Big), \qquad k=1, \cdots, N,\ t\geqq 0 \tag{5.1}$$

where $I(x)=1$ if $x\geqq 0$, $I(x)=0$ if $x<0$. Let $\{\xi_j\}$ be a sequence of random variables with $E\xi_j=0$ and $E|\xi_j|^{2+\delta}<\infty$ for some $\delta\geqq 0$. Then

$$S_N(t)=\frac{1}{\sigma}\sum_{j=1}^{[Nt]}\xi_j \tag{5.2}$$

where σ is some positive constant suitably chosen. Hence, Theorem 2 implies Strassen's version of log log law for the sequence $\{\xi_j\}$ (cf. Strassen [15], Oodaira and Yoshihara [12], Yoshihara [18]). We remark here that analogous results for weighted sums or some weakly dependent random variables such as martingale, mixingale, etc. are easily obtained. (cf. Chow and Teicher [4] and McLeish [9]).

(II) Normalized sums of moving average processes. Let $\{x_j\}$ be a sequence of i. i. d. random variables with zero mean. Define ξ_j by

$$\xi_j=\sum_{k=-\infty}^{\infty}c_{k-j}x_k \tag{5.3}$$

where $\sum_{k=-\infty}^{\infty}c_k^2<\infty$. Further, let X_n be the process defined by

$$X_n(t)=\{\Psi(n)\}^{-1/2}\sum_{j=1}^{k}\xi_j \qquad \text{for } t=t_{n,k}=\frac{\Psi(k)}{\Psi(n)} \quad (k=0, 1, \cdots, n) \tag{5.4}$$

where $\Psi(n)=\mathrm{Var}\,(S_n)\uparrow\infty(n\to\infty)$ and B_γ is the Gaussian process with correlation function

$$B_\gamma(s,\ t)=(s+t-|s^{1/\gamma}-t^{1/\gamma}|^\gamma)/2\,. \tag{5.5}$$

Using the method of the proof of Theorem 2 in Davydov [6] we can prove that if $E|x_j|^{2k}<\infty$, $k\geqq 2$, and $\Psi(n)=n^\gamma h(n)$, $2/(k+2)<\gamma\leqq 1$ where $h(n)$ is a slowly varying function, then $X_n\xrightarrow{D}B_\gamma$ in $D[0, 1]$. Hence, putting $I_{j,k}^{(n)}=[t_{n,k}, t_{n,k+1})$ and

$$H_{nk}(t,\ x)=\frac{n^{1/2}}{\Psi^{1/2}(n)}xI(t-t_{n,k})\,, \tag{5.6}$$

from Theorem 2 we have the following theorem which is new.

THEOREM 3. *Let*

$$f_n(t)=\frac{X_n(t)}{(2N\log\log N)^{1/2}}, \quad t\in T, \quad (n\geqq 1)\,. \tag{5.7}$$

Suppose that the above conditions are satisfied. Then, the sequence $\{f_n\}$ *is rela-*

tively compact and the set of its limit points coincides with the unit ball of the r.k. Hilbert space $H(B_\gamma)$ with r.k. B_γ defined by (5.5).

(III) Functional laws of the iterated logarithm for empirical distribution functions. Next, let

$$H_{Nk}(t, x)=I(t-x), \qquad k=1, \cdots, N, \ 0\leqq t\leqq 1. \tag{5.8}$$

Let $\{\xi_j\}$ be a sequence of random variables distributed uniformly over $[0, 1]$. Then

$$S_N(t)=\sum_{j=1}^{N}\{I(t-\xi_j)-t\}. \tag{5.9}$$

Hence, in this case, Theorem 2 implies a functional law of the iterated logarithm for empirical distribution functions. (cf. Finkelstein [7], Philipp [15], Berkes and Philipp [3] and Yoshihara [19]).

(IV) An estimator of a biometric function. Yang [17] proved a weak convergence theorem for a sequence of estimators $e_x^{(n)}$ of the life expectancy at stage x, i.e.,

$$r_x=\int_x^\infty T(v)dv/T(x) \qquad \text{for } x\in[0, \infty) \tag{5.10}$$

where $T(x)=1-F(x)$ is the survival function.

Let $\{\xi_j\}$ be a sequence of i.i.d. nonnegative random variables each having pdf $f(x)$, $x\geqq 0$ and df F. Suppose that $E|\xi_j|^4<\infty$. Let

$$T_n(x)=\sum_{j=1}^{n} I(\xi_j-x) \qquad \text{for every } x\in[0, \infty) \tag{5.11}$$

and

$$\hat{e}_x^{(n)}=(T_n(x))^{-1}\int_x^\infty (T_n(v))dvI(\xi_{(n)}-x) \tag{5.12}$$

where $\xi_{(n)}=\max_{1\leqq j\leqq n}\xi_j$.

Now, put

$$G_{Nk}(t, x)=xI\left(F(x)-\frac{[tN]}{N}\right), \qquad (0\leqq t\leqq 1)\ (k=1, 2, \cdots, N) \tag{5.13}$$

and

$$S_N^*(t)=\sum_{j=1}^{N}\{G_{Nj}(t, \xi_j)-EG_{Nj}(t, \xi_j)\}. \tag{5.14}$$

Hence, the remark in Section 4 is applicable to the sequence $\{V_n\}$, defined by

$$V_n(t)=n^{-1/2}S_n^*(t), \qquad t\geqq 0. \tag{5.15}$$

Thus, from the proof of Theorem 1 in Yang [17], we have the following theorem which is new.

THOREM 4. *Let $t=F(x)$. Let e_x and $\hat{e}_x^{(n)}$ be as given in* (5.8) *and* (5.10).

Let $U=\{U(t)|t\in[0, b]\}$ $(0<b<1)$ *be a Gaussian process with mean zero and covariance function*

$$\Gamma(s, t)=(1-s)^{-2}(1-t)^{-2}\{(1-s)(1-t)\sigma^2(t, 1)-t(1-s)\theta^2(t, 1)\} \quad 0\leqq s\leqq t\leqq b, \tag{5.16}$$

where

$$\theta(t, u)=E\{\xi_1 I_{(t,u]}(F(\xi_1))\}, \quad \textit{and} \quad \sigma^2(t, u)=\mathrm{Var}\{\xi_1 I_{(t,u]}(F(\xi_1))\} \tag{5.17}$$

and $I_{(t,u]}(\alpha)$ *is the indicator of the semi-closed interval* $(t, u]$.

If the covariance function $\Gamma(s, t)$ *is positive definite, then the* sequence $\{f_N(t), N\geqq 3\}$ *defined by*

$$f_N(t)=(N\log\log N)^{-1/2}(\hat{e}^{(N)}_{F^{-1}_{(t)}}-e_{F^{-1}_{(t)}}) \qquad \text{for } t\in[0, b], \tag{5.18}$$

is with probability one relatively compact in $D[0, \infty)$ *and the set of limit points of the sequence coincides with the set*

$$K=\{h\in H(\Gamma)\,|\,\|h\|_H\leqq 1\}. \tag{5.19}$$

REMARK. It is obvious from the proof of Yang [17] that the above result is easily extended to the mixing case. (cf. Oodaira and Yoshihara [12]).

(V) Normalized sums of induced order statistics. Let $\{Z_j\}=\{(X_j, Y_j): -\infty<j<\infty\}$ be a sequence of i. i. d. two-dimensional random vectors. Let $F(x)$ denote the marginal cdf of X_1 which is continuous. We define induced order statistics $Y_{n1}, \cdots, Y_{nn}$ as $Y_{nk}=Y_j$ if $X_{nk}=X_j$. Let $m(x)$ denote the conditional expectation and $\sigma^2(x)$ the conditional variance of Y_1 given $X_1=x$, and let

$$\Psi(t)=\int_{-\infty}^{F^{-1}(t)}\sigma^2(x)dF(x) \quad \text{and} \quad \varphi(t)=\Psi^{-1}(t\Psi(1)), \qquad 0\leqq t\leqq 1. \tag{5.20}$$

Bhattacharya [1] proved that under some additional conditions the sequence of the processes X_n defined by

$$X_n(t)=(n\Psi(1))^{-1/2}\sum_{j=1}^{[n\varphi(t)]}(Y_{nj}-m(X_{nj})), \qquad 0\leqq t\leqq 1, \tag{5.21}$$

converges weakly to a Brownian motion. Now, define another sequence of the processes Y_n (appeared in a different from in Bhattacharya [2]) by

$$Y_n(t)=(n\Psi(1))^{-1/2}\sum_{F(X_j)\leq\varphi(t)}(Y_j-m(X_j)), \qquad 0\leqq t\leqq 1. \tag{5.22}$$

Then, the sequence $\{Y_n\}$ also converges weakly to a Brownian. Now, we put

$$H_{nj}(t, (x, y))=\Psi(1)^{-1/2}(y-m(x))I(\varphi(t)-F(x)) \tag{5.23}$$

and

$$\eta_{nj}(t)=H_{nj}(t, (X_j, Y_j))-EH_{nj}(t, (X_j, Y_j)). \tag{5.24}$$

The following result is new.

THEOREM 5. *Suppose that* (i) *F is continuous,* (ii) *$\sigma^2(x)$ is of bounded variation,* (iii) *for some M (>0)*

$$\beta_p(x) \leqq M\sigma^2(x), \qquad p=3, 4, 6, \tag{5.25}$$

where

$$\beta_p(x) = E\{|Y_1 - m(x)|^p | X_1 = x\} \tag{5.26}$$

and

$$\text{(iv)} \qquad \int_{\varphi(s)}^{\varphi(t)} \sigma^2(x) dF(x) \leqq K|t-s|.$$

Then, the sequence $\{f_N(t), N \geqq 3\}$ defined by

$$f_N(t) = (2N \log\log N)^{-1/2} \sum_{j=1}^{N} \eta_{Nj}(t), \qquad 0 \leqq t \leqq 1, \tag{5.27}$$

is with probability one relatively compact in $D\,[0, 1]$ and the set of limit points of the sequence coincides with the set

$$K = \left\{ h \in C[0, 1] \,\middle|\, \int_0^1 (h'(t))^2 dt \leqq 1,\ h(0) = 0 \right\}. \tag{5.28}$$

PROOF. To prove Theorem 5, it is enough to show (4.6). In fact, if $|t-s| \leqq 1/N$, then

$$\begin{aligned}
&E|Y_N(t) - Y_N(s)|^6 \\
&\leqq \frac{K}{N^3}\Big[N\int_{\varphi(s)}^{\varphi(t)} \beta_6(x) dF(x) + N^2 \int_{\varphi(s)}^{\varphi(t)} \beta_3(x) dF(x) \int_{\varphi(s)}^{\varphi(t)} \beta_3(x) dF(x) \\
&\quad + N^2 \int_{\varphi(s)}^{\varphi(t)} \beta_4(x) dF(x) \int_{\varphi(s)}^{\varphi(t)} \sigma^2(x) dF(x) \\
&\quad + N^3 \Big\{\int_{\varphi(s)}^{\varphi(t)} \sigma^2(x) dF(x)\Big\}^3\Big] \\
&\leqq K\Big\{\frac{|t-s|}{N^2} + \frac{|t-s|}{N} + |t-s|^3\Big\} \leqq \frac{K}{N^2}|t-s|,
\end{aligned}$$

which implies (4.6). Hence, we have the desired conclusion.

REMARK. When $\{Z_j\}$ is a strong mixing stationary process, then $\{Y_n(t)\}$ converges weakly to a Gaussian process (not necessarily Brownian motion) under some conditions on the mixing coefficient. Hence, in this case, we can also obtain by Theorem 2, a result corresponding to Theorem 5 under suitable additional conditions.

6. Concluding remarks.

Throughout the paper, we have treated the family of random variables defined by (2.3), i.e., $\eta_{Nk}(t) = H_{n,k}(t, \xi_k) - EH_{n,k}(t, \xi_k)$ $(k=1, \cdots, n;\ n=1, 2, \cdots)$.

Yoshihara [20] have obtained some results concerning the weak convergence problem of $\{n^{-1/2}\sum_{j=1}^{n}\eta_{nj}(t),\ t\in[a, b]\}$.

However, in general cases, the sequences $\{\eta_{Nj}(t)\}$ and $\{\eta_{Nj}(t)-\eta_{Nj}(s)\}$ constitute triangular arrays of random variables for every fixed t and s, even if $\{\xi_i\}$ is a sequence of independently and identically distributed random variables. So, it seems to be uneasy to find general methods which assert the validity of Assumptions (IIA) and (IIC).

But, if $\{\eta_{Nj}(t)-\eta_{Nj}(s)\}$ is a sequence of independent random variables or a certain sequence of weakly dependent random variables, then using the known results (especially, convergence rates to normality) we can easily check whether (IIA) holds or not.

For example, let $N=kn$ where $n=O(N^{\nu})(0<\nu<1)$. If for some absolute constant $K>0$, for $|t-s|\geqq N^{-1}$ and for any $m(0\leqq m\leqq N-n)$

$$E|n^{-1}\sum_{j=m+1}^{m+n}(\eta_{Nj}(t)-\eta_{Nj}(s))|^{3}\leqq Kv^{3}(s, t),$$

then the sequence

$$\left\{\frac{1}{\sqrt{n}\,v(s, t)}\sum_{i=(j-1)n+1}^{jn}(\eta_{Nj}(t)-\eta_{Nj}(s)),\ j=1, \cdots, k\right\}$$

becomes a sequence of weakly dependent random variables with uniformly bounded third moments. So, we can obtain the rate of convergence to normality.

Similarly, if for any m-tuple $(\beta_1, \cdots, \beta_m)$ of real numbers and any m-tuple $(t_1, \cdots, t_m)$ $\{\sum_{1}^{m}\beta_i\eta_{nk}(t_i)\}$ constitute a sequence of independent random variables or a certain sequence of weakly dependent random variables, then to the sequence we can apply the known results concerning the laws of the iterated logarithm and ascertain the validity of Assumption (IIC).

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