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Low Frequency Divergence in Quantum Field Theory

By

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Abstract

The well known divergences due to the energy conservations in the intermediate states are considered, from the point of view of the graphical structure and of the field reactions.

§ 1. Introduction

In contrast to the ultraviolet divergence, the low frequency divergences due to the coincidence of poles in transition matrices or in transition probabilities in the quantum field theory are generally regarded as a defect of the perturbation theory. These difficulties should disappear in principle. However, they stand sometimes in the way of computational works with the Feynman-Dyson's covariant S-matrix formalism, especially in the process involving mesons. Therefore, the special discussion of these divergences is also a matter of practical importance.

In quantum electrodynamics, the low frequence divergences are conveniently classified into three cathegories; the first of them appears from the single pole of matrix elements corresponding to the energy conservation in the intermediate states. As is well known, the divergences of this type are of quite formal nature and they raise no difficulties, as they are removed by well known procedure of the amplitude-renormalization of the wave functions, or by other suitable limiting processes, such as the principal value evaluation, etc. Therefore, they are out of our consideration. The second type divergence originates from the matrix element which is finite certainly but not uniformly. A typical example of this divergence is so called "*infra-red catastroph*", that is, the matrix elements of bremsstrahlung are certaintly finite, but their total transition probability diverges at the low frequency limit. The divergence of the third kind comes from the coincidence of poles in a transition matrix. As is well known, the coincidence of the displaced poles corresponds to the excitation of the real process or to other competing processes. Therefore, the divergence of this type is regarded as a "*resonance*

*) The content of the present work was published in Japanese more one or two years ago, separated to several papers. So, we summarize here our main considerations.

catastrophe" and they will disappear whenever the effects of field reactions, i.g., the effect of radiation damping, etc., are properly taken into account.

In the meson theory, it is evident that no infra-red catastrophe can appear on account of the non-vanishing meson mass. The reason why this is so is readily seen if we note the fact that the infra-red catastrophe originates from the vanishing energy denominators in the perturbation theory. As a simplest example we consider the process of bremsstrahlung in Fig. 1. As the real emission of a quantum k on the transition from free particle state p into a free state $p-k$ is prohibited by the conservation law, the matrix element of this process is certainly finite, except only for one case $k=0$, where the energy denominator of the intermediate propagator vanishes. The element is finite in almost everywhere, however, it is not uniform. But in the corresponding meson process, $k=0$ is excluded by non-vanishing meson mass. This is the reason why no infra-red catastrophe appears in the emission process of meson.

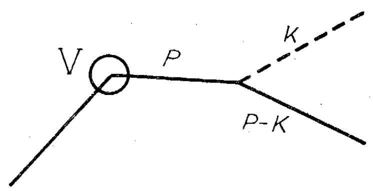

Fig. 1. V: external field of force

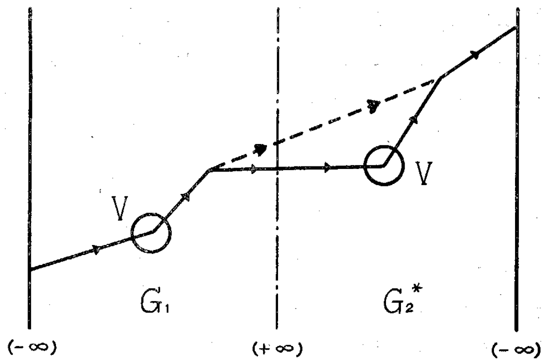

Fig. 2

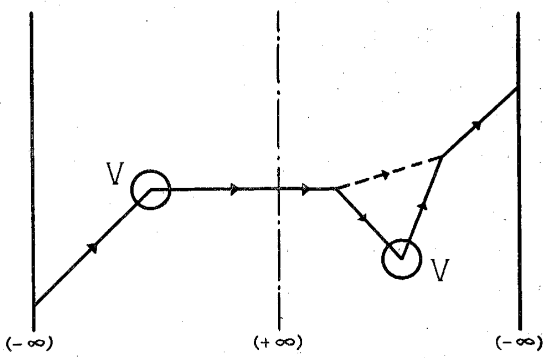

Fig. 3

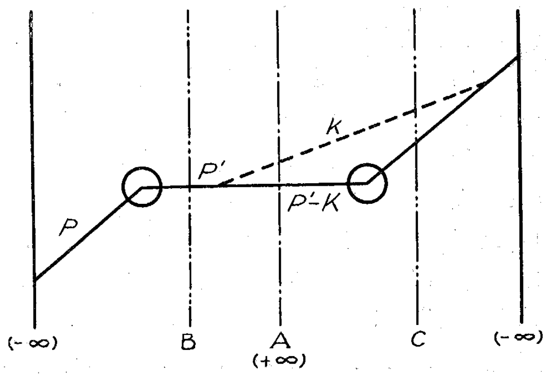

Fig. 4

But in the case of the corresponding process of meson-decay where the meson p in intermediate state decays into two lighter particles, a free particle state is allowed as the intermediate state. Therefore, new divergence corresponding to "*infra-red catastrophe*" re-appears in this decay-process. Let us now consider this "*infra-red-like catastrophe*" divergence. For this purpose, it is convenient to introduce a Feynman-diagram of transition probability. This diagram is constructed from two ordinary Feynman-diagrams G_1 and

G_2, as follows. We define here the conjugate diagram G^* of G, which is obtained by reversals of all direction of the propagating particles in G. Put G_2^* at the right edge of G_1 and join each lines smoothly as illustrated in Fig. 2, then we obtain the Feynman-diagram of the transition probabilities constructed from G_1 and G_2. In the same way, we have Fig. 3 as a Feynman-diagram of a radiative correction term for the transition probability of the elastic scattering by an external field of force. Here it is notable that Fig. 2 has the same structure as that of Fig. 3 with different position of the "*line of observation*" $(+\infty)$. A part of the transition probability, which results from a cross term of the matrix elements H_1 and H_2 corresponding to the graphs G_1 and G_2 respectively,

$$W=\frac{2\pi}{h}\langle H_1 H_2\rangle \tag{1}$$

is readily obtained from these diagrams according to the usual rules of Feynman, provided that the "*internal*" lines which cross the line $(+\infty)$ in Fig. 2 must be regarded as the out-going free waves in the final state, that is to say, the observation- (or projection-) operators to the final states are hidden on this line.

The problem of the infra-red catastrophe has been discussed by many authors,[1] and clarified that the infra-red divergence appeared in the bremsstrahlung is cancelled out by that of the radiative correction of the elastic scattering process. Recently, T. Kinoshita[2] analyzed this problem in more detail and demonstrated that cancellations are accomplished term by term between the parts of transition probabilities that have the Feynman-diagram of same structure. Extending his result, it is shown that the same procedure can be successfully applied to the case of the *infra-red-like catastrophe* divergence in the meson-decay process and finite rate of decrease of initial state probality is obtained in each order approximation of perturbation. This method gives, however, no finite results for each one of the component process. This situation often limits the practical applicability of this method, inspite of its theoretical interest. In order to obtain the finite result for each component process, we have to take into account the effects of the field reactions, such as the effects of radiation damping. The effect of radiation damping is introduced here entirely within the frame-work of Feynman-Dyson's covariant theory of S-matrix. A covariant generalizatien of Heitler's theory of radiation damping has been proposed by T. Miyajima and N. Fukuda.[3] So far as the effect of radiation damping is concerned, our

1) F. Bloch and A. Nordsieck, Phys. Rev. 52 (1937), 1045 (L)
W. Pauli and M. Fierz, Nuovo Cimento 15 (1938), 167.
W. Braunbek and E. Weinmann, Z. S. f. phys. 110 (1938), 360.
2) T. Kinoshita, Prog. Theor. Phys. 5 (1950), 1045 (L).
3) N. Fukuda and T. Miyajima, Prog. Theor. Phys. 5 (1950), 849.

method gives, of course, equivalent results with that of Miyajima and Fukuda. The difference consists in the way of taking into account the virtual processes. As our method is developed completely in conformity with the Dyson's S-matrix theory[4], it is convenient for the renormalization, for S-matrix theoretical characterization and for application to the practical problems.

§ 2. Infra-red-like catastrophe aspects.[5),6)]

In this section, let us show that the *infra-red-like catastrophe* divergence which appeared in the meson-decay process is removed by the same procedure as used in the theory of *infra-red catastrophe.* In order to clarify the correspondence to the ordinary infra-red catastrophe, we consider the meson-analogue of bremsstrahlung, i. e. the process in which a heavy meson (mass M) decays into two lighter mesons (mass m and m') in the course of deflection by an external field of force V. (See Fig. 1) In contrast to the ultra-violet divergence, the type of the relevant particles is of minor importance. Therefore, we may choose the simplest model for each particle without any loss of generality. As an illustrating example, let us consider a part of the transition probability, the Feynman-diagram which is illustrated in Fig. 4. Using following the simplest model;

$U(x)$	: charged heavy scalar field of mass M	
$\psi(x)$	: charged light scalar field of mass m	(2.1)
$\phi(x)$	: neutral scalar field of mass m'	
$V(x)$	: external field	

and their mutual interactions;

$$H(x) = g(\bar{U}\psi + \bar{\psi}U)\phi \tag{2.2}$$

$$K(x) = \bar{U}\,U\,V \tag{2.3}$$

$$K'(x) = \bar{\psi}\,\psi\,V \tag{2.4}$$

the transition probability W_A of the decay process "A"[**)] is calculated as

$$W_A = \frac{g^2}{4(2\pi)^8}\frac{1}{E_p}\int_{\epsilon_k=\mu^2}^{\epsilon_{max}}\frac{dK}{\epsilon_k}\int\frac{dP'}{e_{p'-k}}\frac{|V(P-P')|^2}{[E_p^2-E_{p'}^2][e_{p'-k}-e_{p-k}]}\times$$

$$\times\,\delta(E_p-e_{p'-k}-\epsilon_k) \tag{25}$$

where $(E_p, e_p, \epsilon_p) \equiv \sqrt{P^2+(M^2, m^2, \mu^2)}$ respectively. The integand has two poles. As is easily seen, one of them corresponds to the energy conservation

4) F. J. Dyson, Phys. Rev. 75 (1949), 1736.
5) D. Ito, Prog. Theor. Phys. 6 (1951), 1020 (L) and 1022 (L).
6) J. Edden, Proc. Roy. Soc. 210 (1952), 388.
**) The process "A" in Fig. 4 is the same with Fig. 2.

in the intermediate state B, in Fig. 4, and the other corresponds to the energy conservation in the intermediate state C. In the vicinity of these poles, W_A diverges logarithmically and takes the following asymptotic forms.

$$W_A(B)=\frac{g^2}{4(2\pi)^8}\frac{1}{E_p}\int\frac{dK\,dP'}{2E_{p'}\,e_{p'-k}\,\epsilon_k}\frac{|V(P-P')|^2}{e^2_{p'-k}-e^2_{p-k}}\frac{\delta(E_p-e_{p'-k}-\epsilon_k)}{e_{p'-k}+\epsilon_k-E_{p'}} \tag{2.6}$$

in this case $(\epsilon_k\sim E_{p'}-e_{p'-k})$,

$$W_A(C)=\frac{g^2}{4(2\pi)^8}\frac{1}{E_p}\int\frac{dK\,dP'}{2e_{p-k}\,e_{p'-k}\,\epsilon_k}\frac{|V(P-P')|^2}{E^2_p-E^2_{p'}}\frac{\delta(E_p-e_{p-k}-\epsilon_k)}{e_{p'-k}-e_{p-k}} \tag{2.7}$$

in this case $(e_{p'-k}\doteqdot e_{p-k})$

In analogy with the theory of ordinary infra-red catastrophe, let us examine the transition probability W_B of the elastic scattering process "B" in Fig. 4.**) In this case, we have to assume the line B as the line of the observation. W_B is calculated as follows;

$$W_B=\frac{g^2}{4(2\pi)^8}\frac{1}{E_p}\int_{\epsilon_k=\mu}^{\infty}\frac{dK}{\epsilon_k}\int\frac{dP'}{E_{p'}}\frac{|V(P-P')|^2}{[e^2_{p-k}-(E_p-\epsilon_k)^2][e^2_{p'-k}-(E_{p'}-\epsilon_k)^2]}\times$$
$$\times\delta(E_p-E_{p'}) \tag{2.8}$$

Two poles of the integrand correspond to the conservation of energy in the intermediate states A and B respectively, and the asymptotic forms at these points are

$$W_B(A)=\frac{g^2}{4(2\pi)^8}\frac{1}{E_p}\int\frac{dK\,dP'}{2E_{p'}\,e_{p'-k}\,\epsilon_k}\frac{|V(P-P')|^2}{e^2_{p-k}-e^2_{p'-k}}\frac{\delta(E_p-E_{p'})}{e_{p'-k}+\epsilon_k-E_p} \tag{2.9}$$

in this case $(\epsilon_k\doteqdot E_p-e_{p'-k})$

$$W_B(C)=\frac{g^2}{4(2\pi)^8}\frac{1}{E_p}\int\frac{dK\,dP'}{2E_{p'}\,e_{p-k}\,\epsilon_k}\frac{|V(P-P')|^2}{e^2_{p'-k}-e^2_{p-k}}\frac{\delta(E_p-E_{p'})}{e_{p-k}+\epsilon_k-E_p} \tag{2.10}$$

in this case $(\epsilon_k\doteqdot E_p-e_{p-k})$

Further, we have to take into acount another process whose Feynman-diagram is obtained from Fig. 4, assuming the line C as the observation line. This is a process of meson-decay accompanied by the double scattering by the external field, which has no analogue in the corresponding photon process. The transition probability W_C of this process "C" turns out to be

$$W_C=\frac{g^2}{4(2\pi)^8}\frac{1}{E_p}\int_{\epsilon_k=\mu}^{\epsilon_{max}}\frac{dK}{\epsilon_k}\int\frac{dP}{e_{p-k}}\frac{|V(P-P')|^2}{[E^2_{p'}-E^2_p][e^2_{p'-k}-e^2_{p-k}]}\times$$
$$\times\delta(E_p-e_{p-k}-\epsilon_k) \tag{2.11}$$

and the asymptotic forms at A and B are

**) The process "B" in Fig. 4 is the same with Fig. 3.

$$W_C(A)=\frac{g^2}{4(2\pi)^8}\frac{1}{E_p}\int\frac{dK\,dP'}{2e_{p-k}e_{p'-k}\epsilon_k}\frac{|V(P-P')|^2}{E_{p'}^2-E_p^2}\frac{\delta(E_p-e_{p-k}-\epsilon_k)}{e_{p'-k}-e_{p-k}}$$

in this case $(e_{p'-k}\doteqdot e_{p-k})$ (2.12)

$$W_c(B)=\frac{g^2}{4(2\pi)^8}\frac{1}{E_p}\int\frac{dK\,dP'}{2E_{p'}e_{p-k}\epsilon_k}\frac{|V(P-P')|^2}{e_{p'-k}^2-e_{p-k}^2}\frac{\delta(E_p-e_{p-k}-\epsilon_k)}{E_{p'}-e_{p-k}-\epsilon_k} \quad (2.13)$$

in this case $(\epsilon_k\doteqdot E_{p'}-e_{p-k})$

Comparing these asymptotic forms of divergent integrals, we can readily find out a kind of "reciprocity relation" holding between them,

$$W_A(B)=-W_B(A)$$
$$W_B(C)=-W_C(B)$$
$$W_C(A)=-W_A(C) \quad (2.14)$$

From the view-point of the asymptotic forms of the divergences, we may summarize the above results as follows

$$W_A\sim \qquad W_A(B)+W_A(C)$$
$$W_B\sim W_B(A) \qquad +W_B(C) \quad (2.15)$$
$$W_C\sim W_C(A)+W_C(B)$$

On account of these asymptotic forms and the above reciprocity relations, the infra-red-like catastrophe divergences are cancelled out from the total rate of decrease of the initial state-probability.

In this way, the divergence appeared in the process of our meson-decay is cancelled out by simultaneous consideration of the processes of the elastic scattering and of the decay with double scattering. This is nothing but the way, by which the ordinary infra-red catastrophe has been removed. Therefore, the divergences due to the energy conservation in intermediate states of the meson process are of the same nature as that of the "infra-red catastrophe" and they are removed by natural generalization of the method used in the case of quantum electrodynamics.

Moreover, the above consideration reveals us that the reason why the cancellations of the infra-red-like divergences are possible, consists in the validity of the reciprocity relations. Of course, these relations also have played essential roles in the case of the ordinary infra-red catastrophe, but we could not notice them on account of the over-simplifications of the mechanism due to the simpler nature of photons. Accordingly, our method will be extended to more general cases, if we are able to give a general proof of the validity of the reciprocity relations. In order to give it, let us consider a Feynman-diagram of the transition probability of more general form, as illustrated in Fig. 5, where each one of G_1, G_2 and G represents arbitrary graph. Referring to this graph, the reciprocity relation to be proved is formulated as follows; The asymptotic form of divergence $W_A(B)$ due to the energy conservation in intermediate state B of the process "A" has just the same form and opposite sign as the asymptotic form $W_B(A)$ of similar divergence due to the conservation in A of the process "B". Assuming again all relevant particles are scalar, we obtain the following asymptotic forms after some elementary calculations

$$W_A(B)=\prod_{i=1}^{n}\int\frac{dP_i}{E_i(P_i)}\prod_{j=1}^{n}\int\frac{dq_j}{e_j(q_j)}\frac{f(P_1,E_1(P_1)\cdots;q_1,e_1(q_1)\cdots;)}{\sum e_j(q_j)-\sum E_i(T_i)}\times$$

$$\times\delta(E_{initial}-\sum_{j=1}^{n}e_j(q_j)) \tag{2.16}$$

in this case $(\sum e_j \doteqdot \sum E_i)$

$$W_B(A)=\sum_{i=1}^{n}\int\frac{dP_i}{E_i(P_i)}\prod_{j=1}^{n}\int\frac{dq_j}{e_j(q_j)}\frac{f(P_1,E_1(P_1);\cdots;q_1,e_1(q_1)\cdots;)}{\sum E_i(P_i)-\sum e_j(q_j)}\times$$

$$\times\delta(E_{initial}-\sum_{i=1}^{n}E_i(P_i)) \tag{2.17}$$

in this case $(\sum E \doteqdot \sum e)$

These results show that the reciprocity relation

$$W_A(B)=-W_B(A) \tag{2.18}$$

holds for the most general cases. The possibility of the cancellation of the divergences due to the energy conservations in intermediate states by means of simultaneous consideration of related processes is an immediate consequence of this reciprocity relation. Accordingly, the method once applied to the cancellation of the infra-red catastrophe is now able to be extended to more general cases of such low frequency divergences as arising in the meson theory.

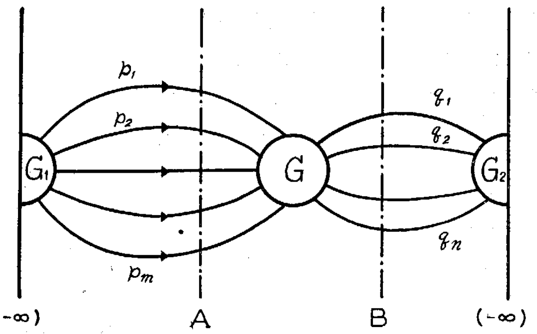

Fig. 5. $E_i(P_i)\equiv\sqrt{P_i^2+M^2}$ $e_j(q_j)\equiv\sqrt{q_j^2+m_j^2}$

§ 3. Consideration of field reaction

In the previous section, it is clarified that the method used in the theory of infra-red catastrophe can also be applied to cancellations of the low frequency divergences appearing in the meson-decay processes. However, we could have only a finite rate of decrease of the initial state-amplitude. The transition probability of each component process which transfers the system in its initial state into each final state is still divergent, therefore, we could have no information about these component processes. This situation limits considerably the applicability of this method, inspite of their theoretical interest. In order to obtain a finite result for each component process, the effects of field reactions must be taken into account.

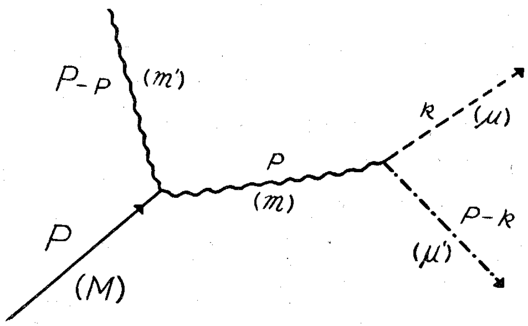

Fig. 6

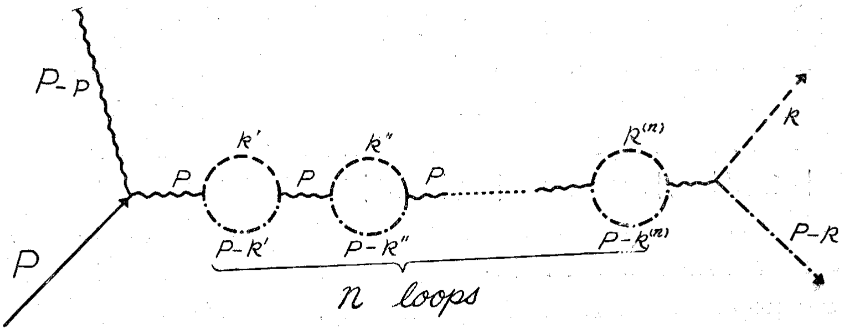

Fig. 7

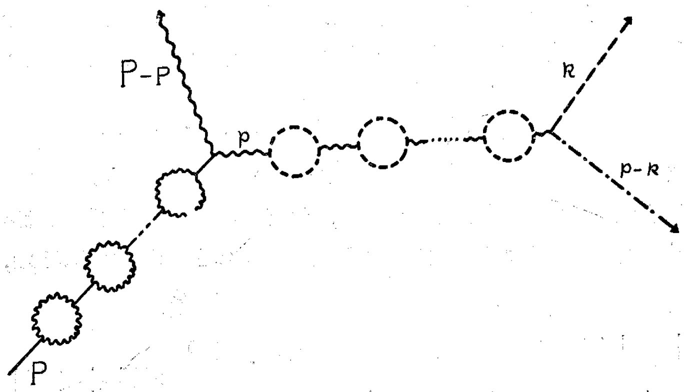

Fig. 8

(1) *general survey*

Let us begin with a simple example; the double decay process of mesons, in which a heavy meson (mass M) decays into two lighter mesons (masses

m and m') and then one of them (*say*, mass m,) decays into two lighter mesons (mass μ and μ'). The Feynman graph of this process is illustrated in Fig. 6. According to the perturbation theory, the lowest order approximation of the total transition probability diverges at the point which corresponds to the free particle state of the intermediate particle m. As mensioned above, the divergence of this type is of the same nature as that of the infra-red catastrophe in quantum electrodynamics and will be removed by the same method.

Apart from the divergence-problem, however, the description by the lowest order approximation of perturbation is expected to be qualitatively very incomplete. Because, unstable mesons are fated to decay sooner or later, and they will certainly decay into lighter mesons in a very long lapse of time. And whenever the rate of decay of the intermediate meson m is greater than that of the creation, the rate of the total process will be determined by the decay probability of the mother meson M alone, that is, the S-matrix element of the total process $M \to m + m' \to m' + \mu + \mu'$ will be in the limiting case independent of the detailed forms of the interaction operator describing the latter process $m \to \mu + \mu'$. It is probably impossible to describe such situations by the lowest order approximation of perturbation alone. The above situation is clearly a result of very long duration of time after the creation of meson m in comparison with their life. The transition probability $W \sim |<\mu\mu'|H_2|m><mm'|H_1|M>|^2 t$ calculated to the lowest order approximation has its meaning only if the duration of time is very small in comparison with the life time. Nevertheless, if we tend t to infinity in order to know about the situation in infinite future, W increases beyond all limits.[7] This is nothing but the space-time interpretation of the infra-red-like catastrophe divergence, as the linear time dependence is a result of the energy conservation in the intermediate states. Therefore, the divergence will disappear in such theory as is able to describe correctly the behavior of transition probability of the total process.

From above considerations, it seems natural to remove the divergence by considering the effect of damping in the intermediate states. In view of the characteristic features of damping, it is expected that the situation will probably be described by the following natural interpolation, at least as an approximation.

$$\left.\begin{array}{ll} W \sim |<mm'|H_1|M>|^2 & (t \gg life) \\ W \sim |<\mu\mu'|H_2|m><mm'|H_1|M>|^2 t & (t \ll life) \end{array}\right\} \to W \sim |<M|H_1|mm'>|^2 \times$$
$$\times \left(1 - e^{-|<m|H_2|\mu\mu'>|^2 t}\right) \tag{3.1}$$

This is not an unfounded expectation. When the proper frequency of the

7) B. A. Lippmann and J. Schwinger, Phys. Rev. 79 (1950), 469.

meson is larger than the uncertainty of energy due to the finite life time $\frac{2}{\gamma}$, that is, when the particles in intermediate state can be regarded as if it were in its free particle- state, the whole process may be a series of the natural radioactivity. In the series A, B, C, of natural radioactivity, the probabilities of finding the atomic nuclei A, B, C, respectively, n_A, n_B and n_C are solutions of the following equation

$$\dot{n}_A=-\Gamma n_A, \qquad \dot{n}_B=-\gamma n_B+\Gamma n_A, \qquad \dot{n}_C=+\gamma n_B$$

with initial condition

$$n_A(0)=\mathbf{1}, \qquad n_B(0)=n_C(0)=0$$

where Γ, and γ represent the transition probabilities of the process $A\to B$, and $B\to C$, respectively. The rate of the total process $W=\dot{n}_C$ is calculated as

$$W=\Gamma(1-e^{-\gamma t}) \tag{3.2}$$

This is just the same form as (3.1). Therefore, the correct theory has to have a solution which reduces to (3.1) when the intermediate state could be regarded as a state of free particle. Such solution is really obtained if we take into account the radiation damping in intermediate state in accordance with the theory of Weisskopf and Wigner.[8)]

(2) *damping effect in intermediate state*

Suggested by the above rough consideration, let us consider the corrections of the internal line by succession of the self-energy loops consisting of the pair of lighter mesons μ and μ', as illustrated in Fig. 7. Assuming all the particles are scalar and, their interactions are

$$H_1=g_1\psi^*\phi\,\phi'+conj. \qquad H_2=g_2\phi^*\,a\,a'+conj. \tag{3.3}$$

where ψ, ϕ, ϕ', a, and a' are fields of meson M, m, m', μ and μ' respectively. We obtain the following expression as total contribution to S-matrix from all graphs having the forms given in Fig. 7.

$$S=\frac{ig_1g_2}{(2\pi)_3}\frac{1}{\sqrt{16E_p(E_p-\epsilon_k-\epsilon'_{p-k})\epsilon_k\epsilon'_{p-k}}}\frac{1}{P^2+m^2-i(\epsilon+\Gamma(p))} \tag{3.4}$$

where
$$\Gamma(p)\equiv\frac{g_2^2}{(2\pi)_4}\int\frac{(dk'')}{[k''^2+\mu^2-i\epsilon][(p-k'')^2+\mu'^2-i\epsilon]} \tag{3.5}$$

represents the self energy operator of the internal line with propagation vector p. The transition probability W is calculated from above S-matrix as

$$W=\frac{(g_1g_2)^2}{(2\pi)^5}\int\frac{dP\,dK}{16E_p(E_p-\epsilon_k-\epsilon'_{p-k})\,\epsilon_k\epsilon'_{p-k}}\frac{1}{|P^2+m^2-i(\epsilon+\Gamma(p))|^2} \tag{3.6}$$

8) V. Weisskopf and E. Wigner, Z. S. f. Phys. 63 (1930), 54 and 65 (1930), 18.

In the decay process, $\Gamma(p)$ is, in general, complex and given by

$$-i\Gamma(p) \equiv \delta m^2(p) + ip_0 \frac{\gamma(p)}{2} \tag{3.7}$$

$$= \frac{g_2^2}{(2\pi)^4} \int (dk'') \left[\pi \left(\frac{\delta((p-k'')^2+\mu'^2)}{k''^2+\mu^2} + \frac{\delta(k''^2+\mu^2)}{(p-k'')^2+\mu'^2} \right) + \right.$$
$$\left. + i\pi^2 \delta((p-k'')^2+\mu'^2)\, \delta(k''^2+\mu^2) \right] \tag{3.8}$$

where the real part $\delta m^2(p)$ represents the self-energy of the intermediate particle containing a correction of mass, an effect of level-shift etc., and was a principal matter of concern in the theory of the renormalization. The imaginary part $ip_0 \frac{\gamma(p)}{2}$ contains in this case a non-vanishing part

$$\gamma(p) = 2\pi\, g_2^2 \int \frac{dK/(2\pi)^3}{8p_0\, \epsilon_{k''}(p_0-\epsilon_{k''})}\, \delta(p_0-\epsilon_{k''}-\epsilon_{p-k''}) \tag{3.9}$$

As is readily seen this is nothing but the total transition probability of the second decay process $m \to \mu + \mu'$. Putting $e_p \equiv \sqrt{P + (m^2 + \delta m^2(p))}$ we have

$$W = \frac{(g_1 g_2)^2}{(2\pi)^5} \int \frac{dP\, dp_0\, dK}{16E_p(E_p-p_0)\epsilon_k \epsilon'_{p-k}}\, \frac{\delta(\epsilon_k + \epsilon'_{p-k} - p_0)}{(e_p^2 - p_0^2)^2 + (p_0\gamma(p)/2)^2} \tag{3.10}$$

As $\gamma(p)$ does not vanish, we obtain always convergent W. The functional forms of $\delta m^2(p)$ and $\gamma(p)$ are usually very complicated and it is not a simple matter to evaluate the integral rigoroutly. It is not our present purpose to calculate the transition probability exactly, but to verify the fact that the result is divergence-free and is insensitive to the detailed nature of the interaction describing the second decay process, whenever the free particle state is dominant in the intermediate state and the latter decay is much faster than the first. Therefore, we calculate it here approximately in the case of "sharp resonance" where $\gamma(p)$ is small enough to replace the argument p of $\delta m(p)$ and $\gamma(p)$ by its resonance value. Then, the evaluation of the integral is elementary and reduces to

$$W = 2\pi g_1^2 \int \frac{(dP)/(2\pi)^3}{8E_p(E_p-e_p)e_p}\; 2\pi g_2^2 \int \frac{dK/(2\pi)^3}{8e_p\, \epsilon_k (e_p - \epsilon_k)} \times$$
$$\times\, \delta(e_p - \epsilon_k - \epsilon'_{p-k}) \Big/ \frac{\gamma(p)}{2} \tag{3.11}$$

On account of the definition of $\gamma(p)$, this reduces further to

$$W = 2\pi g_1^2 \int \frac{(dP)/(2\pi)^3}{8E_p\, e_p(E_p - e_p)} \tag{3.12}$$

This is nothing but the transition probability of the decay process of the first step $M \to m + m'$. Thus, under the condition described above, the transition probability of the total process is determined by the first decay process alone, and does not depend upon the detailed natures of the second decay as expected before.

(3) *time- dependence of transition probability*

The above results are based upon the S-matrix theory and are able to predict only the situation in infinite future. The next task is to examine the time dependence of the transition probability.

The transformation function $U[\sigma, \sigma_0]$ of this process is written as[9)]

$$U[\sigma,\sigma_0]=\sum_{n=1}^{\infty}\int_{\sigma_0}^{\sigma} dx\,dx'\,dx_1\,dx_1'\cdots dx_n\,dx_n'\,A(x)\,\Delta(x-x')\,L(x'-x_1)\times$$
$$\times\Delta(x_1-x_1')\,L(x_1'-x_2)\cdots\times L(x_{n-1}-x_n)\,\Delta(x_n-x_n')\,B(x_n') \qquad (3.13)$$

where

$$A(x)=g_1\,\phi^{*\prime}(x)\,\psi(x) \qquad L(x)=-(g_2/2)^2\,D_F(x)\,D_F'(x)$$
$$B(x)=g_2\,a(x)\,a'(x) \qquad \tfrac{1}{2}D_F(x-x')\equiv\langle P(a(x),a(x'))\rangle_0 \qquad (3.14)$$
$$\Delta(x)=\tfrac{1}{2}\Delta_F(x) \qquad \tfrac{1}{2}D_F'(x-x')\equiv\langle P(a'(x),a'(x'))\rangle_0$$

or in momentum space, it turns out to be

$$U[\sigma,\sigma_0]\sum_{n=0}^{\infty}\int dp\,dp_1\,dp_2\cdots dp_n\,A_{\sigma_0}^{\sigma}(p)\,\Delta(p)\,L_{\sigma_0}^{\sigma}(p,p_1)\,\Delta(p_1)\cdots\times$$
$$\times\cdots L_{\sigma_0}^{\sigma}(p_{n-1},p_n)\,\Delta(p_n)\,B_{\sigma_0}^{\sigma}(p_n) \qquad (3.15)$$

$$A_{\sigma_0}^{\sigma}=\frac{1}{(2\pi)^2}\int_{\sigma_0}^{\sigma}A(x)\,e^{ipx}dx\,,\quad L_{\sigma_0}^{\sigma}(p,q)=\frac{1}{(2\pi)^4}\int_{\sigma_0}^{\sigma}L(x-y)e^{-ipx+iqy}dx\,dy \quad (3.16)$$

$$B_{\sigma_0}^{\sigma}=\frac{1}{(2\pi)^2}\int_{\sigma_0}^{\sigma}B(x)\,e^{-ipx}dx\,,\qquad \Delta(p)=\frac{1}{i(p^2+m^2-i\epsilon)}$$

As the contribution from each one of the loops $L_{\sigma_0}^{\sigma}(p,q)$ can be written as

$$L_{\sigma_0}^{\sigma}(p,q)=\frac{1}{(2\pi)^4}\int_{\sigma_0}^{\sigma}e^{-i(p-q)x}dx\int_0^{(\sigma-\sigma_0)}L(y)\,e^{-iqy}dy\equiv\delta_{\sigma_0}^{\sigma}(p-q)\,L^{(\sigma-\sigma_0)}(q) \qquad (3.17)$$

we have

$$U[\sigma,\sigma_0]=\sum_{n=0}^{\infty}\int dp\,dp_1\cdots dp_n\,\delta_{\sigma_0}^{\sigma}(p-p_1)\cdots\delta_{\sigma_0}^{\sigma}(p_{n-1}-p_n)\,A_{\sigma_0}^{\sigma}(p)\,\Delta(p)\times$$
$$\times L^{(\sigma-\sigma_0)}(p_1)\,\Delta(p_1)\cdots L^{(\sigma-\sigma_0)}(p_n)\,\Delta(p_n)\,B_{\sigma_0}^{\sigma}(p_n) \qquad (3.18)$$

where

$$\delta_{\sigma_0}^{\sigma}(p-q)=\frac{1}{(2\pi)^4}\int_{\sigma_0}^{\sigma}dx^{\,-i(p-q)x}=\delta(p-q)\,\frac{1}{2\pi}\int_{t_0}^{t}e^{i(p_0-q_0)t'}dt' \qquad (3.19)$$

reduces to an ordinary δ-functions when the time distance $t-t_0=T$ is large enough. As we are interested in the effect of the radiation damping, it is

9) D. Ito, H. Tanaka and M. Yamazaki, Prog. Theor. Phys. 7 (1952), 128 (L).

of no use to consider the behavior of the transformation function $U[\sigma, \sigma_0]$ in such a region where the temporal distance T is comparable with the life time $\frac{2}{\gamma}$. Consequently, we consider the case where $mc^2 \gg \frac{\hbar\gamma}{2} \gg \hbar/T$ holds. Then all $\delta^{\sigma}_{\sigma_0}(p)$ may be replaced by ordinary $\delta(p)$, $L^{(\sigma-\sigma_0)}(p)$ by $-\Gamma(p)$, and $\frac{1}{\Delta(p)}+L^{(\sigma-\sigma_0)}(p)$ by $i\left[p^2+(m^2+\delta m^2(p))+ip_0\frac{\gamma(p)}{2}\right]$, so $U[\sigma, \sigma_0]$ becomes

$$U[\sigma, \sigma_0]=\int_{\sigma_0}^{\sigma} dx\, dx'\, A(x)\, \tfrac{1}{2}\, \Delta_D(x-x')\, B(x') \tag{3.20}$$

where

$$\begin{aligned}\Delta_D(x) &= \frac{-2i}{(2\pi)^4}\int \frac{e^{ipx}}{p^2+(m^2+\delta m^2(p))-ip_0\,\gamma(p)/2-i\epsilon}\,(dp) \\ &= e^{-\frac{1}{2}\gamma(p)t}\Delta'_F(x)\end{aligned} \tag{3.21}$$

Performing the space integral, we have

$$\begin{aligned}U[\sigma, \sigma_0] &\cong (2\pi)^3 A(0)\, B(0)\int \delta(P-K-K')dP\int_{-\infty}^{+\infty}\frac{dp_0/2\pi}{e_p^2-p_0^2+ip_0\,\gamma(p)/2}\times \\ &\times\left(\frac{1-e^{i(\epsilon_k+\epsilon_{k'}-P_0)t}}{\epsilon_k+\epsilon'_k-p_0}+\frac{1-e^{-i(\epsilon_k+\epsilon_{k'}-P_0)(T-t)}}{\epsilon_k+\epsilon_{k'}-p_0}\right)\end{aligned} \tag{3.22}$$

When the mass m of the intermediate particle is large, the second term is small enough to be neglected in comparison with the first term in this sharp resonance approximation. Then,

$$\begin{aligned}W(t) &\cong \frac{(g_1 g_2)^2}{(2\pi)^5}\int\frac{dP\, dK}{16\, E_p(E_p-e_p)\,\epsilon_k\,\epsilon'_{p-k}}\,\frac{1}{e_p^2} \\ &\int_{-\infty}^{\infty} dP_0\,\frac{(1-e^{\frac{\gamma t}{2}})^2+4e^{-\frac{\gamma t}{2}}\sin^2(p_0-e_p)\frac{t}{2}}{(p_0-e_p)^2+(\gamma(p)/2)^2}\,\delta(p_0-\epsilon_k-\epsilon'_{p-k})\end{aligned} \tag{3.23}$$

Using the formula

$$\int_{-\infty}^{+\infty}\frac{(1\,;\sin^2\frac{\omega t}{2})}{\omega^2+(\gamma/2)^2}\,d\omega=\frac{2\pi}{\gamma}\left(1\,;\,(1-e^{-\frac{\gamma t}{2}})\right) \tag{3.24}$$

we have

$$W(t)=2\pi\, g_1^2\int\frac{(dP)/(2\pi)^3}{8E_p(E_p-e_p)\, e_p}\,(1-e^{-\gamma(p)t})=\int dP\,\Gamma(P)(1-e^{-\gamma(p)t}) \tag{3.25}$$

This asymptotic time dependence of the transition probability has just the same form as that of (3.1). It is observed here that the effect of the detailed properties of the second decay is fading away exponentially from the expression of the transition probability, and after a long time the expression tends to the result previously calculated from the S-matrix. Thus, we have obtained the divergence-free and physically acceptable result.

(4) *several remarks*

In the same way as above, the incident line in Fig. 7 must be corrected as illustrated in Fig. 8. Then, if the rate of the first decay is greater than that of the second, the detailed nature of the first decay will fade away from the expression of the transition probability as if the history of the creation of intermediate state were completely forgotten. Such situation are frequently met in the theory of the nucler reactions, such as the compound nucleus etc., or in the case of the Hohlraumstrahlung and in the Fermi's theory[10] of multiple production of mesons.

In the nuclear processes, π°-meson can decay into two photons through the virtual as well as the real state. Roughly speaking with our method, the decay constant $\frac{\hbar\gamma}{2}$ is so small compared with the meson mass mc^2 that the decay process through virtual states will be very rare.

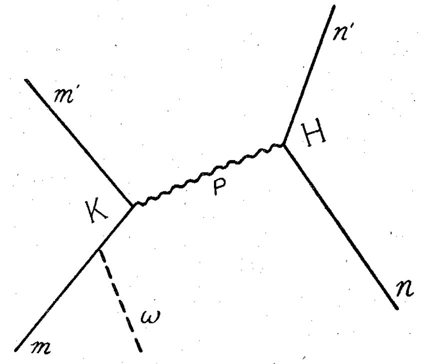

Fig. 9. ——— nucleon line, 〰〰 meson line, ------ photon line.
m, *m'*, *n* and *n'* are nucleon states. *K* and *H* are their interactions.

Next, we consider the photo-nuclear effect. It is often said that in these processes the mesons emitted and then re-absorbed in the nucleus play the important roles.[11),12)] At high energy, the cross-section depends mostly on the character of meson-production only and the detailed nature of meson-absorption is not important. This is just the effect of damping by absorption, and will be treated well according to our method. We take here only the simplest case. (See Fig. 9) The transformation matrix M becomes

$$M=\int K(x)\,\Delta_D(x-x')\,H(x')=\int d\vec{r}\,dt\,K(\vec{r}t)\,e^{i(E_n-E_n')t}\int d\vec{\xi}\,H(\vec{r}+\vec{\xi},0)\,\times$$

$$\times\int_{-\infty}^{\infty}d\tau\,\Delta_D(\vec{\xi},\tau)\,e^{-i(E_n-E_n')\tau}\equiv\int d\vec{r}\,d\vec{r}'\,\overline{K}(\vec{r})\,G(\vec{r}-\vec{r}')\,\overline{H}(\vec{r}') \qquad (3.26)$$

10) E. Fermi, Prog. Theor. Phys. 5 (1950), 570.
11) S. Kikuchi, Phys. Rev. 85 (1951) 1062 and ibid 86 (1952) 41.
12) R. Wilson, Phys. Rev. 86 (1952), 125.

where

$$G(\vec{r}) \equiv \int^{\infty} \Delta_D(\vec{r}, t)\, e^{-i\Delta E t}\, dt = \frac{-2i}{(2\pi)^3} \int dP \frac{e^{ip,r}}{P^2+\mu^2-(\Delta E)^2-i\Gamma(\Delta E)}$$

$$= \frac{i}{4\pi} \frac{e^{ip,r}}{r} \tag{3.27}$$

$$P \equiv \sqrt{(\Delta E)^2-\mu^2+i\Gamma(\Delta E)}, \qquad \Gamma(\Delta E) = \Delta E \cdot \gamma(\Delta E)$$

Assuming the absorbing-source density is uniform in the nuclear radius R, and after some approximate calculation we obtain the following form†), ††)

$$|M|^2 \sim \frac{1}{4\pi} |\bar{K} \cdot \bar{H}|^2 \lambda(\Delta E) (1-e^{-\frac{R}{\lambda(\Delta E)}}) \tag{3.28}$$

where $\frac{1}{\lambda(\Delta E)} \equiv \frac{\gamma(\Delta E)}{v(\Delta E)}$, $v(\Delta E) \equiv \frac{\partial(\Delta E)}{\partial \bar{P}}$ $\bar{P}(\Delta E) \equiv \sqrt{(\Delta E)^2-\mu^2}$

Therefore, a picture of the free path of meson is asymptotically contained in this problem.

In our above treatment we have considered only a correction of the internal line by the lowest order proper graph of self-energy. According to Dyson's integral equation, our method consists in the replacement of Σ^* by its first term Σ_2^* in the expansion. So, this may be regarded as one-particle analogue of the "ladder approximation" introduced by Bethe and Salpeter[13] in their theory of the relativistic two body problem.

†) We expand $P \equiv \sqrt{(\Delta E)^2-\mu^2+i\Gamma(\Delta E)}$ in powers of $\Gamma(\Delta E)$

††) $v(\Delta E) \equiv \frac{\partial(\Delta E)}{\partial \bar{P}}$ has some approximate meaning of (group) velocity of mesons.

13) E. E. Salpeter and H. A. Bethe, Phys. Rev. 84 (1951), 1232.