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A bilevel optimization model for the newsvendor problem with the focus theory of choice

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Abstract The newsvendor problem is formulated with the focus theory of choice in which instead of using the expected utility in the existing approaches, the optimal order quantity is determined as per which order quantity's focus (the most salient demand) is the most preferred. This new formulation is a bilevel optimization problem in which the lower level chooses the focus of each order quantity and the upper level determines the optimal order quantity. The proposed model has maximin upper and lower level programs that are nonsmooth. We derive the optimal order quantity under this new framework and characterize the properties of the optimal solution. The proposed model provides a new perspective to analyze the newsvendor problems.

Keywords Bilevel optimization · Nonsmooth · Focus theory of choice · Newsvendor model

Mathematics Subject Classification (2000) 90B05 · 90C26 · 90C30

1 Introduction

The newsvendor problem is a typical one-time business decision with three characteristics: (i) the demand for the product is uncertain; (ii) the procure-

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ment lead time exceeds the selling season; and (iii) both overstock and understock result in economic loss. Newsvendor problems reflect the essence of many real-world business situations, ranging from manufacturing and retailing fashion and sports apparel, and electronics with time sensitive components, to managing capacity and booking in airline and hotel industries, making final purchase or production decisions, and selling perishable food items. As such, although studies on newsvendor problems date back to the 19^{th} century (Edgeworth 1888; Arrow et al. 1951), significant attention is still paid to this research area and the classic model has been extended along different lines such as adopting alternative objective and utility functions, employing different pricing and return policies, and considering various settings of demand information. For comprehensive reviews, readers are referred to (Petruzzi and Dada 1999; Qin et al. 2011) and the edited book by Choi (2012). More research on this important topic is reported in (Arcelus et al. 2012; Güler 2019; Mitra 2018; Huber et al. 2019; Ji and Kamrad 2019).

This research aims to address this issue by putting it within the focus theory of choice framework, which offers decision aids based on salient information instead of expected value. A growing body of evidence has shown that salience (attention-grabbing) information plays a critical role in human decision making (Lacetera et al. 2012; Busse et al. 2013; Brandstätter and Korner 2014). Guo (2011) argues that an individual evaluates a decision alternative based on some associated event (called the focus of a decision), which is most salient to the decision maker due to its resultant payoff and probability, thereby proposing a one-shot decision theory. The one-shot decision theory has been applied to auction problems (Wang and Guo 2017), production planning (Zhu and Guo 2020a), newsvendor models for innovative products (Guo and Ma 2014; Zhu and Guo 2017; Zhu and Guo 2020b), multistage decision making (Guo and Li 2014; Li and Guo 2015), and duopoly markets of innovative products (Guo 2010a). To further refine the one-shot decision theory, Guo (2019) proposes the focus theory of choice that models and axiomatizes procedural rationality articulated first by Simon (1976) and resolves several well-known anomalies such as the St. Petersburg, Allais, and Ellsberg paradoxes, and violations of stochastic dominance. The core argument of the focus theory of choice is that the most salient event corresponds to the most-preferred action, where salience depends on the decision maker's specific frame of mind and reflects different behavioral patterns in human decision processes. This argument is consistent with the results of the psychological experiments conducted by Stewart et al. (2016). Recently, focus theory of choice is used to solve sequential decision problems under uncertainty (Guo 2022).

In this paper, we assume that the retailer is an optimistic decision maker so that we analyze newsvendor problems under the positive evaluation system of the focus theory of choice. This framework envisages the retailer's decision as a two-step procedure. In the first step, for each potential order quantity, the retailer looks for his/her most salient demand by examining the payoffs of all possible demands and that corresponding probabilities. The demand that generates the relatively high payoff with the relatively high probability is identified as the focus of this order quantity. In the second step, the retailer selects the optimal order quantity by scanning through the foci of all possible order quantities. Following the above idea, the newsvendor problem is formulated as a bilevel optimization problem in which the lower level program is used to choose the focus of each order quantity and the upper level program is to determine the optimal order quantity. In the proposed bilevel optimization model, the upper and lower levels are both maximin optimization problems. It is well known that the bilevel optimization problem is an important hierarchical optimization problem, which plays an important role in many fields (see, e.g., Bard 1998; Colson 2005; Colson et al. 2005; Dempe 2018) and solving a bilevel optimization problem is very hard because its constraint region is implicitly determined by a series of optimization problems. Since the maximin upper and lower level programs are nonsmooth and sometimes even nonconvex, the proposed model cannot be solved by existing optimization methods for bilevel optimization problems. We derive the analytical optimal order quantity under this new framework and obtain the mathematical properties of the optimal solution. Theoretical analysis shows that this research provides a new perspective to examine the retailer ' s ordering decision.

The remainder of this paper is organized as follows. In Section 2, we establish a newsvendor model with the focus theory of choice. It is a bilevel optimization problem with maximin upper and lower level programs. In Section 3, we theoretically obtain the optimal solutions of the proposed bilevel optimization problem, provide the decisional insights on the positive foci and the optimal order quantity, and offer a comparison with the result derived by the classic newsvendor model. In Section 4, we present an illustrative example to show how the proposed model works and offers a comparison with other models. Section 5 concludes this paper with some remarks. All proofs of the theoretical results are given in Appendix.

2 The newsvendor model with the focus theory of choice

Considering a retailer who sells a short life-cycle product, his/her payoff function is given as follows:

$$v(x,q) = \begin{cases} r * x + s * (q - x) - w * q & \text{for } x < q, \\ (r - w) * q - g * (x - q) & \text{for } x \geq q, \end{cases} \tag{1}$$

where x is the market demand, q is the order quantity, $w > 0$ is the wholesale price per unit, $r > 0$ is the retailer price per unit, $s > 0$ is the salvage value per unit, and $g > 0$ is the opportunity cost per unit due to stockout. Without loss of generality, we assume $r > w > s$. Clearly, given an order quantity q, $v(x,q)$ reaches its maximum at $x = q$. We further make the following generic assumptions on the uncertain demand:

(i) The demand lies on an interval $X = [l, h]$;

(ii) The probability density function of demand $f(x)$ is a strictly quasi-concave continuous function and $f(x) > 0$ for any $x \in [l, h]$.

As a rational retailer, his/her order quantity should lie within the demand interval $[l, h]$. We have the retailer's lowest profit $\min_{x,q\in[l,h]} v(x, q) = \min\{v(l, h), v(h, l)\} = \min\{p*l+s(h-l)-w*h, (p-w)*l-g*(h-l)\}$, occurring at one of the two extreme cases when the order quantity mismatches the realized demand. The retailer's highest profit $\max_{x,q\in[l,h]} v(x, q) = v(h, h) = (r - w) * h$ occurs at $x = q = h$.

To reframe the newsvendor model under the paradigm of the focus theory of choice, we next convert the retailer's payoff function to a satisfaction function and the probability density function to a relative likelihood function.

Satisfaction function: Let V be the range of the retailer's payoff function $v(x, q)$, $u : V \to [0, 1]$ is called a satisfaction function if

$$u(v_1) > u(v_2) \iff v_1 > v_2, \quad \forall\, v_1, v_2 \in V,$$

and $\exists\, v_c \in V$ such that $u(v_c) = \max_{v\in V} u(v) = 1$.

For a given order quantity q, $u(x, q)$ represents the retailer's satisfaction level about the resulting payoff if demand arises as x. Clearly, given an order quantity q, $u(x, q)$ reaches its maximum at $x = q$. The satisfaction function is only a value function that converts payoffs to relative satisfaction levels. In this research, we define the satisfaction function as follows:

$$u(x, q) = \frac{v(x, q) - \min_{x,q\in[l,h]} v(x, q)}{\max_{x,q\in[l,h]} v(x, q) - \min_{x,q\in[l,h]} v(x, q)}. \tag{2}$$

Relative likelihood function: $\pi : X \to [0, 1]$ is called a relative likelihood function if

$$\pi(x_1) > \pi(x_2) \iff f(x_1) > f(x_2), \quad \forall\, x_1, x_2 \in X,$$

and $\exists\, x_c \in X$ such that $\pi(x_c) = \max_{x\in X} \pi(x) = 1$.

For any $x \in X$, $\pi(x)$ is called its relative likelihood degree. In this research, we define the relative likelihood function as follows:

$$\pi(x) = \frac{f(x)}{\max_{x\in X} f(x)}. \tag{3}$$

A relative likelihood function is a normalized probability density function to represent the relative likelihood positions of different outcomes. Instead of using the original values of payoffs and probabilities, the focus theory of choice takes the satisfaction and relative likelihood functions as the basic decision input because a mounting body of evidence suggests that relative values are more perceptible (have a higher accessibility) than absolute values and play a more important role in human decision making (Frank 1985; Solnick and Hemenway 1998).

Since $f(x)$ is a strictly quasi-concave continuous function on $[l, h]$, there exists $c_0 \in [l, h]$ such that $f(c_0) = \max_{x\in[l,h]} f(x)$, we know that $\pi(x)$ attains its unique maximum at $x = c_0$, $\pi(x)$ is strictly increasing on $[l, c_0]$ and strictly decreasing on $[c_0, h]$.

The focus theory of choice postulates that a decision maker inherently owns two distinct evaluation systems: positive and negative. Typically, these two systems correspond to different frames of mind and one of them is working for a particular decision situation. Generally, the positive evaluation system is active for an optimistic decision maker and the negative evaluation system is activated when a decision maker is pessimistic. In this paper, we assume that the decision maker is an optimistic retailer. Hence, we only analyze the newsvendor problem and derive its optimal order quantity under the positive evaluation system.

In this case, for a given decision action $q \in Q = [l, h]$, we denote $X_p(q)$ as the set of optimal solutions to the following optimization problem:

$$\max_{x \in X} \ \min\{\varphi * \pi(x), u(x, q)\}, \tag{4}$$

where parameter φ is a positive real number. (4) is derived from the positive focus representation theorem in (Guo 2019). Note that both $\pi(x)$ and $u(x, q)$ are dimensionless and between 0 and 1, and φ serves as a scaling factor that directly affects whether the outcome with a higher relative likelihood or a higher satisfaction arises from the inner minimization operation in (4). Given q, for x_1, $x_2 \in X$, if $\pi(x_1) \geq \pi(x_2)$ and $u(x_1, q) \geq u(x_2, q)$, then we have $\min\{\varphi * \pi(x_1), u(x_1, q)\} \geq \min\{\varphi * \pi(x_2), u(x_2, q)\}$. Clearly, for any decision action q (order quantity in the newsvendor context), (4) seeks the outcome (demand) that brings a relatively high relative likelihood degree and a relatively high satisfaction level. Increasing φ makes $\varphi * \pi(x)$ bigger and allows $u(x, q)$ to arise more easily out of the inner minimization operation in (4), leading to an optimal x therein with a relatively high satisfaction level (payoff) and a relatively low relative likelihood (probability) (Guo 2019). Conversely, decreasing φ makes $\pi(x)$ to take a more prominent role in determining the output of (4), resulting in an optimal x with a relatively high relative likelihood (probability) and a relatively low satisfaction level (payoff). Hence, φ can be interpreted as a weight that a retailer balances his/her emphasis between the satisfaction level and the relative likelihood degree. Increasing φ means that the retailer aims to pursue a higher payoff by somewhat sacrificing the probability. Therefore, φ can be used to measure how optimistic the retailer is: the higher the value of φ, the more optimistic the retailer.

Since $\pi(x)$ and $u(x, q)$ are both strictly quasi-concave continuous functions in x on X, we know that for any given order quantity $q \in Q$, there is a unique element in $X_p(q)$, denoted by $x_p(q)$ and referred to as the positive focus of order quantity q. Next, among all the positive foci for different actions (order quantities), we seek the optimal q^* by the following optimization problem:

$$\max_{q \in Q} \ \min\{\kappa * \pi(x_p(q)), u(x_p(q), q)\}, \tag{5}$$

where κ is a positive real number. (5) follows from the representation theorem for an optimal action under the positive evaluation system in (Guo 2019). The set of q^* is denoted by Q^*.

For any $q_1, q_2 \in Q$, if $u(x_p(q_1), q_1) \geq u(x_p(q_2), q_2)$ and $\pi(x_p(q_1)) \geq \pi(x_p(q_2))$, then we have $\min\{\kappa * \pi(x_p(q_1)), u(x_p(q_1), q_1)\} \geq \min\{\kappa * \pi(x_p(q_2)), u(x_p(q_2), q_2)\}$. This implies that (5) finds an order quantity whose focus has a relatively high relative likelihood degree and yields a relatively high satisfaction level. Similar to the interpretations of parameter φ, a heightened κ elevates the level of $\kappa * \pi(x_p(q))$ relative to $u(x_p(q), q)$ and leads to an order quantity whose focus is relatively high in satisfaction but relatively low in likelihood. Conversely, a reduced κ lowers the level of $\kappa * \pi(x_p(q))$ relative to $u(x_p(q), q)$ and results in an order quantity whose positive focus is relatively high in likelihood but relatively low in satisfaction. Since (5) is used for identifying actions (order quantities) based on their positive foci, κ can be interpreted as the retailer's confidence index on his/her decision: the higher the value of κ, the more confident the decision maker (retailer).

Definition 1 If there is only one element in Q^*, then it is the optimal order quantity under the positive evaluation system, denoted by q_p^*. If there is more than one element in Q^* and there does not exist $q^* \in Q^*$ such that $\pi(x_p(q^*)) > \pi(x_p(q_p^*))$, $u(x_p(q^*), q^*) \geq u(x_p(q_p^*), q_p^*)$ or $\pi(x_p(q^*)) \geq \pi(x_p(q_p^*))$, $u(x_p(q^*), q^*) > u(x_p(q_p^*), q_p^*)$ for $q_p^* \in Q^*$, then q_p^* is the optimal order quantity under the positive evaluation system.

Definition 1 indicates that the optimal order quantity q_p^* weakly dominates all other elements in Q^* if it contains multiple quantities.

It should be noted that the newsvendor model with the focus theory of choice consisting of (4) and (5) is a bilevel optimization problem where (4) is the lower level problem and (5) is the upper level problem. Since the upper and lower level programs are nonsmooth nonconvex maximin problems, the proposed model cannot be solved by existing optimization methods. In the following section, we will theoretically derive the optimal solution and its properties.

3 Theoretical results of the newsvendor model with the focus theory of choice

First, we furnish Lemma 1 to characterize the positive focus for any given order quantity which is the unique optimal solution to the lower level problem (4).

Lemma 1 *For any order quantity $q \in Q$, its positive focus $x_p(q)$ is characterized as follows:*

(i) *If $\varphi > \frac{u(q,q)}{\pi(q)}$, then $x_p(q) = q$;*

(ii) *If $\frac{u(c_0,q)}{\pi(c_0)} \leq \varphi \leq \frac{u(q,q)}{\pi(q)}$ and $q < c_0$, then $x_p(q)$ is the unique solution to the equation $u(x, q) = \varphi * \pi(x)$ in x on $[q, c_0]$;*

(iii) *If* $\frac{u(c_0,q)}{\pi(c_0)} \leq \varphi \leq \frac{u(q,q)}{\pi(q)}$ *and* $q \geq c_0$*, then* $x_p(q)$ *is the unique solution to the equation* $u(x,q) = \varphi * \pi(x)$ *in* x *on* $[c_0, q]$*;*
(iv) *If* $0 < \varphi < \frac{u(c_0,q)}{\pi(c_0)}$*, then* $x_p(q) = c_0$*.*

Lemma 1 confirms the role of φ in determining the focus demand of any given order quantity q: When φ is sufficiently small in case (iv), the relative likelihood function plays a primary part in identifying the focus $x_p(q) = c_0$, which has the highest relative likelihood at a lower satisfaction level. When φ increases into the middling range in case (ii) or (iii), the focus is identified as the unique solution to the equation $u(x,q) = \varphi * \pi(x)$ in x on $[q, c_0]$ or $[c_0, q]$, which has a higher satisfaction level, but a lower relative likelihood degree. When φ further increases into the high range in case (i), the satisfaction function stands out in determining the focus $x_p(q) = q$, which possesses the highest satisfaction level at an even lower relative likelihood. Based on Lemma 1, we obtain the following result.

Theorem 1 $x_p(q)$ *is an increasing and continuous function of* q *on* $[l, h]$*.*

Theorem 1 implies that for any given order quantities q_i and q_j, if $q_i > q_j$, then the positive focus of q_i is larger than or equal to the positive focus of q_j. Since $\pi(x)$ is a continuous function, it is increasing on $[l, c_0]$ and decreasing on $[c_0, h]$, the following result is natural.

Theorem 2 $\pi(x_p(q))$ *is a continuous function of* q *on* $[l, h]$*, it is increasing on* $[l, c_0]$ *and decreasing on* $[c_0, h]$*.*

Theorem 2 means that the relative likelihood function of the positive focus is a quasi-concave continuous function of q. When φ is sufficiently small such that $\varphi < \frac{u(c_0,q)}{\pi(c_0)}$ holds for all $q \in Q$, as per Lemma 1(iv), we have $\pi(x_p(q)) = \pi(c_0)$ for all $q \in Q$. To study the monotonicity of the function $u(x_p(q), q)$, we give the following lemma.

Lemma 2 *For any order quantity* $q \in [c_0, h]$*, its positive focus* $x_p(q)$ *is determined as follows:*

(i) *If* $\varphi > \frac{u(h,h)}{\pi(h)}$*, then* $x_p(q) = q$*;*
(ii) *If* $\frac{u(c_0,c_0)}{\pi(c_0)} \leq \varphi \leq \frac{u(h,h)}{\pi(h)}$*, then there exists a unique solution to the equation* $\varphi * \pi(x) = u(x,x)$ *on* $[c_0, h]$*, denoted by* x_φ*, such that*

$$x_p(q) = \begin{cases} q & \text{for } q \in [c_0, x_\varphi], \\ x_q & \text{for } q \in (x_\varphi, h], \end{cases}$$

where x_q *is the unique solution to the equation* $u(x,q) = \varphi * \pi(x)$ *in* x *on* $[c_0, q]$*;*
(iii) *If* $\frac{u(c_0,h)}{\pi(c_0)} \leq \varphi < \frac{u(c_0,c_0)}{\pi(c_0)}$*, then there is a unique solution to the equation* $\varphi * \pi(c_0) = u(c_0, q)$ *on* $[c_0, h]$*, denoted by* q_φ*, such that*

$$x_p(q) = \begin{cases} c_0 & \text{for } q \in [c_0, q_\varphi], \\ x_q & \text{for } q \in (q_\varphi, h], \end{cases}$$

where x_q *is the unique solution to the equation* $\varphi * \pi(x) = u(x, q)$ *in* x *on* $[c_0, q]$*;*

(iv) *If* $0 < \varphi < \frac{u(c_0,h)}{\pi(c_0)}$, *then* $x_p(q) = c_0$.

Lemma 2 further characterizes the focus of the order quantity $q \in [c_0, h]$. With the assistance of Lemma 1, we can now pinpoint the thresholds of φ, which are independent of q such that different foci arise within different ranges of φ values.

As mentioned earlier, φ reflects how optimistic the retailer is, Lemma 2 can thus be interpreted from a behavioral perspective as follows. When φ is at the low range ($0 < \varphi < \frac{u(c_0,h)}{\pi(c_0)}$, meaning that the retailer is not so optimistic), its scaling effect makes relative likelihood more prominent in determining the focus demand. As such, the retailer will always focus on the most likely demand c_0 regardless of the order quantity q. When φ increases to a slightly higher level such that $\frac{u(c_0,h)}{\pi(c_0)} \leq \varphi < \frac{u(c_0,c_0)}{\pi(c_0)}$, the retailer exhibits a higher optimistic level by balancing the relative likelihood of a demand quantity and its resulting satisfaction. When the order quantity is at the lower end, the focus remains at the most likely demand c_0, but it moves up to a higher level at the solution to $\varphi * \pi(x) = u(x, q)$ in x, which is no more than the order quantity q. When φ further increases to a moderately high level ($\frac{u(c_0,c_0)}{\pi(c_0)} \leq \varphi \leq \frac{u(h,h)}{\pi(h)}$), the retailer becomes even more optimistic by leaning further towards the satisfaction of a focus demand given an order quantity. When the order quantity is at the lower end, the focus demand will be the order quantity. But the focus demand lies at the solution to $\varphi * \pi(x) = u(x, q)$ in x when the order quantity q falls within the upper end. If φ is at a significantly high level ($\varphi > \frac{u(h,h)}{\pi(h)}$), the retailer becomes so optimistic that he/she simply focuses on the most rosy scenario with the highest satisfaction level given any demand quantity, $x_p(q) = q$.

With an increasing φ (implying a more optimistic retailer), his/her attention is gradually shifted from seeking assurance (higher likelihood) to aspiring for profitability (higher satisfaction), leading to a heightened focus demand that approaches the order quantity. These explanations are intuitive and consistent with the behavioral patterns of decision makers at different optimistic levels. Therefore, Lemma 2 sheds insights on how a retailer evaluates the most attractive demand under different order quantities and how this behaviour is linked to his/her personality attributes as reflected by the parameter φ. Lemma 2, together with Lemma 1 and Theorem 1, yields the following result.

Theorem 3 $u(x_p(q), q)$ *is a strictly quasi-concave continuous function of* q *on* $[l, h]$.

(i) *If* $\varphi > \frac{u(h,h)}{\pi(h)}$, *then* $u(x_p(q), q)$ *is strictly increasing on* $[l, h]$.

(ii) *If* $\frac{u(c_0,c_0)}{\pi(c_0)} \leq \varphi \leq \frac{u(h,h)}{\pi(h)}$, *then* $u(x_p(q), q)$ *is strictly increasing on* $[l, x_\varphi]$ *and strictly decreasing on* $[x_\varphi, h]$ *where* x_φ *is the unique solution to the equation* $u(x, x) = \varphi * \pi(x)$ *on* (c_0, h).

(iii) *If* $\frac{u(c_0,h)}{\pi(c_0)} \leq \varphi < \frac{u(c_0,c_0)}{\pi(c_0)}$, *then* $u(x_p(q), q)$ *is strictly increasing on* $[l, c_0]$ *and strictly decreasing on* $[c_0, h]$.

(iv) *If* $0 < \varphi < \frac{u(c_0,h)}{\pi(c_0)}$, *then* $u(x_p(q), q)$ *is strictly increasing on* $[l, c_0]$ *and strictly decreasing on* $[c_0, h]$.

Now we are ready to present the optimal order quantity to the proposed newsvendor model.

Theorem 4 *The optimal order quantity* q_p^* *under the positive evaluation system and its corresponding positive focus* $x_p(q_p^*)$ *are furnished as follows:*

$$q_p^* = x_p(q_p^*) = \begin{cases} h & \textit{if} \ \ \varphi > \frac{u(h,h)}{\pi(h)} \ \textit{and} \ \kappa > \frac{u(h,h)}{\pi(h)}, \\ x_\varphi & \textit{if} \ \ \frac{u(c_0,c_0)}{\pi(c_0)} \le \varphi \le \frac{u(h,h)}{\pi(h)} \ \textit{and} \ \kappa > \varphi, \\ x_\kappa & \textit{if} \ \ \frac{u(c_0,c_0)}{\pi(c_0)} \le \kappa \le \frac{u(h,h)}{\pi(h)} \ \textit{and} \ \varphi \ge \kappa, \\ c_0 & \textit{if} \ \ 0 < \varphi < \frac{u(c_0,c_0)}{\pi(c_0)} \ \textit{or} \ 0 < \kappa < \frac{u(c_0,c_0)}{\pi(c_0)}. \end{cases} \tag{6}$$

Here x_φ *and* x_κ *are unique solutions to the equations* $u(x,x) = \varphi * \pi(x)$ *and* $u(x,x) = \kappa * \pi(x)$ *with respect to* x *on* $[c_0, h]$, *respectively.*

Theorem 4 implies that the optimal order quantity must lie within $[c_0, h]$ under the positive evaluation system. This conclusion is intuitive because the order quantity c_0 brings the highest payoff with the highest relative likelihood when q is confined to $[l, c_0]$.

As mentioned earlier, κ reflects how confident a retailer is in his/her selection of decisions after examining the foci identified at the first stage. We can interpret the behavioral patterns in the results of Theorem 4 as follows. If the retailer is highly optimistic and highly confident (both φ and κ take large values), then he/she will select the highest demand h as the optimal order quantity. In the other extreme case, if the retailer has a low optimism (confidence) level with a small φ (κ), regardless of the value of κ (φ), he/she will take the most likely demand c_0 as the optimal order quantity, which is the lowest possible optimal quantity under the positive evaluation system. If the retailer is moderately optimistic and confident (both φ and κ take values in the middle range), he/she will take a value in-between as the optimal order quantity. These results are intuitive and consistent with common sense. Theorem 4 thus sheds insight on how different characteristics of a retailer affects his/her order decision.

Remark 1: In the classic newsvendor model, the optimal order quantity q_0^* satisfies $\int_l^{q_0^*} f(x)dx = \frac{r-w+g}{r-s+g}$ where $R = \frac{r-w+g}{r-s+g} \in (0,1)$ is referred to as the critical ratio. If the probability density function is symmetric, then the optimal order quantity will be larger than c_0 when $R \ge 0.5$ (high margin) holds. This conclusion is logically consistent with Theorem 4 because a high margin (with a low w and high s) motivates the retailer to think positively, thereby activating his/her positive evaluation system. Under such a mindset, the retailer will order more than c_0. These results support the common recognition that decision problems are inherently context dependent (Kahneman and Tversky

1984). Therefore, $R \geq 0.5$ can be regarded as a rough yardstick for employing the proposed newsvendor model with the positive evaluation system.

Remark 2: If the probability density function is symmetric about the mode c_0, the mean of demand is also c_0. When $R \geq 0.5$, since $\frac{u(c_0,c_0)}{\pi(c_0)} \leq \frac{u(q_0^*,q_0^*)}{\pi(q_0^*)} \leq \frac{u(h,h)}{\pi(h)}$, it is concluded from Theorem 4 that when $\frac{u(c_0,c_0)}{\pi(c_0)} \leq \kappa \leq \frac{u(h,h)}{\pi(h)}$ and $\varphi \geq \kappa$, the optimal order quantity under the positive evaluation system is x_κ which is equal to (smaller than or larger than) q_0^* if κ is equal to (smaller than or larger than) $\frac{u(q_0^*,q_0^*)}{\pi(q^*)}$; when $\frac{u(c_0,c_0)}{\pi(c_0)} \leq \varphi \leq \frac{u(h,h)}{\pi(h)}$ and $\kappa > \varphi$, the optimal order quantity under the positive evaluation system takes the value of x_φ which is equal to (smaller than or larger than) q_0^* if φ is equal to (smaller than or larger than) $\frac{u(q_0^*,q_0^*)}{\pi(q_0^*)}$. Especially when $\frac{u(c_0,c_0)}{\pi(c_0)} \leq \kappa \leq \frac{u(q_0^*,q_0^*)}{\pi(q_0^*)}$ and $\varphi \geq \kappa$ or $\frac{u(c_0,c_0)}{\pi(c_0)} \leq \varphi \leq \frac{u(q_0^*,q_0^*)}{\pi(q_0^*)}$ and $\kappa > \varphi$, the retailer's optimal solution takes a value between the mean of demand and the classic optimal quantity.

Remark 3: Newsvendor models with the one-shot decision theory (OSDT) have been proposed by Guo and Ma (2014). As the focus theory of choice is a further refinement of OSDT, we can establish links between our results in Theorem4 with those obtained with OSDT in Guo and Ma (2014). More specifically, it is clear from Theorem4 that: When φ is equal to 1 and κ is big enough, since $u(h,h) > \pi(h)$ and $\pi(c_0) > u(c_0,c_0)$, the optimal order quantity under the positive evaluation system will take the unique solution to the equation $u(x,x) = \pi(x)$ on $[c_0,h]$ which is the same as the optimal active order quantity of the newsvendor model with OSDT (Theorem 13(1) in Guo and Ma 2014); When both φ and κ are big enough, the optimal order quantity under the positive evaluation system will take the highest demand h which is identical to the optimal daring order quantity of the newsvendor model with OSDT (Theorem 13(2) in Guo and Ma 2014).

4 An illustrative example and comparison with other models

This section presents an illustrative example to show how to apply the proposed newsvendor model and clarify its relationship with the classic model. Consider a retailer selling a seasonal product whose retail price r, wholesale price w, salvage value s and opportunity cost g are 10, 6, 4 and 2 (dollars) per unit, respectively. We have the payoff function as follows:

$$v(x,q) = \begin{cases} 6x - 2q & \text{for } x < q, \\ 6q - 2x & \text{for } x \geq q. \end{cases}$$

The range of uncertain demand is estimated as $[10, 25]$. The probability density function of uncertain demand is assumed to be triangular as follows:

$$f(x) = \begin{cases} \frac{17}{750}x - \frac{13}{60} & \text{for } 10 \leq x \leq 15, \\ -\frac{17}{1500}x + \frac{22}{75} & \text{for } 15 \leq x \leq 25. \end{cases} \tag{7}$$

Clearly, $f(l) = f(10) = \frac{1}{100}$, $f(h) = f(25) = \frac{1}{100}$, $f(c_0) = f(15) = \frac{37}{300}$ and $\int_{10}^{25} f(x)\mathrm{d}x = 1$. We can calculate the mean of demand as $x_0 = 16.79$. In a classic newsvendor model, the optimal order quantity q_0^* satisfies

$$\int_{l}^{q_0^*} f(x)\mathrm{d}x = \frac{p - w + g}{p - s + g}, \tag{8}$$

where $R = \frac{p-w+g}{p-s+g} = 0.75$ is the critical ratio. Plugging (7) into (8), we obtain $q_0^* = 19.18$.

Next, let us consider the optimal order quantity of this newsvendor problem under the positive evaluation system of the focus theory of choice. Since the range of demand is $[10, 25]$, we only need to consider order quantities within the same range. Therefore, the maximal and minimal payoffs are $v(h, h) = v(25, 25) = 100$ and $\min\{v(l, h), v(h, l)\} = \min\{v(10, 25), v(25, 10)\} = 10$, respectively. We adopt (2) to normalize payoffs and obtain the satisfaction function as follows:

$$u(x, q) = \begin{cases} \frac{6x-2q-10}{90} & \text{for } x < q, \\ \frac{6q-2x-10}{90} & \text{for } x \geq q. \end{cases} \tag{9}$$

Clearly, (9) attains the highest satisfaction level of 1 at the maximal payoff 100 when $x = q = h = 25$ and the lowest satisfaction level of 0 at the minimal payoff 10 when $x = h = 25$, $q = l = 10$ or $x = l = 10$, $q = h = 25$. From (9), we know

$$u(x, x) = \frac{4x - 10}{90},$$

and $u(l, l) = u(10, 10) = \frac{1}{3}$, $u(c_0, c_0) = u(15, 15) = \frac{5}{9}$, $u(h, h) = u(25, 25) = 1$.

Similarly, we define the relative likelihood function as per (3) as

$$\pi(x) = \begin{cases} \frac{34}{185}x - \frac{65}{37} & \text{for } 10 \leq x \leq 15, \\ -\frac{17}{185}x + \frac{88}{37} & \text{for } 15 \leq x \leq 25. \end{cases} \tag{10}$$

which is obtained by dividing the probability density function in (7) by its maximum $f(15) = \frac{37}{300}$. From (10), we have $\pi(10) = \pi(25) = \frac{3}{37}$ and $\pi(15) = 1$.

According to Theorem 4, we can directly obtain the following results:

- if $\varphi > \frac{u(h,h)}{\pi(h)} = 12.33$ and $\kappa > \frac{u(h,h)}{\pi(h)} = 12.33$, then $q_p^* = h = 25$;
- if either $0 < \varphi < \frac{u(c_0,c_0)}{\pi(c_0)} = 0.56$ (regardless of κ) or $0 < \kappa < \frac{u(c_0,c_0)}{\pi(c_0)} = 0.56$ (regardless of φ), then $q_p^* = c_0 = 15$.

Briefly speaking, if the retailer is highly optimistic about uncertain market demand and highly confident in his/her decision (both φ and κ are large, or larger than 12.3 in this particular example), the optimal order quantity is the highest possible value $q_p^* = h = 25$, which is identical to the optimal daring order quantity of the newsvendor model with OSDT (Guo and Ma 2014). On the other hand, if the retailer's optimism about demand uncertainty or confidence in his/her decision is at a low level (either φ or κ is small, or in this example, $0 < \varphi < 0.56$ or $0 < \kappa < 0.56$, the optimal order quantity is the most likely demand $q_p^* = c_0 = 15$.

Next, we consider the case when the retailer is moderately optimistic about uncertain demand or moderately confident in his/her decision (φ or κ takes a moderate value, or in this specific example, $0.56 \leq \varphi \leq 12.3$ or $0.56 \leq \kappa \leq 12.3$). Solving equations $u(x,x) = \varphi * \pi(x)$ and $u(x,x) = \kappa * \pi(x)$ on $[c_0, h]$ and denoting their solutions by x_φ and x_κ, respectively, we have

$$x_\varphi = \frac{7920\varphi + 370}{306\varphi + 148}, \tag{11}$$

and

$$x_\kappa = \frac{7920\kappa + 370}{306\kappa + 148}. \tag{12}$$

According to Theorem 4, we have the following results:

- if $0.56 = \frac{u(c_0,c_0)}{\pi(c_0)} \leq \varphi \leq \frac{u(h,h)}{\pi(h)} = 12.3$ and $\kappa > \varphi$, then $q_p^* = x_\varphi$;
- if $0.56 = \frac{u(c_0,c_0)}{\pi(c_0)} \leq \kappa \leq \frac{u(h,h)}{\pi(h)} = 12.3$ and $\varphi \geq \kappa$, then $q_p^* = x_\kappa$.

For example, if $\varphi = 5$ and $\kappa = 10$, we have $q_p^* = x_p(q_p^*) = x_\varphi = 23.82$ by (12); if $\varphi = 15$ and $\kappa = 3$, we have $q_p^* = x_p(q_p^*) = x_\kappa = 22.64$ by (12). Particularly if $\varphi = 1$ and κ is big enough (or in this example $\kappa > 12.3$), we have $q_p^* = x_p(q_p^*) = x_\varphi = 18.26$ by (11), which is the same as the optimal active order quantity of the newsvendor model with OSDT (Guo and Ma 2014).

Given that the optimal order quantity of the classic newsvendor model is $q_0^* = 19.18$, we can properly set the values of φ and κ such that the optimal order quantity under the positive evaluation system is equal to that of the classic model, $q_p^* = q_0^* = 19.18$. More specifically, let $x_\varphi = 19.18$ and $x_\kappa = 19.18$ and solve equations (11) and (12), we have $\varphi = \kappa = 1.20$. According to Theorem 4, we understand that setting $\varphi \geq 1.20$ and $\kappa = 1.20$ or setting $\varphi = 1.20$ and $\kappa > 1.20$ will yield the optimal order quantity under the positive evaluation system equal to that of the classic newsvendor model, that is, $q_p^* = q_0^* = 19.18$. We can also set the values of φ and κ such that the optimal order quantity under the positive evaluation system is equal to the mean of demand, $q_p^* = x_0 = 16.79$. Similarly, let $x_\varphi = 16.79$ and $x_\kappa = 16.79$ and solve equations (11) and (12), we have $\varphi = \kappa = 0.76$. Consequently, when $0.76 \leq \kappa \leq 1.2$ and $\varphi \geq \kappa$ or $0.76 \leq \varphi \leq 1.2$ and $\kappa > \varphi$, the optimal order quantity under the positive evaluation system will take a value between the mean of demand and the theoretical optimal quantity of the classic newsvendor model.

In summary, under the positive evaluation system, this illustrative example clearly shows that the highest possible value h can arise as the optimal order quantity only if the retailer is both highly optimistic and highly confident. If either the optimism or confidence level falls within the middle range, the optimal order quantity can be any value between the most likely demand c_0 and the highest possible value h contingent upon the specific values of φ and κ. If either the optimism or confidence level falls within the low range, the most likely demand c_0 arises as the optimal order quantity. This result is consistent with the general behavioral pattern of decision makers: When a retailer is more optimistic about uncertain demand and more confident in his/her decision, he/she tends to aim high and act more aggressively by ordering a higher quantity.

5 Conclusions

First, this research contributes to the literature by establishing a new newsvendor model with the focus theory of choice. Building upon two axioms in (Guo 2019), the proposed newsvendor model conceives the retailer's ordering decision as a two-stage procedure. Firstly, for each order quantity, the retailer chooses the most salient demand as its focus after assessing the resulting satisfaction and relative likelihood levels of all possible demands. Secondly, the retailer determines the optimal order quantity by selecting the most preferred focus among those of all possible order quantities. This research furnishes a new perspective to understand the retailer's ordering decision by accounting for his/her optimism and confidence levels as well as the underlying behavioral insights behind the choice.

Second, this research obtains the analytical solution to the proposed model. It is a bilevel optimization problem with maximin upper and lower level programs that are nonsmooth and cannot be solved by existing optimization methods. We derive the theoretical results of the optimal order quantity under the positive evaluation system and show the mathematical properties of the optimal order quantity and its positive focus.

This research enriches the literature of newsvendor modeling and bilevel optimization and will be the theoretical basis for building supply chain models with the focus theory of choice framework.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

Proof of Lemma 1: For any $q \in Q$, as $u(q,q) \geq u(c_0,q)$ and $0 < \pi(q) \leq \pi(c_0)$, we have $\frac{u(q,q)}{\pi(q)} \geq \frac{u(c_0,q)}{\pi(c_0)}$.

(i) For any $x \neq q$, as $\varphi * \pi(q) > u(q,q)$, we have $\min\{\varphi * \pi(q), u(q,q)\} = u(q,q) > u(x,q) \geq \min\{\varphi * \pi(x), u(x,q)\}$. Based on (4) and the definition of the positive focus, we have $x_p(q) = q$.

(ii) If $\frac{u(c_0,q)}{\pi(c_0)} \leq \varphi \leq \frac{u(q,q)}{\pi(q)}$ and $q < c_0$, then $u(q,q) \geq \varphi * \pi(q)$ and $u(c_0,q) \leq \varphi * \pi(c_0)$. Since $u(x,q)$ strictly decreases in x on $[q,c_0]$ and $\varphi * \pi(x)$ strictly increases on $[q,c_0]$, there is a unique solution to the equation $u(x,q) = \varphi * \pi(x)$ in x on $[q,c_0]$, denoted by $x_{pl}(q,\varphi)$. For any $x \neq x_{pl}(q,\varphi)$, we have $\min\{\varphi*\pi(x_{pl}(q,\varphi)), u(x_{pl}(q,\varphi),q)\} > \min\{\varphi*\pi(x), u(x,q)\}$. This means $x_p(q) = x_{pl}(q,\varphi)$.

(iii) Similar to case (ii), we have $x_p(q) = x_{pr}(q,\varphi)$ where $x_{pr}(q,\varphi)$ is the unique solution to the equation $u(x,q) = \varphi * \pi(x)$ in x on $[c_0,q]$

(iv) For any $x \neq c_0$, as $\varphi*\pi(c_0) < u(c_0,q)$, we have $\min\{\varphi*\pi(c_0), u(c_0,q)\} = \varphi * \pi(c_0) > \varphi * \pi(x) \geq \min\{\varphi * \pi(x), u(x,q)\}$. So, we have $x_p(q) = c_0$. □

Proof of Theorem 1: Since $x_p(q)$ is the unique optimal solution to the lower level problem (4), as per the stability theory in parametric optimization (Evans and Gould 1970), we know that $x_p(q)$ is a continuous function of q.

It follows from Lemma 1 that $x_p(c_0) = c_0$ whenever $\varphi > 0$. To show the monotonicity of $x_p(q)$, we consider the following two cases.

Let $q_1, q_2 \in [l, c_0]$ and $q_1 < q_2$. By Lemma 1, we know $x_p(q_i) \in [q_i, c_0]$ for $i = 1, 2$. In the following, we use contradictions to show the proof. Suppose $x_p(q_1) > x_p(q_2)$, then we have $l \leq q_1 < q_2 \leq x_p(q_2) < x_p(q_1) \leq c_0$. By the definitions of $v(x,q)$ and $u(x,q)$, it is easy to verify that

$$u(x_p(q_2), q_1) < u(x_p(q_2), q_2), \tag{A.1}$$

and

$$u(x_p(q_2), q_1) > u(x_p(q_1), q_1). \tag{A.2}$$

Considering $x_p(q_2) < c_0$, it follows from Lemma 1 that

$$\text{either} \quad \varphi > \frac{u(q_2,q_2)}{\pi(q_2)} \quad \text{or} \quad \frac{u(c_0,q_2)}{\pi(c_0)} \leq \varphi \leq \frac{u(q_2,q_2)}{\pi(q_2)}, \tag{A.3}$$

and if $\varphi > \frac{u(q_2,q_2)}{\pi(q_2)}$ then $x_p(q_2) = q_2$, otherwise $x_p(q_2)$ satisfies the equation $u(x_p(q_2), q_2) = \varphi * \pi(x_p(q_2))$. In either case of (A.3), we have

$$u(x_p(q_2), q_2) \leq \varphi * \pi(x_p(q_2)). \tag{A.4}$$

(A.4) together with (A.1) leads to $u(x_p(q_2), q_1) < \varphi*\pi(x_p(q_2))$. Since $x_p(q_1) \in X_p(q_1)$ and $x_p(q_1) \neq x_p(q_2)$, we further have

$$\begin{aligned} u(x_p(q_1), q_1) &\geq \min\{\varphi * \pi(x_p(q_1)), u(x_p(q_1), q_1)\} \\ &> \min\{\varphi * \pi(x_p(q_2)), u(x_p(q_2), q_1)\} \\ &= u(x_p(q_2), q_1). \end{aligned} \tag{A.5}$$

It is clear that (A.2) and (A.5) contradict each other. Thus, for any $q_1, q_2 \in [l, c_0]$, if $q_1 < q_2$ then

$$x_p(q_1) \leq x_p(q_2) \leq x_p(c_0). \tag{A.6}$$

Similar to the above proof, we can show that for any $q_3, q_4 \in [c_0, h]$, if $q_3 < q_4$ then

$$x_p(c_0) \leq x_p(q_3) \leq x_p(q_4). \tag{A.7}$$

(A.6) together with (A.7) provides the proof. □

Proof of Lemma 2: From the definitions of the satisfaction and relative likelihood functions, one can see that $\pi(x)$ is strictly decreasing and $u(x,x)$ is strictly increasing on $[c_0,h]$.

(i) Since $\varphi > \frac{u(h,h)}{\pi(h)} \geq \frac{u(q,q)}{\pi(q)}$ holds for any $q \in [c_0,h]$, it follows from Lemma 1(i) that $x_p(q) = q$.

(ii) If $\frac{u(c_0,c_0)}{\pi(c_0)} \leq \varphi \leq \frac{u(h,h)}{\pi(h)}$, then the equation $\varphi * \pi(x) = u(x,x)$ has a unique solution x_φ within $[c_0,h]$ as the monotonicity of $\pi(x)$ and $u(x,x)$ on $[c_0,h]$. For $q \in [c_0,x_\varphi)$, as $\varphi * \pi(q) > u(q,q)$, we have $x_p(q) = q$ as per Lemma 1(i). For $q \in [x_\varphi,h]$, we have $\varphi = \frac{u(x_\varphi,x_\varphi)}{\pi(x_\varphi)} \leq \frac{u(q,q)}{\pi(q)}$ and $\varphi \geq \frac{u(c_0,c_0)}{\pi(c_0)} \geq \frac{u(c_0,q)}{\pi(c_0)}$. From Lemma 1(iii), we know that there is a unique solution to the equation $u(x,q) = \varphi * \pi(x)$ in x on $[c_0,q]$, denoted by x_q, such that $x_p(q) = x_q$. In addition, we have $x_p(x_\varphi) = x_\varphi$.

(iii) If $\frac{u(c_o,h)}{\pi(c_0)} \leq \varphi < \frac{u(c_o,c_o)}{\pi(c_0)}$, as $u(c_0,x)$ is strictly decreasing on $[c_0,h]$, the equation $\varphi * \pi(c_0) = u(c_0,q)$ has a unique solution q_φ within $[c_0,h]$. For $q \in [c_0,q_\varphi)$, as $\varphi*\pi(c_0) < u(c_0,q)$, it follows from Lemma 1(iv) that $x_p(q) = c_0$. For $q \in [q_\varphi,h]$, as $\varphi * \pi(c_0) = u(c_0,q_\varphi)$, we have $\varphi = \frac{u(c_0,q_\varphi)}{\pi(c_0)} \geq \frac{u(c_0,q)}{\pi(c_0)}$ and $\varphi = \frac{u(c_0,q_\varphi)}{\pi(c_0)} \leq \frac{u(c_0,c_0)}{\pi(c_0)} \leq \frac{u(q,q)}{\pi(q)}$. By Lemma 1(iii), we know $x_p(q) = x_q$. In addition, we have $x_p(q_\varphi) = c_0$.

(iv) If $0 < \varphi < \frac{u(c_o,h)}{\pi(c_0)}$, then $\varphi * \pi(c_0) < u(c_0,h) \leq u(c_0,q)$ for $q \in [c_0,h]$. As per Lemma 1(iv), we have $x_p(q) = c_0$. □

Proof of Theorem 3: It follows from (1) that $v(x_p(q),q) = \min\{(r-s) * x_p(q) - (w-s) * q,\ (r-w+g) * q - g * x_p(q)\}$. Since $x_p(q)$ is continuous on $[l,h]$, the function $v(x_p(q),q)$ is also continuous. Considering the definition of satisfaction function, we know that $u(x_p(q),q)$ is a continuous function of q on $[l,h]$.

To show the monotonicity of $u(x_p(q),q)$, we divide $[l,h]$ into two intervals: $[l,c_0]$ and $[c_0,h]$. In what follows, we consider the two cases respectively.

(I) Let $q_1,q_2 \in [l,c_0]$ and $q_1 < q_2$. As per Lemma 1, we have $q_i \leq x_p(q_i) \leq c_0$ for $i = 1,2$. As per Lemma 1, we have $x_p(q_1) \leq x_p(q_2)$. By the definition of (1), we have

$$v(x_p(q_i),q_i) = (r-w+g) * q_i - g * x_p(q_i), \quad i = 1,2. \tag{A.8}$$

In the case $x_p(q_2) = q_2$, it follows from (A.8) that $v(x_p(q_1),q_1) - v(x_p(q_2),q_2) \leq v(q_1,q_1) - v(q_2,q_2) < 0$. By the definition of satisfaction function, we have $u(x_p(q_1),q_1) < u(x_p(q_2),q_2)$.

In the case $x_p(q_2) > q_2$, as per Lemma 1 (i) and (ii), we have $u(x_p(q_2),q_2) \geq \varphi * \pi(x_p(q_2))$. If $x_p(q_1) = x_p(q_2)$, it follows from (A.8) that $v(x_p(q_1),q_1) - v(x_p(q_2),q_2) < 0$ due to $q_1 < q_2$ and hence, we have $u(x_p(q_1),q_1) < u(x_p(q_2),q_2)$. If $x_p(q_1) < x_p(q_2)$, we consider the following two cases: $q_1 = x_p(q_1)$ and $q_1 < x_p(q_1)$. If $x_p(q_1) < x_p(q_2)$ and $q_1 = x_p(q_1)$, as per Lemma 1, we have $u(x_p(q_1),q_1) \leq \varphi * \pi(x_p(q_1))$. Considering $u(x_p(q_2),q_2) \geq \varphi * \pi(x_p(q_2))$, we further have $u(x_p(q_1), q_1) - u(x_p(q_2), q_2) \leq \varphi * \pi(x_p(q_1)) - \varphi * \pi(x_p(q_2)) < 0$. If $x_p(q_1) < x_p(q_2)$ and $q_1 < x_p(q_1)$, we have $q_1 < x_p(q_1) < x_p(q_2) \leq c_0$, as per Lemma 1 (ii), we further have $u(x_p(q_1), q_1) = \varphi * \pi(x_p(q_2)) < \varphi * \pi(x_p(q_2)) \leq u(x_p(q_2), q_2)$.

From the above analysis, we know that $u(x_p(q), q)$ is strictly increasing on $[l, c_0]$ whenever $\varphi > 0$.

(II) It follows from Lemma 1 that $x_p(c_0) = c_0$ whenever $\varphi > 0$. In the following, we consider the monotonicity of $u(x_p(q), q)$ for cases (i), (ii), (iii) and (iv), respectively.

(i) For any $q \in [c_0, h]$, it follows from Lemma 2(i) that $x_p(q) = q$. As noted earlier, $u(x_p(q), q) = u(q, q)$ is strictly increasing on $[c_0, h]$. Considering the continuity of $u(x_p(q), q)$ and case (I), we know that $u(x_p(q), q)$ is strictly increasing on $[l, h]$.

(ii) Since $h \geq x_\varphi > c_0$, we divide $[c_0, h]$ into the following two intervals: $[c_0, x_\varphi]$ and $(x_\varphi, h]$. In what follows, we consider the two cases respectively.

(iia) For any $q \in [c_0, x_\varphi]$, it follows from Lemma 2(ii) that $x_p(q) = q$. Clearly, $u(x_p(q), q) = u(q, q)$ is strictly increasing on $[c_0, x_\varphi]$.

(iib) Let $q_5, q_6 \in (x_\varphi, h]$ and $q_5 < q_6$. As per Lemma 1, we have $x_p(q_i) \leq q_i$ for $i = 5, 6$. By the definition of (1), we know that (A.9) holds for $i = 5, 6$. If $x_p(q_5) < x_p(q_6)$, as per Lemma 2(ii), we have $u(x_p(q_i), q_i) = \varphi * \pi(x_p(q_i))$ for $i = 5, 6$ and hence, we have $u(x_p(q_5), q_5) = \varphi * \pi(x_p(q_5)) > \varphi * \pi(x_p(q_6)) = u(x_p(q_6), q_6)$. If $x_p(q_5) = x_p(q_6)$, (A.9) leads to $u(x_p(q_5), q_5) > u(x_p(q_6), q_6)$ due to $q_5 < q_6$ and hence, we have $u(x_p(q_5), q_5) > u(x_p(q_6), q_6)$.

In summary, cases (I) and (iia) show that the function $u(x_p(q), q)$ is strictly increasing on $[l, x_\varphi]$, case (iib) shows that $u(x_p(q), q)$ is strictly decreasing on $[x_\varphi, h]$.

(iii) As per Lemma 2(iii), we have $q_\varphi \geq c_0$. We divide $[c_0, h]$ into two intervals: $[c_0, q_\varphi]$ and $(q_\varphi, h]$. In what follows, we consider the two cases respectively.

(iiia) For any $q \in [c_0, q_\varphi]$, it follows from Lemma 2(iii) that $x_p(q) = c_0$. By the definition of (1), we know that $v(x_p(q), q) = v(c_0, q)$ is strictly decreasing on $[c_0, q_\varphi]$. That is, $u(x_p(q), q)$ is strictly decreasing on $[c_0, q_\varphi]$

(iiib) Let $q_3, q_4 \in (q_\varphi, h]$ and $q_3 < q_4$. As per Lemma 1, we have $x_p(q_i) \leq q_i$ for $i = 3, 4$. By the definition of (1), for $i = 3, 4$, we have

$$v(x_p(q_i), q_i) = (r - s) * x_p(q_i) - (w - s) * q_i. \tag{A.9}$$

If $x_p(q_3) < x_p(q_4)$, as per Lemma 2(iii), we have $u(x_p(q_i), q_i) = \varphi * \pi(x_p(q_i))$ for $i = 3, 4$ and hence, we have $u(x_p(q_3), q_3) = \varphi * \pi(x_p(q_3)) > \varphi * \pi(x_p(q_4)) = u(x_p(q_4), q_4)$. If $x_p(q_3) = x_p(q_4)$, (A.9) leads to $v(x_p(q_3), q_3) > v(x_p(q_4), q_4)$ due to $q_3 < q_5$ and hence, we have $u(x_p(q_3), q_4) > u(x_p(q_4), q_4)$.

In summary, case (I) shows that the function $u(x_p(q), q)$ is strictly increasing on $[l, c_0]$, cases (iiia) and (iiib) show that $u(x_p(q), q)$ is strictly decreasing on $[c_0, h]$.

(iv) For any $q \in [c_0, h]$, it follows from Lemma 2(iv) that $x_p(q) = c_0$. It is easy to verify from the definition of (1) that $v(x_p(q), q) = v(c_0, q)$ is strictly decreasing on $[c_0, h]$. Considering the definition of satisfaction function, we know that $u(x_p(q), q)$ is strictly decreasing on $[c_0, h]$. In summary, the function $u(x_p(q), q)$ is strictly increasing on $[l, c_0]$ and strictly decreasing on $[c_0, h]$. □

Proof of Theorem 4: Note that (6) can be equivalently divided into the following three cases:
Case 1. When $\varphi > \frac{u(h,h)}{\pi(h)}$,

$$q_p^* = x_p(q_p^*) = \begin{cases} h & \text{if } \kappa > \frac{u(h,h)}{\pi(h)}, \\ x_\kappa & \text{if } \frac{u(c_0,c_0)}{\pi(c_0)} \le \kappa \le \frac{u(h,h)}{\pi(h)}, \\ c_0 & \text{if } 0 < \kappa < \frac{u(c_0,c_0)}{\pi(c_0)}. \end{cases}$$

Case 2. When $\frac{u(c_0,c_0)}{\pi(c_0)} \le \varphi \le \frac{u(h,h)}{\pi(h)}$,

$$q_p^* = x_p(q_p^*) = \begin{cases} x_\varphi & \text{if } \kappa > \varphi, \\ x_\kappa & \text{if } \frac{u(c_0,c_0)}{\pi(c_0)} \le \kappa \le \varphi, \\ c_0 & \text{if } 0 < \kappa < \frac{u(c_0,c_0)}{\pi(c_0)}. \end{cases}$$

Case 3. When $0 < \varphi < \frac{u(c_0,c_0)}{\pi(c_0)}$,

$$q_p^* = x_p(q_p^*) = c_0, \quad \forall\, \kappa > 0.$$

For any $q \in [l, c_0)$, we have $\pi(x_p(c_0)) \ge \pi(x_p(q))$ and $u(x_p(c_0), c_0) > u(x_p(q), q)$ whenever $\varphi > 0$. This means that q_p^* will not lie in the interval $[l, c_0)$. In the following proof, we only need to consider the interval $[c_0, h]$.
Case 1. When $\varphi > \frac{u(h,h)}{\pi(h)}$, we have $x_p(q) = q$ for any $q \in [c_0, h]$ based on Lemma 2(i). As noted earlier, $\pi(x)$ is strictly decreasing on $[c_0, h]$ and $u(x, x)$ is strictly increasing on $[c_0, h]$.

If $\kappa > \frac{u(h,h)}{\pi(h)}$, then we have $\min\{\kappa * \pi(x_p(h)), u(x_p(h), h)\} = u(h, h) > u(x_p(q), q) \ge \min\{\kappa * \pi(x_p(q)), u(x_p(q), q)\}$ for any $q \in [c_0, h)$. Based on Definition 1, we have $q_p^* = x_p(q_p^*) = h$.

If $\frac{u(c_0,c_0)}{\pi(c_0)} \le \kappa \le \frac{u(h,h)}{\pi(h)}$, there is a unique solution to the equation $u(x, x) = \kappa * \pi(x)$ on $[c_0, h]$, denoted by x_κ. For any $q \in [c_0, x_\kappa) \cup (x_\kappa, h]$, $\min\{\kappa * \pi(x_p(x_\kappa)), u(x_p(x_\kappa), x_\kappa)\} = \min\{\kappa*\pi(x_\kappa), u(x_\kappa, x_\kappa)\} > \min\{\kappa*\pi(q), u(q, q)\} = \min\{\kappa * \pi(x_p(q)), u(x_p(q), q)\}$ holds. This means $q_p^* = x_p(q_p^*) = x_\kappa$.

If $0 < \kappa < \frac{u(c_0,c_0)}{\pi(c_0)}$, we have $\min\{\kappa * \pi(x_p(c_0)), u(x_p(c_0), c_0)\} = \kappa * \pi(c_0) > \kappa * \pi(q) \ge \min\{\kappa * \pi(x_p(q)), u(x_p(q), q)\}$ for any $q \in (c_0, h]$. This means $q_p^* = x_p(q_p^*) = c_0$.
Case 2. When $\frac{u(c_0,c_0)}{\pi(c_0)} \le \varphi \le \frac{u(h,h)}{\pi(h)}$, $\pi(x_p(q))$ is decreasing on $[c_0, h]$ as per Theorem 2, $u(x_p(q), q)$ is strictly increasing on $[c_0, x_\varphi]$ and strictly decreasing on $[x_\varphi, h]$ as per Theorem 3. This means that q_p^* will lie in the interval $[c_0, x_\varphi]$. As per Lemma 2(ii), we have $x_p(q) = q$ for any $q \in [c_0, x_\varphi]$.

If $\kappa \ge \varphi$, for any $q \in [c_0, x_\varphi)$, we have $\min\{\kappa * \pi(x_p(x_\varphi)), u(x_p(x_\varphi), x_\varphi)\} \ge \min\{\varphi * \pi(x_p(x_\varphi)), u(x_p(x_\varphi), x_\varphi)\} = u(x_p(x_\varphi), x_\varphi) > u(x_p(q), q) \ge \min\{\kappa * \pi(x_p(q)), u(x_p(q), q)\}$. Thus, we have $q_p^* = x_p(q_p^*) = x_\varphi$.

If $\frac{u(c_0,c_0)}{\pi(c_0)} \leq \kappa \leq \varphi$, then $x_p(q) = q$ for any $q \in [c_0, x_\varphi]$. Since $\kappa * \pi(x_p(c_0)) = \kappa * \pi(c_0) \geq u(c_0, c_0) = u(x_p(c_0), c_0)$ and $\kappa * \pi(x_p(x_\varphi)) = \kappa * \pi(x_\varphi) \leq \varphi * \pi(x_\varphi) = u(x_\varphi, x_\varphi) = u(x_p(x_\varphi), x_\varphi)$, we know that there is a unique solution to the equation $u(x,x) = \kappa * \pi(x)$ on $[c_0, x_\varphi]$, denoted by x_κ such that $\min\{\kappa * \pi(x_p(x_\kappa)), u(x_p(x_\kappa), x_\kappa)\} = \kappa * \pi(x_p(x_\kappa)) = \kappa * \pi(x_\kappa) > \kappa * \pi(q) = \kappa * \pi(x_p(q)) \geq \min\{\kappa * \pi(x_p(q)), u(x_p(q), q)\}$ holds for any $q \in [c_0, x_\kappa)$ and $\min\{\kappa * \pi(x_p(x_\kappa)), u(x_p(x_\kappa), x_\kappa)\} = u(x_p(x_\kappa), x_\kappa) > u(x_p(q), q) \geq \min\{\kappa * \pi(x_p(q)), u(x_p(q), q)\}$ holds for any $q \in (x_\kappa, x_\varphi]$. This means $q_p^* = x_p(q_p^*) = x_\kappa$.

If $0 < \kappa < \frac{u(c_0,c_0)}{\pi(c_0)}$, then $x_p(q) = q$ for any $q \in [c_0, x_\varphi]$. Since $\min\{\kappa * \pi(x_p(c_0)), u(x_p(c_0), c_0)\} = \kappa * \pi(c_0) > \kappa * \pi(q) = \kappa * \pi(x_p(q)) \geq \min\{\kappa * \pi(x_p(q)), u(x_p(q), q)\}$ holds for any $q \in (c_0, x_\varphi]$, we have $q_p^* = x_p(q_p^*) = c_0$.

Case 3. When $0 < \varphi < \frac{u(c_0,c_0)}{\pi(c_0)}$, for any $q \in (c_0, h]$, we have $\pi(x_p(c_0)) \geq \pi(x_p(q))$ as per Theorem 2 and $u(x_p(c_0), c_0) > u(x_p(q), q)$ as per Theorem 3(i-ii). Based on Definition 1, for any $\kappa > 0$, we have $q_p^* = x_p(q_p^*) = c_0$. □

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