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C^r $\mathscr{K}$-versality of the graph deformation of a C^r stable map-germ

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1. *Introduction*

In his celebrated paper [**7**], Martinet showed the equivalence between the infinitesimal versality and the versality for $\mathscr{A}$- and $\mathscr{K}$-morphisms. By using this theorem, he obtained the following Theorem 1·1 which played one of the key roles for the classification of C^∞ stable map-germs ([**1**, **7**]).

THEOREM 1·1. *For any C^∞ stable map-germ f, its graph deformation $F(x,y) = f(x) - y$ is $\mathscr{K}$-versal.*

When we try to study the C^r classification of C^r stable map-germs, it is natural to ask whether the graph deformation of any C^r stable map-germ is C^r $\mathscr{K}$-versal or not ($0 \leqslant r < \infty$).

In this paper, we shall answer this question affirmatively by constructing a C^r $\mathscr{K}$-morphism directly ($0 \leqslant r \leqslant \infty$). In fact, we shall give a very simple proof of the following.

THEOREM 1·2. *Let f: $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a C^∞ map-germ and Φ: $(\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^p, 0)$ be a C^∞ deformation-germ of f. Let r be an element of the set $\{0, 1, \ldots, \infty\}$. Suppose that there exists a C^r $\mathscr{A}$-morphism from Φ to f. Then there exists a C^r $\mathscr{K}$-morphism from Φ to the graph deformation of f.*

COROLLARY 1·1. *For any C^r stable map-germ f, its graph deformation is C^r $\mathscr{K}$-versal for $0 \leqslant r \leqslant \infty$.*

For $1 \leqslant r < \infty$, the uniqueness of C^r $\mathscr{K}$-versal deformation-germ of a given map-germ may be proved easily by a slight modification of Martinet's proof of the uniqueness of C^∞ $\mathscr{K}$-versality (pp. 155–156 of [**1**]; pp. 21–22 of [**7**]), because in order to prove the uniqueness we need only one implication, *the C^r versality implies the infinitesimal C^{r-1} versality*, which is clear. Thus, by using Martinet's argument (p. 158 of [**1**]; p. 28 of [**7**]), we see that Corollary 1·1 yields a C^r generalization of Mather's classification theorem ($1 \leqslant r \leqslant \infty$) without any difficulty.

COROLLARY 1·2. *Let f, g : $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be C^r stable map-germs ($1 \leqslant r \leqslant \infty$). Suppose that there exists a germ of C^∞ diffeomorphism s : $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map-germ M: $(\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that $f(x) = M(x)g(s(x))$. Then f and g are C^r right-left equivalent.*

Corollary 1·2 may be proved also by using a criterion for C^r right-left equivalence of two given map-germs ($1 \leqslant r \leqslant \infty$) (see [**11**, corollary 1·1]).

Problem 1·1. Let Φ_1, Φ_2: $(\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^p, 0)$ be C^0 $\mathscr{K}$-versal deformation-germs of the C^∞ map-germ f: $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$. Then, does there exist a C^0 equivalent C^0 $\mathscr{K}$-morphism from Φ_1 to Φ_2?

Problem 1·2. Let f, g: $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be C^0 stable map-germs. Suppose that there exists a germ of C^∞ diffeomorphism s: $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map-germ M: $(\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that $f(x) = M(x)g(s(x))$. Then, are they C^0 right-left equivalent?

A partial affirmative solution to Problem 1·2 is known.

THEOREM 1·3 ([**10**]). *Let f, g: $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be Thom stable map-germs. Suppose that there exists a germ of C^∞ diffeomorphism s: $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map-germ M: $(\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that $f(x) = M(x)g(s(x))$. Then, they are C^0 right-left equivalent.*

For the definition of Thom stability, [**10**].

Note that, in contrast with the case that $r = \infty$, C^r $\mathscr{K}$-equivalence of C^r stable map-germs can not necessarily be extended to the corresponding one of C^r $\mathscr{K}$-versal deformations of these map-germs for $0 \leqslant r < \infty$. This can be seen as follows. We let $f_k(x) = x^k$ and $F_{k,m}$: $(\mathbf{R} \times \mathbf{R}^m, (0,0)) \to (\mathbf{R} \times \mathbf{R}^m, (0,0))$ be their C^∞ stable unfolding germs. Then, we see that $F_{k.m}$ is topologically $\mathscr{K}$-equivalent to $F_{k',m}$ if $k \equiv k'$ (mod 2) ([**9**]). However, it is well-known that $F_{k.m}$ is not topologically right-left equivalent to $F_{k',m}$ if $k \neq k'$ ([**2**, **3**]).

In Section 2, we recall several definitions. The definition of C^r stability, ($0 \leqslant r \leqslant \infty$), which we adopt in this paper is given in Definition 2·4. The proof of Theorem 1·2 is given in Section 3.

2. *Several definitions*

For a C^∞ map-germ f: $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, a C^∞ map-germ Φ: $(\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^p, 0)$ such that $\Phi(x, 0) = f(x)$ is called a C^∞ *deformation-germ* of f. The *graph deformation* F: $(\mathbf{R}^n \times \mathbf{R}^p, (0,0)) \to (\mathbf{R}^p, 0)$ of a given C^∞ map-germ f: $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ is the C^∞ deformation-germ of f given by $F(x, y) = f(x) - y$.

Definition 2·1. Let f: $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a C^∞ map-germ and Φ_i: $(\mathbf{R}^n \times \mathbf{R}^{k(i)}, (0,0)) \to (\mathbf{R}^p, 0)$ a C^∞ deformation-germ of f ($i = 1, 2$). For any r ($0 \leqslant r \leqslant \infty$), if there exist C^r map-germs h : $(\mathbf{R}^n \times \mathbf{R}^{k(1)}, (0,0)) \to (\mathbf{R}^n \times \mathbf{R}^{k(2)}, (0,0))$, H: $(\mathbf{R}^n \times \mathbf{R}^{k(1)} \times \mathbf{R}^p, (0,0,0)) \to (\mathbf{R}^n \times \mathbf{R}^{k(2)} \times \mathbf{R}^p, (0,0,0))$ and φ: $(\mathbf{R}^{k(1)}, 0) \to (\mathbf{R}^{k(2)}, 0)$ such that the following conditions (2·1·1), (2·1·2), (2·1·3) and (2·1·4) hold, then we say $\{h, H, \varphi\}$ is a C^r $\mathscr{K}$-*morphism* from Φ_1 to Φ_2.

(2·1·1) For any representatives $\widetilde{h}$ of h and $\widetilde{H}$ of H, there exist neighbourhoods U of the origin in $\mathbf{R}^n$, V of the origin in $\mathbf{R}^{k(1)}$ and W of the origin in $\mathbf{R}^p$ such that the restrictions $\widetilde{h}|_{U \times \{\lambda\}}$ and $\widetilde{H}|_{U \times \{\lambda\} \times W}$ are C^r diffeomorphisms for any $\lambda \in V$.

(2·1·2) For any representative $\widetilde{H}$ of H, there exist neighbourhoods U of the origin in $\mathbf{R}^n$ and V of the origin in $\mathbf{R}^{k(1)}$ such that

$$\widetilde{H}(U \times V \times \{0\}) \subset \mathbf{R}^n \times \mathbf{R}^{k(2)} \times \{0\}.$$

(2·1·3) The following diagram commutes.

$$\begin{array}{ccccc}
(\mathbf{R}^n \times \mathbf{R}^{k(1)} \times \mathbf{R}^p, (0,0,0)) & \xrightarrow{\pi_{x,\lambda(1)}} & (\mathbf{R}^n \times \mathbf{R}^{k(1)}, (0,0)) & \xrightarrow{\pi_{\lambda(1)}} & (\mathbf{R}^{k(1)}, 0) \\
{\scriptstyle H}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \varphi}\downarrow \\
(\mathbf{R}^n \times \mathbf{R}^{k(2)} \times \mathbf{R}^p, (0,0,0)) & \xrightarrow{\pi_{x,\lambda(2)}} & (\mathbf{R}^n \times \mathbf{R}^{k(2)}, (0,0)) & \xrightarrow{\pi_{\lambda(2)}} & (\mathbf{R}^{k(2)}, 0)
\end{array}$$

(2·1·4) The following diagram commutes.

$$\begin{array}{ccc}
(\mathbf{R}^n \times \mathbf{R}^{k(1)}, (0,0)) & \xrightarrow{(\pi_{x,\lambda(1)}, \Phi_1)} & (\mathbf{R}^n \times \mathbf{R}^{k(1)} \times \mathbf{R}^p, (0,0,0)) \\
{\scriptstyle h}\downarrow & & {\scriptstyle H}\downarrow \\
(\mathbf{R}^n \times \mathbf{R}^{k(2)}, (0,0)) & \xrightarrow{(\pi_{x,\lambda(2)}, \Phi_2)} & (\mathbf{R}^n \times \mathbf{R}^{k(2)} \times \mathbf{R}^p, (0,0,0))
\end{array}$$

Here $\pi_{x,\lambda(i)}$: $(\mathbf{R}^n \times \mathbf{R}^{k(i)} \times \mathbf{R}^p, (0,0,0)) \to (\mathbf{R}^n \times \mathbf{R}^{k(i)}, (0,0))$ and $\pi_{\lambda(i)}$: $(\mathbf{R}^n \times \mathbf{R}^{k(i)}, (0,0)) \to (\mathbf{R}^{k(i)}, 0)$ are canonical projections.

We remark that the conditions (2·1·1), (2·1·2) and (2·1·3) in Definition 2·1 imply

(2·1·5) For any representative $\widetilde{H}$ of H, there exist neighbourhoods U of the origin in $\mathbf{R}^n$, V of the origin in $\mathbf{R}^{k(1)}$ and W of the origin in $\mathbf{R}^p$ such that

$$\widetilde{H}(U \times V \times (W - \{0\})) \subset \mathbf{R}^n \times \mathbf{R}^{k(2)} \times (\mathbf{R}^p - \{0\})$$

and the condition (2·1·4) implies

(2·1·6) For any representative $\widetilde{H}$ of H, there exist neighbourhoods U of the origin in $\mathbf{R}^n$, V of the origin in $\mathbf{R}^{k(1)}$ and W of the origin in $\mathbf{R}^p$ such that

$$\widetilde{H}(\mathrm{graph}\,(\Phi_1) \cap (U \times V \times W)) \subset \mathrm{graph}\,(\Phi_2).$$

Definition 2·2. Let f: $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a C^∞ map-germ and Φ_i: $(\mathbf{R}^n \times \mathbf{R}^{k(i)}, (0,0)) \to (\mathbf{R}^p, 0)$ a C^∞ deformation-germ of f $(i = 1, 2)$. For any r $(0 \leqslant r \leqslant \infty)$, if there exist C^r map-germs $h : (\mathbf{R}^n \times \mathbf{R}^{k(1)}, (0,0)) \to (\mathbf{R}^n \times \mathbf{R}^{k(2)}, (0,0))$, $H : (\mathbf{R}^p \times \mathbf{R}^{k(1)}, (0,0)) \to (\mathbf{R}^p \times \mathbf{R}^{k(2)}, (0,0))$ and $\varphi : (\mathbf{R}^{k(1)}, 0) \to (\mathbf{R}^{k(2)}, 0)$ such that the following conditions (2·2·1) and (2·2·2) hold, then we say $\{h, H, \varphi\}$ is a C^r *$\mathscr{A}$-morphism* from Φ_1 to Φ_2.

(2·2·1) For any representatives $\widetilde{h}$ of h and $\widetilde{H}$ of H, there exist neighbourhoods U of the origin in $\mathbf{R}^n$, V of the origin in $\mathbf{R}^{k(1)}$ and W of the origin in $\mathbf{R}^p$ such that the restrictions $\widetilde{h}|_{U \times \{\lambda\}}$ and $\widetilde{H}|_{W \times \{\lambda\}}$ are C^r diffeomorphisms for any $\lambda \in V$.

(2·2·2) The following diagram commutes.

$$\begin{tikzcd}
(\mathbf{R}^n \times \mathbf{R}^{k(1)}, (0,0)) \arrow[r, "(\Phi_1, \pi_\lambda)"] \arrow[d, "h"'] & (\mathbf{R}^p \times \mathbf{R}^{k(1)}, (0,0)) \arrow[r, "\pi_\lambda"] \arrow[d, "H"'] & (\mathbf{R}^{k(1)}, 0) \arrow[d, "\varphi"'] \\
(\mathbf{R}^n \times \mathbf{R}^{k(2)}, (0,0)) \arrow[r, "(\Phi_2, \pi_\lambda)"] & (\mathbf{R}^p \times \mathbf{R}^{k(2)}, (0,0)) \arrow[r, "\pi_\lambda"] & (\mathbf{R}^{k(2)}, 0)
\end{tikzcd}$$

A C^r $\mathscr{A}$-morphism $\{h, H, \varphi\}$ from Φ_1 to Φ_2 is said to be C^r *equivalent* if φ is a germ of C^r diffeomorphism. If there exists a C^r equivalent C^r $\mathscr{A}$-morphism from the given deformation $\Phi: (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^p, 0)$ to the trivial deformation $f: (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^p, 0)$ given by $f(x, \lambda) = f(x)$, then we say Φ is C^r *trivial*.

Definition 2·3. Let $\Phi: (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^p, 0)$ be a C^∞ deformation-germ of the C^∞ map-germ and $f: (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$. We say Φ is C^r *$\mathscr{K}$-versal* if for any C^∞ deformation-germ $\tilde{\Phi}: (\mathbf{R}^n \times \mathbf{R}^j, (0,0)) \to (\mathbf{R}^p, 0)$ of f there exist C^r map-germs $h: (\mathbf{R}^n \times \mathbf{R}^j, (0,0)) \to (\mathbf{R}^n \times \mathbf{R}^k, (0,0))$, $H: (\mathbf{R}^n \times \mathbf{R}^j \times \mathbf{R}^p, (0,0,0)) \to (\mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^p, (0,0,0))$ and $\varphi: (\mathbf{R}^j, 0) \to (\mathbf{R}^k, 0)$ such that $\{h, H, \varphi\}$ is a C^r $\mathscr{K}$-morphism from $\tilde{\Phi}$ to Φ.

The definition of C^∞ $\mathscr{K}$-versality is equivalent to the V-versality of Martinet's definition ([7]); and the definiton of C^r $\mathscr{K}$-versality is its C^r analogue.

Concerning the definition of C^r stability, we adopt the following in this paper.

Definition 2·4. For $0 \leqslant r \leqslant \infty$, a C^∞ map-germ f is said to be C^r *stable* if any C^∞ deformation-germ of f is C^r trivial.

The C^0 stability of Definition 2·4 is same as the strongly topological stability in [**5**] and also as the P-C^0-stability in [**12**]. MT-stable map-germs, which have been used to prove the density of topological stable mappings in the space of all proper C^∞ mappings ([**5**, **8**]), are C^0 stable in the sense of Definition 2·4. Examples of Looijenga [**6**] and Damon [**4**] also are so.

3. *Proof of Theorem* 1·2

Since there exists a C^r $\mathscr{A}$-morphism from Φ to f, there exist C^r map-germs $h: (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^n \times \mathbf{R}^k, (0,0))$, $H: (\mathbf{R}^p \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^p \times \mathbf{R}^k, (0,0))$ and $\varphi: (\mathbf{R}^k, 0) \to (\mathbf{R}^k, 0)$ such that the following (3·1) and (3·2) hold.

(3·1) The condition (2·2·1) holds with $k(1) = k(2) = k$.

(3·2) The condition (2·2·2) holds with $k(1) = k(2) = k$ and Φ_1, Φ_2 replaced by Φ, f.

By (3·2), we may write

$$h = (h_1, \varphi) \quad \text{and} \quad H = (H_1, \varphi).$$

Let $\varphi'_H: (\mathbf{R}^k, 0) \to (\mathbf{R}^p, 0)$ be the C^r map-germ given by

$$\varphi'_H(\lambda) = H_1(0, \lambda).$$

We set also $h': (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^n \times \mathbf{R}^p, (0,0))$ as

$$h'(x, \lambda) = (h_1(x, \lambda), \varphi'_H(\lambda))$$

and H': $(\mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^p, (0,0,0)) \to (\mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^p, (0,0,0))$ as

$$H'(x, \lambda, y) = (h'(x, \lambda), H_1(y, \lambda) - \varphi'_H(\lambda)).$$

Then we see

(3·3) h' and H' are C^r map-germs.

(3·4) The condition (2·1·1) holds with $k(1) = k$, $k(2) = p$ and h, H replaced by h', H'.

(3·5) The condition (2·1·2) holds with $k(1) = k$, $k(2) = p$ and H replaced by H'.

(3·6) The condition (2·1·3) holds with $k(1) = k$, $k(2) = p$ and h, H, φ replaced by h', H', φ'_H.

Let F be the graph deformation of f. We see

$$\begin{aligned} F(h'(x, \lambda)) &= F(h_1(x, \lambda), \varphi'_H(\lambda)) \\ &= f(h_1(x, \lambda)) - \varphi'_H(\lambda) \\ &= H_1(\Phi(x, \lambda), \lambda) - \varphi'_H(\lambda). \end{aligned}$$

Hence, we have

(3·7) The condition (2·1·4) also holds with $k(1) = k$, $k(2) = p$ and Φ_1, Φ_2, h, H replaced by Φ, F, h', H'.

Therefore $\{h', H', \varphi'_H\}$ is a C^r $\mathscr{K}$-morphism from the given C^∞ deformation germ Φ to the graph deformation F.

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