Nonlinearity

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To cite this article: Takashi Nishimura 2022 *Nonlinearity* **35** 2587

OPEN ACCESS
IOP Publishing | London Mathematical Society Nonlinearity
Nonlinearity **35** (2022) 2587–2620 https://doi.org/10.1088/1361-6544/ac61a0

Hyperplane families creating envelopes

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Received 4 April 2021, revised 22 March 2022
Accepted for publication 28 March 2022
Published 29 April 2022

Abstract
A simple geometric mechanism: 'the locus of intersections of perpendicular bisectors and normal lines', often arises in many guises in nonlinear sciences. In this paper, a new application of this simple geometric mechanism is given. Namely, we show that this mechanism gives answers to all four basic problems on envelopes created by hyperplane families (existence problem, representation problem, equivalence problem of definitions, uniqueness problem) at once.

Keywords: hyperplane family, envelope, frontal, mirror-image mapping, anti-orthotomic, orthotomic, creative
Mathematics Subject Classification numbers: 57R45, 58C25.

(Some figures may appear in colour only in the online journal)

1. Introduction

Throughout this paper, let n be a positive integer. Moreover, all manifolds, functions and mappings are of class C^∞ unless otherwise stated.

A simple geometric mechanism: 'the locus of intersections of perpendicular bisectors and normal lines', often arises in many guises in physical sciences. For example, as Richard Feynman elegantly explained in [9], the orbit of a planet around the Sun can be understood as a consequence of this mechanism under the assumption of the inverse-square law (see figure 1 where the circle is the hodograph of the velocity vectors of a planet, that is to say, the circle is a curve drawn by the end points of the vectors that are parallel to the velocity vectors and start at a fixed point P. The orbit of the planet is similar to the locus of intersections B_t of the perpendicular bisectors of velocity vectors $\overrightarrow{PA_t}$ and the normal lines to the circle at A_t). This is

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Recommended by Dr Richard Montgomery.

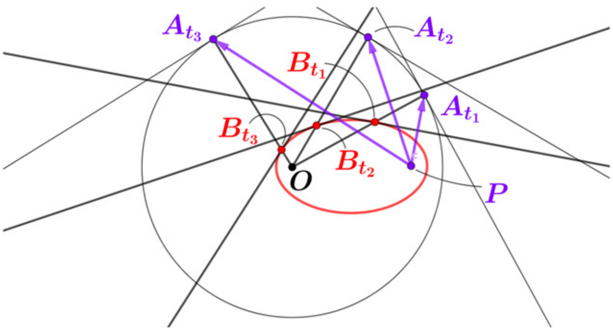


Figure 1. Locus similar to the orbit of a planet.

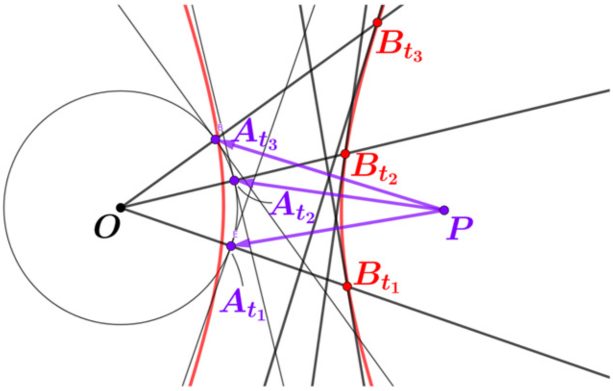


Figure 2. Locus of an α particle.

an example in celestial mechanics. In the same book [9], one can find that even the historical discovery of atomic nucleus due to Ernest Rutherford can be explained as a consequence of this simple geometric mechanism (see figure 2 where the centre of circle O is an atomic nucleus. The orbit of an α particle is the locus of intersections B_t of the perpendicular bisectors of the segment $\overline{PA_t}$ and the normal lines to the circle at A_t). This is an example in nuclear physics.

In crystallography, one can find such the mechanism in the so-called *Wulff construction* for the equilibrium shape of a crystal. A brief explanation of the Wulff construction is as follows. Given an equilibrium crystal, take an arbitrary point P inside the crystal and fix it. Georg Wulff discovered in [20] the so-called Gibbs–Wulff theorem which asserts that the length from the fixed point P to the foot of the perpendicular to the tangent space to the face of the crystal is proportional to its surface energy density of the face. Let $\gamma : S^2 \to \mathbb{R}$ be the surface energy density function of the equilibrium crystal. The graph of γ with respect to the polar coordinates about the point P defines the mapping $g : S^2 \to \mathbb{R}^3$. The mapping g is often called the *polar plot of* γ or the γ-*plot* or the *Wulff plot*. Set $f = 2g$ and suppose that the image $f(S^2)$ has the well-defined normal vectors at any point $f(x)$. Then, by the Gibbs–Wulff theorem, the accurate shape of the crystal surface is proportional to the shape obtained by our simple geometric mechanism: 'the locus of the intersection of the perpendicular bisector of the vector $\overrightarrow{Pf(x)}$ and the normal line to $f(S^2)$ at $f(x)$'. This is the Wulff construction and the constructed shape is

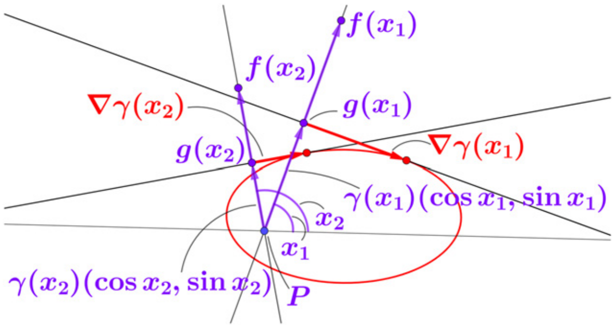


Figure 3. The Wulff construction and the Cahn–Hoffman vector formula in the plane.

called the *Wulff shape*. Notice that in general f is a continuous mapping and thus from the viewpoint of rigorous mathematics, the Wulff construction is not a well-defined construction method in general. Nevertheless, Hoffman and Cahn showed in [11] that if $\gamma : S^2 \to \mathbb{R}$ is differentiable, then the image $f(S^2)$ has a well-defined normal vector at each point $f(x)$ and the set $\{\nabla\gamma(x) + \gamma(x)x | x \in S^2\}$ is exactly the shape obtained by our simple geometric mechanism for the point P and the surface $f(S^2)$. The Wulff construction and the Cahn–Hoffman formula in the plane is depicted in figure 3. For details on the Wulff construction and Wulff shapes, see for instance [8, 10].

Moreover, it is a surprising fact that our simple geometric mechanism: 'the locus of intersections of perpendicular bisectors and normal lines' can be applied even to seismic survey (see 7.14 (9) of [6]).

In mathematics, our simple geometric mechanism: 'the locus of intersections of perpendicular bisectors and normal lines' is called the *anti-orthotomic* of a mapping f having a well-defined normal vector to its image at each point (for details on anti-orthotomics, see 7.14 of [6]. See also [15] where anti-orthotomics are generalized to frontals and [16] where more elementary explanation on anti-orthotomics can be found.). In mathematics as well, there are examples where anti-orthotomics are effectively applied (see [6]).

In order to understand better the powerfulness of the simple geometric mechanism, we would like to have more striking examples in mathematics where anti-orthotomics are effectively applied. Namely, we want to seek mathematical problems which can be geometrically solved by our simple geometric mechanism though it seems difficult to solve them by other methods. This is the primitive motivation of this paper. In this paper, we show that the existence and uniqueness problem of envelopes for a given hyperplane family is one of such problems. Namely, we give a necessary and sufficient condition (see definition 2) for a given hyperplane family to create an envelope. And then, we give a necessary and sufficient condition for the uniqueness of created envelopes if the given hyperplane family creates an envelope. It seems difficult to prove that the condition given in definition 2 is actually a sufficient condition to create envelopes by other methods. In order to apply our simple geometric mechanism, we need some geometric objects to which the normal line can be reasonably well-defined at each point. Hyperplane families themselves are far from the reasonable geometric objects for our purpose. The reasonable geometric objects are frontals (the definition of frontal is given in the next paragraph). In order to obtain a frontal from a given hyperplane family, the mirror-image mapping will be locally introduced. Then, it turns out that if the given hyperplane family is creative (see definition 2 below), then the mirror-image mapping is actually a frontal such that the normal line at each point intersects the corresponding hyperplane. Thus, we can apply the anti-orthotomic method developed in [15] to obtain theorems 1 and 2. The existence and uniqueness problem of envelopes for a given hyperplane family can be easily interpreted as the existence and uniqueness problem of solutions for a certain type of system of first order differential equations with one constraint condition. In the author's opinion, one of the most attractive features of our simple geometric mechanism is that it can make all solutions and their precise expressions clear in one shot by geometry without the need to solve the corresponding system of differential equations with one constraint condition.

Let S^n be the n-dimensional unit sphere in the $(n+1)$-dimensional vector space $\mathbb{R}^{n+1}$. Given a point P of $\mathbb{R}^{n+1}$ and an $(n+1)$-dimensional unit vector $\mathbf{n} \in S^n \subset \mathbb{R}^{n+1}$, the *hyperplane* $H_{(P,\mathbf{n})}$ relative to P and $\mathbf{n}$ is naturally defined as follows, where the dot in the centre stands for the standard scalar product of two vectors $(X-P)$ and $\mathbf{n}$ in the vector space $\mathbb{R}^{n+1}$.

$$H_{(P,\mathbf{n})} = \{X \in \mathbb{R}^{n+1} \mid (X-P)\cdot\mathbf{n} = 0\}.$$

Let N be an n-dimensional manifold without boundary. Given two mappings $\widetilde{\varphi}: N \to \mathbb{R}^{n+1}$ and $\widetilde{\nu}: N \to S^n$, the *hyperplane family* $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ relative to $\widetilde{\varphi}$ and $\widetilde{\nu}$ is naturally defined as follows.

$$\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})} = \left\{H_{(\widetilde{\varphi}(x),\widetilde{\nu}(x))}\right\}_{x\in N}.$$

A mapping $\widetilde{f}: N \to \mathbb{R}^{n+1}$ is called a *frontal* if there exists a mapping $\widetilde{\nu}: N \to S^n$ such that $d\widetilde{f}_x(\mathbf{v})\cdot\widetilde{\nu}(x) = 0$ for any $x \in N$ and any $\mathbf{v} \in T_xN$, where two vector spaces $T_{\widetilde{f}(x)}\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1}$ are identified. By definition, it is natural to call $\widetilde{\nu}: N \to S^n$ a *Gauss mapping* of the frontal $\widetilde{f}$. The notion of frontal has been recently investigated (for instance, see [13]). In this paper, as the definition of envelope created by a hyperplane family, the following is adopted.

Definition 1. Let $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ be a hyperplane family. A mapping $\widetilde{f}: N \to \mathbb{R}^{n+1}$ is called an *envelope created by* $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ if the following two conditions are satisfied.

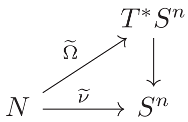
(a) $\widetilde{f}(x) \in H_{(\widetilde{\varphi}(x),\widetilde{\nu}(x))}$ for any $x \in N$.
(b) $d\widetilde{f}_x(\mathbf{v})\cdot\widetilde{\nu}(x) = 0$ for any $x \in N$ and any $\mathbf{v} \in T_xN$.

In other words, an envelope created by $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is a mapping $\widetilde{f}: N \to \mathbb{R}^{n+1}$ giving a solution of the following system of first order differential equations with one constraint condition, where $(U,(x_1,\ldots,x_n))$ is an arbitrary coordinate neighbourhood of N such that $x \in U$.

$$\begin{cases} \dfrac{\partial\widetilde{f}}{\partial x_1}(x)\cdot\widetilde{\nu}(x) = 0, \\ \qquad\qquad\vdots \\ \dfrac{\partial\widetilde{f}}{\partial x_n}(x)\cdot\widetilde{\nu}(x) = 0, \\ \left(\widetilde{f}(x) - \widetilde{\varphi}(x)\right)\cdot\widetilde{\nu}(x) = 0. \end{cases}$$

By definition, any envelope $\widetilde{f}: N \to \mathbb{R}^{n+1}$ created by a hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ must be a frontal with Gauss mapping $\widetilde{\nu}: N \to S^n$. For details on envelopes created by families of plane regular curves, refer to [6]. In chapter 5 of [6], several definitions for envelope are given. For a hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$, definition 1 is a generalization of their definition E_2 from a viewpoint of parametrization (E_2 envelope is a variety tangent to all lines of the given line family. Thus, in the case of plane, an envelope defined by definition 1 is the same notion of E_2 envelope. For details on the definition E_2, see 5.12 of [6]). The following definition, which may be regarded as a higher dimensional generalization of E_1 from a viewpoint of parametrization (E_1 envelope is the set of the limits of intersections with nearby members of the given line family. For details on the definition E_1, see 5.8 of [6] and for the relation between definition 2 in the plane case and E_1, see subsection 2.3), is the key notion for this paper.

Definition 2. Let N be an n-dimensional manifold without boundary and let $\widetilde{\varphi} : N \to \mathbb{R}^{n+1}$, $\widetilde{\nu} : N \to S^n$ be mappings. Let $\widetilde{\gamma} : N \to \mathbb{R}$ be the function defined by $\widetilde{\gamma}(x) = \widetilde{\varphi}(x) \cdot \widetilde{\nu}(x)$. Let T^*S^n be the cotangent bundle of S^n. A hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is said to be *creative* if there exists a mapping $\widetilde{\Omega} : N \to T^*S^n$ with the form $\widetilde{\Omega}(x) = (\widetilde{\nu}(x), \widetilde{\omega}(x))$ such that for any $x_0 \in N$ the equality $\mathrm{d}\widetilde{\gamma} = \widetilde{\omega}$ holds as germs of one-form at x_0.



Namely, $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is creative if there exists a one-form $\widetilde{\Omega}$ along $\widetilde{\nu}$ such that for any $x_0 \in N$ by using of a coordinate neighbourhood $(U, (x_1, \ldots, x_n))$ of N at x_0 and a *normal* coordinate neighbourhood $(V, (\Theta_1, \ldots, \Theta_n))$ of S^n at $\widetilde{\nu}(x_0)$, the one-form germ $\mathrm{d}\widetilde{\gamma}$ at x_0 is expressed as follows.

$$\mathrm{d}\widetilde{\gamma} = \sum_{i=1}^{n} \left(\widetilde{\omega}(x) \left(\Pi_{(\widetilde{\nu}(x),\widetilde{\nu}(x_0))} \left(\frac{\partial}{\partial \Theta_i} \right) \right) \right) \mathrm{d}\,(\Theta_i \circ \widetilde{\nu}),$$

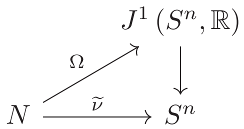
where a normal coordinate neighbourhood $(V, (\Theta_1, \ldots, \Theta_n))$ is a local coordinate neighbourhood at $\widetilde{\nu}(x_0)$ obtained by the inverse mapping of the exponential mapping at $\widetilde{\nu}(x_0)$, S^n inherits its metric from the ambient space $\mathbb{R}^{n+1}$ and $\Pi_{(\widetilde{\nu}(x),\widetilde{\nu}(x_0))} : T_{\widetilde{\nu}(x_0)}S^n \to T_{\widetilde{\nu}(x)}S^n$ is the Levi-Civita translation. Notice that our objective manifold is the unit sphere S^n with metric inherited from $\mathbb{R}^{n+1}$. Therefore, the Levi-Civita translation $\Pi_{(\widetilde{\nu}(x),\widetilde{\nu}(x_0))}$ is the restriction of the rotation $R : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ satisfying $R(\widetilde{\nu}(x_0)) = \widetilde{\nu}(x)$ to the tangent space $T_{\widetilde{\nu}(x_0)}S^n$. In particular, in the case $n = 1$, a normal coordinate Θ at $\widetilde{\nu}(x)$ is nothing but the *radian* (or, its negative) between two unit vectors $\widetilde{\nu}(x_0)$ and $\widetilde{\nu}(x)$ and the Levi-Civita translation $\Pi_{(\widetilde{\nu}(x),\widetilde{\nu}(x_0))}$ is just the restriction of the plane rotation through Θ to the tangent space $T_{\widetilde{\nu}(x_0)}S^1$.

Remark 1.1.

(a) It is reasonable to say that $\widetilde{\gamma}$ is *totally differentiable with respect to* $\widetilde{\nu}$ if $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is creative.

(b) For a creative hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$, the map-germ $(\widetilde{\nu}, \widetilde{\gamma}) : (N, x_0) \to S^n \times \mathbb{R}$ at any $x_0 \in N$ is called an *opening* of $\widetilde{\nu} : (N, x_0) \to S^n$ (for opening germs, see for example [12]). Thus, definition 2 may be regarded as a globalization of the notion of opening.

(c) Definition 2 may be interpreted as follows. Let θ be a canonical contact one-form on $J^1(S^n, \mathbb{R})$, namely at any $(X_0, Y_0, P_0) \in J^1(S^n, \mathbb{R})$ the one-form germ θ is expressed as $\theta = \mathrm{d}Y - \sum_{i=1}^{n} C_i\, \mathrm{d}\Theta_i$, where $(\Theta_1, \ldots, \Theta_n)$ is a *normal* coordinate system at X_0 and $(\Theta_1, \ldots, \Theta_n, Y, C_1, \ldots, C_n)$ is a canonical coordinate system of $J^1(S^n, \mathbb{R})$ at (X_0, Y_0, P_0). Then, a hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is creative if there exists a mapping $\Omega : N \to J^1(S^n, \mathbb{R})$ with the form $\Omega(x) = \left(\widetilde{\nu}(x), \widetilde{\gamma}(x), \widetilde{c}_1(x), \ldots, \widetilde{c}_n(x)\right)$ such that $\Omega^*\theta = 0$, where $\widetilde{c}_1, \ldots, \widetilde{c}_n : N \to \mathbb{R}$ are some functions.



Notice that in Legendrian singularity theory, at any point $x_0 \in N$, the map-germ $\Omega : (N, x_0) \to J^1(S^n, \mathbb{R})$ is assumed to be immersive and it is called a *Legendrian immersion*; and for Legendrian immersion Ω, the mapping $N \ni x \mapsto (\widetilde{\nu}(x), \widetilde{\gamma}(x))$ is called a *wavefront* or *front* (for details on Legendrian singularity theory and fronts, see for instance [1, 2, 17]). On the other hand, in definition 2, Ω is not assumed to be immersive in general and the mapping Ω is called a *Legendrian mapping* (for details on Legendrian mappings, see for instance [12, 13, 18]). Thus, in definition 2, in general, the set-germ $(\Omega(N), \Omega(x_0))$ may be singular at some point $x_0 \in N$ (for example, see example 4.1(d)).

(d) Notice that the one-form $\widetilde{\Omega}$ along $\widetilde{\nu}$ in definition 2 is not necessarily the pullback of a one-form over S^n by $\widetilde{\nu}$ (for example, see example 4.1(c) and (d)) and the 'creativeness' does not depend on the particular choice of $\widetilde{\varphi}, \widetilde{\nu}$ and depends only on the hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$. In the case that $N = S^n$ and $\widetilde{\nu} : S^n \to S^n$ is the identity mapping, for any $\widetilde{\varphi} : S^n \to \mathbb{R}^{n+1}$ the hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is always creative by the following equality.

$$\mathrm{d}\widetilde{\gamma} = \sum_{i=1}^{n} \frac{\partial \widetilde{\gamma}}{\partial \Theta_i} \mathrm{d}\Theta_i.$$

More generally, if $\widetilde{\gamma} : U \to \mathbb{R}$ may be expressed as the composition of $\widetilde{\nu} : U \to S^n$ and a certain function $\xi : S^n \to \mathbb{R}$ over an open set $U \subset N$, then the hyperplane family $\mathcal{H}_{(\widetilde{\varphi}|_U,\widetilde{\nu}|_U)}$ is creative. However, there are examples showing that there does not exist a function $\widetilde{\alpha} : S^n \to \mathbb{R}$ such that $\widetilde{\gamma} = \widetilde{\alpha} \circ \widetilde{\nu}$ although $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is creative (for example, see example 4.1(c) and (d)). Moreover, there are many examples such that $\mathcal{H}_{(\widetilde{\varphi}|_U,\widetilde{\nu}|_U)}$ is not creative. For instance, for any constant mapping $\widetilde{\nu} : \mathbb{R} \to S^1$, the line family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is not creative where $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}^2$ is defined by $\widetilde{\varphi}(t) = t^2\widetilde{\nu}(t)$. And, it is clear in this case that $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ does not create an envelope in the sense of definition 1. However, it is easily seen that

$$\begin{aligned}\mathcal{D} &= \left\{(X_1, X_2) \in \mathbb{R}^2 \mid \exists t \text{ s.t. } F(X_1, X_2, t) = \frac{\partial F}{\partial t}(X_1, X_2, t) = 0\right\} \\ &= \left\{(X_1, X_2) \in \mathbb{R}^2 \mid (X_1, X_2) \cdot \widetilde{\nu}(0) = 0\right\} \neq \emptyset,\end{aligned}$$

where $F(X_1, X_2, t) = ((X_1, X_2) - \widetilde{\varphi}(t)) \cdot \widetilde{\nu}(t)$. Thus, for this example, the envelope defined by definition 1 is different from the envelope in the sense of classical definition (see 5.3 of [6]). For more examples on creative/non-creative hyperplane families and on comparison of definition 2 with the classical envelope $\mathcal{D}$, see section 4. Therefore, it seems that the current situation on both the definitions of envelope and the relation of the creative condition (definition 2) with an envelope seems to be wrapped in mystery.

By definition, any frontal $\widetilde{f} : N \to \mathbb{R}^{n+1}$ with Gauss mapping $\widetilde{\nu} : N \to S^n$ is an envelope created by $\mathcal{H}_{(\widetilde{f},\widetilde{\nu})}$. Therefore, the notion of envelope created by a hyperplane family is the same as the notion of frontal. Moreover, it is clear that for any mapping $\widetilde{\nu} : N \to S^n$, a constant mapping $\widetilde{f} : N \to \mathbb{R}^{n+1}$ is an envelope created by $\mathcal{H}_{(\widetilde{f},\widetilde{\nu})}$. On the other hand, for a constant mapping $\widetilde{\nu} : \mathbb{R} \to S^1$, if the line family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ does not create an envelope then $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}^2$ must be not constant. From these elementary observations, it is natural to ask to obtain a necessary and sufficient condition for a given hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ to create an envelope $\widetilde{f} : N \to \mathbb{R}^{n+1}$ in terms of $\widetilde{\gamma} : N \to \mathbb{R}$ and $\widetilde{\nu} : N \to S^n$. Moreover, it is also desirable to solve the following two incidentally. 'Suppose that a given hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ creates an envelope $\widetilde{f} : N \to \mathbb{R}^{n+1}$. Then, obtain a representation formula of $\widetilde{f}$.' 'Suppose that $n = 1$. Then, find the precise relation

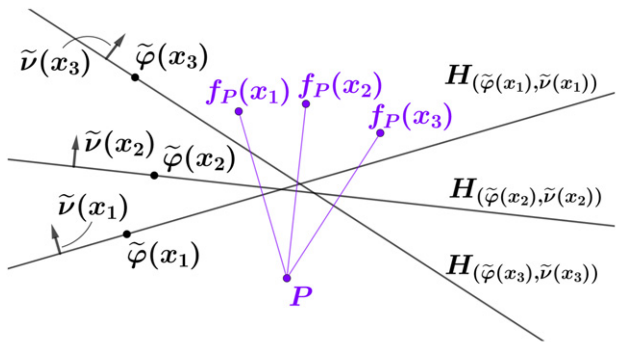


Figure 4. The mirror-image mapping $f_P : U_P \to \mathbb{R}^{n+1}$.

between E_1 envelope and E_2 envelope.' In this paper, as an application of our simple geometric mechanism, all of these problems are solved as follows.

Theorem 1. *Let N be an n-dimensional manifold without boundary and let* $\widetilde{\varphi} : N \to \mathbb{R}^{n+1}$, $\widetilde{\nu} : N \to S^n$ *be mappings. Then, the following three hold.*

(*a*) *The hyperplane family* $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ *creates an envelope if and only if it is creative.*

(*b*) *Suppose that the hyperplane family* $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ *creates an envelope* $\widetilde{f} : N \to \mathbb{R}^{n+1}$. *Then, for any* $x \in N$, *under the canonical identifications* $T^*_{\widetilde{\nu}(x)}S^n \cong T_{\widetilde{\nu}(x)}S^n \subset T_{\widetilde{\nu}(x)}\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$, *the* $(n+1)$*-dimensional vector* $\widetilde{f}(x)$ *is represented as follows.*

$$\widetilde{f}(x) = \widetilde{\omega}(x) + \widetilde{\gamma}(x)\widetilde{\nu}(x),$$

where the $(n+1)$*-dimensional vector* $\widetilde{\omega}(x)$ *is identified with the corresponding n-dimensional cotangent vector* $\widetilde{\omega}(x)$ *under these identifications.*

(*c*) *Suppose that* $n = 1$. *Then, the line family* $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ *creates an envelope* (E_2*-envelope*) *if and only if it creates an* E_1 *envelope. Moreover, these two envelopes are exactly the same.*

By theorem 1, it is natural to call $\widetilde{\omega}$ the *creator* for an envelope $\widetilde{f}$ created by $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$. Recall that E_1 envelope (resp., E_2 envelope) is the set of the limit of intersections with nearby lines (resp., a parametrization tangent to all members of the given family). Thus, even in the case of plane, E_2 envelope is exactly the same as the envelope in definition 1.

The key idea for the proof of theorem 1 is to regard the given hyperplane family as a moving mirror parametrized by $x \in N$. Then, for any parameter $x_0 \in N$, by taking a point $P \in \mathbb{R}^{n+1}$ outside the mirror $H_{(\widetilde{\varphi}(x_0),\widetilde{\nu}(x_0))}$, the mirror-image

$$f_P(x) = 2\left((\widetilde{\varphi}(x) - P) \cdot \widetilde{\nu}(x)\right)\widetilde{\nu}(x) + P$$

of P by the mirror $H_{(\widetilde{\varphi}(x),\widetilde{\nu}(x))}$ must have the same information as the mirror since the mirror is reconstructed as the perpendicular bisectors of the segment $\overline{Pf_P(x)}$, where x is a point in a sufficiently small neighbourhood U_P of x_0. Hence, investigation of the given hyperplane family $\mathcal{H}_{\left(\widetilde{\varphi}|_{U_P},\widetilde{\nu}|_{U_P}\right)}$ may be replaced with analysing the associated *mirror-image mapping* $f_P : U_P \to \mathbb{R}^{n+1}$ (see figure 4). This suggests applicability of results in [15] to the problem of this paper.

A sketch of the proof of theorem 1(a) may be given as follows. Suppose that the hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is creative. Then, by definition, there exists a mapping $\widetilde{\Omega} : N \to T^*S^n$ having

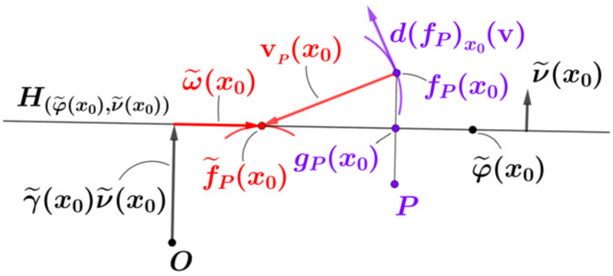


Figure 5. The location $\widetilde{f}_P(x_0)$ does not depend on the particular choice of P.

the form $\widetilde{\Omega}(x) = (\widetilde{\nu}(x), \widetilde{\omega}(x))$ such that the equality $\mathrm{d}\widetilde{\gamma} = \widetilde{\omega}$ holds as germs of one-form at x_0. Then, by investigating the Jacobian matrix of the mirror-image mapping $f_P : U_P \to \mathbb{R}^{n+1}$ at $x \in U_P$ directly, it turns out that for any $x \in U_P$ the non-zero vector

$$\mathbf{v}_P(x) = \sum_{i=1}^{n} \left((\widetilde{\omega}\,(x) - P) \left(\frac{\partial}{\partial \Theta_{(i,\widetilde{\nu}(x))}} \right) \right) \frac{\partial}{\partial \Theta_{(i,\widetilde{\nu}(x))}} - ((\widetilde{\varphi}(x) - P) \cdot \widetilde{\nu}\,(x))\,\widetilde{\nu}\,(x)$$

is perpendicular to the vector $\mathrm{d}(f_P)_x\,(\mathbf{v})$ for any $\mathbf{v} \in T_xN$, where $\mathbb{R}^{n+1}$, $T_{\widetilde{\nu}(x)}\mathbb{R}^{n+1}$ and $T^*_{\widetilde{\nu}(x)}\mathbb{R}^{n+1}$ are identified and $\frac{\partial}{\partial \Theta_{(i,\widetilde{\nu}(x))}} = P_{(\widetilde{\nu}(x),\widetilde{\nu}(x_0))}\left(\frac{\partial}{\partial \Theta_i}\right)$. Thus, $f_P : U_P \to \mathbb{R}^{n+1}$ is a frontal. From the construction, the mapping $\widetilde{f}_P = \mathbf{v}_P + f_P : U_P \to \mathbb{R}^{n+1}$ must be exactly the same as the mapping $\widetilde{f}_P$ given in theorem 1 of [15]. Therefore, by theorem 1 of [15] asserting that $\widetilde{f}_P$ satisfies both conditions (a) and (b) of definition 1, $\widetilde{f}_P$ is an envelope created by the hyperplane family $\mathcal{H}_{\left(\widetilde{\varphi}|_{U_P},\widetilde{\nu}|_{U_P}\right)}$. The mapping $\widetilde{f}_P : U_P \to \mathbb{R}^{n+1}$ is called the *anti-orthotomic* of $f_P : U_P \to \mathbb{R}^{n+1}$ relative to P. Calculation shows

$$\widetilde{f}_P(x_0) = \widetilde{\omega}\,(x_0) + \widetilde{\gamma}\,(x_0)\,\widetilde{\nu}\,(x_0)\,. \qquad (*)$$

Thus, unlike $f_P(x_0)$, the location $\widetilde{f}_P(x_0)$ does not depend on the particular choice of P. In other words, in order to discover the formula $(*)$, the role of P is merely an auxiliary point just like an auxiliary line in elementary geometry (see figure 5). Since x_0 is an arbitrary point of N, the hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ creates an envelope $\widetilde{f} : N \to \mathbb{R}^{n+1}$.

Conversely, suppose that the given hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ creates an envelope $\widetilde{f} : N \to \mathbb{R}^{n+1}$. Then, the mirror-image mapping $f_P : U_P \to \mathbb{R}^{n+1}$ (resp., the mapping $g_P : U_P \to \mathbb{R}^{n+1}$ defined by $g_P(x) = \left(\widetilde{f}(x) - P\right) \cdot \widetilde{\nu}(x) + P)$ is called the *orthotomic* (resp., *pedal*) of $\widetilde{f}|_{U_P}$ relative to the point P. It is known that both the orthotomic f_P and the pedal g_P are frontals (see proposition 1 and corollary 1 of [15]). We prefer to investigate the orthotomic f_P rather than the pedal g_P because its Gauss mapping $\nu_P : U_P \to S^n$ has characteristic properties: $\nu_P(x) = \frac{\widetilde{f}(x) - f_P(x)}{\|\widetilde{f}(x) - f_P(x)\|}$ and $\widetilde{\nu}(x) \cdot \nu_P(x) \neq 0$ for any $x \in U_P$, and thus we can take a bird's eye view of $\widetilde{f}(x)$. Set $\widetilde{\omega}(x) = \widetilde{f}(x) - \widetilde{\gamma}(x)\widetilde{\nu}(x)$ and $\widetilde{\Omega}(x) = (\widetilde{\nu}(x), \widetilde{\omega}(x))$ for any $x \in U_P$. Then, under the identification of $\mathbb{R}^{n+1}$ and $T^*_{\widetilde{\nu}(x)}\mathbb{R}^{n+1}$, $\widetilde{\Omega}$ having the form $\widetilde{\Omega}(x) = (\widetilde{\nu}(x), \widetilde{\omega}(x))$ is a well-defined mapping $U_P \to T^*S^n$. By investigating the Jacobian matrix of the mirror image mapping f_P at $x \in U_P$ directly again, it turns out that $\widetilde{\omega}$ is actually the creator for the envelope $\widetilde{f}|_{U_P}$. Since the vector $\widetilde{\omega}(x_0)$ does not depend on the particular choice of P and the point x_0 is an arbitrary point of N, $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is creative.

Theorem 1(b) is a direct by-product of the proof of theorem 1(a) (see figure 5). Theorem 1(c) seems to be not a direct by-product of the proof of theorem 1(a) although it can be proved relatively easily by using the above argument (see subsection 2.3).

When $N = S^n$ and $\widetilde{\nu} : S^n \to S^n$ is the identity mapping, it is easily seen $\widetilde{\omega}(x) = \nabla\widetilde{\gamma}(x)$. Therefore, in the case that $N = S^n$ and $\widetilde{\nu} : S^n \to S^n$ is the identity mapping, theorem 1(b) has been known as the Cahn–Hoffman vector formula ([11]). Theorem 1(b) is a comprehensive generalization of their formula. Any Wulff shape is clearly a convex body and conversely it is known that any convex body can be constructed by the Wulff construction (for instance, see [19]). There are many Wulff shapes such that the surface energy density functions $\gamma : S^n \to \mathbb{R}$ are not differentiable (convex polytopes are typical examples). Thus, for studying Wulff shapes having non-smooth surface energy functions, it is very significant to answer the following two problems: '(a) generalize Cahn–Hoffman vector formula to the corresponding formula for any $\widetilde{\nu} : N \to S^n$' and '(b) resolution of singularities of the boundary of a convex body having non-smooth boundary by a frontal $\widetilde{f} : S^n \to \mathbb{R}^{n+1}$'. By theorem 1(b), the problem (a) is completely solved. As for the problem (b), to the best of author's knowledge, only the boundary of a square has been realized as a frontal $\widetilde{f} : S^1 \to \mathbb{R}^2$ so far (see [15]). Although there are apparently no published proofs at present, it is a comparatively straightforward generalization of this result to show that the boundary of a convex polygon is realized as a frontal $\widetilde{f} : S^1 \to \mathbb{R}^2$. However, even in the plane case, the problem (b) for the boundary of a convex body in general seems to be wrapped in mystery.

Moreover, theorem 1(b) might be useful even for the study of force problems in higher dimensional vector spaces. In [4], Petr Blaschke discovered that pedal coordinates are more suitable settings to study force problems in $\mathbb{R}^2$. Readers who want to confirm their usefulness are recommended to refer to [4] (see also 7.24 (6) of [6] though this is not a force problem but a very suitable problem for understanding how useful pedal coordinates are). Theorem 1(b) may be regarded as a higher dimensional generalization of pedal coordinates. Hence, it is expected that theorem 1(b) is a very suitable expression to study force problems etc in all finite-dimensional vector spaces over $\mathbb{R}$. Example 4.2(b) might be regarded as examples in which higher dimensional version of pedal coordinates are effectively used.

As an application of theorem 1, a characterization for a hyperplane family to create a unique envelope is given as follows.

Theorem 2. *Let $\widetilde{\varphi} : N \to \mathbb{R}^{n+1}$, $\widetilde{\nu} : N \to S^n$ be mappings. Then, the hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ creates a unique envelope if and only if it is creative and the set consisting of regular points of $\widetilde{\nu}$ is dense in N.*

Under the assumption that Ω in remark 1.1(b) is immersive and some conditions are satisfied, a unique existence result of envelopes for hyperplane families has been obtained in [7]. Since their assumptions clearly imply that the creative condition defined in definition 2 is satisfied and the set consisting of regular points of $\widetilde{\nu}$ is dense, their result follows from theorems 1 and 2.

Notice that non-unique existence cases, too, are intriguing cases since theorem 1 may be effectively applied even in such cases (see example 4.2(a) and (b)).

This paper is organized as follows. Theorems 1 and 2 are proved in sections 2 and 3 respectively. In section 4, examples are given. An alternative proof of theorem 1 except for theorem 1(c) is given in appendix. The alternative proof is a proof by a gauge theoretic approach. In order to avoid unnecessary complication, the alternative proof is given only in the case $n = 1$. The author has no idea on how to prove theorem 1(c) by using the alternative proof.

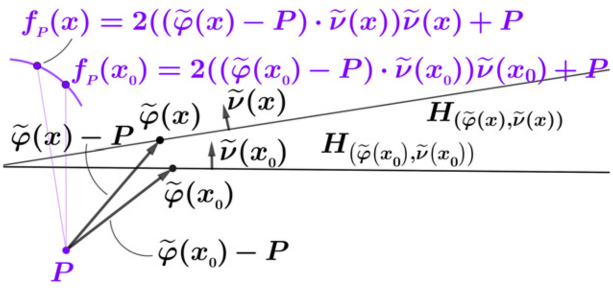


Figure 6. The mirror-image mapping $f_P : U_P \to \mathbb{R}^{n+1}$.

2. Proof of theorem 1

2.1. Proof of theorem 1(a)

2.1.1. Proof of 'if' part. Let x_0 be an arbitrary point of N. Take one point P of $\mathbb{R}^{n+1} - H_{(\widetilde{\varphi}(x_0),\widetilde{\nu}(x_0))}$ and fix it. It follows $(\widetilde{\varphi}(x_0) - P)\cdot\widetilde{\nu}(x_0) \neq 0$. Let $\widetilde{U}_P$ be the set of points $x \in N$ satisfying

$$(\widetilde{\varphi}(x) - P)\cdot\widetilde{\nu}(x) \neq 0. \tag{2.1}$$

Then, it is clear that $\widetilde{U}_P$ is an open neighbourhood of x_0 and the mirror image of the fixed point P by the mirror $H_{(\widetilde{\varphi}(x),\widetilde{\nu}(x))}$ is given by

$$2\left((\widetilde{\varphi}(x) - P)\cdot\widetilde{\nu}(x)\right)\widetilde{\nu}(x) + P$$

for any $x \in \widetilde{U}_P$.

Since the hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is assumed to be creative, there exists a mapping $\widetilde{\Omega} : N \to T^*S^n$ with the form $\widetilde{\Omega}(x) = (\widetilde{\nu}(x), \widetilde{\omega}(x))$ such that for any $x \in N$ the following equality holds as one-form germs at x.

$$\mathrm{d}\widetilde{\gamma} = \widetilde{\omega}.$$

Let $(V, (\Theta_1, \ldots, \Theta_n))$ be a normal coordinate neighbourhood of S^n at $\widetilde{\nu}(x_0)$. Set $U_P = \widetilde{U}_P \cap \widetilde{\nu}^{-1}(V)$. Consider the mirror-image mapping $f_P : U_P \to \mathbb{R}^{n+1}$ defined by

$$f_P(x) = 2\left((\widetilde{\varphi}(x) - P)\cdot\widetilde{\nu}(x)\right)\widetilde{\nu}(x) + P$$

for any $x \in U_P$ (figure 6). In order to show that f_P is a frontal, it is sufficient to construct a Gauss mapping with respect to f_P. By using the mapping $\widetilde{\Omega}|_{U_P}$, a Gauss mapping for f_P is constructed as follows. For any $x \in U_P$ set $X = \widetilde{\nu}(x)$. Let $\Pi_{(X,X_0)} : T_{X_0}S^n \to T_XS^n$ be the Levi-Civita translation. For any i $(1 \leqslant i \leqslant n)$, set $\frac{\partial}{\partial\Theta_{(i,X)}} = \Pi_{(X,X_0)}\left(\frac{\partial}{\partial\Theta_i}\right)$. Then notice that for any $x \in U_P$, under the identification of $\mathbb{R}^{n+1}$ and $T_{f_P(x)}\mathbb{R}^{n+1}$,

$$\left\langle \frac{\partial}{\partial\Theta_{(1,X)}}, \ldots, \frac{\partial}{\partial\Theta_{(n,X)}}, \widetilde{\nu}(x) \right\rangle$$

is an orthonormal basis of the tangent vector space $T_{f_P(x)}\mathbb{R}^{n+1}$.

Lemma 2.1. *For any $x \in U_P$, the following equality holds.*

$$\mathrm{d}\left(P \cdot \widetilde{\nu}\right) = \sum_{i=1}^{n} \left(P \cdot \frac{\partial}{\partial \Theta_{(i,X)}}\right) \mathrm{d}\left(\Theta_i \circ \widetilde{\nu}\right).$$

Proof of lemma 2.1.

$$\begin{aligned}
\mathrm{d}\left(P \cdot \widetilde{\nu}\right) &= \sum_{j=1}^{n} \frac{\partial\left(P \cdot \widetilde{\nu}\right)}{\partial x_j}(x)\mathrm{d}x_j \\
&= \sum_{j=1}^{n} \left(P \cdot \left(\sum_{i=1}^{n} \frac{\partial\left(\Theta_i \circ \widetilde{\nu}\right)}{\partial x_j}(x) \frac{\partial}{\partial \Theta_{(i,X)}}\right)\right) \mathrm{d}x_j \\
&= \sum_{i=1}^{n} \left(P \cdot \frac{\partial}{\partial \Theta_{(i,X)}}\right) \left(\sum_{j=1}^{n} \frac{\partial\left(\Theta_i \circ \widetilde{\nu}\right)}{\partial x_j}(x)\mathrm{d}x_j\right) \\
&= \sum_{i=1}^{n} \left(P \cdot \frac{\partial}{\partial \Theta_{(i,X)}}\right) \mathrm{d}\left(\Theta_i \circ \widetilde{\nu}\right).
\end{aligned}$$

□

By lemma 2.1, under the identification of $T_{\widetilde{\nu}(x)}S^n$ and $T^*_{\widetilde{\nu}(x)}S^n$, it follows

$$\begin{aligned}
\mathrm{d}\left(\left(\widetilde{\varphi} - P\right) \cdot \widetilde{\nu}\right) &= \mathrm{d}\left(\widetilde{\varphi} \cdot \widetilde{\nu}\right) - \mathrm{d}\left(P \cdot \widetilde{\nu}\right) \\
&= \mathrm{d}\widetilde{\gamma} - \mathrm{d}\left(P \cdot \widetilde{\nu}\right) \\
&= \widetilde{\omega} - \mathrm{d}\left(P \cdot \widetilde{\nu}\right) \\
&= \sum_{i=1}^{n} \left(\widetilde{\omega}(x) \cdot \frac{\partial}{\partial \Theta_{(i,X)}}\right) \mathrm{d}\left(\Theta_i \circ \widetilde{\nu}\right) - \sum_{i=1}^{n} \left(P \cdot \frac{\partial}{\partial \Theta_{(i,X)}}\right) \mathrm{d}\left(\Theta_i \circ \widetilde{\nu}\right) \\
&= \sum_{i=1}^{n} \left(\left(\widetilde{\omega}(x) - P\right) \cdot \frac{\partial}{\partial \Theta_{(i,X)}}\right) \mathrm{d}\left(\Theta_i \circ \widetilde{\nu}\right)
\end{aligned}$$

for any $x \in U_P$. Set

$$\mathbf{v}_P(x) = \sum_{i=1}^{n} \left(\left(\widetilde{\omega}(x) - P\right) \cdot \frac{\partial}{\partial \Theta_{(i,X)}}\right) \frac{\partial}{\partial \Theta_{(i,X)}} - \left(\left(\widetilde{\varphi}(x) - P\right) \cdot \widetilde{\nu}(x)\right) \widetilde{\nu}(x)$$

for any $x \in U_P$ where $\mathbb{R}^{n+1}$ and $T_{f_P(x)}\mathbb{R}^{n+1}$ are identified and $T_{f_P(x)}S^n$ and $T^*_{f_P(x)}S^n$ are identified. By section 2.1, $\mathbf{v}_P(x)$ is not the zero vector. Moreover, the following holds.

Lemma 2.2. *For any $\mathbf{v} \in T_{x_0}N$, $\mathbf{v}_P(x_0)$ is perpendicular to $\mathrm{d}(f_P)_{x_0}(\mathbf{v})$.*

Proof of lemma 2.2. Calculation of the product of the vector $\mathbf{v}_P(x_0)$ and the Jacobian matrix of f_P at x_0 (denoted by $J(f_P)_{x_0}$) is carried out as follows, where $\mathbb{R}^{n+1}$ and $T_{f_P(x_0)}\mathbb{R}^{n+1}$ are identified and $T_{f_P(x_0)}S^n$ and $T^*_{f_P(x_0)}S^n$ are identified.

$$
\begin{aligned}
&\mathbf{v}_P(x_0)\, J(f_P)_{x_0} \\
&\quad = 2\sum_{i=1}^{n}\left((\widetilde{\omega}(x_0)-P)\cdot\frac{\partial}{\partial\Theta_i}\right)((\widetilde{\varphi}(x_0)-P)\cdot\widetilde{\nu}(x_0))\,\mathrm{d}(\Theta_i\circ\widetilde{\nu}) \\
&\qquad -2\,((\widetilde{\varphi}(x_0)-P)\cdot\widetilde{\nu}(x_0))\,\mathrm{d}((\widetilde{\varphi}-P)\cdot\widetilde{\nu})_{\text{at }x_0} \\
&\quad = 2\,((\widetilde{\varphi}(x_0)-P)\cdot\widetilde{\nu}(x_0))\sum_{i=1}^{n}\left((\widetilde{\omega}(x_0)-P)\cdot\frac{\partial}{\partial\Theta_i}\right)\mathrm{d}(\Theta_i\circ\widetilde{\nu}) \\
&\qquad -2\,((\widetilde{\varphi}(x_0)-P)\cdot\widetilde{\nu}(x_0))\sum_{i=1}^{n}\left((\widetilde{\omega}(x_0)-P)\cdot\frac{\partial}{\partial\Theta_i}\right)\mathrm{d}(\Theta_i\circ\widetilde{\nu}) \\
&\quad = 0.
\end{aligned}
$$

□

We may consider that the point x_0 is an arbitrary point of U_P. Thus we have the following.

Lemma 2.3. *The mapping $f_P : U_P \to \mathbb{R}^{n+1}$ is a frontal with Gauss mapping $\nu_P : U_P \to S^n$ such that $\nu_P(x)\cdot\widetilde{\nu}(x) \neq 0$, where $\nu_P(x) = \frac{\mathbf{v}_P(x)}{\|\mathbf{v}_P(x)\|}$.*

By lemma 2.3, the hyperplane $H_{(\widetilde{\varphi}(x),\widetilde{\nu}(x))}$ and the line $\ell_x = \{f_P(x) + t\nu_P(x)\,|t\in\mathbb{R}\}$ must intersect only at one point for any $x \in U_P$. Define the mapping $\widetilde{f}_P : U_P \to \mathbb{R}^{n+1}$ by

$$\left\{\widetilde{f}_P(x)\right\} = H_{(\widetilde{\varphi}(x),\widetilde{\nu}(x))}\cap\ell_x.$$

Then, from the construction, $\widetilde{f}_P$ must have the following form (see p 7 of [15]).

$$\widetilde{f}_P(x) = f_P(x) - \frac{\|f_P(x)-P\|^2}{2\,(f_P(x)-P)\cdot\nu_P(x)}\nu_P(x).$$

By theorem 1 of [15] (more precisely, by 3.1 in p 9 of [15]) and lemma 2.3, we have the following.

Lemma 2.4. *The mapping $\widetilde{f}_P$ is a frontal with Gauss mapping $\widetilde{\nu}|_{U_P} : U_P \to S^n$. In other words, $\widetilde{f}_P : U_P \to \mathbb{R}^{n+1}$ is an envelope created by the hyperplane family $\mathcal{H}_{\left(\widetilde{\varphi}|_{U_P},\widetilde{\nu}|_{U_P}\right)}$.*

On the other hand, it is easily seen that $(f_P(x_0) + \mathbf{v}_P(x_0) - \widetilde{\varphi}(x_0))\cdot\widetilde{\nu}(x_0) = 0$ (see figure 7). Thus, the vector $f_P(x_0) + \mathbf{v}_P(x_0)$ must belong to $H_{(\widetilde{\varphi}(x_0),\widetilde{\nu}(x_0))}$. From the construction and by using the equality $P = \sum_{i=1}^{n}\left(P\cdot\frac{\partial}{\partial\Theta_i}\right)\frac{\partial}{\partial\Theta_i} + (P\cdot\widetilde{\nu}(x_0))\,\widetilde{\nu}(x_0)$, we have the following.

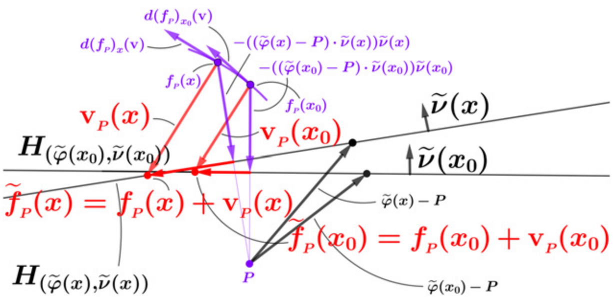


Figure 7. Figure for proof of 'if' part.

$$\begin{aligned}
\widetilde{f}_P(x_0) &= f_P(x) + \mathbf{v}_P(x_0)\\
&= 2\left((\widetilde{\varphi}(x_0) - P)\cdot \widetilde{\nu}(x_0)\right)\widetilde{\nu}(x_0) + P\\
&\quad + \sum_{i=1}^{n}\left((\widetilde{\omega}(x_0) - P)\cdot \frac{\partial}{\partial \Theta_i}\right)\frac{\partial}{\partial \Theta_i} - \left((\widetilde{\varphi}(x_0) - P)\cdot \widetilde{\nu}(x_0)\right)\widetilde{\nu}(x_0)\\
&= \left((\widetilde{\varphi}(x_0) - P)\cdot \widetilde{\nu}(x_0)\right)\widetilde{\nu}(x_0) + P + \sum_{i=1}^{n}\left((\widetilde{\omega}(x_0) - P)\cdot \frac{\partial}{\partial \Theta_i}\right)\frac{\partial}{\partial \Theta_i}\\
&= \left(\widetilde{\varphi}(x_0)\cdot \widetilde{\nu}(x_0)\right)\widetilde{\nu}(x_0) + \sum_{i=1}^{n}\left(\widetilde{\omega}(x_0)\cdot \frac{\partial}{\partial \Theta_i}\right)\frac{\partial}{\partial \Theta_i}\\
&= \widetilde{\gamma}(x_0)\widetilde{\nu}(x_0) + \widetilde{\omega}(x_0).
\end{aligned}$$

This proves the following lemma.

Lemma 2.5. *The following equality holds.*

$$\widetilde{f}_P(x_0) = \widetilde{\gamma}(x_0)\widetilde{\nu}(x_0) + \widetilde{\omega}(x_0).$$

Lemma 2.5 shows that $\widetilde{f}_P(x_0)$ does not depend on the particular choice of $P \in \mathbb{R}^{n+1} - H_{(\widetilde{\varphi}(x_0),\widetilde{\nu}(x_0))}$. Define the mapping $\widetilde{f} : N \to \mathbb{R}^{n+1}$ by $\widetilde{f}(x) = \widetilde{\gamma}(x)\widetilde{\nu}(x) + \widetilde{\omega}(x)$. Since x_0 is an arbitrary point of N, by lemmas 2.4 and 2.5, it follows that the mapping $\widetilde{f} : N \to \mathbb{R}^{n+1}$ is an envelope created by $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$. This completes the proof of 'if' part. □

2.1.2. Proof of 'only if' part. Suppose that the hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ creates an envelope $\widetilde{f} : N \to \mathbb{R}^{n+1}$. Then, by definition, $\widetilde{f}$ is a frontal such that the inclusion $\widetilde{f}(x) + d\widetilde{f}_x(T_xN) \subset H_{(\widetilde{\varphi}(x),\widetilde{\nu}(x))}$ holds for any $x \in N$. Let $\widetilde{\omega} : N \to \mathbb{R}^{n+1}$ be the mapping defined by $\widetilde{\omega}(x) = \widetilde{f}(x) - \widetilde{\gamma}(x)\widetilde{\nu}(x)$ (see figure 8). It is sufficient to show that under some identifications, $\widetilde{\omega}$ is actually a creator for the envelope $\widetilde{f}$.

It is easily seen that $\widetilde{\omega}(x)\cdot\widetilde{\nu}(x) = 0$ for any $x \in N$. Thus, under the identification of $\mathbb{R}^{n+1}$ and $T^*_{\widetilde{\nu}(x)}\mathbb{R}^{n+1}$, we have

Lemma 2.6. *For any $x \in N$, $\widetilde{\omega}(x) \in T^*_{\widetilde{\nu}(x)}S^n$ holds.*

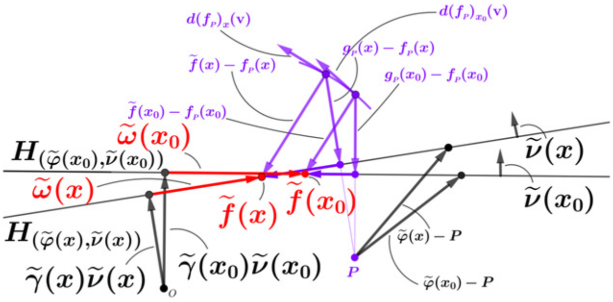


Figure 8. Figure for proof of 'only if' part.

Let $\widetilde{\Omega} : N \to T^*S^n$ be the mapping defined by $\widetilde{\Omega}(x) = (\widetilde{\nu}(x), \widetilde{\omega}(x))$. Let x_0 be an arbitrary point of N and let P be a point of $\mathbb{R}^{n+1} - H_{(\widetilde{\varphi}(x_0),\widetilde{\nu}(x_0))}$. Again, we consider the mirror-image mapping $f_P : \widetilde{U}_P \to \mathbb{R}^{n+1}$ defined by

$$f_P(x) = 2\left((\widetilde{\varphi}(x) - P) \cdot \widetilde{\nu}(x)\right)\widetilde{\nu}(x) + P,$$

where $\widetilde{U}_P = \{x \in N \mid (\widetilde{\varphi}(x) - P) \cdot \widetilde{\nu}(x) \neq 0\}$. The mapping f_P is exactly the orthotomic of $\widetilde{f}|_{\widetilde{U}_P}$ relative to the point P. Thus, by proposition 1 of [15] (more precisely, by 2.1 in pp 7–8 of [15]), f_P is a frontal and the mapping $\nu_P : \widetilde{U}_P \to S^n$ define by

$$\nu_P(x) = \frac{\widetilde{f}(x) - f_P(x)}{\|\widetilde{f}(x) - f_P(x)\|}$$

is its Gauss mapping. In particular, we have the following.

Lemma 2.7. *For any $x \in \widetilde{U}_P$ and any $\mathbf{v} \in T_xN$, the following holds.*

$$\left(\widetilde{f}(x) - f_P(x)\right) \cdot \mathrm{d}(f_P)_x(\mathbf{v}) = 0.$$

For any $x \in \widetilde{U}_P$, set

$$g_P(x) = \frac{1}{2}\left(f_P(x) - P\right) + P = \left((\widetilde{\varphi}(x) - P) \cdot \widetilde{\nu}(x)\right)\widetilde{\nu}(x) + P.$$

Then, since $f_P(x)$ is the mirror-image of P with respect to the mirror $H_{(\widetilde{\varphi}(x),\widetilde{\nu}(x))}$, the following clearly holds.

Lemma 2.8. *The vector $\widetilde{f}(x) - g_P(x)$ is perpendicular to the vector $g_P(x) - f_P(x) = -\left((\widetilde{\varphi}(x) - P) \cdot \widetilde{\nu}(x)\right)\widetilde{\nu}(x)$ for any $x \in \widetilde{U}_P$.*

Thus,

$$\widetilde{f}(x) - f_P(x) = \left(\widetilde{f}(x) - g_P(x)\right) + \left(g_P(x) - f_P(x)\right)$$

is an orthogonal decomposition of $\widetilde{f}(x) - f_P(x)$ for any $x \in \widetilde{U}_P$ (see figure 8).

In order to decompose the vector $\widetilde{f}(x) - g_P(x)$ reasonably, the open neighbourhood $\widetilde{U}_P$ of x_0 is reduced as follows. Let $(V, (\Theta_1, \ldots, \Theta_n))$ be a normal coordinate neighbourhood of S^n at $\widetilde{\nu}(x_0)$. Set again $U_P = \widetilde{U}_P \cap \widetilde{\nu}^{-1}(V)$. Notice that $\langle \mathrm{d}\Theta_1, \ldots, \mathrm{d}\Theta_n \rangle$ is an orthonormal basis of the cotangent space $T^*_{\widetilde{\nu}(x_0)}S^n$.

Lemma 2.9. *The equality*

$$\widetilde{f}(x_0) - g_P(x_0) = \widetilde{\omega}(x_0) - \sum_{i=1}^{n} \left(P \cdot \frac{\partial}{\partial \Theta_i} \right) \frac{\partial}{\partial \Theta_i},$$

holds where three vector spaces $\mathbb{R}^{n+1}$, $T_{\widetilde{\nu}(x_0)}\mathbb{R}^{n+1}$ *and* $T^*_{\widetilde{\nu}(x_0)}\mathbb{R}^{n+1}$ *are identified.*

Proof of lemma 2.9.

$$\begin{aligned}
\widetilde{f}(x_0) - g_P(x_0) &= \widetilde{f}(x_0) - (((\widetilde{\varphi}(x_0) - P) \cdot \widetilde{\nu}(x_0))\widetilde{\nu}(x_0) + P) \\
&= \left(\widetilde{f}(x_0) - (\widetilde{\varphi}(x_0) \cdot \widetilde{\nu}(x_0))\widetilde{\nu}(x_0)\right) + ((P \cdot \widetilde{\nu}(x_0))\widetilde{\nu}(x_0) - P) \\
&= \left(\widetilde{f}(x_0) - \widetilde{\gamma}(x_0)\widetilde{\nu}(x_0)\right) + ((P \cdot \widetilde{\nu}(x_0))\widetilde{\nu}(x_0) - P) \\
&= \widetilde{\omega}(x_0) - \sum_{i=1}^{n} \left(P \cdot \frac{\partial}{\partial \Theta_i} \right) \frac{\partial}{\partial \Theta_i}.
\end{aligned}$$

□

By lemma 2.9, the following holds.

$$\begin{aligned}
\widetilde{f}(x_0) - f_P(x_0) &= \left(\widetilde{f}(x_0) - g_P(x_0)\right) + (g_P(x_0) - f_P(x_0)) \\
&= \widetilde{\omega}(x_0) - \sum_{i=1}^{n} \left(P \cdot \frac{\partial}{\partial \Theta_i} \right) \frac{\partial}{\partial \Theta_i} - ((\widetilde{\varphi}(x_0) - P) \cdot \widetilde{\nu}(x_0))\widetilde{\nu}(x_0).
\end{aligned}$$

Hence, by lemmas 2.1 and 2.7, the germ of one-form $\mathrm{d}\widetilde{\gamma}$ at x_0 is calculated as follows, where $X = \widetilde{\nu}(x)$, $\frac{\partial}{\partial \Theta_{(i,X)}} = P_{(X,X_0)} \left(\frac{\partial}{\partial \Theta_i} \right)$. and $P_{(X,X_0)} : T_{X_0}S^n \to T_X S^n$ is the Levi-Civita translation.

$$\begin{aligned}
\mathrm{d}\widetilde{\gamma} &= \mathrm{d}\widetilde{\gamma} - \mathrm{d}(P \cdot \widetilde{\nu}) + \mathrm{d}(P \cdot \widetilde{\nu}) \\
&= \mathrm{d}((\widetilde{\varphi} - P) \cdot \widetilde{\nu}) + \mathrm{d}(P \cdot \widetilde{\nu}) \\
&= \sum_{i=1}^{n} \left((\widetilde{\omega} - P) \cdot \frac{\partial}{\partial \Theta_{(i,X)}} \right) \mathrm{d}(\Theta_i \circ \widetilde{\nu}) + \sum_{i=1}^{n} \left(P \cdot \frac{\partial}{\partial \Theta_{(i,X)}} \right) \mathrm{d}(\Theta_i \circ \widetilde{\nu}) \\
&= \left(\widetilde{\omega} - \sum_{i=1}^{n} \left(P \cdot \frac{\partial}{\partial \Theta_{(i,X)}} \right) \mathrm{d}(\Theta_i \circ \widetilde{\nu}) \right) + \sum_{i=1}^{n} \left(P \cdot \frac{\partial}{\partial \Theta_{(i,X)}} \right) \mathrm{d}(\Theta_i \circ \widetilde{\nu}) \\
&= \widetilde{\omega}.
\end{aligned}$$

This calculation proves the following lemma.

Lemma 2.10. *The equality*

$$\mathrm{d}\widetilde{\gamma} = \widetilde{\omega}$$

holds as germs of one-form at x_0.

Since x_0 is an arbitrary point of N, by lemma 2.10, $\widetilde{\omega}$ is actually the creator for the given envelope $\widetilde{f}: N \to \mathbb{R}^{n+1}$. This completes the proof of 'only if' part. □

2.2. Proof of theorem 1(b)

Theorem 1(b) is a direct by-product of the proof of theorem 1(a). □

2.3. Proof of theorem 1(c)

Recall that the line family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is said to *create an* E_1 *envelope* [denoted by (E_1) in this subsection] if for any fixed $t_0 \in N$ and any $t \in N$ near t_0 the limit $\lim_{t\to t_0} H_{(\widetilde{\varphi}(t),\widetilde{\nu}(t))} \cap H_{(\widetilde{\varphi}(t_0),\widetilde{\nu}(t_0))}$ exists. On the other hand, the line family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is said to *create an* E_2 *envelope* [denoted by (E_2) in this subsection] if it creates an envelope in the sense of definition 1.

$(E_1) \Rightarrow (E_2)$. Let t_0 be a point of N and let $t_i \in N$ $(i = 1, 2, \ldots)$ be a sequence conversing to t_0. Since (E_1) is assumed, we can assume that a point X_{t_i} can be taken from the intersection $H_{(\widetilde{\varphi}(t),\widetilde{\nu}(t))} \cap H_{(\widetilde{\varphi}(t_0),\widetilde{\nu}(t_0))}$ such that $\lim_{t_i\to t_0} X_{t_i}$ exists. Denote the limit by X_{t_0}. Then, we have the following.

$$\left(X_{t_i} - \widetilde{\varphi}(t_i)\right) \cdot \widetilde{\nu}(t_i) = 0,$$
$$\left(X_{t_i} - \widetilde{\varphi}(t_0)\right) \cdot \widetilde{\nu}(t_0) = 0.$$

This implies

$$X_{t_i} \cdot (\widetilde{\nu}(t_i) - \widetilde{\nu}(t_0)) = \widetilde{\gamma}(t_i) - \widetilde{\gamma}(t_0).$$

Thus we have

$$X_{t_0} \cdot \frac{\partial \widetilde{\nu}}{\partial t}(t_0) = \frac{\partial \widetilde{\gamma}}{\partial t}(t_0).$$

This implies that there exists a real number $\alpha(t_0)$ such that the following identity holds where $\mathrm{d}\,(\Theta \circ \widetilde{\nu})$ and $\mathrm{d}\widetilde{\gamma}$ stand for the one-dimensional cotangent vectors in $T^*_{t_0}N$, namely the following identity is nothing but the identity of two real numbers.

$$\alpha(t_0)\mathrm{d}\,(\Theta \circ \widetilde{\nu}) = \mathrm{d}\widetilde{\gamma}.$$

It is not difficult to see that the function $\alpha: N \to \mathbb{R}$ is of class C^∞. This means that the line family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is creative. Therefore, by theorem 1(a), the line family creates an E_2 envelope. □

$(E_2) \Rightarrow (E_1)$. For the proof of this implication, it is used the notions and notations introduced in the proof of theorem 1(a). The assumption (E_2) implies that $\widetilde{\gamma}$ is totally differentiable with respect to $\widetilde{\nu}$. Take an arbitrary point $t_0 \in N$ and fixed it. Since $\widetilde{\gamma}$ is totally differentiable with respect to $\widetilde{\nu}$ at t_0, for any t near t_0 if the length of the vector $\overrightarrow{f_P(t_0)f_P(t)}$ is positive, then the horizontal vector of $\overrightarrow{f_P(t_0)f_P(t)}$ must be non-zero, where P is a point taken outside the line $H_{(\widetilde{\varphi}(t_0),\widetilde{\nu}(t_0))}$ and f_P is a mirror-image mapping introduced in the proof of theorem 1(a). Denote the intersection of the perpendicular bisector of $\overrightarrow{f_P(t_0)f_P(t)}$ and the line $H_{(\widetilde{\varphi}(t_0),\widetilde{\nu}(t_0))}$ by J_t. Then,

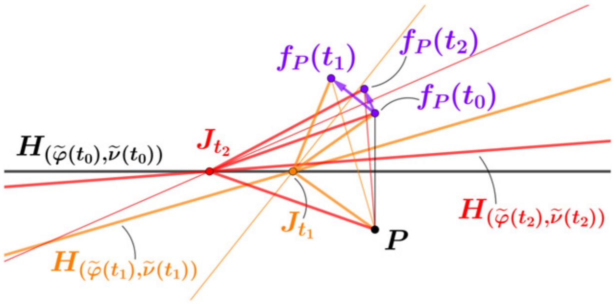


Figure 9. Figure for $(E_2) \Rightarrow (E_1)$.

from the construction, it follows that the triangle $\triangle J_t f_P(t_0) f_P(t)$ is an isosceles triangle with legs $J_t f_P(t_0)$ and $J_t f_P(t)$. This implies the following (see figure 9).

$$J_t \in H_{(\widetilde{\varphi}(t),\widetilde{\nu}(t))} \cap H_{(\widetilde{\varphi}(t_0),\widetilde{\nu}(t_0))}.$$

Notice that $\lim_{t\to t_0} \|J_t f_P(t_0)\|$ is positive. Thus, we have

$$\lim_{t\to t_0} \angle J_t f_P(t_0) f_P(t) = \lim_{t\to t_0} \angle J_t f_P(t) f_P(t_0) = \frac{\pi}{2}.$$

By proposition 1 of [15] asserting that f_P is a frontal with its Gauss mapping $\frac{f_P(t_0)-\widetilde{f}_P(t_0)}{\|f_P(t_0)-\widetilde{f}_P(t_0)\|}$, it follows

$$\lim_{t\to t_0} J_t = \widetilde{f}_P(t_0),$$

where $\widetilde{f}_P$ is the anti-orthotomic of f_P relative to the point P introduced in the proof of theorem 1(a). Since t_0 is an arbitrary point of N, the given E_2 envelope must be an E_1 envelope by theorem 1(a). □

3. Proof of theorem 2

Proof of 'if' part. Since the hyperplane $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is creative, by theorem 1, it creates an envelope. Let $\widetilde{f}_1, \widetilde{f}_2 : N \to \mathbb{R}^{n+1}$ be envelopes created by $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$.

Let $x_0 \in N$ be a regular point of $\widetilde{\nu}$. Then, there exists an open coordinate neighbourhood $(U, (x_1, \ldots, x_n))$ such that $x_0 \in U$ and $\widetilde{\nu}|_U : U \to \widetilde{\nu}(U)$ is a diffeomorphism. Then, the germ of one-form $\mathrm{d}(\widetilde{\varphi}\cdot\widetilde{\nu})$ at $x_0 \in U$ is

$$\begin{aligned}
\mathrm{d}(\widetilde{\varphi}\cdot\widetilde{\nu}) &= \sum_{j=1}^{n} \frac{\partial(\widetilde{\varphi}\cdot\widetilde{\nu})}{\partial x_j}(x)\mathrm{d}x_j \\
&= \sum_{j=1}^{n} \frac{\partial(\widetilde{\varphi}\cdot\widetilde{\nu})}{\partial x_j}(x)\left(\sum_{i=1}^{n} \frac{\partial\left(x_j\circ\widetilde{\nu}^{-1}\right)}{\partial\Theta_{(i,\widetilde{\nu}(x))}}(\widetilde{\nu}(x))\,\mathrm{d}\Theta_i\right) \\
&= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \frac{\partial(\widetilde{\varphi}\cdot\widetilde{\nu})}{\partial x_j}(x)\frac{\partial\left(x_j\circ\widetilde{\nu}^{-1}\right)}{\partial\Theta_{(i,\widetilde{\nu}(x))}}(\widetilde{\nu}(x))\right)\mathrm{d}\Theta_i.
\end{aligned}$$

Let $\widetilde{\Omega} : N \to T^*S^n$ be the mapping with the form $\widetilde{\Omega}(x) = (\widetilde{\nu}(x), \widetilde{\omega}(x))$ such that $\widetilde{\omega}$ is the creator for $\widetilde{f}$. Then, by the above calculation, $\widetilde{\omega}|_U$ must have the following form.

$$\widetilde{\omega}|_U(x) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial\left(\widetilde{\varphi}\cdot\widetilde{\nu}\right)}{\partial x_j}(x) \frac{\partial\left(x_j \circ \widetilde{\nu}^{-1}\right)}{\partial \Theta_{(i,\widetilde{\nu}(x))}} (\widetilde{\nu}(x)) \right) \mathrm{d}\Theta_i.$$

Hence, by theorem 1(b), we have the following.

Lemma 3.1. *At a regular point $x_0 \in N$ of $\widetilde{\nu}$, the equality $\widetilde{f}_1(x_0) = \widetilde{f}_2(x_0)$ holds.*

Let $x_0 \in N$ be a singular point of $\widetilde{\nu}$. Then, since we have assumed that the set of regular points of $\widetilde{\nu}$ is dense, there exists a point-sequence $\{y_i\}_{i=1,2,\ldots} \subset N$ such that y_i is a regular point of $\widetilde{\nu}$ for any $i \in \mathbb{N}$ and $\lim_{i\to\infty} y_i = x_0$. Then, by lemma 3.1, we have

$$\widetilde{f}_1(x_0) = \widetilde{f}_1\left(\lim_{i\to\infty} y_i\right) = \lim_{i\to\infty} \widetilde{f}_1(y_i) = \lim_{i\to\infty} \widetilde{f}_2(y_i) = \widetilde{f}_2\left(\lim_{i\to\infty} y_i\right) = \widetilde{f}_2(x_0).$$

Thus, we have the following.

Lemma 3.2. *Even at a singular point $x_0 \in N$ of $\widetilde{\nu}$, the equality $\widetilde{f}_1(x_0) = \widetilde{f}_2(x_0)$ holds.*

□

Proof of 'only if' part. Suppose that the hyperplane $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is creative and the set of regular points of $\widetilde{\nu}$ is not dense in N. Then, there exists an open set U of N such that any point $x \in U$ is a singular point of $\widetilde{\nu}$. Then, there exist an integer k $(0 \leqslant k < n)$ and an open set U_k such that $U_k \subset U$ and the rank of $\widetilde{\nu}$ at x is k for any $x \in U_k$. Let x_0 be a point of U_k. We may assume that U_k is sufficiently small open neighbourhood of x_0. Then, by the rank theorem (for the rank theorem, see for example [5]), we have the following.

Lemma 3.3. *There exist functions $\eta_1, \ldots, \eta_k : N \to \mathbb{R}$ such that the following three hold.*

(*a*) *For any i $(1 \leqslant i \leqslant n)$, $\eta_i(x) = 0$ if $x \notin U_k$.*
(*b*) *There exists an i $(1 \leqslant i \leqslant n)$ such that $\eta_i(x_0) \neq 0$.*
(*c*) *The following equality holds for any $x \in N$.*

$$\sum_{i=1}^{n} \eta_i(x)\mathrm{d}\left(\Theta_i \circ \widetilde{\nu}\right) = 0.$$

Since we have assumed that $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is creative, there exists a mapping $\widetilde{\Omega} : N \to T^*S^n$ with the form $\widetilde{\Omega}(x) = (\widetilde{\nu}(x), \widetilde{\omega}(x))$ such that $\mathrm{d}\left(\widetilde{\varphi}\cdot\widetilde{\nu}\right) = \widetilde{\omega}$. By lemma 3.3, the following holds.

Lemma 3.4. *For any function $\alpha : N \to \mathbb{R}$ and any $x \in N$, the following equality holds as germs of one-form at x.*

$$\mathrm{d}\left(\widetilde{\varphi}\cdot\widetilde{\nu}\right) = \widetilde{\omega}(x) + \alpha(x)\sum_{i=1}^{n} \eta_i(x)\mathrm{d}\left(\Theta_i \circ \widetilde{\nu}\right).$$

Therefore, by theorem 1(b), uncountably many distinct envelopes $\widetilde{f}$ are created by the same hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$. □

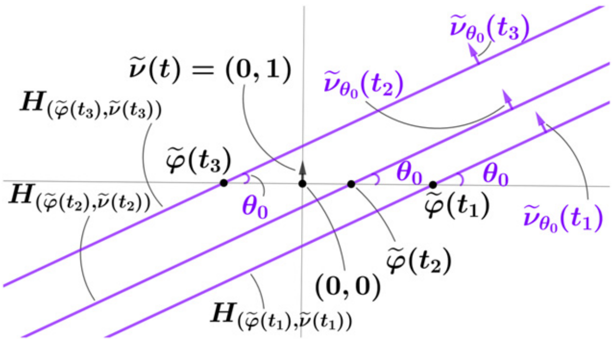


Figure 10. Figure for example 4.1(a).

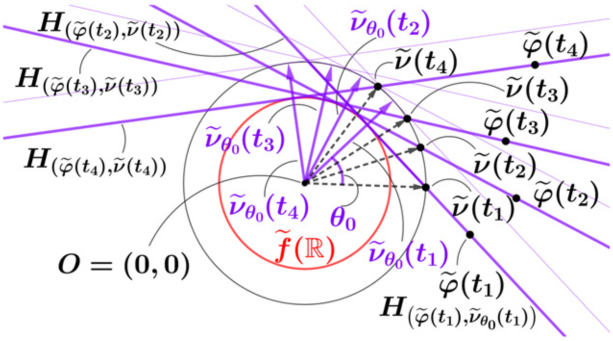


Figure 11. Figure for example 4.1(b).

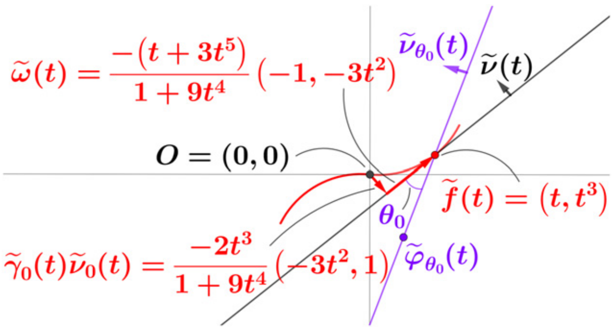


Figure 12. Figure for example 4.1(c) in the case $\theta_0 \in 2\pi\mathbb{Z}$.

4. Examples

Example 4.1 (uniform spin of affine tangent lines)**.**

(a) Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a non-constant function. Notice that α is of class C^∞ as stated at the top of section 1. Let $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}^2$ be the mapping defined by $\widetilde{\varphi}(t) = (\alpha(t), 0)$. Let $\widetilde{\nu} : \mathbb{R} \to S^1$ be the constant mapping $\widetilde{\nu}(t) = (0, 1)$. For any fixed $\theta_0 \in \mathbb{R}$, let $R_{\theta_0} : \mathbb{R}^2 \to \mathbb{R}^2$

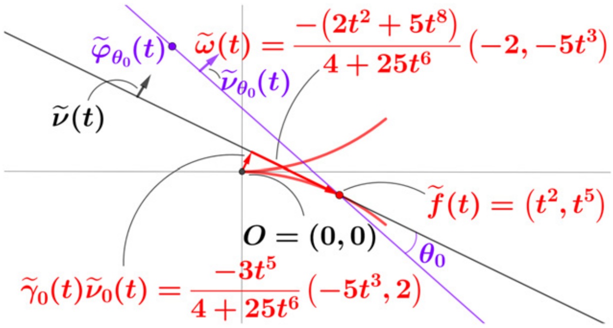


Figure 13. Figure for example 4.1(d) in the case $\theta_0 \in 2\pi\mathbb{Z}$.

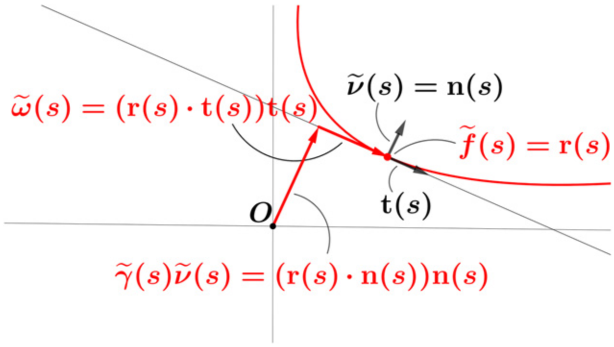


Figure 14. Example 4.2(a).

be the linear mapping representing the rotation through angle θ_0. Set $\widetilde{\nu}_{\theta_0}(t) = R_\theta \circ \widetilde{\nu}(t) = (-\sin\theta_0, \cos\theta_0)$ and $\widetilde{\gamma}_{\theta_0}(t) = \widetilde{\varphi}(t)\cdot\widetilde{\nu}_{\theta_0}(t) = -\alpha(t)\sin\theta_0$. Figure is depicted in figure 10. It follows $\mathrm{d}\left(\Theta\circ\widetilde{\nu}_{\theta_0}\right) \equiv 0$ and $\mathrm{d}\gamma_{\theta_0} = -\sin\theta_0\,\mathrm{d}\alpha$. Since α is non-constant, there exists a regular point of α, that is to say, there exists a $t\in\mathbb{R}$ such that $\alpha'(t)\neq 0$. Therefore, by theorem 1, the line family $\mathcal{H}_{\left(\widetilde{\varphi},\widetilde{\nu}_{\theta_0}\right)}$ creates an envelope if and only if $\theta_0 \in \pi\mathbb{Z}$. Suppose that $\theta_0 \in \pi\mathbb{Z}$. In this case, by theorem 2, uncountably many distinct envelope $\widetilde{f}:\mathbb{R}\to\mathbb{R}^2$ can be created by the given line family $\mathcal{H}_{\left(\widetilde{\varphi},\widetilde{\nu}_{\theta_0}\right)}$. Let $\beta:\mathbb{R}\to\mathbb{R}$ be a function. Since $\mathrm{d}\left(\Theta\circ\widetilde{\nu}_{\theta_0}\right)\equiv 0$ and $\mathrm{d}\gamma_{\theta_0}\equiv 0$ in this case, the one-form $t\mapsto\beta(t)\mathrm{d}\left(\Theta\circ\widetilde{\nu}_{\theta_0}\right)$ along $\widetilde{\nu}_{\theta_0}$ may be a creator $\widetilde{\omega}$ for the line family. By theorem 1(b), the envelope $\widetilde{f}$ has the following form.

$$\widetilde{f}(t) = \widetilde{\omega}(t) + \left(\widetilde{\gamma}_{\theta_0}(t)\cdot\widetilde{\nu}_{\theta_0}(t)\right)\widetilde{\nu}_{\theta_0}(t) = (\pm\beta(t),0) + (0,0) = (\pm\beta(t),0),$$

where double sign should be read in the same order and $\beta(t)\mathrm{d}\left(\Theta\circ\widetilde{\nu}_{\theta_0}\right)$, $\beta(t)R_{\frac{\pi}{2}}\circ\widetilde{\nu}_{\theta_0}(t)$ are identified (both are denoted by the same symbol $\widetilde{\omega}(t)$).

Set $F_{\theta_0}(X_1,X_2,t) = (X_1-\alpha(t),X_2)\cdot\widetilde{\nu}_{\theta_0}(t)$. Suppose that $\theta_0\notin\pi\mathbb{Z}$. In this case, the classical common definition of envelope $\mathcal{D}$ relative to F_{θ_0} is as follows.

$$\mathcal{D} = \left\{(X_1,X_2)\mid \exists t \text{ s.t. } \alpha'(t)=0, X_1 = \cot\theta_0 X_2 + \alpha(t)\right\}.$$

Therefore, in this case, $\mathcal{D} = E_1 = E_2 = \emptyset$ if and only if α is non-singular. Suppose that $\theta_0 \in \pi\mathbb{Z}$. Then,

$$\mathcal{D} = \{(X_1, X_2) \mid X_2 = 0\}.$$

Therefore, in this case, $E_1 = E_2 = \mathcal{D}$ if and only if β is surjective.

(b) Let $\widetilde{\nu} : \mathbb{R} \to S^1$ be the mapping given by $\widetilde{\nu}(t) = (\cos t, \sin t)$. Set $\widetilde{\nu}_{\theta_0} = R_{\theta_0} \circ \widetilde{\nu}$, where R_{θ_0} is the rotation defined in the above example. Then, since $\frac{\mathrm{d}\left(\Theta\circ\widetilde{\nu}_{\theta_0}\right)}{\mathrm{d}t}(t) = 1$, it follows $\mathrm{d}\left(\Theta \circ \widetilde{\nu}_{\theta_0}\right) = \mathrm{d}t$. Thus, by theorems 1(a) and 2, for any $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}^2$ the line family $\mathcal{H}_{\left(\widetilde{\varphi},\widetilde{\nu}_{\theta_0}\right)}$ creates a unique envelope $\widetilde{f}_{\theta_0}$. For any $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}^2$, set $\widetilde{\gamma}_{\theta_0}(t) = \widetilde{\varphi}(t) \cdot \widetilde{\nu}_{\theta_0}(t)$. Since $\mathrm{d}\widetilde{\gamma}_{\theta_0} = \frac{\mathrm{d}\widetilde{\gamma}_{\theta_0}}{\mathrm{d}t}(t)\mathrm{d}\left(\Theta \circ \widetilde{\nu}_{\theta_0}\right)$, by theorem 1(b), it follows

$$\begin{aligned}\widetilde{f}(t) &= \frac{\mathrm{d}\widetilde{\gamma}_{\theta_0}}{\mathrm{d}t}(t)R_{\pi/2} \circ \widetilde{\nu}_{\theta_0}(t) + \widetilde{\gamma}_{\theta_0}(t)\widetilde{\nu}_{\theta_0}(t) \\ &= \frac{\mathrm{d}\widetilde{\gamma}_{\theta_0}}{\mathrm{d}t}(t)R_{\pi/2} \circ \widetilde{\nu}_{\theta_0}(t) + \widetilde{\gamma}_{\theta_0}(t)\left(\cos(t+\theta_0), \sin(t+\theta_0)\right),\end{aligned}$$

where the one-form $\mathrm{d}(\Theta \circ \widetilde{\nu})$ and the vector field $R_{\pi/2} \circ \widetilde{\nu}_{\theta_0}(t)$ are identified. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a function and set $\widetilde{\varphi}(t) = \widetilde{\nu}(t) + \alpha(t)R_{\pi/2} \circ \widetilde{\nu}_{\theta_0}(t)$. Then, it follows $\frac{\mathrm{d}\widetilde{\gamma}_{\theta_0}}{\mathrm{d}t}(t) \equiv 0$. Thus, as expected, the envelope created by the line family $\mathcal{H}_{\left(\widetilde{\varphi},\widetilde{\nu}_{\theta_0}\right)}$ in this case is actually the circle with radius $|c|$ centred at the origin, where $c = \widetilde{\gamma}_{\theta_0}(t) = \cos\theta_0$ (see figure 11).

(c) Let $\widetilde{\nu} : \mathbb{R} \to S^1$ be the mapping defined by $\widetilde{\nu}(t) = \frac{1}{\sqrt{1+9t^4}}\left(-3t^2, 1\right)$. Set $\widetilde{\nu}_{\theta_0} = R_{\theta_0} \circ \widetilde{\nu}$ where R_{θ_0} is as above. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a function and set $\widetilde{\varphi}_{\theta_0}(t) = (t, t^3) + \alpha(t)R_{\pi/2} \circ \widetilde{\nu}_{\theta_0}(t)$. Set $\widetilde{\gamma}_{\theta_0}(t) = \widetilde{\varphi}_{\theta_0}(t) \cdot \widetilde{\nu}_{\theta_0}(t)$. It is easily seen that 0 is a singular point of $\widetilde{\gamma}_{\theta_0}$ if and only if $\theta_0 \in \pi\mathbb{Z}$. On the other hand, by calculation, we have $\frac{\mathrm{d}\left(\Theta\circ\widetilde{\nu}_{\theta_0}\right)}{\mathrm{d}t}(t) = \frac{6t}{1+9t^4}$ and thus 0 is a unique singular point of $\widetilde{\nu}_{\theta_0}$ for any θ_0. Therefore, by theorem 1, the hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu}_\theta)}$ does not create an envelope if $\theta_0 \notin \pi\mathbb{Z}$.

Next, suppose that $\theta_0 \in \pi\mathbb{Z}$. Then, calculations show

$$\mathrm{d}\left(\widetilde{\gamma}_{\theta_0}\right) = \frac{\mp(6t^2 + 18t^6)}{(1+9t^4)^{\frac{3}{2}}}\mathrm{d}t = \frac{\mp(t+3t^5)}{\sqrt{1+9t^4}}\frac{\mathrm{d}\left(\Theta \circ \widetilde{\nu}_{\theta_0}\right)}{\mathrm{d}t}(t)\mathrm{d}t = \frac{\mp(t+3t^5)}{\sqrt{1+9t^4}}\mathrm{d}\left(\Theta \circ \widetilde{\nu}_{\theta_0}\right),$$

where double sign should be read in the same order. Set $\widetilde{\omega}(t) = \frac{\mp(t+3t^5)}{\sqrt{1+9t^4}}\mathrm{d}\left(\Theta \circ \widetilde{\nu}_{\theta_0}\right)$. By theorems 1 and 2, the hyperplane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu}_{\theta_0})}$ creates a unique envelope with the desired form

$$\begin{aligned}\widetilde{f}(t) &= \widetilde{\omega}(t) + \widetilde{\gamma}_{\theta_0}(t)\widetilde{\nu}_{\theta_0}(t) \\ &= \frac{\mp(t+3t^5)}{1+9t^4}\left(\mp 1, \mp 3t^2\right) \mp \frac{2t^3}{1+9t^4}(\mp 3t^2, \pm 1) \\ &= \frac{1}{1+9t^4}\left(t + 3t^5 + 6t^5, 3t^3 + 9t^7 - 2t^3\right) \\ &= \left(t, t^3\right),\end{aligned}$$

where for each $t \in \mathbb{R}$ the cotangent vector $\frac{\mp(t+3t^5)}{\sqrt{1+9t^4}}\mathrm{d}\left(\Theta \circ \widetilde{\nu}_{\theta_0}\right)$ and the vector $\frac{\mp(t+3t^5)}{\sqrt{1+9t^4}}R_{\pi/2} \circ \widetilde{\nu}_{\theta_0}(t)$ in the vector space $\mathbb{R}^2$ are identified (see figure 12).

Set $U = \mathbb{R} - \{0\}$. It is easily seen that $\widetilde{\nu}_{\theta_0}|_U$ is non-singular even in the case $\theta_0 \notin \pi\mathbb{Z}$. Hence, by theorems 1 and 2, the hyperplane family $\mathcal{H}_{\left(\widetilde{\varphi}|_U, \widetilde{\nu}_{\theta_0}|_U\right)}$ creates a unique envelope $\widetilde{f}_{\theta_0} : U \to \mathbb{R}^2$ even when $\theta_0 \notin \pi\mathbb{Z}$ and $\lim_{t\to 0}\|\widetilde{f}_{\theta_0}(t)\| = \infty$ when $\theta_0 \notin \pi\mathbb{Z}$.

(d) Let $\widetilde{\nu} : \mathbb{R} \to S^1$ be the mapping defined by $\widetilde{\nu}(t) = \frac{1}{\sqrt{4+25t^6}}\left(-5t^3, 2\right)$. Set $\widetilde{\nu}_{\theta_0} = R_{\theta_0} \circ \widetilde{\nu}$ where R_{θ_0} is as above. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a function and set $\widetilde{\varphi}_{\theta_0}(t) = (t^2, t^5) + \alpha(t)R_{\pi/2} \circ \widetilde{\nu}_{\theta_0}(t)$. Set $\widetilde{\gamma}_{\theta_0}(t) = \widetilde{\varphi}_{\theta_0}(t) \cdot \widetilde{\nu}_{\theta_0}(t) = \frac{-3t^5\ \cos\ \theta_0 - 2t^2\ \sin\ \theta_0 - 5t^8\ \sin\ \theta_0}{\sqrt{4+25t^6}}$. By calculation, we have $\frac{\mathrm{d}\left(\Theta\circ\widetilde{\nu}_{\theta_0}\right)}{\mathrm{d}t}(t) = \frac{30t^2}{4+25t^6}$. Therefore, the hyperplane family $\mathcal{H}_{(\widetilde{\varphi}, \widetilde{\nu}_\theta)}$ is not creative if $\theta \notin \pi\mathbb{Z}$ and it creates no envelope in this case by theorem 1.

Next, suppose that $\theta_0 \in \pi\mathbb{Z}$. Then, calculation shows

$$\begin{aligned}\mathrm{d}\left(\widetilde{\gamma}_{\theta_0}\right) &= \frac{\mp 30t^2\left(2t^2+5t^8\right)}{(4+25t^6)\sqrt{4+25t^6}}\mathrm{d}t \\ &= \frac{\mp(2t^2+5t^8)}{\sqrt{4+25t^6}}\frac{\mathrm{d}\left(\Theta\circ\widetilde{\nu}_{\theta_0}\right)}{\mathrm{d}t}(t)\mathrm{d}t = \frac{\mp(2t^2+5t^8)}{\sqrt{4+25t^6}}\mathrm{d}\left(\Theta\circ\widetilde{\nu}_{\theta_0}\right),\end{aligned}$$

where double sign should be read in the same order. Therefore, the hyperplane family $\mathcal{H}_{(\widetilde{\varphi}, \widetilde{\nu}_\theta)}$ is creative. Set $\widetilde{\omega}(t) = \frac{\mp(2t^2+5t^8)}{\sqrt{4+25t^6}}.\mathrm{d}\left(\Theta\circ\widetilde{\nu}_{\theta_0}\right)$. By theorems 1 and 2, $\mathcal{H}_{(\widetilde{\varphi}, \widetilde{\nu}_{\theta_0})}$ creates a unique envelope with the desired form

$$\begin{aligned}\widetilde{f}(t) &= \widetilde{\omega}(t) + \widetilde{\gamma}_{\theta_0}(t)\widetilde{\nu}_{\theta_0}(t) \\ &= \frac{\mp(2t^2+5t^8)}{4+25t^6}\left(\mp 2, \mp 5t^3\right) + \frac{\mp 3t^5}{4+25t^6}(\mp 5t^3, \pm 2) \\ &= \frac{1}{4+25t^6}\left(4t^2+10t^8+15t^8, 10t^5+25t^{11}-6t^5\right) \\ &= \left(t^2, t^5\right),\end{aligned}$$

where for each $t \in \mathbb{R}$ the cotangent vector $\frac{\mp(2t^2+5t^8)}{\sqrt{4+25t^6}}\mathrm{d}\left(\Theta\circ\widetilde{\nu}_{\theta_0}\right)$ and the vector $\frac{\mp(2t^2+5t^8)}{\sqrt{4+25t^6}}R_{\pi/2}\circ\widetilde{\nu}_{\theta_0}(t)$ in the vector space $\mathbb{R}^2$ are identified (see figure 13). In the case $\theta_0 = 0$, consider the mapping $\widetilde{\Omega} : \mathbb{R} \to T^*S^1$ given in definition 2 and $\Omega : \mathbb{R} \to J^1\left(S^1, \mathbb{R}\right)$ given in remark 1.1(a). Namely, consider the following two mappings.

$$\begin{aligned}\widetilde{\Omega}(t) &= \left(\frac{1}{\sqrt{4+25t^6}}\left(\mp 5t^3, \pm 2\right), \frac{\mp 30t^2(2t^2+5t^8)}{\left(4+25t^6\right)^{\frac{3}{2}}}\right), \\ \Omega(t) &= \left(\frac{1}{\sqrt{4+25t^6}}\left(\mp 5t^3, \pm 2\right), \frac{\mp 3t^5}{\sqrt{4+25t^6}}, \frac{\mp 30t^2(2t^2+5t^8)}{\left(4+25t^6\right)^{\frac{3}{2}}}\right).\end{aligned}$$

Since $\mathrm{d}\left(\widetilde{\gamma}_{\theta_0}\right) = \frac{\mp(2t^2+5t^8)}{\sqrt{4+25t^6}}\mathrm{d}\left(\Theta\circ\widetilde{\nu}_{\theta_0}\right)$, the map-germ of Ω at any t is nothing but an opening of the map-germ $\widetilde{\Omega} : (\mathbb{R}, t) \to T^*S^1$. At $t = 0$, the map-germ of each of them is not immersive and has singular images.

Set $U = \mathbb{R} - \{0\}$. It is easily seen that $\widetilde{\nu}_{\theta_0}|_U$ is non-singular even in the case $\theta_0 \notin \pi\mathbb{Z}$. Hence, by theorems 1 and 2, the hyperplane family $\mathcal{H}_{\left(\widetilde{\varphi}|_U, \widetilde{\nu}_{\theta_0}|_U\right)}$ creates a unique envelope $\widetilde{f}_{\theta_0} : U \to \mathbb{R}^2$ even when $\theta_0 \notin \pi\mathbb{Z}$ and $\lim_{t\to 0}\|\widetilde{f}_{\theta_0}(t)\| = \infty$ when $\theta_0 \notin \pi\mathbb{Z}$.

Example 4.2 (unit speed curves)**.**

(a) Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ be a unit speed curve. As usual, set $\mathbf{t}(s) = \mathbf{r}'(s)$ and $\mathbf{n}(s)$ is defined from $\mathbf{t}(s)$ by rotating anticlockwise through $\frac{\pi}{2}$. The Serret–Frenet formulas for the plane curve $\mathbf{r}$ is as follows.

$$\begin{cases}\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s)\\ \mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s).\end{cases}$$

Set $\widetilde{\varphi} = \mathbf{r}$ and $\widetilde{\nu} = \mathbf{n}$. Then, the line family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})} = \mathcal{H}_{(\mathbf{r},\mathbf{n})}$ is the affine tangent line family of the curve $\mathbf{r}$. In this case, the correspondence $\mathbf{r} \mapsto \mathcal{H}_{(\mathbf{r},\mathbf{n})}$ may be regarded as the Legendre transformation of the given curve $\mathbf{r}$. Set $\widetilde{\gamma}(s) = \widetilde{\varphi}(s)\cdot\widetilde{\nu}(s)$. Then,

$$\widetilde{\gamma}'(s) = \mathbf{r}(s)\cdot(-\kappa(s)\mathbf{t}(s)) = -\left(\mathbf{r}(s)\cdot\mathbf{t}(s)\right)(\Theta_{\mathbf{t}}\circ\widetilde{\nu})'(s),$$

where $\widetilde{\nu}(s) = (\cos\,\Theta_{\mathbf{t}}\circ\widetilde{\nu}(s), \sin\,\Theta_{\mathbf{t}}\circ\widetilde{\nu}(s))$. Therefore, by theorem 1, the line family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ creates an envelope.

Suppose that the set of regular points of $\widetilde{\nu}$ is dense, that is to say, the set $\{s \in \mathbb{R} \mid \kappa(s) \neq 0\}$ is dense. Then, by theorem 2, the created envelopes are unique. By theorem 1, the unique envelope is as follows (see figure 14).

$$\begin{aligned}\widetilde{f}(s) &= \widetilde{\omega}(s) + \widetilde{\gamma}(s)\cdot\widetilde{\nu}(s)\\ &= \left(\mathbf{r}(s)\cdot\mathbf{t}(s)\right)\mathbf{t}(s) + \left(\mathbf{r}(s)\cdot\mathbf{n}(s)\right)\mathbf{n}(s)\\ &= \mathbf{r}(s).\end{aligned}$$

Notice that if there is a point $s \in \mathbb{R}$ such that $\kappa(s) = 0$, then the full discriminant of the line family is different from the unique desired envelope since the full discriminant includes the affine tangent line at s. This is one of advantages of our method. The correspondence

$$\mathcal{H}_{(\mathbf{r},\mathbf{n})} \mapsto \mathbf{r}$$

may be regarded as the inverse Legendre transformation for plane curves.

Next, suppose that the set of regular points of $\widetilde{\nu}$ is not dense. Then, there exists an open interval (a, b) such that $\kappa(s) = 0$ for any $s \in (a, b)$. Then, for any $s \in (a, b)$ and any function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(\mathbb{R} - (a, b)) = \{0\}$, it follows

$$\widetilde{\gamma}'(s) = \alpha(s)(\Theta_{\mathbf{t}}\circ\widetilde{\nu})'(s).$$

By theorem 1,

$$\begin{aligned}\widetilde{f}(s) &= \widetilde{\omega}(s) + \widetilde{\gamma}(s)\cdot\widetilde{\nu}(s)\\ &= \alpha(s)\mathbf{t}(s) + \left(\mathbf{r}(s)\cdot\mathbf{n}(s)\right)\mathbf{n}(s)\\ &= \left((\alpha(s) - (\mathbf{r}(s)\cdot\mathbf{t}(s))) + (\mathbf{r}(s)\cdot\mathbf{t}(s))\right)\mathbf{t}(s) + \left(\mathbf{r}(s)\cdot\mathbf{n}(s)\right)\mathbf{n}(s)\\ &= \mathbf{r}(s) + \beta(s)\mathbf{t}(s),\end{aligned}$$

where $\beta(s) = \alpha(s) - (\mathbf{r}(s)\cdot\mathbf{t}(s))$. Hence, in this case, the inverse Legendre transformation does not work well.

(b) Let $\mathbf{r}:\mathbb{R}\to\mathbb{R}^3$ be a unit speed space curve. As usual, set $\mathbf{t}(s) = \mathbf{r}'(s)$ and assume $||\mathbf{t}'(s)|| > 0$ for any $s\in\mathbb{R}$ so that the principal normal vector $\mathbf{n}(s)$ can be defined by $\mathbf{t}'(s) = ||\mathbf{t}'(s)||\mathbf{n}(s)$. As usual, the binormal vector $\mathbf{b}(s)$ is defined by $\det(\mathbf{t}(s),\mathbf{n}(s),\mathbf{b}(s)) = 1$. The Serret–Frenet formulas for the space curve $\mathbf{r}$ is as follows.

$$\begin{cases} \mathbf{t}'(s) = \kappa(s)\mathbf{n}(s) \\ \mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s) \\ \mathbf{b}'(s) = -\tau(s)\mathbf{n}(s). \end{cases}$$

Define $\widetilde{\varphi}:\mathbb{R}^2\to\mathbb{R}^3$ and $\widetilde{\nu}:\mathbb{R}^2\to S^2$ by $\widetilde{\varphi}(s,u) = \mathbf{r}(s)$ and $\widetilde{\nu}(s,u) = \mathbf{b}(s)$ respectively. Then, the plane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is the family of osculating planes of the space curve $\mathbf{r}$. Set $\widetilde{\gamma}(s,u) = \mathbf{r}(s)\cdot\mathbf{b}(s)$. Then, all of the following six identities are clear.

$$\frac{\partial\widetilde{\gamma}}{\partial s}(s,u) = \mathbf{r}(s)\cdot(-\tau(s)\mathbf{n}(s)), \qquad \frac{\partial\widetilde{\gamma}}{\partial u}(s,u) = 0, \qquad \frac{\partial(\Theta_{\mathbf{t}}\circ\widetilde{\nu})}{\partial s}(s,u) = 0,$$

$$\frac{\partial(\Theta_{\mathbf{t}}\circ\widetilde{\nu})}{\partial u}(s,u) = 0, \qquad \frac{\partial(\Theta_{\mathbf{n}}\circ\widetilde{\nu})}{\partial s}(s,u) = -\tau(s), \qquad \frac{\partial(\Theta_{\mathbf{n}}\circ\widetilde{\nu})}{\partial u}(s,u) = 0.$$

Therefore, we have the following.

$$\begin{aligned} \frac{\partial\widetilde{\gamma}}{\partial s}(s,u) &= \alpha_1(s,u)\frac{\partial(\Theta_{\mathbf{t}}\circ\widetilde{\nu})}{\partial s}(s,u) + (\mathbf{r}(s)\cdot\mathbf{n}(s))\frac{\partial(\Theta_{\mathbf{n}}\circ\widetilde{\nu})}{\partial s}(s,u), \\ \frac{\partial\widetilde{\gamma}}{\partial u}(s,u) &= \alpha_2(s,u)\frac{\partial(\Theta_{\mathbf{t}}\circ\widetilde{\nu})}{\partial u}(s,u) + \alpha_3(s,u)\frac{\partial(\Theta_{\mathbf{n}}\circ\widetilde{\nu})}{\partial u}(s,u), \end{aligned}$$

where $\alpha_1,\alpha_2,\alpha_3:\mathbb{R}^2\to\mathbb{R}$ are arbitrary functions. Thus, by theorem 1, the plane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ creates an envelope if and only if $(\mathbf{r}(s)\cdot\mathbf{n}(s)) = \alpha_3(s,u)$ and $\alpha_1(s,u) = \alpha_2(s,u)$. Therefore, again by theorem 1, we have the following concrete expression of the created envelopes.

$$\begin{aligned} \widetilde{f}(s,u) &= \widetilde{\omega}(s,u) + \widetilde{\gamma}(s)\widetilde{\nu}(s) \\ &= (\mathbf{r}(s)\cdot\mathbf{n}(s))\,\mathbf{n}(s) + \alpha(s,u)\mathbf{t}(s) + (\mathbf{r}(s)\cdot\mathbf{b}(s))\,\mathbf{b}(s) \\ &= (\mathbf{r}(s)\cdot\mathbf{n}(s))\,\mathbf{n}(s) + (\mathbf{r}(s)\cdot\mathbf{t}(s))\,\mathbf{t}(s) \\ &\quad + (\alpha(s,u) - (\mathbf{r}(s)\cdot\mathbf{t}(s)))\,\mathbf{t}(s) + (\mathbf{r}(s)\cdot\mathbf{b}(s))\,\mathbf{b}(s) \\ &= \mathbf{r}(s) + \beta(s,u)\mathbf{t}(s), \end{aligned}$$

where $\alpha(s,u) = \alpha_1(s,u) = \alpha_2(s,u)$ and $\beta(s,u) = \alpha(s,u) - (\mathbf{r}(s)\cdot\mathbf{t}(s))$. All envelopes created by the osculating family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ can be exactly expressed as above. Hence, for example, both the tangent developable of $\mathbf{r}$ (in the case $\beta(s,u) = u$) and the space curve $\mathbf{r}$ (in the case $\beta(s,u) = 0$) are envelopes of $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$. Not only these two, there are uncountably many envelopes created by $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$. All envelopes for the osculating plane family are created only by the given curve $\mathbf{r}$ and its unit tangent curve $\mathbf{t}$.

Next, we consider envelopes created by $\mathcal{H}_{(\mathbf{r},\mathbf{b})}$ and $\mathcal{H}_{(\widetilde{f},\mathbf{n})}$. Namely, we obtain all solutions $\widetilde{g}(s,u)$ for the following system of PDEs (Partial Differential Equation) with one constraint condition.

$$\begin{cases} \dfrac{\partial \widetilde{g}}{\partial s}(s,u)\cdot \mathbf{b}(s)=0, \\ \dfrac{\partial \widetilde{g}}{\partial u}(s,u)\cdot \mathbf{b}(s)=0, \\ \dfrac{\partial \widetilde{g}}{\partial s}(s,u)\cdot \mathbf{n}(s)=0, \\ \dfrac{\partial \widetilde{g}}{\partial u}(s,u)\cdot \mathbf{n}(s)=0, \\ \left(\widetilde{g}(s,u)-\mathbf{r}(s)\right)\cdot \mathbf{b}(s)=0. \end{cases}$$

Since $\kappa(s) > 0$ for any $s \in \mathbb{R}$ and

$$\frac{\partial \widetilde{f}}{\partial s}(s,u)=\mathbf{t}(s)+\frac{\partial \beta}{\partial s}(s,u)\mathbf{t}(s)+\beta(s,u)\left(\kappa(s)\mathbf{n}(s)\right),$$
$$\frac{\partial \widetilde{f}}{\partial u}(s,u)=\frac{\partial \beta}{\partial u}(s,u)\mathbf{t}(s),$$

if $\widetilde{f}$ itself is a solution of the above system of PDEs, then $\beta(s,u)$ must be constant 0. Conversely, it is clear that $\mathbf{r}$ itself is a solution of the above system of PDEs with one constraint condition. Therefore, for the above system of PDEs with one constraint condition, there are no solutions except for the trivial solution $\mathbf{r}$. This implies that even for a space curve $\mathbf{r}:\mathbb{R}\to\mathbb{R}^3$, the inverse Legendre transformation

$$\mathcal{H}_{(\mathbf{r},\{\mathbf{b},\mathbf{n}\})} \mapsto \mathbf{r}$$

works well.

Finally, we consider envelopes created by $\mathcal{H}_{(\mathbf{r},\mathbf{b})}$ and $\mathcal{H}_{\left(\widetilde{f},\mathbf{t}\right)}$. Namely, we obtain all solutions $\widetilde{g}(s,u)$ for the following system of PDEs with one constraint condition.

$$\begin{cases} \dfrac{\partial \widetilde{g}}{\partial s}(s,u)\cdot \mathbf{b}(s)=0, \\ \dfrac{\partial \widetilde{g}}{\partial u}(s,u)\cdot \mathbf{b}(s)=0, \\ \dfrac{\partial \widetilde{g}}{\partial s}(s,u)\cdot \mathbf{t}(s)=0, \\ \dfrac{\partial \widetilde{g}}{\partial u}(s,u)\cdot \mathbf{t}(s)=0, \\ \left(\widetilde{g}(s,u)-\mathbf{r}(s)\right)\cdot \mathbf{b}(s)=0. \end{cases}$$

By the above calculations, if $\widetilde{f}$ is a solution of the above system of PDEs, then both $1+\frac{\partial\beta}{\partial s}(s,u)=0$ and $\frac{\partial\beta}{\partial u}(s,u)=0$ must be satisfied. It follows $\beta(s,u)=-s+c$ $(c\in\mathbb{R})$. It is easily seen that for any $c\in\mathbb{R}$, the space curve $s\mapsto \mathbf{r}(s)+(-s+c)\mathbf{t}(s)$ is a solution of the above system of PDEs with one constraint condition. Thus, in this case, the system of PDEs with one constraint condition has uncountably many solutions.

Example 4.3.

(a) (The shoe surface: example 1 of [3].) In this example, along the general theory developed in this paper, we start from making several general formulas for the envelope created by the affine tangent plane family of the surface having the form $\widetilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}^3$, $\widetilde{\varphi}(x, y) = (x,\ y,\ \widetilde{\varphi}_1(x, y))$ such that the origin $(0, 0)$ is a singular point of the function $\widetilde{\varphi}_1 : \mathbb{R}^2 \to \mathbb{R}$ and there are no other singular points of $\widetilde{\varphi}_1$. Then, by calculating the obtained general formulas in the case of the shoe surface $\widetilde{\varphi}(x, y) = \left(x,\ y,\ \frac{1}{3}x^3 - \frac{1}{2}y^2\right)$, just by calculations, we confirm that the concrete representation form of the envelope created by the affine tangent plane family of the shoe surface $\widetilde{\varphi}$ is actually the shoe surface itself.

Let $\widetilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}^3$ be the mapping having the form $\widetilde{\varphi}(x, y) = (x, y, \widetilde{\varphi}_1(x, y))$, where the function $\widetilde{\varphi}_1 : \mathbb{R}^2 \to \mathbb{R}$ has a unique singularity at the origin, namely $\frac{\partial\widetilde{\varphi}_1}{\partial x}(0, 0) = \frac{\partial\widetilde{\varphi}_1}{\partial y}(0, 0) = 0$ and $\left(\frac{\partial\widetilde{\varphi}_1}{\partial x}(x, y),\ \frac{\partial\widetilde{\varphi}_1}{\partial y}(x, y)\right) \neq (0, 0)$ for any $(x, y) \in \mathbb{R}^2 - \{(0, 0)\}$. Then, the mapping $\widetilde{\nu} : \mathbb{R}^2 \to S^2$ defined by

$$\widetilde{\nu}(x, y) = \frac{\frac{\partial\widetilde{\varphi}_1}{\partial x}(x, y) \times \frac{\partial\widetilde{\varphi}_1}{\partial y}(x, y)}{\|\frac{\partial\widetilde{\varphi}_1}{\partial x}(x, y) \times \frac{\partial\widetilde{\varphi}_1}{\partial y}(x, y)\|} = \frac{\left(-\frac{\partial\widetilde{\varphi}_1}{\partial x},\ -\frac{\partial\widetilde{\varphi}_1}{\partial y},\ 1\right)}{\sqrt{\left(\frac{\partial\widetilde{\varphi}_1}{\partial x}\right)^2 + \left(\frac{\partial\widetilde{\varphi}_1}{\partial y}\right)^2 + 1}}$$

is a Gauss mapping of the tangent plane family of $\widetilde{\varphi}$. Here, the tangent plane family of $\widetilde{\varphi}$ is $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$. Let (x_0, y_0) be an arbitrary point of $\mathbb{R}^2 - \{(0, 0)\}$. Then, by the assumption on the function $\widetilde{\varphi}_1$, it follows that $\widetilde{\nu}(x_0, y_0) \neq (0,\ 0,\ 1)$. Set

$$\begin{aligned}
\mathbf{v}_0(x_0, y_0) &= \widetilde{\nu}(x_0, y_0),\\
\mathbf{v}_1(x_0, y_0) &= \frac{(0,\ 0,\ 1) - ((0,\ 0,\ 1) \cdot \mathbf{v}_0(x_0, y_0))\,\mathbf{v}_0(x_0, y_0)}{\|(0,\ 0,\ 1) - ((0,\ 0,\ 1) \cdot \mathbf{v}_0(x_0, y_0))\,\mathbf{v}_0(x_0, y_0)\|},\\
\mathbf{v}_2(x_0, y_0) &= \mathbf{v}_0(x_0, y_0) \times \mathbf{v}_1(x_0, y_0).
\end{aligned}$$

Then, $\langle \mathbf{v}_0(x_0, y_0), \mathbf{v}_1(x_0, y_0), \mathbf{v}_2(x_0, y_0)\rangle$ is an orthonormal basis of $\mathbb{R}^3$, and under the identification of two vector spaces $\mathbb{R}^3$ and $T_{\widetilde{\nu}(x_0,y_0)}\mathbb{R}^3$, $\langle \mathbf{v}_1(x_0, y_0), \mathbf{v}_2(x_0, y_0)\rangle$ is an orthonormal basis of the tangent vector space $T_{\widetilde{\nu}(x_0,y_0)}S^2$. Let ε be a sufficiently small positive number and denote the set $\{\Theta_1\mathbf{v}_1(x_0, y_0) + \Theta_2\mathbf{v}_2(x_0, y_0) \mid -\varepsilon < \Theta_1, \Theta_2 < \varepsilon\}$ by V'. Let $\exp : V' \to S^2$ be the restriction of the exponential mapping at $\widetilde{\nu}(x_0, y_0)$ to V' and set $V = \exp(V')$. Let $(V, (\Theta_1, \Theta_2))$ be the normal coordinate neighbourhood at $\widetilde{\nu}(x_0, y_0)$ defined by $\exp^{-1} : V \to V'$. Set

$$\widetilde{\gamma}(x, y) = \widetilde{\varphi}(x, y) \cdot \widetilde{\nu}(x, y) = \frac{-x\frac{\partial\widetilde{\varphi}_1}{\partial x} - y\frac{\partial\widetilde{\varphi}_1}{\partial y} + \widetilde{\varphi}_1(x, y)}{\sqrt{\left(\frac{\partial\widetilde{\varphi}_1}{\partial x}\right)^2 + \left(\frac{\partial\widetilde{\varphi}_1}{\partial y}\right)^2 + 1}}.$$

Since $\widetilde{\nu} : \mathbb{R}^2 \to S^2$ is a Gauss mapping of $\widetilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}^3$, we have

$$\begin{aligned}
\frac{\partial\widetilde{\gamma}}{\partial x}(x_0, y_0) &= \widetilde{\varphi}(x_0, y_0) \cdot \frac{\partial\widetilde{\nu}}{\partial x}(x_0, y_0)\\
&= (\widetilde{\varphi}(x_0, y_0) \cdot \mathbf{v}_1(x_0, y_0))\,\frac{\partial\,(\Theta_1 \circ \widetilde{\nu})}{\partial x}(x_0, y_0)\\
&\quad + (\widetilde{\varphi}(x_0, y_0) \cdot \mathbf{v}_2(x_0, y_0))\,\frac{\partial\,(\Theta_2 \circ \widetilde{\nu})}{\partial x}(x_0, y_0)
\end{aligned}$$

and

$$\frac{\partial\widetilde{\gamma}}{\partial y}(x_0,y_0)=\widetilde{\varphi}(x_0,y_0)\cdot\frac{\partial\widetilde{\nu}}{\partial y}(x_0,y_0)$$
$$=(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_1(x_0,y_0))\frac{\partial\left(\Theta_1\circ\widetilde{\nu}\right)}{\partial y}(x_0,y_0)$$
$$+(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_2(x_0,y_0))\frac{\partial\left(\Theta_2\circ\widetilde{\nu}\right)}{\partial y}(x_0,y_0).$$

Thus, as the equality of two-dimensional cotangent vectors of $T^*_{(x_0,y_0)}\mathbb{R}^2$, we have the following equality.

$$\mathrm{d}\widetilde{\gamma}=\frac{\partial\widetilde{\gamma}}{\partial x}(x_0,y_0)\mathrm{d}x+\frac{\partial\widetilde{\gamma}}{\partial y}(x_0,y_0)\mathrm{d}y$$
$$=(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_1(x_0,y_0))\,\mathrm{d}\left(\Theta_1\circ\widetilde{\nu}\right)+(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_2(x_0,y_0))\,\mathrm{d}\left(\Theta_2\circ\widetilde{\nu}\right).$$

Set $U=\mathbb{R}^2-\{(0,0)\}$ and assume that the singular set of $\widetilde{\nu}$ is of Lebesgue measure zero. Then, since (x_0,y_0) is an arbitrary point of U, by theorems 1(a) and 2, it follows that $\mathcal{H}_{(\widetilde{\varphi}|_U,\widetilde{\nu}|_U)}$ creates a unique envelope. Set

$$\widetilde{\omega}(x_0,y_0)=(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_1(x_0,y_0))\,\mathrm{d}\left(\Theta_1\circ\widetilde{\nu}\right)+(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_2(x_0,y_0))\,\mathrm{d}\left(\Theta_2\circ\widetilde{\nu}\right).$$

Then, under the canonical identifications

$$T^*_{\widetilde{\nu}(x_0,y_0)}S^2\cong T_{\widetilde{\nu}(x_0,y_0)}S^2\subset T_{\widetilde{\nu}(x_0,y_0)}\mathbb{R}^3\cong\mathbb{R}^3,$$

the two-dimensional cotangent vector

$$\widetilde{\omega}(x_0,y_0)=(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_1(x_0,y_0))\,\mathrm{d}\Theta_1+(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_2(x_0,y_0))\,\mathrm{d}\Theta_2$$

may be regarded as the following three-dimensional vector (denoted by the same symbol $\widetilde{\omega}(x_0,y_0)$).

$$\widetilde{\omega}(x_0,y_0)=(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_1(x_0,y_0))\,\mathbf{v}_1(x_0,y_0)$$
$$+(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_2(x_0,y_0))\,\mathbf{v}_2(x_0,y_0).$$

Therefore, by theorem 1(b), the envelope vector at (x_0,y_0) must have the following form:

$$\widetilde{f}(x_0,y_0)=\widetilde{\omega}(x_0,y_0)+\widetilde{\gamma}(x_0,y_0)\widetilde{\nu}(x_0,y_0)$$
$$=(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_1(x_0,y_0))\,\mathbf{v}_1(x_0,y_0)+(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_2(x_0,y_0))\,\mathbf{v}_2(x_0,y_0)$$
$$+(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_0(x_0,y_0))\,\mathbf{v}_0(x_0,y_0)$$
$$=\widetilde{\varphi}(x_0,y_0).$$

By continuity, it follows that $\widetilde{f}=\widetilde{\varphi}$ is the unique envelope created by the given plane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$.

Next, we apply the above formulas to the shoe surface. The shoe surface is the image of $\widetilde{\varphi}:\mathbb{R}^2\to\mathbb{R}^3$ defined by $\widetilde{\varphi}(x,y)=\left(x,y,\frac{1}{3}x^3-\frac{1}{2}y^2\right)$. Set $\widetilde{\varphi}_1(x,y)=\frac{1}{3}x^3-\frac{1}{2}y^2$. Then, the origin $(0,0)$ is a unique singular point of $\widetilde{\varphi}_1$. For the given $\widetilde{\varphi}$, we have

$\widetilde{\nu}(x,y) = \frac{\frac{\partial\widetilde{\varphi}}{\partial x}(x,y)\times\frac{\partial\widetilde{\varphi}}{\partial y}(x,y)}{\|\frac{\partial\widetilde{\varphi}}{\partial x}(x,y)\times\frac{\partial\widetilde{\varphi}}{\partial y}(x,y)\|} = \frac{(-x^2,\,y,\,1)}{\sqrt{x^4+y^2+1}}$. It is easily confirmed that the set consisting of regular points of $\widetilde{\nu}$ is dense. In fact, it is known that any singularity of $\widetilde{\nu}$ is a fold singularity (see [3]). Set $U = \mathbb{R}^2 - \{(0,0)\}$ and take an arbitrary point (x_0, y_0) of U. For the shoe surface $\widetilde{\varphi}$, we set

$$\mathbf{v}_0(x_0,y_0) = \widetilde{\nu}(x_0,y_0) = \frac{\left(-x_0^2,\, y_0,\, 1\right)}{\sqrt{x_0^4+y_0^2+1}},$$

$$\mathbf{v}_1(x_0,y_0) = \frac{(0,\,0,\,1) - ((0,\,0,\,1)\cdot\mathbf{v}_0(x_0,y_0))\,\mathbf{v}_0(x_0,y_0)}{\|(0,\,0,\,1) - ((0,\,0,\,1)\cdot\mathbf{v}_0(x_0,y_0))\,\mathbf{v}_0(x_0,y_0)\|} = \frac{\left(x_0^2,\, -y_0,\, x_0^4+y_0^2\right)}{\sqrt{\left(x_0^4+y_0^2\right)\left(x_0^4+y_0^2+1\right)}},$$

$$\mathbf{v}_2(x_0,y_0) = \mathbf{v}_0(x_0,y_0)\times\mathbf{v}_1(x_0,y_0) = \frac{\left(y_0,\, x_0^2,\, 0\right)}{\sqrt{x_0^4+y_0^2}}.$$

By calculation, we have

$$\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_1(x_0,y_0) = \frac{x_0^3 - y_0^2 + \left(\frac{1}{3}x_0^3 - \frac{1}{2}y_0^2\right)\left(x_0^4+y_0^2\right)}{\left(x_0^4+y_0^2\right)^{\frac{1}{2}}\left(x_0^4+y_0^2+1\right)^{\frac{1}{2}}},$$

$$\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_2(x_0,y_0) = \frac{x_0y_0 + x_0^2y_0}{\left(x_0^4+y_0^2\right)^{\frac{1}{2}}}.$$

Let $(V, (\Theta_1, \Theta_2))$ be the normal coordinate neighbourhood of S^2 defined above. By calculations using the following two identities

$$\frac{\partial\widetilde{\nu}}{\partial x}(x_0,y_0) = \mathbf{v}_1(x_0,y_0)\frac{\partial\left(\Theta_1\circ\widetilde{\nu}\right)}{\partial x}(x_0,y_0) + \mathbf{v}_2(x_0,y_0)\frac{\partial\left(\Theta_2\circ\widetilde{\nu}\right)}{\partial x}(x_0,y_0),$$

$$\frac{\partial\widetilde{\nu}}{\partial y}(x_0,y_0) = \mathbf{v}_1(x_0,y_0)\frac{\partial\left(\Theta_1\circ\widetilde{\nu}\right)}{\partial y}(x_0,y_0) + \mathbf{v}_2(x_0,y_0)\frac{\partial\left(\Theta_2\circ\widetilde{\nu}\right)}{\partial y}(x_0,y_0),$$

we have the following.

$$\frac{\partial\left(\Theta_1\circ\widetilde{\nu}\right)}{\partial x}(x_0,y_0) = \frac{-2x_0^3}{\left(x_0^4+y_0^2\right)^{\frac{1}{2}}\left(x_0^4+y_0^2+1\right)},$$

$$\frac{\partial\left(\Theta_2\circ\widetilde{\nu}\right)}{\partial x}(x_0,y_0) = \frac{-2x_0y_0 - 2x_0y_0^3 - 2x_0^5y_0}{\left(x_0^4+y_0^2\right)^{\frac{1}{2}}\left(x_0^4+y_0^2+1\right)^{\frac{3}{2}}},$$

$$\frac{\partial\left(\Theta_1\circ\widetilde{\nu}\right)}{\partial y}(x_0,y_0) = \frac{-y_0}{\left(x_0^4+y_0^2\right)^{\frac{1}{2}}\left(x_0^4+y_0^2+1\right)},$$

$$\frac{\partial\left(\Theta_2\circ\widetilde{\nu}\right)}{\partial y}(x_0,y_0) = \frac{x_0^2}{\left(x_0^4+y_0^2\right)^{\frac{1}{2}}\left(x_0^4+y_0^2+1\right)^{\frac{1}{2}}}.$$

On the other hand, from the form $\widetilde{\gamma}(x,y) = \widetilde{\varphi}(x,y)\cdot\widetilde{\nu}(x,y) = \frac{-\frac{2}{3}x^3+\frac{1}{2}y^2}{\sqrt{x^4+y^2+1}}$, we have

$$\frac{\partial\gamma}{\partial x}(x_0,y_0) = \frac{-2x_0^2-2x_0^2y_0^2-x_0^3y_0^2-\frac{2}{3}x_0^6}{\left(x_0^4+y_0^2+1\right)^{\frac{3}{2}}},$$
$$\frac{\partial\gamma}{\partial y}(x_0,y_0) = \frac{y_0+\frac{1}{2}y_0^3+\frac{2}{3}x_0^3y_0+x_0^4y_0}{\left(x_0^4+y_0^2+1\right)^{\frac{3}{2}}}.$$

Thus, we have the following desired identity at (x_0, y_0).

$$\begin{aligned}
\mathrm{d}\widetilde{\gamma} &= \frac{\partial\widetilde{\gamma}}{\partial x}(x_0,y_0)\mathrm{d}x + \frac{\partial\widetilde{\gamma}}{\partial y}(x_0,y_0)\mathrm{d}y \\
&= \frac{-2x_0^2-2x_0^2y_0^2-x_0^3y_0^2-\frac{2}{3}x_0^6}{\left(x_0^4+y_0^2+1\right)^{\frac{3}{2}}}\mathrm{d}x + \frac{y_0+\frac{1}{2}y_0^3+\frac{2}{3}x_0^3y_0+x_0^4y_0}{\left(x_0^4+y_0^2+1\right)^{\frac{3}{2}}}\mathrm{d}y \\
&= \left(\frac{x_0^3-y_0^2+\left(\frac{1}{3}x_0^3-\frac{1}{2}y_0^2\right)\left(x_0^4+y_0^2\right)}{\left(x_0^4+y_0^2\right)^{\frac{1}{2}}\left(x_0^4+y_0^2+1\right)^{\frac{1}{2}}}\frac{-2x_0^3}{\left(x_0^4+y_0^2\right)^{\frac{1}{2}}\left(x_0^4+y_0^2+1\right)}\right. \\
&\quad \left.+\frac{\left(x_0y_0+x_0^2y_0\right)}{\left(x_0^4+y_0^2\right)^{\frac{1}{2}}}\frac{\left(-2x_0y_0-2x_0y_0^3-2x_0^5y_0\right)}{\left(x_0^4+y_0^2\right)^{\frac{1}{2}}\left(x_0^4+y_0^2+1\right)^{\frac{3}{2}}}\right)\mathrm{d}x \\
&\quad +\left(\frac{x_0^3-y_0^2+\left(\frac{1}{3}x_0^3-\frac{1}{2}y_0^2\right)\left(x_0^4+y_0^2\right)}{\left(x_0^4+y_0^2\right)^{\frac{1}{2}}\left(x_0^4+y_0^2+1\right)^{\frac{1}{2}}}\frac{-y_0}{\left(x_0^4+y_0^2\right)^{\frac{1}{2}}\left(x_0^4+y_0^2+1\right)}\right. \\
&\quad \left.+\frac{\left(x_0y_0+x_0^2y_0\right)}{\left(x_0^4+y_0^2\right)^{\frac{1}{2}}}\frac{x_0^2}{\left(x_0^4+y_0^2\right)^{\frac{1}{2}}\left(x_0^4+y_0^2+1\right)^{\frac{1}{2}}}\right)\mathrm{d}y \\
&= \left(\left(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_1(x_0,y_0)\right)\frac{\partial\left(\Theta_1\circ\widetilde{\nu}\right)}{\partial x}(x_0,y_0)\right. \\
&\quad \left.+\left(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_2(x_0,y_0)\right)\frac{\partial\left(\Theta_2\circ\widetilde{\nu}\right)}{\partial x}(x_0,y_0)\right)\mathrm{d}x \\
&\quad +\left(\left(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_1(x_0,y_0)\right)\frac{\partial\left(\Theta_1\circ\widetilde{\nu}\right)}{\partial y}(x_0,y_0)\right. \\
&\quad \left.+\left(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_2(x_0,y_0)\right)\frac{\partial\left(\Theta_2\circ\widetilde{\nu}\right)}{\partial y}(x_0,y_0)\right)\mathrm{d}y \\
&= \left(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_1(x_0,y_0)\right)\mathrm{d}\left(\Theta_1\circ\widetilde{\nu}\right) + \left(\widetilde{\varphi}(x_0,y_0)\cdot\mathbf{v}_2(x_0,y_0)\right)\mathrm{d}\left(\Theta_2\circ\widetilde{\nu}\right).
\end{aligned}$$

Hence, by theorems 1(a) and 2, the plane family $\mathcal{H}_{(\widetilde{\varphi}|_U,\widetilde{\nu}|_U)}$ for the shoe surface $\widetilde{\varphi}(x,y) = \left(x,y,\frac{1}{3}x^3-\frac{1}{2}y^2\right)$ has a unique envelope $\widetilde{f}: U\to\mathbb{R}^3$, where $U = \mathbb{R}^2-\{(0,0)\}$. Then, under the canonical identifications

$$T^*_{\widetilde{\nu}(x_0,y_0)}S^2 \cong T_{\widetilde{\nu}(x_0,y_0)}S^2 \subset T_{\widetilde{\nu}(x_0,y_0)}\mathbb{R}^3 \cong \mathbb{R}^3,$$

the two-dimensional cotangent vector

$$\widetilde{\omega}(x_0, y_0) = (\widetilde{\varphi}(x_0, y_0) \cdot \mathbf{v}_1(x_0, y_0))\, \mathrm{d}\Theta_1 + (\widetilde{\varphi}(x_0, y_0) \cdot \mathbf{v}_2(x_0, y_0))\, \mathrm{d}\Theta_2$$

is identified with the following three-dimensional vector (denoted by the same symbol $\widetilde{\omega}(x_0, y_0)$).

$$\begin{aligned}\widetilde{\omega}(x_0, y_0) &= (\widetilde{\varphi}(x_0, y_0) \cdot \mathbf{v}_1(x_0, y_0))\, \mathbf{v}_1(x_0, y_0) + (\widetilde{\varphi}(x_0, y_0) \cdot \mathbf{v}_2(x_0, y_0))\, \mathbf{v}_2(x_0, y_0) \\ &= \frac{\left(x_0^3 - y_0^2 + \left(\frac{1}{3}x_0^3 - \frac{1}{2}y_0^2\right)\left(x_0^4 + y_0^2\right)\right)}{\left(x_0^4 + y_0^2\right)^{\frac{1}{2}}\left(x_0^4 + y_0^2 + 1\right)^{\frac{1}{2}}} \frac{\left(x_0^2,\, -y_0,\, x_0^4 + y_0^2\right)}{\left(x_0^4 + y_0^2\right)^{\frac{1}{2}}\left(x_0^4 + y_0^2 + 1\right)^{\frac{1}{2}}} \\ &\quad + \frac{\left(x_0 y_0 + x_0^2 y_0\right)}{\left(x_0^4 + y_0^2\right)^{\frac{1}{2}}} \frac{\left(y_0,\, x_0^2,\, 0\right)}{\left(x_0^4 + y_0^2\right)^{\frac{1}{2}}}.\end{aligned}$$

Therefore, by theorem 1(b), the unique envelope $\widetilde{f}$ must have the following desired parametric representation on $U = \mathbb{R}^2 - \{(0,0)\}$.

$$\begin{aligned}\widetilde{f}(x_0, y_0) &= \widetilde{\omega}(x_0, y_0) + \widetilde{\gamma}(x_0, y_0)\widetilde{\nu}(x_0, y_0) \\ &= \frac{\left(x_0^3 - y_0^2 + \left(\frac{1}{3}x_0^3 - \frac{1}{2}y_0^2\right)\left(x_0^4 + y_0^2\right)\right)}{\left(x_0^4 + y_0^2\right)^{\frac{1}{2}}\left(x_0^4 + y_0^2 + 1\right)^{\frac{1}{2}}} \frac{\left(x_0^2,\, -y_0,\, x_0^4 + y_0^2\right)}{\left(x_0^4 + y_0^2\right)^{\frac{1}{2}}\left(x_0^4 + y_0^2 + 1\right)^{\frac{1}{2}}} \\ &\quad + \frac{\left(x_0 y_0 + x_0^2 y_0\right)}{\left(x_0^4 + y_0^2\right)^{\frac{1}{2}}} \frac{\left(y_0,\, x_0^2,\, 0\right)}{\left(x_0^4 + y_0^2\right)^{\frac{1}{2}}} + \frac{\left(-\frac{2}{3}x_0^3 + \frac{1}{2}y_0^2\right)}{\left(x_0^4 + y_0^2 + 1\right)^{\frac{1}{2}}} \frac{\left(-x_0^2,\, y_0,\, 1\right)}{\left(x_0^4 + y_0^2 + 1\right)^{\frac{1}{2}}} \\ &= \left(x_0,\, y_0,\, \frac{1}{3}x_0^3 - \frac{1}{2}y_0^2\right) \\ &= \widetilde{\varphi}(x_0, y_0).\end{aligned}$$

By continuity, it follows that the given shoe surface $\widetilde{\varphi}$ itself is the unique envelope created by the tangent plane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$.

The set called the *parabolic line* of $\widetilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}^3$ consists of points $(x, y) \in \mathbb{R}^2$ at which $\widetilde{\nu}$ is singular. For the shoe surface, the parabolic line is the y-axis $\{(0, y) \,|\, y \in \mathbb{R}\}$. Thus, as similar as the case of unit speed plane curves $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ with inflection points, the full discriminant of the tangent plane family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ for the shoe surface $\widetilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}^3$ is different from the unique desired envelope $\widetilde{\varphi}$ itself, since the full discriminant includes an affine tangent line $\left\{ \left(\lambda, y, -\frac{1}{2}y^2\right) \;\middle|\; \lambda \in \mathbb{R}\right\}$ at any point $(0, y)$. Therefore, even in the case of surfaces in $\mathbb{R}^3$, by our method, one can distinguish the envelope in the sense of definition 1 and the full discriminant. This means that, in the case of surfaces in $\mathbb{R}^3$ as well, our method has an advantage.

(b) (Example 4.1 of [14].) Let $\widetilde{\nu} : \mathbb{R}^n \to S^n \subset \mathbb{R}^{n+1}$ be the mapping defined by $\widetilde{\nu}(p_1, \ldots, p_n) = \frac{1}{\sqrt{\sum_{i=1}^n p_i^2 + 1}}(p_1, \ldots, p_n, -1)$. Then, $\widetilde{\nu}$ is non-singular and its inverse mapping $\widetilde{\nu}^{-1} : \widetilde{\nu}\left(\mathbb{R}^{n+1}\right) \to \mathbb{R}^{n+1}$ is the central projection relative to the south pole $(0, \ldots, 0, -1)$ of S^n. Let $\widetilde{\varphi} : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be an arbitrary mapping. Set $\widetilde{\gamma}(p) = \widetilde{\varphi}(p) \cdot \widetilde{\nu}(p)$ where $p = (p_1, \ldots, p_n)$ be a point of $\mathbb{R}^{n+1}$. Let $(X = (X_1, \ldots, X_n)\,, Y)$ be a point of $\mathbb{R}^n \times \mathbb{R}$. Since $J^1(\mathbb{R}^n, \mathbb{R})$ and $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ are identified, (X, Y, p) may be regarded as the canonical coordinate system of $J^1(\mathbb{R}^n, \mathbb{R})$. Since $\frac{X_i \circ \widetilde{\nu}(p)}{Y \circ \widetilde{\nu}(p)} = -p_i$ for any i $(1 \leqslant i \leqslant n)$ and any $p \in \mathbb{R}^{n+1}$, considering the first order differential equation

$$((X, Y) - \widetilde{\varphi}(p)) \cdot \widetilde{\nu}(p) = 0$$

is exactly the same as considering the following Clairaut equation

$$Y = \sum_{i=1}^{n} X_i p_i + \frac{\widetilde{\varphi}(p) \cdot \widetilde{\nu}(p)}{Y \circ \widetilde{\nu}(p)}.$$

Thus, for each $x \in \mathbb{R}^n$ the hyperplane $H_{(\widetilde{\varphi}(x), \widetilde{\nu}(x))}$ is a complete solution of the above Clairaut equation. Since $\widetilde{\nu}$ is non-singular, by theorems 1 and 2, the above Clairaut equation has a unique singular solution $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^{n+1}$. By theorem 1 again, the unique singular solution $\widetilde{f}$ has the following expression where x is an arbitrary point of $\mathbb{R}^n$ and $(V, (\Theta_1, \ldots, \Theta_n))$ is a sufficiently small normal coordinate neighbourhood of S^n at $\widetilde{\nu}(x)$.

$$\widetilde{f}(x) = \sum_{i=1} \frac{\partial \left(\widetilde{\gamma} \circ \widetilde{\nu}^{-1}\right)}{\partial \Theta_i} (\widetilde{\nu}(x)) \frac{\partial}{\partial \Theta_i} + \widetilde{\gamma}(x)\widetilde{\nu}(x).$$

By this expression, for instance, it is easily seen that when $\widetilde{\gamma}(x) \equiv c(\neq 0)$ for any $x \in \mathbb{R}^{n+1}$, then the unique singular solution $Y : U_c \to \mathbb{R}$ must be an explicit solution with the following expression where $U_c = \{X \mid \|X\| < |c|\}$.

$$Y(X) = \begin{cases} -\sqrt{|c|^2 - \sum_{i=1}^{n} X_i^2} & (\text{if } c > 0) \\ \sqrt{|c|^2 - \sum_{i=1}^{n} X_i^2} & (\text{if } c < 0). \end{cases}$$

Acknowledgments

The author is most grateful to two anonymous reviewers. Their comments/suggestions are hard to replace. The author would like to thank Richard Montgomery for appropriate comments. His suggestions improved this paper.

Appendix. Alternative proof of theorem 1 except for the assertion (c) in the case $n = 1$

Let N be a one-dimensional manifold and let $\widetilde{\varphi} : N \to \mathbb{R}^2, \widetilde{\nu} : N \to S^1$ be mappings. Define the function $\widetilde{\Theta} : N \to \mathbb{R}$ by $\widetilde{\nu}(t) = \left(\cos \widetilde{\Theta}(t), \sin \widetilde{\Theta}(t)\right)$. Define also $\widetilde{\tau}(t) := \left(\sin \widetilde{\Theta}(t), -\cos \widetilde{\Theta}(t)\right)$. Then, the following trivially holds.

Fact A.1. For any $h : N \to \mathbb{R}^2$,

$$h(t) = (h(t) \cdot \widetilde{\tau}(t)) \widetilde{\tau}(t) + (h(t) \cdot \widetilde{\nu}(t)) \widetilde{\nu}(t).$$

We first show that the creative condition can be naturally obtained from an envelope by introducing a gauge theoretic approach. Suppose that $\widetilde{f} : N \to \mathbb{R}^2$ is an envelope created by the line family $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$. Then, we have the following.

$$\widetilde{\gamma}'(t) = \left(\widetilde{f}(t)\cdot\widetilde{\nu}(t)\right)' = \widetilde{f}'(t)\cdot\widetilde{\nu}(t) + \widetilde{f}(t)\cdot\widetilde{\nu}'(t) = 0 - \left(\widetilde{f}(t)\cdot\widetilde{\tau}(t)\right)\widetilde{\Theta}'(t).$$

Let $h : N \to N$ be a bijective mapping. Then, notice that

$$\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})} = \mathcal{H}_{(\widetilde{\varphi}\circ h,\widetilde{\nu}\circ h)}$$

and

$$(\widetilde{\gamma}\circ h)'(t) = -\left(\widetilde{f}(h(t))\cdot\widetilde{\tau}(h(t))\right)\widetilde{\Theta}'(h(t))h'(t).$$

From these simple observations, we see that it is important to extract a significant quantity which does not depend on the particular choice of h. Then, we naturally reach the following setting.

$$\widetilde{\omega}(t) := -\left(\widetilde{f}(t)\cdot\widetilde{\tau}(t)\right)\mathrm{d}\widetilde{\Theta}$$

and we trivially have $\mathrm{d}\widetilde{\gamma} = \widetilde{\omega}$. Take an arbitrary point t_0 of N and fix it. Let (V, Θ) be a normal coordinate neighbourhood of S^1 at $\widetilde{\nu}(t_0)$ such that $\Theta\left(\widetilde{\nu}(t_0)\right) = 0$ and $\widetilde{\Theta}(t) = \left(\Theta\circ\widetilde{\nu}\right)(t)$ for any $t \in \widetilde{\nu}^{-1}(V)$. In other words, $\left(\Theta\circ\widetilde{\nu}\right)(t)$ $\left(t \in \widetilde{\nu}^{-1}(V)\right)$ is just the radian (or its negative) between two unit vectors $\widetilde{\nu}(t_0)$ and $\widetilde{\nu}(t)$. By using the function $\Theta : V \to \mathbb{R}$, the one-form $\widetilde{\omega}(t)$ may be written as follows.

$$\widetilde{\omega}(t) = -\left(\widetilde{f}(t)\cdot\widetilde{\tau}(t)\right)\widetilde{\nu}^*\,\mathrm{d}\Theta,$$

where $\widetilde{\nu}^*\,\mathrm{d}\Theta$ stands for the pullback of the one-form $\mathrm{d}\Theta$ by $\widetilde{\nu}$. Hence, we naturally reach the following one-form which is denoted by the same symbol $\widetilde{\omega}$.

$$\widetilde{\omega}(t) = -\left(\widetilde{f}(t)\cdot\widetilde{\tau}(t)\right)\mathrm{d}\Theta.$$

It is easily seen that for any $t \in \widetilde{\nu}^{-1}(V)$, under the canonical identifications

$$T^*_{\widetilde{\nu}(t)}S^1 \cong T_{\widetilde{\nu}(t)}S^1 \subset T_{\widetilde{\nu}(t)}\mathbb{R}^2 \cong \mathbb{R}^2,$$

the one-dimensional cotangent vector

$$\widetilde{\omega}(t) = -\left(\widetilde{f}(t)\cdot\widetilde{\tau}(t)\right)\mathrm{d}\Theta \in T^*_{\widetilde{\nu}(t)}S^1$$

is identified with the two-dimensional vector

$$\widetilde{\omega}(t) = \left(\widetilde{f}(t)\cdot\widetilde{\tau}(t)\right)\widetilde{\tau}(t) \in \mathbb{R}^2.$$

Since t_0 is an arbitrary point of N, we naturally see that the creative condition is satisfied for $\mathcal{H}_{\left(\widetilde{\varphi,\nu}\right)}$ and the following horizontal–vertical decomposition formula holds for any $t \in N$.

Fact A.2.

$$\widetilde{f}(t) = \left(\widetilde{f}(t)\cdot\widetilde{\tau}(t)\right)\widetilde{\tau}(t) + \left(\widetilde{f}(t)\cdot\widetilde{\nu}(t)\right)\widetilde{\nu}(t) = \widetilde{\omega}(t) + \widetilde{\gamma}(t)\widetilde{\nu}(t).$$

Conversely, suppose that $\mathcal{H}_{(\widetilde{\varphi},\widetilde{\nu})}$ is creative. Then, there exists a function $\alpha : N \to \mathbb{R}$ such that $\mathrm{d}\widetilde{\gamma} = \alpha\, \mathrm{d}\widetilde{\Theta}$. Set $\widetilde{\omega} = \alpha\, \mathrm{d}\widetilde{\Theta}$. Let $t_0 \in N$ be an arbitrary point. Then, under the canonical identifications

$$T^*_{\widetilde{\nu}(t)}S^1 \cong T_{\widetilde{\nu}(t)}S^1 \subset T_{\widetilde{\nu}(t)}\mathbb{R}^2 \cong \mathbb{R}^2,$$

the one-dimensional cotangent vector

$$\widetilde{\omega}(t) = \alpha(t)\mathrm{d}\Theta \in T^*_{\widetilde{\nu}(t)}S^1$$

is identified with the two-dimensional vector

$$\widetilde{\omega}(t) = -\alpha(t)\widetilde{\tau}(t) \in \mathbb{R}^2,$$

where (V, Θ) is a normal coordinate system of S^1 at $\widetilde{\nu}(t_0)$ such that $\Theta(\widetilde{\nu}(t_0)) = 0$ and $t \in \widetilde{\nu}^{-1}(V)$. Set

$$\widetilde{f}(t) = \widetilde{\omega}(t) + \widetilde{\gamma}(t)\widetilde{\nu}(t) = -\alpha(t)\widetilde{\tau}(t) + \widetilde{\gamma}(t)\widetilde{\nu}(t).$$

Then, $\widetilde{f}$ clearly satisfies the condition (b) of definition 1 for any $t \in \widetilde{\nu}^{-1}(V)$. Moreover we have the following.

Lemma A.1. *For any $t \in \widetilde{\nu}^{-1}(V)$, $\widetilde{f}'(t) \cdot \widetilde{\nu}(t) = 0$ holds.*

Proof of lemma A.1. We have

$$\widetilde{\gamma}'(t) = \left(\widetilde{f}(t) \cdot \widetilde{\nu}(t)\right)' = \widetilde{f}'(t) \cdot \widetilde{\nu}(t) - \left(\widetilde{f}(t) \cdot \widetilde{\tau}(t)\right) \widetilde{\Theta}'(t) = \widetilde{f}'(t) \cdot \widetilde{\nu}(t) + \alpha(t)\widetilde{\Theta}'(t).$$

Thus, we have the following.

$$\begin{aligned}\widetilde{\omega}(t) = \mathrm{d}\widetilde{\gamma} = \widetilde{\gamma}'(t)\mathrm{d}t &= \left(\widetilde{f}'(t) \cdot \widetilde{\nu}(t)\right) \mathrm{d}t + \alpha(t)\widetilde{\Theta}'(t)\mathrm{d}t \\ &= \left(\widetilde{f}'(t) \cdot \widetilde{\nu}(t)\right) \mathrm{d}t + \alpha(t)\mathrm{d}\widetilde{\Theta} \\ &= \left(\widetilde{f}'(t) \cdot \widetilde{\nu}(t)\right) \mathrm{d}t + \widetilde{\omega}(t).\end{aligned}$$

It follows $\left(\widetilde{f}'(t) \cdot \widetilde{\nu}(t)\right) \mathrm{d}t = 0$. Since t is a coordinate function on an open set $\widetilde{\nu}^{-1}(V)$ of N, for any fixed $t \in \widetilde{\nu}^{-1}(V)$, the one-dimensional cotangent vector $\mathrm{d}t$ at t is not zero. Therefore, the number $\left(\widetilde{f}'(t) \cdot \widetilde{\nu}(t)\right)$ is always zero for any $t \in \widetilde{\nu}^{-1}(V)$. Since t_0 is an arbitrary point of N, theorem 1(a) holds. By the above decomposition of $\widetilde{f}(t)$, theorem 1(b) holds as well. □

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