The Interest Rate Determination when Economic Variables are Partially Observable*

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January, 2021

Abstract

While recent studies based on factor models with no-arbitrage restrictions provide evidence of a positive correlation between the nominal interest rates and real activity, there are few dynamic general equilibrium models which can successfully explain this positive relationship. This paper provides a dynamic general equilibrium model that naturally generates a positive correlation between the nominal interest rates and excess consumption. To this end, we focus on the partial observability of economic variables in a pure exchange economy and derive closed form solutions for the equilibrium interest rates. Our empirical analysis based on the results indeed indicates the positive correlation between the nominal interest rates and excess consumption. Moreover, the time series of the model-implied nominal yield captures many of the short- and long-run fluctuations in the actual data.

JEL classification: D51, E43, G12

Key Words: Term structures, Incomplete information, Dynamic general equilibrium, Exchange economy, Kalman filter

*The authors would like to thank the editor and two anonymous referees for there constructive comments and suggestions, which improve the paper considerably. We also thank participants at the YNU and Nanzan University finance workshop, the Daiwa Young Researchers' International Workshop on Finance, the Annual meeting of Asian Financial Association, and Paris Financial Management Conference, and seminar participants at the University of Tokyo and Bank of Japan for their helpful comments and suggestions. Declarations of interest: none.

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1. Introduction

The study of the joint behavior of the term structures of interest rates and macroeconomic variables has drawn much attention from theorists, practitioners, and policymakers. In particular, a growing body of literature has analyzed the relation between yield curve and real economy based on factor models with no-arbitrage restrictions. For instance, Ang and Piazzesi (2003) construct a real economic factor by extracting the first principal component from real activity measures, including the index of Help Wanted Advertising in Newspapers (HELP) and industrial production growth. Along with many other findings, they demonstrate that nominal interest rates positively react to real economic factor shocks. In addition, Bikbov and Chernov (2010) use HELP as a proxy for real activity, and suggest the existence of a positive relationship between the nominal interest rates and real activity. Further, Ang, Don, and Piazzesi (2007) estimate the Taylor rule with no-arbitrage restrictions, indicating that the nominal short rate increases after a positive shock to the output gap.

While these studies provide evidence of a positive correlation between the nominal interest rates and real activity, few dynamic general equilibrium models can successfully explain this positive relationship. For instance, following Campbell and Cochrane (1999), Wachter (2006) explores a consumption-based model of the term structure of interest rates with habit persistence. Although her model effectively captures many features of the yield curve, including the upward sloping yield curve and the failure of expectations hypothesis, the results imply a negative correlation between the nominal and real interest rates and real economic activity.[1] Further, Buraschi and Jiltsov (2007) propose a new class of non-affine models that link the macroeconomic variables and the yield curve when preferences are subject to habit persistence. Similar to Wachter's model, their model can reproduce many characteristics of the term structures of interest rates but suggests a negative correlation between the real economic activity and interest rates. More precisely, models with habit formation, like that of Wachter (2006) and Buraschi and Jiltsov (2007), can produce a positive or negative correlation between interest rates and real economic activity. However, they require a negative correlation between them to fit an upward sloping yield curve, which is usually observed in reality. In this sense, the models with habit persistence are appropriate for explaining the upward sloping yield curve but not the comovements between interest rates

[1] The measure for real economic activity in Wachter (2006) is surplus consumption, which is defined as the consumption over the habit normalized by the current consumption level and close to the excess consumption defined as the (log of) consumption level over the weighted average of the past consumption, which is the measure of real economic activity in this paper.

and real activity.

The main objective of this paper is to provide an alternative dynamic general equilibrium model that naturally generates a positive correlation between the nominal interest rates and real economic activity without sacrificing an upward sloping yield curve.[2] To this end, we focus on the partial observability of economic variables in a pure exchange economy, where the aggregate endowment and its price follow a system of Gaussian processes. In a complete information model of pure exchange economy, where the aggregate endowment follows a Gaussian process, the instantaneous expected rate of change in endowments is the state variable of the term structure of interest rates. On the other hand, in the real world, the agents in the economy cannot observe the expected change in income. Rather, they infer the expected change in income based on available information, including the past income stream. This is also true for the price level. From this point of view, we set up a general equilibrium model, where the level of aggregate endowment and its price are observable but their instantaneous expected growth rates are not.

Several studies have explored the role of partially observed income on consumption. For instance, Wang (2004) considers the optimal consumption rule when the agent can only observe her total income, and not individual components, and shows that partial observability of individual components of income gives rise to additional precautionary saving due to estimation risk. Guvenen (2007) proposes two stochastic income processes for lifecycle consumption behavior, and provides a systematic comparison of these implications on US data. In addition, Wang (2009) investigates an individual's optimal consumption-saving and portfolio choice problem with unobservable income growth. Unlike these works, this paper examines the market equilibrium interest rates when some economic variables are partially observable.

The effects of an unobservable factor on the market equilibrium have also been studied by a number of researchers. For instance, Detemple (1986), Dothan and Feldman (1986), and Björk, Davis, and Landen (2010) study an economy wherein an unobservable state variable exists, but these works concentrate on methodological aspects. Other studies have focused on the term structure of interest rates and have investigated the functional relationship between the interest rates and economic agents' estimates of an unobservable factor. These include Feldman (1989), Feldman (2003), and Riedel (2000a, 2000b). The present paper differs from these previous studies in two important ways. First, our model includes two

[2]Many of the aforementioned previous papers have analyzed the nominal interest rates. Following those studies, we focus on nominal interest rates in this paper.

unobservable factors, but we still derive a closed-form solution for the equilibrium interest rates. As a consequence, our model can include the two most important economic variables: income (consumption) growth and inflation rates. This is an important extension because it makes the term structure behaviors more realistic without losing tractability. Second, this study investigates the functional relation between the estimates of unobservable factors and observable variables in detail to obtain the equilibrium interest rates as a function of observable variables. This is relevant, as we are able to directly see the relation between the interest rates and the observable macroeconomic variables.

The contributions of the paper are summarized as follows. First, we derive closed-form solutions for nominal equilibrium interest rates. In our model of partial observability, the resulting nominal term structure model turns out to be a two-factor purely Gaussian affine model in which the state variables are the economic agents' estimates of instantaneous expected growth rates of endowment and its price. In addition, with the stationary error process assumption, we show that these state variables can be expressed as weighted sums of excess consumption and price level.[3] To the best of our knowledge, this characterization of the agents' estimates is not found in the relevant literature.

Second, with the characterization mentioned above, our model with partial observability demonstrates an additional role of excess consumption in determining interest rates compared with the surplus consumption in the consumption habit models. For economic agents engaging in the Bayesian inference, the excess consumption is not only the surplus consumption giving them felicity, but also constitutes an economic indicator of real economic activity, thereby helping infer the current trend in income growth. The intuition behind the positive relationship between nominal interest rates and real economic activity is as follows. The standard argument in microeconomics implies that positive excess consumption induces lower current marginal utility. However, positive excess consumption also brings agents an expectation that they may enjoy a much higher income growth in the next period. By engaging in the Bayesian inference, the agents update their estimate of the income growth trend, which motivates them to save less or borrow more. As a result, the market clearing interest rates are determined at a higher level. As we show in the following sections, agents tend to infer the higher expected rate of income growth even when the expected rate of consumption growth and the level of consumption are negatively correlated under some mild conditions on parameters. Moreover, when the positive excess price level is observed, agents update

[3] Based on this result, we consider excess consumption as a measure of real economic activity in this paper.

their estimate of the expected rate of inflation. This motivates agents to save less or borrow more, since a higher price level in the future makes the relative price of future consumption more expensive. Thus, a positive excess price level induces higher nominal interest rates.

Third, we conduct an empirical analysis based on our model. This is an important contribution because previous papers on incomplete information equilibrium have mostly focused on theoretical issues, but not empirical implementations. The parameters for the system of stochastic differential equations are estimated from real consumption and inflation data, while the preference parameters are estimated using the time series of interest rates. These results indicate reasonable values for all parameters and, more importantly, the positive correlation between the implied nominal interest rates and excess consumption. In addition, the time series of the nominal yield implied by the model captures many of the short- and long-run fluctuations in the actual data for all maturities, particularly the short term.

This paper is organized as follows. Section 2 describes a pure exchange economy where the representative agent exists. Section 3 investigates how the level of endowment affects the estimate of expected rate of change in endowment and shows that the equilibrium interest rates can be monotonically increasing in excess consumption. Section 4 explains data and discusses the empirical results, and then Section 5 concludes.

2. Model

Consider a pure exchange economy of a single perishable consumption good. The time span of this economy is $[t_0, \tau]$. Let $(\Omega, \mathcal{F}, Q)$ be a complete probability space. Filtration $\{\mathcal{F}_t : t \in [t_0, \tau]\}$ denotes the $Q-$augmentation of natural filtration generated by four Brownian motions, $W_{y,t}$, $W_{p,t}$, $\widehat{W}_{y,t}$, and $\widehat{W}_{p,t}$. These Brownian motions are mutually independent except that $\widehat{W}_{y,t}$, $\widehat{W}_{p,t}$ are correlated. The correlation between $\widehat{W}_{y,t}$ and $\widehat{W}_{p,t}$ is described by $E\left(d\widehat{W}_{y,t}d\widehat{W}_{p,t}\middle|\mathcal{F}_t\right) = \hat{\rho}dt$ where $\hat{\rho}$ is constant.

The economy is endowed with a flow of the consumption good. The rate of aggregate endowment flow in real term and its price are denoted as y_t and p_t, $t \in [t_0, \tau]$. In this paper, it is assumed that y_t, p_t, and their instantaneous expected change $\mu_{i,t}$ $(i = y, p)$ follow the system of stochastic differential equations,

$$\frac{dy_t}{y_t} = \mu_{y,t}dt + \sigma_y dW_{y,t}, \tag{1}$$

$$d\mu_{y,t} = \kappa_y(\bar{\mu}_y - \mu_{y,t})dt + \upsilon_y dW_{y,t} + \hat{\upsilon}_y d\widehat{W}_{y,t}, \tag{2}$$

$$\frac{dp_t}{p_t} = \mu_{p,t}dt + \sigma_p dW_{p,t}, \tag{3}$$

$$d\mu_{p,t} = \kappa_p(\bar{\mu}_p - \mu_{p,t})dt + \upsilon_p dW_{p,t} + \hat{\upsilon}_p d\widehat{W}_{p,t}, \tag{4}$$

where κ_i, $\bar{\mu}_i$, σ_i, $\hat{\upsilon}_i$ $(i = y, p)$ are positive constants. We do not restrict the sign of constants, υ_i $(i = y, p)$, allowing for negative correlation between changes in level and instantaneous expected change. Although infinitesimal changes in endowment and price are independent, this does not necessarily mean that discrete time changes in both variables are independent since $\mu_{y,t}$ and $\mu_{p,t}$ are correlated.[4]

Throughout this paper, we consider a model where y_t and p_t are observable but $\mu_{y,t}$ and $\mu_{p,t}$ are not. It is assumed that the true value for each parameter is known to all of the agents. Thus, the agents infer $\mu_{y,t}$ and $\mu_{p,t}$, given the history of endowment and price levels up to time t.

Let us denote the $Q-$augmentation of natural filtration generated by y_t and p_t as $\{\mathcal{F}_t^{y,p} : t \in [t_0, \tau]\}$. It is assumed that the distribution of μ_{y,t_0} and μ_{p,t_0} conditional on $\mathcal{F}_{t_0}^{y,p}$ is normal. This is an important assumption for optimal filtering used in this paper. The estimates of $\mu_{y,t}$ and $\mu_{p,t}$ inferred by economic agents are denoted as $m_{y,t}$ and $m_{p,t}$. That is, $m_{i,t} = E\left(\mu_{i,t} \middle| \mathcal{F}_t^{y,p}\right)$ $(i = y, p)$. Covariances of filtering errors are denoted as $\phi_{ij,t} = E\left[(\mu_{i,t} - m_{i,t})(\mu_{j,t} - m_{j,t}) \middle| \mathcal{F}_t^{y,p}\right]$ $(i, j = y, p)$.

It is assumed that the representative agent exists and she is assumed to have preference over the consumption flows,

$$E\left[\int_{t_0}^{\tau} u(c_s, s)ds \middle| \mathcal{F}_{t_0}^{y,p}\right],$$

where felicity function is defined by $u(c_t, t) = e^{-\delta(t-t_0)}\frac{c_t^{1-\gamma}}{1-\gamma}$ for $\gamma > 0, \gamma \neq 1$ or $u(c_t, t) = e^{-\delta(t-t_0)} \ln c_t$ for $\gamma = 1$. It is also assumed that the market is frictionless and the finite number of securities in zero net supply are traded continuously in time. The rigorous formulation of economy and optimization problem for representative agent which lead to obtaining the nominal bond prices are shown in the appendix.

[4]Under the general local covariance structure, we can only numerically solve the matrix Riccati equation satisfied by the filtering error defined below. Our local covariance structure allows us to solve the matrix Riccati equation analytically and give an economic interpretation to bond prices.

3. Term Structure of Interest Rates in Equilibrium

3.1. Estimation problem for the representative agent

The representative agent draws inferences about $\mu_{i,t}$ $(i = y, p)$ by observing y_t and p_t. She forms a posterior distribution and continuously update it. Let us denote the filtering error process as $\Phi_t = [\phi_{ij,t}]$ $(i, j = y, p)$. Our model implies that the process $\overline{W}_t = \left(\overline{W}_{y,t}, \overline{W}_{p,t}\right)$ defined below innovates estimates $m_{i,t}$ of $\mu_{i,t}$ $(i = y, p)$, and $\overline{W}_t$, $m_{i,t}$ $(i = y, p)$, and Φ_t are the unique continuous $\mathcal{F}_t^{y,p}$-measurable solutions of equations:[5]

$$
\begin{aligned}
dm_{y,t} &= \kappa_y(\bar{\mu}_y - m_{y,t})dt + \left(\upsilon_y + \frac{\phi_{yy,t}}{\sigma_y}\right) d\overline{W}_{y,t} + \frac{\phi_{yp,t}}{\sigma_p} d\overline{W}_{p,t}, && (5)\\
dm_{p,t} &= \kappa_p(\bar{\mu}_p - m_{p,t})dt + \frac{\phi_{py,t}}{\sigma_y} d\overline{W}_{y,t} + \left(\upsilon_p + \frac{\phi_{pp,t}}{\sigma_p}\right) d\overline{W}_{p,t}, && (6)\\
d\overline{W}_{y,t} &= \left(\frac{1}{\sigma_y}\right)\left(d\ln y_t - \left(m_{y,t} - (1/2)\sigma_y^2\right) dt\right), && (7)\\
d\overline{W}_{p,t} &= \left(\frac{1}{\sigma_p}\right)\left(d\ln p_t - \left(m_{p,t} - (1/2)\sigma_p^2\right) dt\right), && (8)\\
\frac{d}{dt}\Phi_t &= K\Phi_t + \Phi_t K^\top - \Phi_t G \Phi_t^\top + H, && (9)
\end{aligned}
$$

where each element in 2×2 matrix $K = [k_{ij}], H = [h_{ij}], G = [g_{ij}]$ $(i, j = 1, 2)$ is defined as

$$
\begin{aligned}
k_{11} &= -\kappa_y - \upsilon_y/\sigma_y,\ k_{12} = k_{21} = 0,\ k_{22} = -\kappa_p - \upsilon_p/\sigma_p,\\
h_{11} &= \hat{\upsilon}_y^2,\ h_{12} = h_{21} = \hat{\upsilon}_y\hat{\upsilon}_p\hat{\rho},\ h_{22} = \hat{\upsilon}_p^2,\\
g_{11} &= 1/\sigma_y^2,\ g_{12} = g_{21} = 0,\ g_{22} = 1/\sigma_p^2.
\end{aligned}
$$

Innovation process $\left\{\overline{W}_t : t \in [t_0, \tau]\right\}$ is endogenously determined to be a vector of independent Wiener processes. Since we assume that the distribution of μ_{i,t_0} $i = y, p$ conditional on $\mathcal{F}_{t_0}^{y,p}$ is normal, the innovation process generates the economy, that is, $\mathcal{F}_t^{y,p} = \mathcal{F}_t^{\overline{W}, y_{t_0}, p_{t_0}}$ for all $t \in [t_0, \tau]$ where $\{\mathcal{F}_t^{\overline{W}, y_{t_0}, p_{t_0}} : t \in [t_0, \tau]\}$ is Q-augmentation of natural filtration generated by y_{t_0}, p_{t_0}, and $\left\{\overline{W}_t : t \in [t_0, \tau]\right\}$.[6]

By solving the matrix Riccati equation (9), we can obtain the stationary level of filtering error, $\lim_{t\to\infty}\Phi_t = \overline{\Phi} = [\bar{\phi}_{ij}]$ $(i, j = y, p)$, as[7]

$$
\begin{aligned}
\bar{\phi}_{yy} &= (\kappa_y^* - \kappa_y)\sigma_y^2 - \upsilon_y\sigma_y,\ \bar{\phi}_{pp} = (\kappa_p^* - \kappa_p)\sigma_p^2 - \upsilon_p\sigma_p,\\
\bar{\phi}_{yp} &= \bar{\phi}_{py} = \frac{\hat{\upsilon}_y\hat{\upsilon}_p\hat{\rho}}{\lambda_1 + \lambda_2},
\end{aligned}
$$

[5]See Proposition 12.7 in Liptser and Shiryaev (2001).
[6]See Proposition 12.5 in Liptser and Shiryaev (2001).
[7]The derivation of the solution and its limit is given in the appendix.

where[8]

$$
\begin{aligned}
\kappa_y^* &= \frac{k_{11}^2 + h_{11}/\sigma_y^2 + \lambda_1\lambda_2}{\lambda_1+\lambda_2}, \kappa_p^* = \frac{k_{22}^2 + h_{22}/\sigma_p^2 + \lambda_1\lambda_2}{\lambda_1+\lambda_2}, \\
\lambda_1 &= \left(\frac{(k_{11}^2 + h_{11}/\sigma_y^2) + (k_{22}^2 + h_{22}/\sigma_p^2) + D^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}, \\
\lambda_2 &= \left(\frac{(k_{11}^2 + h_{11}/\sigma_y^2) + (k_{22}^2 + h_{22}/\sigma_p^2) - D^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}, \\
D &= \left((k_{11}^2 + h_{11}/\sigma_y^2) - (k_{22}^2 + h_{22}/\sigma_p^2)\right)^2 + \frac{4h_{12}^2}{\sigma_y^2\sigma_p^2}.
\end{aligned}
$$

It is easy to show that κ_y^* and κ_p^* are strictly positive.

3.2. Time homogeneous model of term structure of interest rates

In our economy, the representative agent solves an optimization problem using the filter equations in the previous subsection. As a result, the equilibrium nominal bond prices is given by,[9]

$$
B(t,T) = \frac{E\left[u_c(y_T,T)/p_T | \mathcal{F}_t^{\overline{W},y_{t_0},p_{t_0}}\right]}{u_c(y_t,t)/p_t}. \tag{10}
$$

So far, initial values of filtering error ϕ_{ij,t_0} and estimates m_{i,t_0} can be any arbitrary numbers. However, arbitrary choice of initial values generates some deterministic components in the process of filtering error and estimates. To have a simple structure in our model, we remove these components in our analysis. This enable us to have a time homogeneous model of term structure. For this purpose, we set the following assumption.

Assumption 1 *The initial value of filtering error process is given by its stationary level. That is,* $\phi_{ij,t_0} = \bar{\phi}_{ij}$ *for* $i,j = y,p$.

Obviously, the assumption implies that $\phi_{ij,t} = \bar{\phi}_{ij}, (i,j = y,p)$ for any $t \geq t_0$.

Under Assumption 1, the term structure of interest rates has the form of time homogeneous models in the sense that it depend on t only through time to maturity $s - t$.

[8]A sufficient condition for λ_2 to be a real number is $h_{11}h_{22} - h_{12}^2 \geq 0$. But this inequality always holds. Thus, λ_2 is a real number.

[9]In the appendix, re-representation of the optimization problem and derivation of equilibrium bond prices in the economy is stated.

Proposition 1 *Under Assumption* 1, *the bond price in equilibrium is given by* $B(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$ *where instantaneous forward rates* $f(t,s)$ *are given by*

$$\begin{aligned}
f(t,s) &= \delta + \gamma\left(m_{y,t}e^{-\kappa_y(s-t)} + \bar{\mu}_y\left(1-e^{-\kappa_y(s-t)}\right) - \sigma_y^2/2\right) - \gamma^2\sigma_y^2/2 \\
&\quad + m_{p,t}e^{-\kappa_p(s-t)} + \bar{\mu}_p\left(1-e^{-\kappa_p(s-t)}\right) - \sigma_p^2 \\
&\quad - \left(\frac{1-e^{-\kappa_y(s-t)}}{\kappa_y}\right)\left(\gamma^2\sigma_y^2(\kappa_y^* - \kappa_y) + \gamma\bar{\phi}_{py}\right) \\
&\quad - \left(\frac{1-e^{-\kappa_p(s-t)}}{\kappa_p}\right)\left(\sigma_p^2(\kappa_p^* - \kappa_p) + \gamma\bar{\phi}_{yp}\right) \\
&\quad - \frac{1}{2}\gamma^2\left(\frac{1-e^{-\kappa_y(s-t)}}{\kappa_y}\right)^2\left(\sigma_y^2(\kappa_y^*-\kappa_y)^2 + \left(\frac{\bar{\phi}_{yp}}{\sigma_p}\right)^2\right) \\
&\quad - \frac{1}{2}\left(\frac{1-e^{-\kappa_p(s-t)}}{\kappa_p}\right)^2\left(\sigma_p^2(\kappa_p^*-\kappa_p)^2 + \left(\frac{\bar{\phi}_{py}}{\sigma_y}\right)^2\right) \\
&\quad - \gamma\left(\frac{1-e^{-\kappa_y(s-t)}}{\kappa_y}\right)\left(\frac{1-e^{-\kappa_p(s-t)}}{\kappa_p}\right) \\
&\qquad \times\left((\kappa_y^*-\kappa_y)\bar{\phi}_{py} + (\kappa_p^*-\kappa_p)\bar{\phi}_{yp}\right).
\end{aligned} \tag{11}$$

Thus, our equilibrium model can be identified as a two-factor completely affine term structure model where state variables $m_{y,t}$ and $m_{p,t}$ follow (5) and (6) with each filtering error replaced by its stationary level and the market prices of risk for $\overline{W}_{y,t}$ and $\overline{W}_{p,t}$ are $\gamma\sigma_y$ and σ_p.[10] As in Feldman (1989), we can decompose the instantaneous forward rate into three parts. The sum of the first two lines in (11) is the expectation of future short rate, and the sum of the next two lines is the risk premium. The remaining terms correspond to the Jensen's inequality bias.

Let us consider the initial date of economy as a distant time in the past, i.e. let t_0 go to $-\infty$. Then, the effect of initial values y_{t_0}, p_{t_0}, m_{t_0} tends to diminish and state variables do not depend on calender time in the limit. More importantly, they are shown to be a function of excess consumption and excess price level. The next proposition states this.

Proposition 2 *Under Assumption* 1, *the stationary processes of* $m_{y,t}$ *and* $m_{p,t}$ *are deter-*

[10]The term structure in our model does not explode as in Riedel (2000a). This confirms the results of Feldman (2003). Also, since the market prices of risk are constant, the equilibrium avoids the pitfalls identified in Kraft (2009).

mined by excess consumption and excess price level as

$$
\begin{aligned}
\lim_{t_0 \to -\infty} m_{y,t} &= \frac{\kappa_p^* b_y - (\bar{\phi}_{yp}/\sigma_p^2) b_p}{a_1 a_2} \\
&+ \frac{(a_1 - \kappa_y)(\kappa_y^* - a_2)}{a_1 - a_2} \left(\ln y_t - \int_{-\infty}^{t} a_1 e^{-a_1(t-s)} \ln y_s \, ds \right) \\
&- \frac{(a_2 - \kappa_y)(\kappa_y^* - a_1)}{a_1 - a_2} \left(\ln y_t - \int_{-\infty}^{t} a_2 e^{-a_2(t-s)} \ln y_s \, ds \right) \\
&+ \frac{(a_1 - \kappa_p)(\bar{\phi}_{yp}/\sigma_p^2)}{a_1 - a_2} \left(\ln p_t - \int_{-\infty}^{t} a_1 e^{-a_1(t-s)} \ln p_s \, ds \right) \\
&- \frac{(a_2 - \kappa_p)(\bar{\phi}_{yp}/\sigma_p^2)}{a_1 - a_2} \left(\ln p_t - \int_{-\infty}^{t} a_2 e^{-a_2(t-s)} \ln p_s \, ds \right), \qquad (12) \\
\lim_{t_0 \to -\infty} m_{p,t} &= \frac{\kappa_y^* b_p - (\bar{\phi}_{py}/\sigma_y^2) b_y}{a_1 a_2} \\
&+ \frac{(a_1 - \kappa_p)(\kappa_p^* - a_2)}{a_1 - a_2} \left(\ln p_t - \int_{-\infty}^{t} a_1 e^{-a_1(t-s)} \ln p_s \, ds \right) \\
&- \frac{(a_2 - \kappa_p)(\kappa_p^* - a_1)}{a_1 - a_2} \left(\ln p_t - \int_{-\infty}^{t} a_2 e^{-a_2(t-s)} \ln p_s \, ds \right) \\
&+ \frac{(a_1 - \kappa_y)(\bar{\phi}_{py}/\sigma_y^2)}{a_1 - a_2} \left(\ln y_t - \int_{-\infty}^{t} a_1 e^{-a_1(t-s)} \ln y_s \, ds \right) \\
&- \frac{(a_2 - \kappa_y)(\bar{\phi}_{py}/\sigma_y^2)}{a_1 - a_2} \left(\ln y_t - \int_{-\infty}^{t} a_2 e^{-a_2(t-s)} \ln y_s \, ds \right), \qquad (13)
\end{aligned}
$$

where parameters a_1, a_2, b_y, b_p are defined as

$$
\begin{aligned}
a_1 &= \frac{\kappa_y^* + \kappa_p^* + \sqrt{(\kappa_y^* - \kappa_p^*)^2 + 4(\bar{\phi}_{yp}/\sigma_p^2)(\bar{\phi}_{py}/\sigma_y^2)}}{2}, \\
a_2 &= \frac{\kappa_y^* + \kappa_p^* - \sqrt{(\kappa_y^* - \kappa_p^*)^2 + 4(\bar{\phi}_{yp}/\sigma_p^2)(\bar{\phi}_{py}/\sigma_y^2)}}{2}, \\
b_y &= \kappa_y \bar{\mu}_y + \frac{1}{2} \left(\upsilon_y \sigma_y + \bar{\phi}_{yy} + \bar{\phi}_{yp} \right), \\
b_p &= \kappa_p \bar{\mu}_p + \frac{1}{2} \left(\upsilon_p \sigma_p + \bar{\phi}_{pp} + \bar{\phi}_{py} \right).
\end{aligned}
$$

Since a_1 and a_2 are strictly positive, the first and second integral in the right hand side of (12) (or the third and fourth integral in (13)) can be interpreted as weighted averages of the past (natural log of) aggregate consumptions where heavy weights are put on the recent consumptions. Thus, if time t is sufficiently distant from the initial date of the economy t_0, then the estimation value of $\mu_{y,t}$ can be approximated by an affine function of excess aggregate consumption.

Are the interest rates positively related to the excess consumption? Since instantaneous forward rates are increasing in $m_{y,t}$ and $m_{p,t}$, this question is equivalent to asking whether $m_{y,t}$ and $m_{p,t}$ are increasing in the excess consumption. To answer this question, let us consider the special case of $\hat{\rho} = 0$ at first. This argument helps us understand the general case of non-zero correlation. If $\hat{\rho} = 0$, then (12) and (13) are reduced to

$$\lim_{t_0 \to -\infty} m_{y,t} = \frac{b_y}{\kappa_y^*} + (\kappa_y^* - \kappa_y) \left(\ln y_t - \int_{-\infty}^{t} \kappa_y^* e^{-\kappa_y^*(t-s)} \ln y_s ds \right), \tag{14}$$

$$\lim_{t_0 \to -\infty} m_{p,t} = \frac{b_p}{\kappa_p^*} + (\kappa_p^* - \kappa_p) \left(\ln p_t - \int_{-\infty}^{t} \kappa_p^* e^{-\kappa_p^*(t-s)} \ln p_s ds \right). \tag{15}$$

Thus, the interest rates depend on the excess consumption only through $\lim_{t_0 \to -\infty} m_{y,t}$ in the case of $\hat{\rho} = 0$. Let us define the correlation coefficient between changes in y_t and $\mu_{y,t}$ as ρ_{y,μ_y}, that is, $E\left[d\ln y_t \, d\mu_{y,t} \middle| \mathcal{F}_t \right] = \rho_{y,\mu_y} dt$. Then we can establish the following proposition.

Proposition 3 *If $\hat{\rho} = 0$, $\lim_{t_0 \to -\infty} m_{y,t}$ is increasing in the excess consumption if and only if the following inequality holds*

$$\rho_{y,\mu_y} \geq -\frac{\left(\upsilon_y^2 + \hat{\upsilon}_y^2\right)^{\frac{1}{2}}}{2\kappa_y \sigma_y}. \tag{16}$$

Note that the numerator of the right hand side of (16) is the volatility of $\mu_{y,t}$. The interesting property of the condition (16) is that the lower bound for the correlation coefficient is strictly negative. This is the important effect of unobservability of $\mu_{y,t}$ on the equilibrium interest rates. To understand this, let us consider the case in which the inequalities

$$-\frac{\upsilon_y^2 + \hat{\upsilon}_y^2}{2\kappa_y} < \upsilon_y \sigma_y < 0$$

hold. The second inequality means that changes in μ_t and y_t are locally negatively correlated. But changes in $m_{y,t}$ is increasing in the excess consumption since the condition in Proposition 3 is met by the first inequality. This interesting result holds, because $\mu_{y,t}$ is unobservable and an increase in the excess consumption, for instance, makes agents infer that $\mu_{y,t}$ has become high even under the negative correlation between changes in $\mu_{y,t}$ and y_t.

As a corollary, we can state a sufficient condition for $m_{y,t}$ to be increasing in the excess consumption for any correlation coefficient ρ_{y,μ_y}.

Corollary 1 *Suppose that $\hat{\rho} = 0$ and the following condition is met;*

$$\kappa_y \leq \frac{1}{2} \left(\upsilon_y^2 + \hat{\upsilon}_y^2\right)^{\frac{1}{2}} / \sigma_y. \tag{17}$$

Then, $\lim_{t_0 \to -\infty} m_{y,t}$ is increasing in the excess consumption for any correlation coefficient $\rho_{y,\mu_y} \in [-1, 1]$.

The fraction in the right hand side of (17) is the volatility ratio which measures the relative size of the volatility of $\mu_{y,t}$ to the volatility of y_t. When this ratio is large, it is likely that the variation in the level of consumption is mainly caused by the changes in $\mu_{y,t}$. Hence, the estimates are likely to be increasing in the level of consumption. Of course, when the speed of mean reversion is very fast, the past observed consumption levels are not informative, because the drift converges to its long run mean immediately. Hence, κ_y emerges in the left hand side of (17) as the bound for the volatility ratio.

One of the major approaches to asset pricing today is pricing risky assets via consumption habit. In this approach, consumption habit is incorporated into the preference on consumption flows, and prices of risky assets in equilibrium are derived. One of the important properties common to these models is that equilibrium asset returns or interest rates are determined by surplus consumption typically defined as the consumption over the habit normalized by the current consumption level. Although the models with habit formation, like Wachter (2006) and Buraschi and Jiltsov (2007), can produce a positive or negative correlation between interest rates and the real economic activity but they require a negative correlation between them to fit an upward sloping yield curve, which is usually observed in reality. Therefore, it is difficult to explain the both upward sloping yield curve and negative correlation between interest rates and real economic activity. In this case, partial observability of economic variables may become a candidate which explains both aspects of the interest rates through excess consumption.

In the general case of $\hat{\rho} \neq 0$, both estimates $\lim_{t_0 \to -\infty} m_{y,t}$ and $\lim_{t_0 \to -\infty} m_{p,t}$ are expressed as the weighted sum of excess consumption and excess price. Each excess level has two types, which are in excess of the weighted average with weights a_1 and a_2. Thus to see how the level of interest rates depends on excess consumptions, we should check the sign of four coefficients of excess consumptions in (12) and (13).

As is easily seen, it never happens that all of four coefficients are positive or negative. When $\kappa_y < a_2$, three coefficients are positive. Conversely, when $a_1 < \kappa_y$, only one coefficient is positive. When κ_y is between a_1 and a_2, the number of positive coefficients varies depending on the sign of covariance component of filtering error $\bar{\phi}_{py}$. Therefore, the overall effect of excess consumptions should be examined empirically. Nonetheless three things deserve to be mentioned. First, when covariance component $\bar{\phi}_{yp}$ is relatively small, $a_2 \approx \min\{\kappa_y^*, \kappa_p^*\}$, $a_1 \approx \max\{\kappa_y^*, \kappa_p^*\}$, and $\kappa_y^* \approx (k_{11}^2 + h_{11}/\sigma_y^2)^{\frac{1}{2}}$. In this case, the condition (17) in Corollary 1 is helpful in understanding each case intuitively. When κ_y is sufficiently small compared to the volatility ratio $(\upsilon_y^2 + \hat{\upsilon}_y^2)^{\frac{1}{2}}/\sigma_y$, $\kappa_y < \kappa_y^*$ is likely to hold and the estimate of $\mu_{y,t}$ is likely to be increasing in both types of excess consumption. Conversely, when the local correlation between $\mu_{y,t}$ and y_t is negative and κ_y is relatively large, $\kappa_y > \kappa_y^*$ is likely to hold and excess consumptions have negative impacts on the estimate of $\mu_{y,t}$, thus the level of interest rates. Again, note that negative local correlation between $\mu_{y,t}$ and y_t does not necessarily imply that $\kappa_y > (k_{11}^2 + h_{11}/\sigma_y)^{\frac{1}{2}} \approx \kappa_y^*$.

Second, the similar result can be obtained by examining the sign of $\partial \lim_{t_0 \to -\infty} m_{y,t}/\partial \ln y_t$. Simple calculation shows that the necessary and sufficient condition for strict positivity of the derivative is $\kappa_y < \kappa_y^*$. It is clear that $\kappa_y < a_2$ is sufficient for this inequality.[11]

Third, the necessary and sufficient condition for $\partial \lim_{t_0 \to -\infty} m_{p,t}/\partial \ln y_t > 0$ is $\hat{\rho} > 0$. When the drift of aggregate consumption process is negatively correlated with the drift of price level, an increase in the aggregate consumption makes the estimate of expected inflation rate updated to the lower level. The magnitude of this effect is determined by the relative size of covariance component in the filtering error $\bar{\phi}_{yp}/\sigma_y$. When this filtering error is small, changes in $\ln y_t$ mainly affect the level of interest rates via the estimate of consumption growth.

4. Empirical Analysis

In the last section, we showed that under the stationary error process assumption the nominal bond yields can be expressed as a function of weighted sums of past excess consumptions and price levels. The analysis also indicated that the bond yields heavily depends on model parameters which we have to estimate from the data. In this section, we estimate the parameters for the economy represented by the system of stochastic differential equations (1)-(4) from the real consumption and CPI data. Then we estimate the preference parameters by minimizing the distance between theoretical and observed time series of interest rates.

Our empirical analysis is based on the quarterly data on consumption and price level from the first quarter of 1952 to the second quarter of 2007. Our sample ends before the Global Financial Crisis to avoid the zero lower bound periods, which is not the scope of the paper. Data on the real per-capita consumption are constructed by adding the seasonally

[11] This argument can apply to the comparative statistics with respect to excess consumptions as far as the excess consumptions are nearly zero at time t, since the dynamics of excess consumptions is given by

$$d \ln y_t - a_i \left(\ln y_t - \int_{-\infty}^{t} a_i e^{-a_i(t-s)} \ln y_s ds \right) \ (i = 1, 2).$$

adjusted real consumption of nondurables and services, then dividing by the population. For the price level, we use the seasonally adjusted consumer price index (CPI). The nominal yield data are quarterly treasury constant maturity rates with maturities of one, two, three, five, seven, and ten years. These data are from the second quarter of 1962 for all maturities. Interest rates are obtained from the Global Financial Data and other data are taken from the Federal Reserve Economic Data (FRED).

4.1. The state-space representation and Kalman filter

To estimate the parameters for the economy represented by the system of stochastic differential equations (1)-(4), we employ the Kalman filter to treat unobservable state variables. To this end, we first derive the state-space representation for the discretized versions of the system of stochastic differential equations, (1)-(4). By the Euler approximation,[12] the system can be discretized as

$$\begin{aligned}
\log(y_{t+1}/y_t) &= -\frac{\sigma_y^2}{2} + \mu_{y,t} + \sigma_y u_{y,t}, && (18)\\
\mu_{y,t+1} &= \kappa_y \bar{\mu}_y + (1-\kappa_y)\mu_{y,t} + \upsilon_y u_{y,t} + \hat{\upsilon}_y \hat{u}_{y,t}, && (19)\\
\log(p_{t+1}/p_t) &= -\frac{\sigma_p^2}{2} + \mu_{p,t} + \sigma_p u_{p,t}, && (20)\\
\mu_{p,t+1} &= \kappa_p \bar{\mu}_p + (1-\kappa_p)\mu_{p,t} + \upsilon_p u_{p,t} + \hat{\upsilon}_p \hat{u}_{p,t}, && (21)
\end{aligned}$$

where $u_{y,t}, u_{p,t}, \hat{u}_{y,t}$, and $\hat{u}_{p,t}$ are mutually independently distributed as standard normal distribution except that $\hat{u}_{y,t}$ and $\hat{u}_{p,t}$ have correlation $\hat{\rho}$. Also we set $\Delta t = 1$ for notational simplicity. Since local trends of endowment and price level, $\mu_{y,t}$ and $\mu_{p,t}$, are not observable, we employ the Kalman filter to estimate this system, rewriting this system as a state-space model as follows. The transition equation consists of equations (19) and (21), and can be written in vector notation as

$$x_t = d_x + F x_{t-1} + G w_{t-1} + v_t, \tag{22}$$

where $x_t = \begin{bmatrix} \mu_{y,t} \\ \mu_{p,t} \end{bmatrix}$, $d_x = \begin{bmatrix} \bar{\mu}_y \kappa_y \\ \bar{\mu}_p \kappa_p \end{bmatrix}$, $F = \begin{bmatrix} 1-\kappa_y & 0 \\ 0 & 1-\kappa_p \end{bmatrix}$, $G = \begin{bmatrix} \upsilon_y/\sigma_y & 0 \\ 0 & \upsilon_p/\sigma_p \end{bmatrix}$, $w_t = \begin{bmatrix} \sigma_y u_{y,t} \\ \sigma_p u_{p,t} \end{bmatrix}$, and $v_t = \begin{bmatrix} \hat{\upsilon}_y \hat{u}_{y,t-1} \\ \hat{\upsilon}_p \hat{u}_{p,t-1} \end{bmatrix}$. The other two equations, (18) and (20), form the observation (measurement) equation, which can be expressed in vector notation as

$$z_t = d_z + x_t + w_t, \tag{23}$$

[12]See Kloeden and Platen (1995) for the Euler approximation of the stochastic differential equations.

where $z_t = \begin{bmatrix} \log(y_{t+1}/y_t) \\ \log(p_{t+1}/p_t) \end{bmatrix}$, $d_z = \begin{bmatrix} -\sigma_y^2/2 \\ -\sigma_p^2/2 \end{bmatrix}$. Note that the disturbances $[v_t',\ w_t']'$ are jointly Gaussian with mean 0 and covariance matrix $\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$, where $Q = \begin{bmatrix} \hat{\upsilon}_y^2 & \hat{\rho}\hat{\upsilon}_y\hat{\upsilon}_p \\ \hat{\rho}\hat{\upsilon}_y\hat{\upsilon}_p & \hat{\upsilon}_p^2 \end{bmatrix}$, $R = \begin{bmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_p^2 \end{bmatrix}$.

Given the state-space representation, (22) and (23), the Kalman filter can be used to construct a likelihood function for the observed data. From some initial conditions, the filter iterates between the prediction equations,

$$\begin{aligned} x_{t|t-1} &= d_x + F\hat{x}_{t-1} + G(z_{t-1} - d_z - H\hat{x}_{t-1}), \\ P_{t|t-1} &= (F - GH)\hat{P}_{t-1}(F - GH) + Q \end{aligned}$$

and the updating equations,

$$\begin{aligned} \hat{x}_t &= x_{t|t-1} + P_{t|t-1}H(HP_{t|t-1}H + R)^{-1}(z_t - d_z - Hx_{t|t-1}), \\ \hat{P}_t &= P_{t|t-1} - P_{t|t-1}H(HP_{t|t-1}H + R)^{-1}HP_{t|t-1}, \end{aligned}$$

where

$$\begin{aligned} x_{t|t-1} &= E(x_t|\mathcal{F}_{t-1}), \\ \hat{x}_t &= E(x_t|\mathcal{F}_t), \\ P_{t|t-1} &= E[(x_t - x_{t|t-1})(x_t - x_{t|t-1})'|\mathcal{F}_{t-1}], \\ \hat{P}_t &= E[(x_t - \hat{x}_t)(x_t - \hat{x}_t)'|\mathcal{F}_t]. \end{aligned}$$

The derivation of the Kalman filter can be found in the appendix. To start the Kalman filter, we use the unconditional mean $E(x_t) = [\bar{\mu}_y,\ \bar{\mu}_p]'$ for the initial values of state vector $\hat{x}_0$. Since we have analyzed the equilibrium interest rates in detail under Assumption 1, we also impose this assumption to estimate the model. Thus, we use the stationary value of error process $\overline{\Phi}$ for $\hat{P}_t$ for all t including the initial value $\hat{P}_0$.

4.2. Estimation results

Estimation results for the state-space model, (22) and (23), are reported in Table 1. All parameters are in quarterly units, and means and standard deviations are expressed in percent. Thus, mean quarterly consumption growth over the period is 0.58% with the standard deviation of the error term about 0.32%, while mean inflation is 0.76% with the disturbance standard deviation about 0.37%. The results indicate that the mean reversion rate of expected consumption is moderate at 0.19. On the other hand, the expected inflation is highly persistent with the mean reversion rate of 0.071. The results also show that the correlations between observable and unobservable factors are essentially zero with statistically insignificant estimates of υ_y and υ_p. Lastly, the correlation between innovations to consumption growth and innovations to inflation is estimated as -0.60.

[INSERT TABLE 1 HERE]

By taking the temporal dependence in the realized consumption growth and inflation rates into account, Wachter (2006) estimates a vector ARMA$(1,1)$ model for consumption growth and inflation and obtains similar results with some differences. Our result of consumption growth 0.58% is similar to 0.55% obtained by Wachter. In contrast, our mean inflation estimate 0.76% is somehow smaller than hers (0.92%). In terms of persistence, our inflation result is comparable with Wachter's results, while she obtains less persistent consumption growth.

There remain only two preference parameters that need to be identified to determine the term structures of interest rates. We estimate these parameters so that the implied time series of interest rates based on (11) have the minimum squared errors. The maturities used for this estimation are one, five, and ten years, which roughly correspond to short-, middle-, and long-term of interest rates. The estimation result for the discount rate δ is 0.0010 in quarterly units with a standard error of 0.0004, while the relative risk aversion γ is estimated at 1.205 with a standard error of 0.066. These results are reasonable, suggesting the plausibility of our model.

4.3. Implied time series of the equilibrium interest rates

In this subsection, we provide the implied time series of the equilibrium interest rates based on (11), given parameter estimates and estimated state variables in the previous subsection. In addition, we examine whether our empirical results imply positive relationship between the real activity and equilibrium interest rates.

Table 2 reports the means and standard deviations of theoretical nominal bond yields based on (11) with the parameters and state variables estimated above. Data moments for bond yields are provided for comparison. As can be seen, the means of the model-implied nominal one-year yield are fairly close to those in the data with less than 1% error for all maturities. One the other hand, the model-implied standard deviations are uniformly much smaller than those in the data with more than 1% error for all maturities, in particular the longer maturities.

[INSERT TABLE 2 HERE]

In terms of the slope of the yield curve, the model-implied average yield curve is downward sloping from 7.1% for one-year yield to 6.5% for ten-year yield, contradicting the upward sloping yield curve in the data. However, this does not mean that our model cannot replicate the upward sloping yield curve. One reason for the downward sloping yield curve is the extremely high interest rates in the early 1980s after the second oil shock. Given that our model for the interest rates is a mean-reverting process, if there is a period with extremely high interest rates, such as the early 1980s, the long-term rates would be underestimated, making the slope of the yield curve negative. In addition, the actual yield curve is mostly flat or downward sloping between 1978Q3 and 1982Q2. As a consequence, the average yield curve is downward sloping for the whole sample. However, if we exclude this period, the average yield curve could be upward sloping. For instance, if we use the post 1985 data, the average yield curve would be slightly upward sloping, as shown in Table 2. The result indicates the possibility of our model describing the upward sloping yield curve.

In sum, the moment comparison reveals the relatively poor performance of our model to reproduce the fluctuation and the upward average yield curve in the data. These results are less satisfactory than those of Wachter's (2006) with more precise replication of moments and the upward sloping yield curve. However, these results do not necessarily mean that our model is unattractive to explain the time series behaviors of nominal interest rates as we will show next.

Fig. 1 illustrates the model-implied time series of one-year nominal yields along with actual data. Note that following Wachter (2006), we de-mean both series. As can be seen, the model-implied yields closely follow the actual data, capturing many of the short- and long-run fluctuations in the actual data for all maturities. Although the magnitude of fluctuations is somewhat smaller, the entire shape of the graph is similar to that of the actual data. Indeed, the correlations between the theoretical and observed yields is 0.72. This is considerably higher than 0.50 obtained by Wachter (2006) for the three-month yield.

[INSERT FIGURE 1 HERE]

We also plot the de-meaned yield spreads on the ten-year nominal bond over the one-year nominal bond implied by the model, and the same series from the data in Fig. 2. Again, the model matches many of the short- and long-run fluctuations in the nominal yield spread from the data. The correlation between the yield spread implied by the model and that in the data is 0.46, which is again higher than 0.34 obtained by Wachter (2006) for the yield spread between three-month and five-year bond.

[INSERT FIGURE 2 HERE]

One possible explanation for these higher correlations is the relation between nominal bond yields and real activity measured in excess consumption. As emphasized in the previous sections, consumption habit models usually induce a negative correlation between interest rates and the excess consumption, while our model can generate a positive correlation more naturally. To show this is the case, Fig. 3 plots the change in one-, five- and ten-year yields against the change in excess consumption, assuming that two types of excess consumption vary the same amount.[13] As can be seen, all three yields increase with excess consumption, but the short yield is the most sensitive to excess consumption. The short-term rates increase by 0.23% as excess consumption increases by 1%, while the middle- and long-yields increase by 0.13% to 0.06%. This is relevant because consumption growth is estimated as a mean reversion process, and the temporal negative/positive shock to the excess consumption is expected to diminish eventually.

[INSERT FIGURE 3 HERE]

The effect of excess price level on nominal bond yields can be analyzed in the same manner. Figure 4 plots the change in one-, five-, and ten-year yields against the change in excess price, assuming that two types of excess price vary at the same amount. As shown in Figure 3, all three yields increase with the positive excess price level. However, in contrast to excess consumption, the sensitivity of bond yields to excess price does not vary much across the maturities. The short-term rates increase by 0.31% as excess price level increases by 1%. The middle- and long-yields increase by 0.30% to 0.28%. Thus, excess price affects the nominal bond yields uniformly across maturity. In other words, the nominal yield curve shows a parallel shift when some excess price level is observed. As suggested by our empirical results, the speed of mean reversion of the price level κ_p is lower than that of consumption κ_y, implying that the price level is more persistent than the consumption. Therefore, it is natural that the price level is not the main factor that changes the slope of the bond yield curve.

The monetary authority is not assumed to be one of the players in our model. However, one possible implication on the monetary policy is that the effect of monetary easing is

[13] Natural interpretation of these changes is that they occur due to the change of current consumption.

weakened by the response of bond markets through the observation of excess price level. Suppose the central bank lowers the nominal short rates and an increase in the inflation rate is observed. After observing a positive excess price level, the market clearing bond yields will be adjusted to a higher level. This means that the interest rates are pushed back to a higher level, making the monetary transmission channel less effective.

Moreover, although we have not tried to address the lower bound of the interest rates, our results have some implications on the recent environment of low inflation and low interest rates. Specifically, the low inflation lowers the excess price level, yielding a lower pressure on the interest rates. Therefore, the lower inflation tends to keep the interest rates low. This could be happening in many advanced countries, including the US and Japan.

[INSERT FIGURE 4 HERE]

To conclude the empirical section, we examine whether the positive relation between the equilibrium nominal interest rates and excess consumption can be observed in the data. To this end, we conduct regression analysis motivated by Wachter (2006) who regresses the ex post real interest rate on surplus consumption proxy to see the negative relation between real interest rates and surplus consumption. Following this idea, we regress the nominal interest rates for several maturities on excess consumption and price level proxies. Thus, the regressions we estimated are

$$r_{t+1}(n) = \alpha_0 + \alpha_1 \sum_{j=1}^{40} \psi^j \Delta \ln y_{t-j} + \alpha_2 \sum_{j=1}^{40} \psi^j \Delta \ln p_{t-j} + \varepsilon_{t+1}, \tag{24}$$

where $r_{t+1}(n)$ is the nominal yield with maturity n. Following Wachter (2006), ψ is set to equal 0.97. Table 3 reports the estimates and Newey-West standard errors for several maturities. In contrast with Wachter's result of negative estimates for α_1, the parameter α_1 is estimated to be positive and statistically significant for all short maturities, one, two, and three years. This result suggests that if we consider the nominal rate as a dependent variable and treat excess price level as an explanatory variable, we can find a positive relation between the nominal short interest rate and the excess consumption. Also, our estimation results indicates α_2 is estimated to be positive and statistically significant for all maturities, suggesting a great explanatory power of excess price level on the nominal yield. This can be one of the reasons why our model has great advantage in explaining the dynamics of nominal interest rates over the last four decades.

[INSERT TABLE 3 HERE]

5. Conclusion

In this paper, we propose a new general equilibrium model that naturally generates the positive correlation between the nominal interest rates and real economic activity. To this end, we focus on the partial observability of economic variables in a pure exchange economy. Departing from the previous studies, we considered the dynamic general equilibrium model with two unobservable factors. Even with this complexity, we have derived closed form solutions for the nominal equilibrium interest rates. The resulting nominal term structure model turns out to be a two-factor purely Gaussian affine model in which state variables can be expressed as a weighted sum of excess consumptions and price levels under stationary error process assumption.

This result allows us to give an additional role of excess consumption in determining interest rates compared with the surplus consumption in the consumption habit models. For economic agents engaging in the Bayesian inference, the excess consumption is not only the surplus consumption giving them felicity, but also constitutes an economic indicator for of the real activity, thereby helping infer the current trend in income growth. Naturally, the economic agents' estimate, hence the equilibrium interest rates, can be increasing in the real economic activity measured by excess consumption under some mild conditions on parameters.

Our empirical analysis also supports this view. The estimation results indicate reasonable values for all parameters and, more importantly, the positive correlation between the implied nominal interest rates and excess consumption. As a consequence, the time series of the nominal yield implied by the model captures many of the short- and long-run fluctuations in the actual data without scarifying the upward sloping yield curve. This is a contrast with models with habit formation, which require a negative correlation between real economic activity and interest rates to fit an upward sloping yield curve.

Although the paper shows a great potential of partial observability to explain the dynamics of interest rates, there remain some issues to be considered. For instance, the fit gets worse in the longer maturities. More importantly, the model cannot solve the failure of expectations hypothesis documented by Campbell and Shiller (1991) and Fama and Bliss (1987), which is one of the main focuses in the recent term structure literature. As emphasized in Dai and Singleton (2002), the key to explain the failure of expectations hypothesis is time-varying risk premia; however our model does not allow it. One possibility to deal with this challenge is to introduce the habit persistence of Campbell and Cochrane (1999) and Wachter (2006) into our model. This is relevant because the consumption habit naturally generates time-varying risk premia. The main difficulty in incorporating the habit persistence into our model is losing model's tractability considerably; however this is surely an interesting future topic.

Appendix A: Formulation of Securities Market and optimization problem for the representative agent

It is assumed that the market is frictionless and the finite number, say N, of securities in zero net supply are traded continuously in time. We assume that these securities do not pay dividends for notational simplicity, but they could be pure discount bonds.[14] $\mathcal{F}_t^{y,p}$-measurable random variable $S_{n,t}$ denotes time t price of nth security ($n = 1, \cdots, N$). Though we use $S_{n,t}$ for general formulation here, when we analyze the term structure of interest rates, we adopt different notation $B(t,T)$ as time t price of pure discount bond which promises to pay one unit of currency at time $T \in (t_0, \tau]$.

The representative agent takes as given N-dimensional security price process $\{(S_{1,t}, \cdots, S_{N,t}) : t \in [t_0, \tau]\}$ which is adapted to the filtration $\{\mathcal{F}_t^{y,p} : t \in [t_0, \tau]\}$. This allows one to define cumulative gains from trade for a predictable portfolio process $\theta_t = (\theta_{1,t}, \cdots, \theta_{N,t})$. It is assumed that a portfolio process satisfies some regularity conditions which imply that the gain-from-trade integral $\sum_{n=1}^{N} \int_{t_0}^{t} \theta_{n,s} dS_{n,s}, (t_0 < t \leq \tau)$ is well defined and θ_t is square integrable.[15] Then, we can state the optimization problem for the representative agent. That is, given a security price process $\{(S_{1,t}, \cdots, S_{N,t}) : t \in [t_0, \tau]\}$, the representative agent solves the following optimization problem,

$$\max_{\{c_s\},\{\theta_s\}} E\left[\int_{t_0}^{\tau} u(c_s, s) ds \middle| \mathcal{F}_{t_0}^{y,p} \right] \tag{25}$$

[14] In the next appendix, this formulation is re-represented as the one having the same structure as complete information models. The same exposition as the past literature on these models is taken to show directly that issues on existence and uniqueness of the solution are covered.

[15] For these conditions, see Duffie and Zame (1989).

where

$$\frac{dy_t}{y_t} = \mu_{y,t}dt + \sigma_y dW_{y,t}, \tag{26}$$

$$d\mu_{y,t} = \kappa_y(\bar{\mu}_y - \mu_{y,t})dt + \upsilon_y dW_{y,t} + \hat{\upsilon}_y d\widehat{W}_{y,t}, \tag{27}$$

$$\frac{dp_t}{p_t} = \mu_{p,t}dt + \sigma_p dW_{p,t}, \tag{28}$$

$$d\mu_{p,t} = \kappa_p(\bar{\mu}_p - \mu_{p,t})dt + \upsilon_p dW_{p,t} + \hat{\upsilon}_p d\widehat{W}_{p,t}, \tag{29}$$

$$\text{s.t. } \sum_{n=1}^{N} \theta_{n,t} S_{n,t} = \sum_{n=1}^{N} \int_{t_0}^{t} \theta_{n,s} dS_{n,s} + \int_{t_0}^{t} p_s(y_s - c_s)ds, \;\; t_0 < t \leq \tau, \tag{30}$$

$$\theta_{n,\tau} = 0, \;\; n = 1, \cdots, N. \tag{31}$$

An equilibrium for the economy is a collection $(\{(S_{1,t}, \cdots, S_{N,t}) : t \in [t_0, \tau]\}, (\{c_t : t \in [t_0, \tau]\}, \{\theta_t : t \in [t_0, \tau]\}))$ such that, given the security price process $\{(S_{1,t}, \cdots, S_{N,t}) : t \in [t_0, \tau]\}$, the plan $(\{c_t : t \in [t_0, \tau]\}, \{\theta_t : t \in [t_0, \tau]\})$ solves the above optimization problem and markets clear, i.e. $c_t = y_t$ and $\theta_t = 0$ for all $t \in [t_0, \tau]$.

Appendix B: Re-representation of the optimization problem and the equilibrium bond prices

Using the filter equations, the representative agent reformulates her optimization problem. Given a security price process $\{(S_{1,t}, \cdots, S_{N,t}) : t \in [t_0, \tau]\}$ which is adapted to the filtration $\{\mathcal{F}_t^{\overline{W}, y_{t_0}, p_{t_0}} : t \in [t_0, \tau]\}$, she solves the following problem,

$$\max_{\{c_s\},\{\theta_s\}} E\left[\int_{t_0}^{\tau} u(c_s, s)ds \middle| \mathcal{F}_{t_0}^{\overline{W}, y_{t_0}, p_{t_0}} \right] \tag{32}$$

where

$$\frac{dy_t}{y_t} = m_{y,t}dt + \sigma_y d\overline{W}_{y,t}, \tag{33}$$

$$dm_{y,t} = \kappa_y(\bar{\mu}_y - m_{y,t})dt + \left(\upsilon_y + \frac{\phi_{yy,t}}{\sigma_y}\right) d\overline{W}_{y,t} + \frac{\phi_{yp,t}}{\sigma_p} d\overline{W}_{p,t}, \tag{34}$$

$$\frac{dp_t}{p_t} = m_{p,t}dt + \sigma_p d\overline{W}_{p,t}, \tag{35}$$

$$dm_{p,t} = \kappa_p(\bar{\mu}_p - m_{p,t})dt + \frac{\phi_{py,t}}{\sigma_y} d\overline{W}_{y,t} + \left(\upsilon_p + \frac{\phi_{pp,t}}{\sigma_p}\right) d\overline{W}_{p,t}, \tag{36}$$

$$\frac{d}{dt}\Phi_t = K\Phi_t + \Phi_t K^\top - \Phi_t G \Phi_t^\top + H, \tag{37}$$

$$\text{s.t. } \sum_{n=1}^{N} \theta_{n,t} S_{n,t} = \sum_{n=1}^{N} \int_{t_0}^{t} \theta_{n,s} dS_{n,s} + \int_{t_0}^{t} p_s(y_s - c_s)ds, \;\; t_0 < t \leq \tau, \tag{38}$$

$$\theta_n^\tau = 0, \;\; n = 1, \cdots, N. \tag{39}$$

Note that given the same security price process $\{(S_{1,t}, \cdots, S_{N,t}) : t \in [t_0, \tau]\}$, the solution to this optimization problem (32) − (39) solves the original optimization problem (25) − (31), because σ-fields $\mathcal{F}_t^{y,p}$ and $\mathcal{F}_t^{\overline{W}, y_{t_0}, p_{t_0}}$ are equivalent.[16]

The optimization problem in the endogenous σ-field equivalent economy stated above is structurally the same as the standard optimization problems in complete information economies. Thus, under our assumption on preference, standard argument shows that an equilibrium exists and the equilibrium nominal bond prices is given by,[17]

$$B(t,T) \;=\; \frac{E\left[u_c(y_T, T)/p_T | \, \mathcal{F}_t^{\overline{W}, y_{t_0}, p_{t_0}} \right]}{u_c(y_t, t)/p_t}.$$

Appendix C: Proof of Proposition 2

Under Assumption 1, the stochastic differential equations for $m_{y,t}$ and $m_{p,t}$ can be expressed as

$$\begin{aligned}
dm_{y,t} &= \left(b_y - \kappa_y^* m_{y,t} - (\bar{\phi}_{yp}/\sigma_p^2) m_{p,t} \right) dt \\
&\quad + (\kappa_y^* - \kappa_y) d \ln y_t + (\bar{\phi}_{yp}/\sigma_p^2) d \ln p_t, \\
dm_{p,t} &= \left(b_p - (\bar{\phi}_{py}/\sigma_y^2) m_{y,t} - \kappa_p^* m_{p,t} \right) dt \\
&\quad + (\bar{\phi}_{py}/\sigma_y^2) d \ln y_t + (\kappa_p^* - \kappa_p) d \ln p_t,
\end{aligned}$$

where parameters b_y and b_p are defined as

$$\begin{aligned}
b_y &= \kappa_y \bar{\mu}_y + \frac{1}{2} \left(\upsilon_y \sigma_y + \bar{\phi}_{yy} + \bar{\phi}_{yp} \right), \\
b_p &= \kappa_p \bar{\mu}_p + \frac{1}{2} \left(\upsilon_p \sigma_p + \bar{\phi}_{pp} + \bar{\phi}_{py} \right).
\end{aligned}$$

[16] This is a pure exchange economy version of re-representation theorem in Feldman (2007).

[17] For instance, Theorem 1 in Duffie and Zame (1989) applies to our endogenous σ-field equivalent economy.

The closed form solution of the system of equations above is given by

$$
\begin{aligned}
m_{y,t} &= \frac{a_1 e^{-a_2(t-t_0)} - a_2 e^{-a_1(t-t_0)}}{a_1 - a_2} m_{y,t_0} \\
&\quad + \frac{\frac{1-e^{-a_2(t-t_0)}}{a_2} - \frac{1-e^{-a_1(t-t_0)}}{a_1}}{a_1 - a_2} \left(\kappa_p^* b_y - b_p(\bar{\phi}_{yp}/\sigma_p^2)\right) \\
&\quad + \frac{e^{-a_2(t-t_0)} - e^{-a_1(t-t_0)}}{a_1 - a_2} \left(b_y - \kappa_y^* m_{y,t_0} - (\bar{\phi}_{yp}/\sigma_p^2) m_{p,t_0}\right) \\
&\quad + \frac{(a_1 - \kappa_y)(\kappa_y^* - a_2)}{a_1 - a_2} \int_{t_0}^{t} e^{-a_1(t-s)} d\ln y_s \\
&\quad - \frac{(a_2 - \kappa_y)(\kappa_y^* - a_1)}{a_1 - a_2} \int_{t_0}^{t} e^{-a_2(t-s)} d\ln y_s \\
&\quad + \frac{(a_1 - \kappa_p)(\bar{\phi}_{yp}/\sigma_p^2)}{a_1 - a_2} \int_{t_0}^{t} e^{-a_1(t-s)} d\ln p_s \\
&\quad - \frac{(a_2 - \kappa_p)(\bar{\phi}_{yp}/\sigma_p^2)}{a_1 - a_2} \int_{t_0}^{t} e^{-a_2(t-s)} d\ln p_s, \\
m_{p,t} &= \frac{a_1 e^{-a_2(t-t_0)} - a_2 e^{-a_1(t-t_0)}}{a_1 - a_2} m_{p,t_0} \\
&\quad + \frac{\frac{1-e^{-a_2(t-t_0)}}{a_2} - \frac{1-e^{-a_1(t-t_0)}}{a_1}}{a_1 - a_2} \left(\kappa_y^* b_p - b_y(\bar{\phi}_{py}/\sigma_y^2)\right) \\
&\quad + \frac{e^{-a_2(t-t_0)} - e^{-a_1(t-t_0)}}{a_1 - a_2} \left(b_p - \kappa_p^* m_{p,t_0} - (\bar{\phi}_{py}/\sigma_y^2) m_{y,t_0}\right) \\
&\quad + \frac{(a_1 - \kappa_p)(\kappa_p^* - a_2)}{a_1 - a_2} \int_{t_0}^{t} e^{-a_1(t-s)} d\ln p_s \\
&\quad - \frac{(a_2 - \kappa_p)(\kappa_p^* - a_1)}{a_1 - a_2} \int_{t_0}^{t} e^{-a_2(t-s)} d\ln p_s \\
&\quad + \frac{(a_1 - \kappa_y)(\bar{\phi}_{py}/\sigma_y^2)}{a_1 - a_2} \int_{t_0}^{t} e^{-a_1(t-s)} d\ln y_s \\
&\quad - \frac{(a_2 - \kappa_y)(\bar{\phi}_{py}/\sigma_y^2)}{a_1 - a_2} \int_{t_0}^{t} e^{-a_2(t-s)} d\ln y_s,
\end{aligned}
$$

where a_1 and a_2 are given by

$$
\begin{aligned}
a_1 &= \frac{\kappa_y^* + \kappa_p^* + \sqrt{(\kappa_y^* - \kappa_p^*)^2 + 4(\bar{\phi}_{yp}/\sigma_p^2)(\bar{\phi}_{py}/\sigma_y^2)}}{2}, \\
a_2 &= \frac{\kappa_y^* + \kappa_p^* - \sqrt{(\kappa_y^* - \kappa_p^*)^2 + 4(\bar{\phi}_{yp}/\sigma_p^2)(\bar{\phi}_{py}/\sigma_y^2)}}{2}.
\end{aligned}
$$

Clearly, the inequalities, $a_2 \leq \min\{\kappa_y^*, \kappa_p^*\} \leq \max\{\kappa_y^*, \kappa_p^*\} \leq a_1$, hold. When $\hat{\rho}$ approaches to 0, the weight a_1 converges to $\max\{\kappa_y^*, \kappa_p^*\}$ and a_2 converges to $\min\{\kappa_y^*, \kappa_p^*\}$. It is not difficult to show that $a_2 > 0$.

By integral by parts, each integral in the closed form solution of $m_{y,t}$ and $m_{p,t}$ is re-expressed. For instance,

$$\int_{t_0}^{t} e^{-a_1(t-s)} d\ln y_s = \ln y_t - e^{-a_1(t-t_0)} \ln y_{t_0} - \int_{t_0}^{t} a_1 e^{-a_1(t-s)} \ln y_s ds,$$

Since the second term in the right hand side converges to zero when $t_0 \to -\infty$, by taking this limit, we obtain two equations in the proposition.

Appendix D: Proof of Proposition 3

If $\hat{\rho} = 0$, κ_y^* is given by

$$\kappa_y^* = \left((\kappa_y + \upsilon_y/\sigma_y)^2 + (\hat{\upsilon}_y/\sigma_y)^2\right)^{\frac{1}{2}}.$$

Thus, $\kappa_y^* - \kappa_y$ is positive if and only if

$$2\kappa_y \upsilon_y \sigma_y + \upsilon_y^2 + \hat{\upsilon}_y^2 \geq 0. \tag{40}$$

Since the correlation coefficient is given by

$$\rho_{y,\mu_y} = \frac{\upsilon_y}{\left(\upsilon_y^2 + \hat{\upsilon}_y^2\right)^{\frac{1}{2}}},$$

substituting this into (40) yields (16).

Appendix E: Proof of Corollary 1

The inequality (17) is equivalent to the inequality $-1 \geq -\frac{\left(\upsilon_y^2+\hat{\upsilon}_y^2\right)^{\frac{1}{2}}}{2\kappa_y\sigma_y}$. Combining this inequality with $\rho_{y,\mu_y} \geq -1$ yields (16). This concludes the proof.

Appendix F: Derivation of filtering error process

Let us consider a system of matrix linear differential equations

$$\begin{aligned}
\frac{d}{dt}U_t &= KU_t + HV_t,\ U_{t_0} = \Phi_{t_0}, \\
\frac{d}{dt}V_t &= GU_t - K^\top V_t,\ V_{t_0} = I,
\end{aligned} \tag{41}$$

where I is 2×2 identity matrix. It is well-known that the solution of (9) is given by $\Phi_t = U_t V_t^{-1}$. Define S_t as

$$S_t = \begin{bmatrix} U_t \\ V_t \end{bmatrix}.$$

Then, the system of equations (41) is expressed as

$$\frac{d}{dt}S_t = A\,S_t, \tag{42}$$

where the 4×4 matrix A is defined by

$$A = \begin{bmatrix} K & H \\ G & -K^\top \end{bmatrix}.$$

Let us denote the eigen values for A by λ_i $(i = 1, 2, 3, 4)$. Then these values are given by

$$\begin{aligned}
\lambda_1 &= \left(\frac{(k_{11}^2 + h_{11}/\sigma_y^2) + (k_{22}^2 + h_{22}/\sigma_p^2) + D^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}, \\
\lambda_2 &= \left(\frac{(k_{11}^2 + h_{11}/\sigma_y^2) + (k_{22}^2 + h_{22}/\sigma_p^2) - D^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}, \\
\lambda_3 &= -\lambda_1, \\
\lambda_4 &= -\lambda_2, \\
\text{where } D &= \left((k_{11}^2 + h_{11}/\sigma_y^2) - (k_{22}^2 + h_{22}/\sigma_p^2)\right)^2 + \frac{4h_{12}^2}{\sigma_y^2\sigma_p^2}.
\end{aligned}$$

The corresponding eigen vectors x_i $(i = 1, 2, 3, 4)$ are given as

$$x_i = \begin{bmatrix} \frac{\lambda_i + k_{11}}{g_{11}} \\ \left(\frac{\lambda_i + k_{11}}{h_{12}g_{22}}\right)\left(\frac{\lambda_i^2 - k_{11}^2}{g_{11}} - h_{11}\right) \\ 1 \\ \frac{1}{h_{12}}\left(\frac{\lambda_i^2 - k_{22}^2}{g_{11}} - h_{11}\right) \end{bmatrix}, \quad i = 1, 2, 3, 4.$$

Next, we construct the matrix $[\xi_1 x_1,\ \xi_2 x_2,\ \xi_3 x_3,\ \xi_4 x_4]$ where constants ξ_i $(i = 1, 2, 3, 4)$ satisfy

$$\begin{aligned}
\xi_1\xi_3 &= \frac{(\lambda_2^2 - \kappa_y^2 - h_{11}g_{11})g_{11}}{2(\lambda_2^2 - \lambda_1^2)\lambda_1}, \\
\xi_2\xi_4 &= -\frac{(\lambda_1^2 - \kappa_y^2 - h_{11}g_{11})g_{11}}{2(\lambda_2^2 - \lambda_1^2)\lambda_2}.
\end{aligned}$$

Denote this matrix as

$$R = \begin{bmatrix} Y & Z \\ X & W \end{bmatrix},$$

where W, X, Y, Z are 2×2 submatrices. Constant scalars ξ_i $(i = 1, 2, 3, 4)$ are for the normalization of matrix in the sense that the inverse of R is given by

$$R^{-1} = \begin{bmatrix} W^\top & -Z^\top \\ -X^\top & Y^\top \end{bmatrix}.$$

Denote a diagonal matrix defined by eigen values as

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 \\ 0 & 0 & 0 & -\lambda_2 \end{bmatrix}.$$

Apparently, $A = R\Lambda R^{-1}$ holds and (42) can be arranged to

$$\frac{d}{dt} R^{-1} S_t = \Lambda R^{-1} S_t.$$

Since Λ is diagonal, $R^{-1}S_t = e^{\Lambda(t-t_0)} R^{-1} S_{t_0}$ and we obtain

$$S_t = R e^{\Lambda(t-t_0)} R^{-1} S_{t_0}.$$

This yields the next two equations,

$$\begin{aligned} U_t &= \left(Y e^{\Lambda_1(t-t_0)} W^\top - Z e^{\Lambda_2(t-t_0)} X^\top \right) \Phi_{t_0} \\ &\quad + Z e^{\Lambda_2(t-t_0)} Y^\top - Y e^{\Lambda_1(t-t_0)} Z^\top, \\ V_t &= \left(X e^{\Lambda_1(t-t_0)} W^\top - W e^{\Lambda_2(t-t_0)} X^\top \right) \Phi_{t_0} \\ &\quad + W e^{\Lambda_2(t-t_0)} Y^\top - X e^{\Lambda_1(t-t_0)} Z^\top, \end{aligned}$$

where Λ_i $(i = 1, 2)$ are submatrices of Λ defined by

$$\Lambda_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \; \Lambda_2 = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}.$$

Finally, the solution for the matrix Riccati equation is obtained by $\Phi_t = U_t V_t^{-1}$. Especially, the limit, $\overline{\Phi} = \lim_{t\to\infty}$, is given by YX^{-1}, since $e^{\Lambda_2(t-t_0)} \to 0$ as $t \to \infty$. Each element of this limit matrix is obtained as

$$\begin{aligned} \bar{\phi}_{yy} &= \left(\kappa_y^* + k_{11}\right) \sigma_y^2 = (\kappa_y^* - \kappa_y)\sigma_y^2 - \upsilon_y \sigma_y, \\ \bar{\phi}_{pp} &= \left(\kappa_p^* + k_{22}\right) \sigma_p^2 = (\kappa_p^* - \kappa_p)\sigma_p^2 - \upsilon_p \sigma_p, \\ \bar{\phi}_{yp} &= \bar{\phi}_{py} = \frac{\hat{\upsilon}_y \hat{\upsilon}_p \hat{\rho}}{\lambda_1 + \lambda_2}, \\ \text{where } \kappa_y^* &= \frac{k_{11}^2 + h_{11}/\sigma_y^2 + \lambda_1\lambda_2}{\lambda_1 + \lambda_2}, \; \kappa_p^* = \frac{k_{22}^2 + h_{22}/\sigma_p^2 + \lambda_1\lambda_2}{\lambda_1 + \lambda_2}. \end{aligned}$$

Appendix G: Derivation of the Kalman Filter

First note that

$$E(w_{t-1}|\mathcal{F}_{t-1}) = E(z_{t-1} - d_z - x_{t-1}|\mathcal{F}_{t-1}) = z_{t-1} - d_z - \hat{x}_{t-1}.$$

From this result and transition equation (22), we have

$$x_{t|t-1} \quad = \quad d_x + F\hat{x}_{t-1} + G(z_{t-1} - d_z - \hat{x}_{t-1}).$$

Hence,

$$\begin{aligned}
x_t - x_{t|t-1} &= (d_x + Fx_{t-1} + Gw_{t-1} + v_t) - (d_x + F\hat{x}_{t-1} + G(z_{t-1} - d_z - \hat{x}_{t-1})) \\
&= F(x_{t-1} - \hat{x}_{t-1}) - G(z_{t-1} - d_z - w_{t-1} - \hat{x}_{t-1}) + v_t \\
&= F(x_{t-1} - \hat{x}_{t-1}) - G(x_{t-1} - \hat{x}_{t-1}) + v_t \\
&= (F - G)(x_{t-1} - \hat{x}_{t-1}) + v_t
\end{aligned}$$

The third equality follows from the observation equation (23). Therefore,

$$\begin{aligned}
P_{t|t-1} &= E\left[(x_t - x_{t|t-1})(x_t - x_{t|t-1})'\right] \\
&= E\left[\{(F - G)(x_{t-1} - \hat{x}_{t-1}) + v_t\}\{(F - G)(x_{t-1} - \hat{x}_{t-1}) + v_t\}'\right] \\
&= (F - G)E[(x_{t-1} - \hat{x}_{t-1})(x_{t-1} - \hat{x}_{t-1})'](F - G)' + E[v_t v_t'] \\
&= (F - G)\hat{P}_{t-1}(F - G) + Q.
\end{aligned}$$

To get the fourth equality, we use the fact that v_t is independent of x_{t-1}. Furthermore, since $\hat{x}_{t-1}$ is a linear function of $z_1, \ldots, z_{t-1}$, it must be independent of v_t. Also, the last equality follows from that F and G are diagonal.

The updating equations can be obtained as follows. By the formula for updating a linear projection we can get[18]

$$\hat{x}_t = x_{t|t-1} + E[(x_t - x_{t|t-1})(z_t - z_{t|t-1})']\left\{E[(z_t - z_{t|t-1})(z_t - z_{t|t-1})']\right\}^{-1}(z_t - z_{t|t-1}). \quad (43)$$

Notice that

$$z_{t|t-1} = d_z + x_{t|t-1}, \quad (44)$$

and so

$$z_t - z_{t|t-1} = x_t - x_{t|t-1} + w_t.$$

Using this result we can calculate

$$\begin{aligned}
E[(z_t - z_{t|t-1})(z_t - z_{t|t-1})'] &= E[\{(x_t - x_{t|t-1}) + w_t\}\{(x_t - x_{t|t-1}) + w_t\}'] \\
&= E[(x_t - x_{t|t-1})(x_t - x_{t|t-1})'] + E[w_t w_t'] \\
&= P_{t|t-1} + R. \quad (45)
\end{aligned}$$

[18]See, for example, Hamilton (1994, p. 99, equation [4.5.30]).

Here the second equality follows from the fact $E[(x_t - x_{t|t-1})w_t'] = 0$. Similarly,

$$\begin{aligned} E[(x_t - x_{t|t-1})(z_t - z_{t|t-1})'] &= E[(x_t - x_{t|t-1})\{(x_t - x_{t|t-1}) + w_t\}'] \\ &= E[(x_t - x_{t|t-1})(x_t - x_{t|t-1})'] \\ &= P_{t|t-1}. \end{aligned} \tag{46}$$

Substituting (44), (45) and (46) into (43) gives

$$\hat{x}_t = x_{t|t-1} + P_{t|t-1}(P_{t|t-1} + R)^{-1}(z_t - d_z - x_{t|t-1})$$

The MSE associated with this updated projection, $\hat{P}_t$, can be found from the formula for the MSE of updated linear projection:[19]

$$\begin{aligned} \hat{P}_t &= E[(x_t - \hat{x}_t)(x_t - \hat{x}_t)'] \\ &= E[(x_t - x_{t|t-1})(x_t - x_{t|t-1})'] \\ &\quad - E[(x_t - x_{t|t-1})(z_t - z_{t|t-1})']E[(z_t - z_{t|t-1})(z_t - z_{t|t-1})']E[(z_t - z_{t|t-1})(x_t - x_{t|t-1})'] \\ &= P_{t|t-1} - P_{t|t-1}(P_{t|t-1} + R)^{-1}P_{t|t-1} \end{aligned}$$

[19]See, for example, Hamilton (1994, p. 99, equation [4.5.31]).

References

[1] Ang, A., Dong, S., Piazzesi, M., 2007. No-arbitrage Taylor rules. NBER Working Paper No. 13448.

[2] Ang, A., Piazzesi, M., 2003. A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. Journal of Monetary Economics 50, 745-787.

[3] Bikbov, R., Chernov, M., 2010. No-arbitrage macroeconomic determinants of the yield curve. Journal of Econometrics 159, 166-182.

[4] Björk, T., Davis, M. H. A., Landén, C., 2010. Optimal investment under partial information, Mathematical Methods of Operations Research 71, 371-399.

[5] Buraschi, A., Jiltsov, A., 2007. Habit formation and macro-models of the term structure of interest rates. Journal of Finance 62, 3009-3063.

[6] Campbell, J. Y., Cochrane, J. H., 1999. By force of habit: A consumption-based explanation of aggregate stock market behavior. Journal of Political Economy 107, 205-251.

[7] Campbell, J. Y., Shiller, R. J., 1991. Yield spreads and interest rate movements: A bird's eye view. Review of Economic Studies 58, 495-514.

[8] Dai, Q., Singleton, K., 2002. Expectation puzzles, time-varying risk premia, and affine models of the term structure. Journal of Financial Economics 63, 415-441.

[9] Detemple, J. B., 1986. Asset pricing in a production economy with incomplete information. Journal of Finance 41, 383-391.

[10] Dothan, M. U., Feldman, D., 1986. Equilibrium interest rates and multiperiod bonds in a partially observable economy. Journal of Finance 41, 369-382.

[11] Duffie, D., Zame, W., 1989. The consumption-based capital asset pricing model. Econometrica 57(6), 1279-1297.

[12] Fama, E. F., Bliss, R. R., 1987. The information in long-maturity forward rates. American Economic Review 77(4), 680-692.

[13] Feldman, D., 1989. The term structure of interest rates in a partially observable economy. Journal of Finance 44, 789-812.

[14] Feldman, D., 2003. The term structure of interest rates: Bounded or falling? Review of Finance 7, 103-113.

[15] Feldman, D., 2007. Incomplete information equilibrium: Separation theorems and other myths, Annals of Operations Research 151, 119-149.

[16] Guvenen, F., 2007. Learning your earning: Are labor income shocks really very persistent? American Economic Review 97, 687-712.

[17] Hamilton, J. D., 1994. Time Series Analysis. Princeton University Press.

[18] Kloeden, P. E., Platen, E., 1995. Numerical Solution of Stochastic Differential Equations, Berlin: Springer.

[19] Kraft, H., 2009. Optimal portfolio with stochastic short rate: Pitfalls when the short rate is non-Gaussian or the market price of risk is unbounded. International Journal of Theoretical and Applied Finance 12, 767-796.

[20] Langtieg, T. C., 1980. A multivariate model of the term structure. Journal of Finance 35, 71-97.

[21] Liptser, R. S. and Shiryaev, A. N., 2001. Statistics of Random Process 2 (second edition), New York: Springer-Verlag.

[22] Riedel, F., 2000a. Decreasing yield curves in a model with an unknown constant growth rate. Review of Finance 4, 51-67.

[23] Riedel, F., 2000b. Imperfect Information and Investor Heterogeneity in the Bond Market, Physica-Verlag.

[24] Wachter, J., 2006. A consumption-based model of the term structure of interest rates. Journal of Financial Economics 79, 365-399.

[25] Wang, N., 2004. Precautionary saving with partially observed income, Journal of Monetary Economics 51, 1645-1681.

[26] Wang, N., 2009. Optimal consumption and asset allocation with unknown income growth, Journal of Monetary Economics 56, 524-534.

Table 1. Estimates of the Model Parameters

This table presents the parameter estimates for the system of aggregate endowment flow in real term y_t and its price level p_t:

$$\begin{aligned}
\frac{dy_t}{y_t} &= \mu_{y,t}dt + \sigma_y dW_{y,t}, \\
d\mu_{y,t} &= \kappa_y(\bar{\mu}_y - \mu_{y,t})dt + \upsilon_y dW_{y,t} + \hat{\upsilon}_y d\widehat{W}_{y,t}, \\
\frac{dp_t}{p_t} &= \mu_{p,t}dt + \sigma_p dW_{p,t}, \\
d\mu_{p,t} &= \kappa_p(\bar{\mu}_p - \mu_{p,t})dt + \upsilon_p dW_{p,t} + \hat{\upsilon}_p d\widehat{W}_{p,t},
\end{aligned}$$

where $W_{y,t}$, $W_{p,t}$, $\widehat{W}_{y,t}$, and $\widehat{W}_{p,t}$ are four Wiener processes. These processes are mutually independent except that $\widehat{W}_{y,t}$ and $\widehat{W}_{p,t}$ are correlated. The correlation between $\widehat{W}_{y,t}$ and $\widehat{W}_{p,t}$ is described by $E\left(d\widehat{W}_{y,t} d\widehat{W}_{p,t} \middle| \mathcal{F}_t \right) = \hat{\rho}dt$. This system is discretized by the Euler approximation and estimated by MLE via the Kalman filter. Data are quarterly, begin in the first quarter of 1952, and end in the second quarter of 2007.

Parameter	Estimate	Std. error
$\bar{\mu}_y$	0.578	0.064
κ_y	0.186	0.000
σ_y	0.316	0.044
υ_y	−0.006	0.049
$\hat{\upsilon}_y$	0.201	0.031
$\bar{\mu}_p$	0.758	0.139
κ_p	0.071	0.000
σ_p	0.374	0.013
υ_p	−0.010	0.045
$\hat{\upsilon}_p$	0.234	0.034
$\hat{\rho}$	−0.600	0.000

Table 2. Means and standard deviations of nominal yields in the model and in the data

This table reports the means and standard deviations of nominal bond yields in the model and in the data. Columns marked "Model" give statistics for nominal yields on nominal bonds in the model; columns marked "Data" give statistics for nominal yields on nominal bonds in the data. Yields are in annual percentages. Maturity is in years. Data are quarterly, begin in the first quarter of 1962, and end in the second quarter of 2007.

	Whole sample				Post 1985 sample			
Maturity	Mean		Std. dev.		Mean		Std. dev.	
	Model	Data	Model	Data	Model	Data	Model	Data
1	7.05	6.29	1.87	2.91	5.98.	5.19	0.82	2.06
2	6.95	6.54	1.68	2.85	6.03	5.56	0.72	2.05
3	6.86	6.68	1.51	2.76	6.06	5.76	0.63	2.00
5	6.71	6.87	1.22	2.67	6.08	6.07	0.51	1.89
7	6.60	7.02	1.01	2.61	6.09	6.30	0.42	1.84
10	6.47	7.08	0.78	2.55	6.09	6.42	0.32	1.78

Table 3. Estimates for the Coefficient of the Excess Consumption in the Regression (24)

This table presents the estimates and Newey-West standard errors for the following regression model:

$$r_{t+1}(n) = \alpha_0 + \alpha_1 \sum_{j=1}^{40} \phi^j \Delta \ln y_{t-j} + \alpha_2 \sum_{j=1}^{40} \phi^j \Delta \ln p_{t-j} + \varepsilon_{t+1}$$

where $r_{t+1}(n)$ is the nominal yield with maturity n. Following Wachter (2006), ϕ is set to equal 0.97. Note also that * and ** indicate results are significant at the 5% and 1% significance levels, respectively.

Maturity	Estimate of α_1	Std. error of α_1	Estimate of α_2	Std. error of α_2
1 year	0.192**	0.082	0.205**	0.024
2 years	0.146**	0.041	0.208**	0.012
3 years	0.122*	0.071	0.205**	0.020
5 years	0.087	0.062	0.203**	0.018
7 years	0.060	0.056	0.201**	0.017
10 years	0.044	0.051	0.199**	0.015

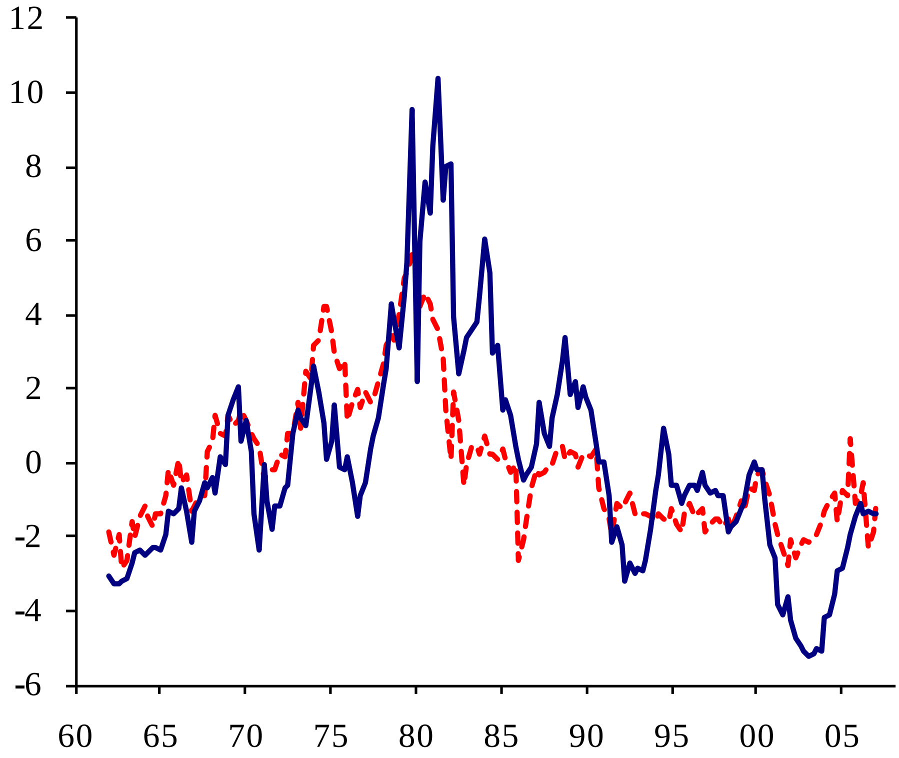

Fig. 1. Time series of the one-year yield in the data and predicted by in the data. The figure plots the time series of one-year yield in the data and predicted by the model. The solid line shows the time series of the nominal one-month yield in quarterly data. The broken line shows the implied time series by the model. Both series are de-meaned. Data are quarterly, begin in the first quarter of 1962, and end in the second quarter of 2007.

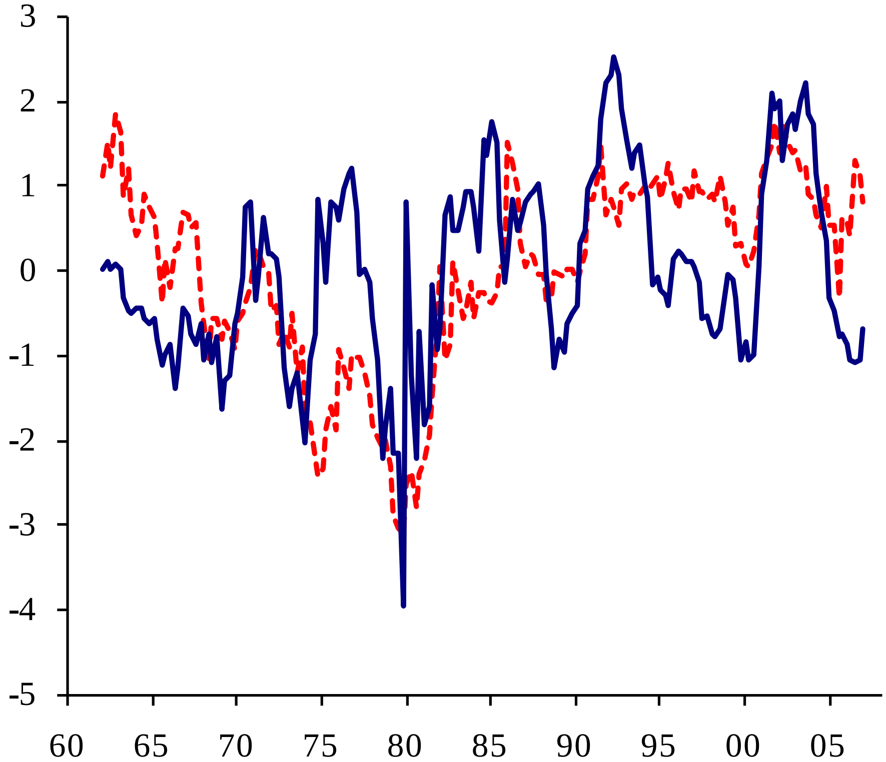

Fig. 2. Time series of the yield spread in the data and predicted by the model. The figure plots the time series of the yield spread in the data and predicted by the model. The yield spread is the difference in yields between the ten-year nominal bond and the one-year bond. The solid line shows the time series of the yield spread between bonds in the data. The broken line shows the implied time series by the model. Both series are de-meaned. Data are quarterly, begin in the first quarter of 1962, and end in the second quarter of 2007.

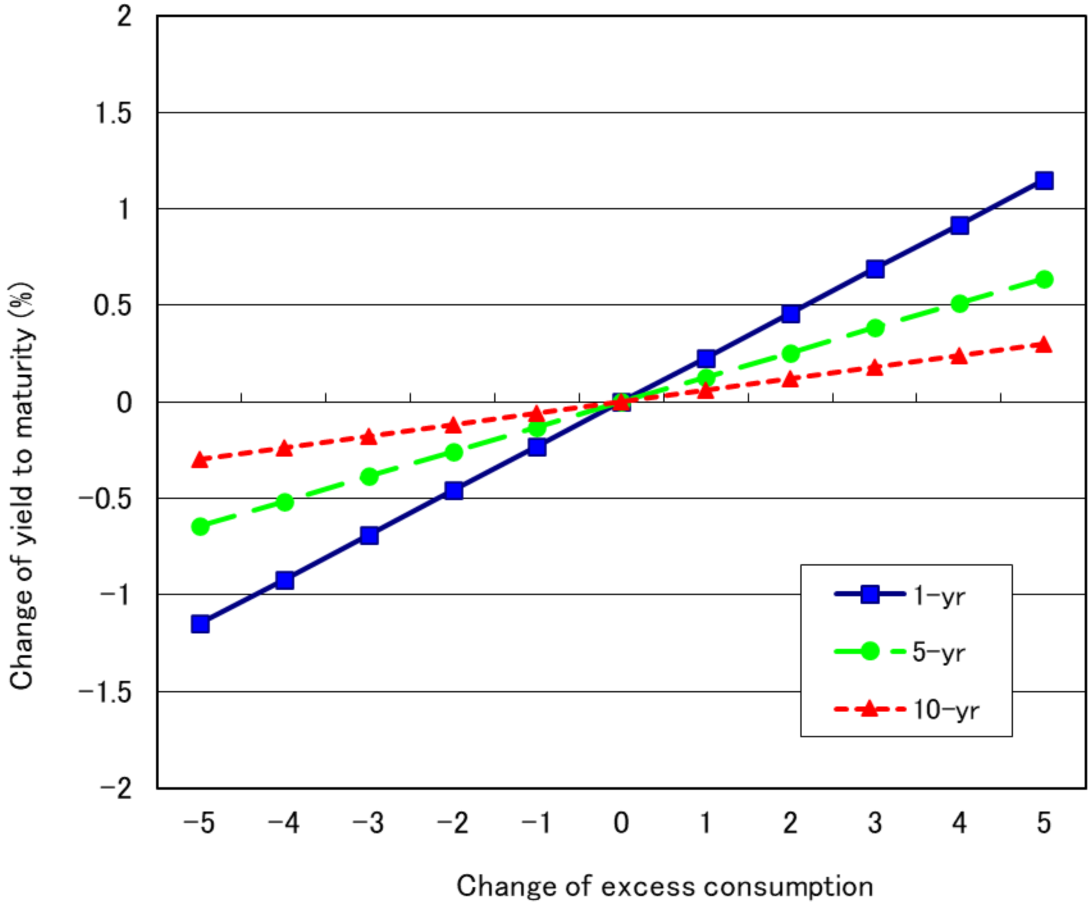

Fig. 3. Effects of the excess consumptions on the short-, middle-, and long-yields. This figure plots the change in one-, five- and ten-year yields against the change in excess consumption, assuming that two types of excess consumption vary the same amount.

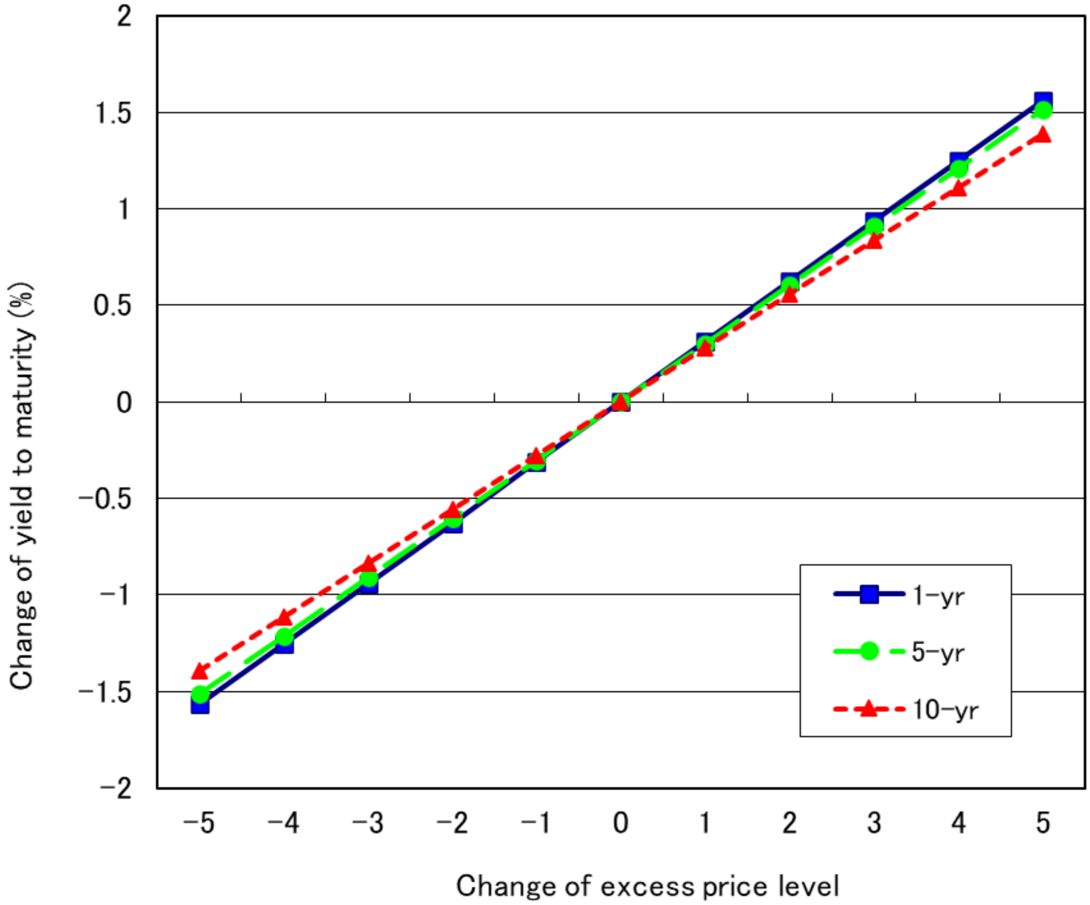

Fig. 4. Effects of the excess price level on the short-, middle-, and long-yields. This figure plots the change in one-, five- and ten-year yields against the change in excess price level, assuming that two types of excess price level vary the same amount.