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THE BEST CONSTANT OF THE DISCRETE ℓ^p SOBOLEV INEQUALITY ON THE COMPLETE GRAPH

By

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Abstract. Let p and q be real numbers > 1 satisfying $1/p + 1/q = 1$. We determine the best constant of the discrete ℓ^p Sobolev inequality on the complete graph K_N with N vertices. We introduce the discrete Laplacian $\boldsymbol{A}$, which is an $N \times N$ real symmetric positive-semidefinite matrix. $\boldsymbol{A}$ has an eigenvalue 0 whose eigenvector is $\boldsymbol{1} = {}^t(1, \cdots, 1) \in \mathbf{C}^N$. We define the pseudo Green's matrix $\boldsymbol{G}_*$ so that it satisfies $\boldsymbol{A}\boldsymbol{G}_* = \boldsymbol{G}_*\boldsymbol{A} = \boldsymbol{I} - N^{-1}\boldsymbol{1}\,{}^t\boldsymbol{1}$ and $\boldsymbol{G}_*$ also has the property of a reproducing kernel. Applying the Hölder inequality to the reproducing relation, we have the discrete ℓ^p Sobolev inequality. The discrete ℓ^p Sobolev inequality shows that the maximum of $u(i)$ is estimated in constant multiples of the potential energy. The potential energy is the ℓ^p norm of the difference between $u(i)$ and $u(j)$, where the vertices i and j are connected to an edge. The best constant is $N^{-1}(N-1)^{1/q}$. The equality holds for any column of $\boldsymbol{G}_*$.

1. Discrete Laplacian of complete graph

For any fixed $N = 2, 3, 4, \cdots$, we set the indices of vertices on the complete graph K_N as Figure 1.

We introduce the edge set

$$\begin{aligned} e = \{(i,j) \,|\, &(0,1), (0,2), (0,3), \cdots, (0, N-1), \\ &(1,2), (1,3), \cdots, (1, N-1), \\ &(2,3), \cdots, (2, N-1), \\ &\cdots, \\ &(N-2, N-1)\}, \end{aligned}$$

that is

$$e = \{(i,j) \,|\, 0 \le i < j \le N-1\},$$

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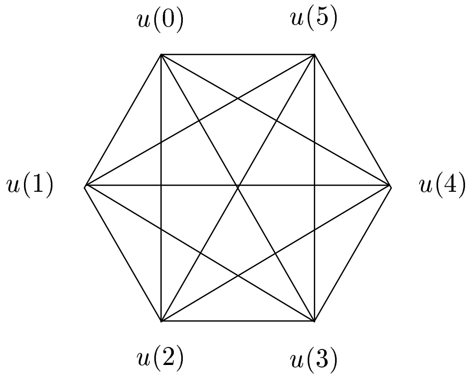

Figure 1 Complete graph K_6.

where the vertices i and j are connected to an edge. We introduce

$$\boldsymbol{B}\boldsymbol{u} = \begin{pmatrix} u(i) - u(j) \end{pmatrix}_{(i,j)\in e} = \begin{pmatrix} u(0)-u(1) \\ u(0)-u(2) \\ \vdots \\ u(0)-u(N-1) \\ u(1)-u(2) \\ \vdots \\ u(1)-u(N-1) \\ \vdots \\ u(N-2)-u(N-1) \end{pmatrix}_{\frac{N(N-1)}{2}\times 1},$$

where

$$\boldsymbol{B} = \begin{pmatrix} 1 & -1 & & & \\ 1 & & -1 & & \\ \vdots & & & \ddots & \\ 1 & & & & -1 \\ 0 & 1 & -1 & & \\ & \vdots & & \ddots & \\ & 1 & & & -1 \\ & & & & \vdots \\ & & & 1 & -1 \end{pmatrix}_{\frac{N(N-1)}{2}\times N}, \qquad \boldsymbol{u} = \begin{pmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(N-1) \end{pmatrix}_{N\times 1}.$$

The discrete Laplacian [6] $\boldsymbol{A}$ is defined as

$$\boldsymbol{A} = \boldsymbol{B}^*\boldsymbol{B} = \begin{pmatrix} N\delta(i-j) - 1 \end{pmatrix}_{0\le i,j\le N-1}, \tag{1.1}$$

where the delta function

$$\delta(i) = \begin{cases} 1 & (\mathrm{Mod}(i,N)=0) \\ 0 & (\mathrm{Mod}(i,N)\neq 0) \end{cases} \qquad (i \in \mathbf{Z}).$$

Here, we show the concrete form of $\boldsymbol{A} = \boldsymbol{A}_N$ $(N = 2, 3, 4)$ as follows:

$$\boldsymbol{A}_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \boldsymbol{A}_3 = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

$$\boldsymbol{A}_4 = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

Here $\boldsymbol{A}$ is an $N \times N$ real symmetric positive-semidefinite matrix which has an eigenvalue 0 and whose eigenvector is $\mathbf{1} = {}^t(1, 1, \cdots, 1) \in \mathbf{C}^N$. We introduce the pseudo Green's matrix

$$\boldsymbol{G}_* = \lim_{a \to +0} \left(\boldsymbol{G}(a) - a^{-1}\boldsymbol{E}_0 \right), \tag{1.2}$$

where $\boldsymbol{G}(a) = (\boldsymbol{A} + a\boldsymbol{I})^{-1}$ is a Green's matrix and $\boldsymbol{E}_0 = N^{-1}\mathbf{1}{}^t\mathbf{1}$ is a projection matrix to the eigenspace corresponding to the eigenvalue 0 of $\boldsymbol{A}$. $\boldsymbol{G}_*$ satisfies $\boldsymbol{A}\boldsymbol{G}_* = \boldsymbol{G}_*\boldsymbol{A} = \boldsymbol{I} - \boldsymbol{E}_0$, $\boldsymbol{G}_*\boldsymbol{E}_0 = \boldsymbol{E}_0\boldsymbol{G}_* = \boldsymbol{O}$. Here, $\boldsymbol{I}$ is the $N \times N$ identity matrix and $\boldsymbol{O}$ is the $N \times N$ zero matrix. From [11, Proposition 2.3 ($M = 1$)], using $\boldsymbol{E}_0$, $\boldsymbol{A}$ and $\boldsymbol{G}_*$ are rewritten as

$$\boldsymbol{A} = N(\boldsymbol{I} - \boldsymbol{E}_0), \qquad \boldsymbol{G}_* = \frac{1}{N}(\boldsymbol{I} - \boldsymbol{E}_0) = \frac{1}{N^2}\boldsymbol{A}.$$

In fact, we have

$$\boldsymbol{A}\boldsymbol{G}_* = N(\boldsymbol{I} - \boldsymbol{E}_0)\frac{1}{N}(\boldsymbol{I} - \boldsymbol{E}_0) = \boldsymbol{I}^2 - 2\boldsymbol{E}_0 + \boldsymbol{E}_0^2 = \boldsymbol{I} - \boldsymbol{E}_0,$$
$$\boldsymbol{G}_*\boldsymbol{E}_0 = \frac{1}{N}(\boldsymbol{I} - \boldsymbol{E}_0)\boldsymbol{E}_0 = \frac{1}{N}(\boldsymbol{E}_0 - \boldsymbol{E}_0^2) = \boldsymbol{O}.$$

$\boldsymbol{G}_*\boldsymbol{A} = \boldsymbol{I} - \boldsymbol{E}_0$ and $\boldsymbol{E}_0\boldsymbol{G}_* = \boldsymbol{O}$ hold similarly. Thus, $\boldsymbol{G}(a)$ is an inverse matrix of $\boldsymbol{A} + a\boldsymbol{I}$ and $\boldsymbol{G}_*$ is a Penrose-Moore generalized inverse matrix [12] of $\boldsymbol{A}$. The concrete form of $\boldsymbol{G}_*$ is

$$\boldsymbol{G}_* = \frac{1}{N^2}\Bigg(N\delta(i-j) - 1 \Bigg)_{0 \le i,j \le N-1}.$$

This paper is composed of four sections. In section 2, we introduce ℓ^p norm and show Theorem 2.1. In section 3, we present a reproducing relation and prove Theorem 2.1. In section 4, we see the concrete case K_4, that is, the regular tetrahedron. In section 5, we consider the discrete ℓ^p Sobolev inequality corresponding to [11, Theorem 1.2 ($M=1$)].

2. Discrete ℓ^p Sobolev inequality

For $\boldsymbol{u},\boldsymbol{v}\in\mathbf{C}^N$, we introduce the inner products

$$(\boldsymbol{u},\,\boldsymbol{v})=\boldsymbol{v}^*\boldsymbol{u},\qquad \|\,\boldsymbol{u}\,\|^2=(\boldsymbol{u},\boldsymbol{u}),$$

where $\boldsymbol{u}^*$ denotes $\boldsymbol{u}^*={}^t\overline{\boldsymbol{u}}$. For $1/p+1/q=1,\ p,q>1$, we introduce the ℓ^p norm and ℓ^q norm

$$\|\,\boldsymbol{u}\,\|_p=\left(\sum_{j=0}^{N-1}|u(j)|^p\right)^{1/p},\qquad \|\,\boldsymbol{u}\,\|_q=\left(\sum_{j=0}^{N-1}|u(j)|^q\right)^{1/q}.$$

For $\boldsymbol{u},\boldsymbol{v}\in\mathbf{C}_0^N:=\{\,\boldsymbol{u}\,|\,\boldsymbol{u}\in\mathbf{C}^N \text{ and } {}^t\mathbf{1}\boldsymbol{u}=0\,\}$, we introduce the inner product

$$(\boldsymbol{u},\,\boldsymbol{v})_A=(\boldsymbol{A}\boldsymbol{u},\,\boldsymbol{v})=\boldsymbol{v}^*\boldsymbol{A}\boldsymbol{u}=\boldsymbol{v}^*\boldsymbol{B}^*\boldsymbol{B}\boldsymbol{u}=(\boldsymbol{B}\boldsymbol{u},\,\boldsymbol{B}\boldsymbol{v}),$$
$$\|\,\boldsymbol{u}\,\|_A^2=(\boldsymbol{u},\boldsymbol{u})_A=(\boldsymbol{B}\boldsymbol{u},\,\boldsymbol{B}\boldsymbol{u})=\|\,\boldsymbol{B}\boldsymbol{u}\,\|^2.$$

$(\cdot,\cdot)_A$ are proved to be an inner product afterwards. We rewrite $\|\,\boldsymbol{B}\boldsymbol{u}\,\|_p$ as

$$\|\,\boldsymbol{B}\boldsymbol{u}\,\|_p=\left(\sum_{0\le i<j\le N-1}|\,u(i)-u(j)\,|^p\right)^{1/p}.$$

We introduce the N-dimensional vector

$$\boldsymbol{\delta}_j={}^t(\quad\delta(i-j)\quad)_{0\le i\le N-1},\qquad \delta(i-j)=\begin{cases}1 & (i=j)\\ 0 & (i\ne j)\end{cases}.$$

In particular, $\|\boldsymbol{B}\boldsymbol{G}_*\boldsymbol{\delta}_j\|_p$ is equal to the best constant of the discrete ℓ^p Sobolev inequality. The most important fact is that $\|\boldsymbol{B}\boldsymbol{G}_*\boldsymbol{\delta}_j\|_p$ is independent of j. For any j_0 $(0\le j_0\le N-1)$, we have

$$\|\,\boldsymbol{B}\boldsymbol{G}_*\boldsymbol{\delta}_{j_0}\,\|_p=\frac{(N-1)^{1/p}}{N},\qquad \|\,\boldsymbol{B}\boldsymbol{G}_*\boldsymbol{\delta}_{j_0}\,\|_q=\frac{(N-1)^{1/q}}{N}.\tag{2.1}$$

In this paper, we obtain the best constants of the discrete ℓ^p Sobolev inequality on K_N as the following theorem.

Theorem 2.1. *For any* $\boldsymbol{u} = {}^t(u(0), u(1), \cdots, u(N-1)) \in \mathbf{C}_0^N$*, there exists a positive constant* C *which is independent of* $\boldsymbol{u}$*, such that the discrete* ℓ^p *Sobolev inequality*

$$\max_{0\le j\le N-1} |\, u(j)\,| \le C \,\|\, \boldsymbol{Bu}\,\|_p = C \left(\sum_{0\le i<j\le N-1} |\, u(i) - u(j)\,|^p \right)^{1/p} \tag{2.2}$$

holds. Among such C*, the best constant is*

$$C_0 = \max_{0\le j\le N-1} \|\, \boldsymbol{BG}_*\boldsymbol{\delta}_j \,\|_q = \|\, \boldsymbol{BG}_*\boldsymbol{\delta}_{j_0} \,\|_q = \frac{(N-1)^{1/q}}{N},$$

where any number j_0 $(0 \le j_0 \le N-1)$*. If we replace* C *by* C_0 *in* (2.2)*, the equality holds if and only if* $\boldsymbol{u}$ *is parallel to*

$$\boldsymbol{G}_*\boldsymbol{\delta}_{j_0} = \frac{1}{N^2}\Big(N\delta(i-j_0) - 1 \Big)_{0\le i\le N-1}.$$

We have obtained the best constant of the discrete ℓ^2 Sobolev inequality (2.2) on various graphs. In our previous papers [11, Theorem 1.1] on the complete graph K_N, in [4, 5, 10] on N-sided polygons, in [2, 8, 9] on regular polyhedra, in [1, 3, 7] on truncated regular polyhedra. In this paper, we extend the discrete ℓ^2 Sobolev inequality into the discrete ℓ^p Sobolev inequality on [11].

3. Proof of Theorem

First, we show that $\boldsymbol{G}_*$ is a reproducing matrix for the inner product $(\cdot,\cdot)_A$.

Lemma 3.1. *For any* $\boldsymbol{u} \in \mathbf{C}_0^N$ *and fixed* j $(0 \le j \le N-1)$*, we have the following reproducing relation:*

$$u(j) = (\boldsymbol{u},\, \boldsymbol{G}_*\boldsymbol{\delta}_j)_A = (\boldsymbol{Bu},\, \boldsymbol{BG}_*\boldsymbol{\delta}_j). \tag{3.1}$$

Proof of Lemma 3.1. Noting $\boldsymbol{G}_*^* = \boldsymbol{G}_*$, we have

$$(\boldsymbol{u},\, \boldsymbol{G}_*\boldsymbol{\delta}_j)_A = {}^t\boldsymbol{\delta}_j \boldsymbol{G}_* \boldsymbol{A} \boldsymbol{u} = {}^t\boldsymbol{\delta}_j (\boldsymbol{I} - \boldsymbol{E}_0)\boldsymbol{u} = {}^t\boldsymbol{\delta}_j \boldsymbol{u} - \frac{1}{N}\boldsymbol{1}\,{}^t\boldsymbol{1}\boldsymbol{u} = u(j).$$

This shows Lemma 3.1. ■

Second, we prove the main theorem.

Proof of Theorem 2.1. For any $\boldsymbol{u} \in \mathbf{C}_0^N$, applying the Hölder inequality to (3.1), we have

$$|\, u(j)\,| = |\, (\boldsymbol{Bu},\, \boldsymbol{BG}_*\boldsymbol{\delta}_j)\,| \le \|\, \boldsymbol{Bu}\,\|_p \|\, \boldsymbol{BG}_*\boldsymbol{\delta}_j \,\|_q. \tag{3.2}$$

Here, for any j_0 $(0 \le j_0 \le N-1)$, we put

$$C_0 = \max_{0\le j\le N-1} \| \boldsymbol{B}\boldsymbol{G}_*\boldsymbol{\delta}_j \|_q = \| \boldsymbol{B}\boldsymbol{G}_*\boldsymbol{\delta}_{j_0} \|_q.$$

Taking the maximum with respect to j on (3.2), we obtain the discrete ℓ^p Sobolev inequality

$$\max_{0\le j\le N-1} | u(j) | \le C_0 \| \boldsymbol{B}\boldsymbol{u} \|_p. \tag{3.3}$$

If we take $\boldsymbol{u} = \boldsymbol{G}_*\boldsymbol{\delta}_{j_0}$ in (3.3) and use (2.1), then we have

$$\max_{0\le j\le N-1} | {}^t\boldsymbol{\delta}_j \boldsymbol{G}_*\boldsymbol{\delta}_{j_0} | \le \| \boldsymbol{B}\boldsymbol{G}_*\boldsymbol{\delta}_{j_0} \|_q \| \boldsymbol{B}\boldsymbol{G}_*\boldsymbol{\delta}_{j_0} \|_p = \frac{N-1}{N^2}.$$

Combining this with the trivial inequality

$$\frac{N-1}{N^2} = | {}^t\boldsymbol{\delta}_{j_0} \boldsymbol{G}_*\boldsymbol{\delta}_{j_0} | \le \max_{0\le j\le N-1} | {}^t\boldsymbol{\delta}_j \boldsymbol{G}_*\boldsymbol{\delta}_{j_0} |,$$

we have

$$\max_{0\le j\le N-1} | {}^t\boldsymbol{\delta}_j \boldsymbol{G}_*\boldsymbol{\delta}_{j_0} | = C_0 \| \boldsymbol{B}\boldsymbol{G}_*\boldsymbol{\delta}_{j_0} \|_p.$$

This shows that C_0 is the best constant of (3.3) and the equality holds for any column of $\boldsymbol{G}_*$. This completes the proof of Theorem 2.1. ■

4. The case of $N = 4$

In this section, we state the concrete case of $N = 4$. The plane graph expression of the regular tetrahedron in Figure 2 is the same as K_4. The regular tetrahedron consists of 4 vertices $(v = 4)$, 4 faces $(f = 4)$ and 6 edges $(e = 6)$, which satisfies Euler polyhedron theorem $v + f = e + 2$.

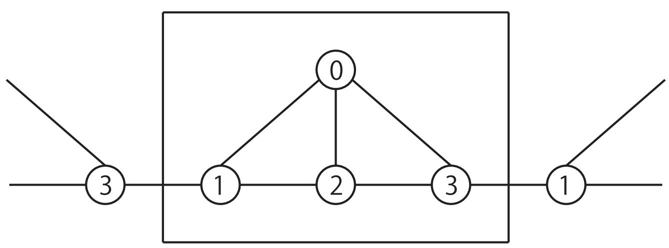

Figure 2 Plane graph of regular tetrahedron.

Theorem 4.1. *For any* $\boldsymbol{u} = {}^t(u(0), u(1), u(2), u(3))$ *and* $u(0)+u(1)+u(2)+u(3)=0$*, there exists a positive constant* C *which is independent of* $\boldsymbol{u}$*, such that the discrete* ℓ^p *Sobolev inequality*

$$\max_{0\le j\le 3} |\,u(j)\,| \le \frac{3^{1/q}}{4}\Big[|\,u(0)-u(1)\,|^p + |\,u(0)-u(2)\,|^p + |\,u(0)-u(3)\,|^p + \\ |\,u(1)-u(2)\,|^p + |\,u(1)-u(3)\,|^p + |\,u(2)-u(3)\,|^p\Big]^{1/p}$$

holds. The equality holds if and only if $\boldsymbol{u}$ *is parallel to*

$$\begin{pmatrix} 3\\ -1\\ -1\\ -1\end{pmatrix} \ \mathit{or}\ \begin{pmatrix} -1\\ 3\\ -1\\ -1\end{pmatrix} \ \mathit{or}\ \begin{pmatrix} -1\\ -1\\ 3\\ -1\end{pmatrix} \ \mathit{or}\ \begin{pmatrix} -1\\ -1\\ -1\\ 3\end{pmatrix}.$$

Physical meaning of Theorem 4.1 is as follows. We assume a classical mechanical model of the regular tetrahedron. We put the 4 atoms on vertices of the regular tetrahedron. Its neighboring two atoms are connected to a linear spring. The spring constant is 1. If $u(i)$ takes a real value, then $u(i)$ represents a deviation from the steady state. The discrete ℓ^p Sobolev inequality shows that the maximum of deviation $|u(i)|$ is estimated in a constant multiples of the potential energy $\|\,\boldsymbol{B}\boldsymbol{u}\,\|_p$. Hence, the best contant $3^{1/q}/4$ represents rigidity of the mechanical model on regular tetrahedron.

5. Discussion

In [11, Theorem 1.2 ($M=1$)], the best constant of the following inequality

$$\max_{0\le j\le N-1} |\,u(j)\,| \le C\left(\sum_{0\le i<j\le N-1} |\,u(i)-u(j)\,|^p + a\sum_{i=0}^{N-1} |\,u(i)\,|^p\right)^{1/p} \tag{5.1}$$

is shown for $p=2$ and some constant $0<a<\infty$. We show that the proof of Theorem 2.1 of the present paper can not be applicable to obtain the best constant of the inequality (5.1) for the case $p>1$ and $p\neq 2$.

We introduce the Green's matrix [11, Proposition 2.2 ($M=1$)]

$$\boldsymbol{G}(a) = \\ \frac{1}{N+a}\boldsymbol{I} + \left(\frac{1}{a} - \frac{1}{N+a}\right)\boldsymbol{E}_0 = \frac{1}{a(N+a)}\Big(a\delta(i-j)+1\Big)_{0\le i,j\le N-1}.$$

We use $\boldsymbol{G} = \boldsymbol{G}(a)$ for short. From [11, Lemma 3.3 ($M = 1$)], for any $\boldsymbol{u} \in \mathbf{C}^N$, the reproducing relation

$$\begin{aligned} u(j) &= ((\boldsymbol{A} + a\boldsymbol{I})\boldsymbol{u},\, \boldsymbol{G}\boldsymbol{\delta}_j) = (\boldsymbol{A}\boldsymbol{u},\, \boldsymbol{G}\boldsymbol{\delta}_j) + a(\boldsymbol{u},\, \boldsymbol{G}\boldsymbol{\delta}_j) \\ &= (\boldsymbol{B}\boldsymbol{u},\, \boldsymbol{B}\boldsymbol{G}\boldsymbol{\delta}_j) + a(\boldsymbol{u},\, \boldsymbol{G}\boldsymbol{\delta}_j) \end{aligned}$$

holds. Applying the Hölder inequality to the above reproducing relation, we have

$$\begin{aligned} |\, u(j)\,| &\le |(\boldsymbol{B}\boldsymbol{u},\, \boldsymbol{B}\boldsymbol{G}\boldsymbol{\delta}_j)| + a|(\boldsymbol{u},\, \boldsymbol{G}\boldsymbol{\delta}_j)| \\ &\le \|\, \boldsymbol{B}\boldsymbol{u}\,\|_p \|\, \boldsymbol{B}\boldsymbol{G}\boldsymbol{\delta}_j\,\|_q + a\|\,\boldsymbol{u}\,\|_p \|\,\boldsymbol{G}\boldsymbol{\delta}_j\,\|_q \\ &= \begin{pmatrix} \|\,\boldsymbol{B}\boldsymbol{u}\,\|_p \\ a^{1/p}\|\,\boldsymbol{u}\,\|_p \end{pmatrix} \cdot \begin{pmatrix} \|\,\boldsymbol{B}\boldsymbol{G}\boldsymbol{\delta}_j\,\|_q \\ a^{1/q}\|\,\boldsymbol{G}\boldsymbol{\delta}_j\,\|_q \end{pmatrix} \\ &\le \left(\|\,\boldsymbol{B}\boldsymbol{u}\,\|_p^p + a\|\,\boldsymbol{u}\,\|_p^p\right)^{1/p} \left(\|\,\boldsymbol{B}\boldsymbol{G}\boldsymbol{\delta}_j\,\|_q^q + a\|\,\boldsymbol{G}\boldsymbol{\delta}_j\,\|_q^q\right)^{1/q}. \end{aligned} \tag{5.2}$$

We note the second and the third inequality of (5.2) do not simultaneously hold the equality. Without loss of generality, we can fix $j = 0$. Suppose to the contrary, the second and the third inequality of (5.2) simultaneously hold the equality. From the second inequality of (5.2), we consider the following two equalities:

$$|(\boldsymbol{B}\boldsymbol{u},\, \boldsymbol{B}\boldsymbol{G}\boldsymbol{\delta}_0)| = \|\,\boldsymbol{B}\boldsymbol{u}\,\|_p \|\,\boldsymbol{B}\boldsymbol{G}\boldsymbol{\delta}_0\,\|_q, \tag{5.3}$$

$$|(\boldsymbol{u},\, \boldsymbol{G}\boldsymbol{\delta}_0)| = \|\,\boldsymbol{u}\,\|_p \|\,\boldsymbol{G}\boldsymbol{\delta}_0\,\|_q. \tag{5.4}$$

We put $g(i)$ $(0 \le i \le N-1)$ as

$$\boldsymbol{G}\boldsymbol{\delta}_0 = {}^t(g(0), \cdots, g(N-1)) = \frac{1}{a(a+N)}\,{}^t(a+1, 1, \cdots, 1) \in \mathbf{C}^N.$$

From (5.4), we have

$$\frac{|\, g(0)\,|^q}{|\, u(0)\,|^p} = \frac{|\, g(1)\,|^q}{|\, u(1)\,|^p} = \cdots = \frac{|\, g(N-1)\,|^q}{|\, u(N-1)\,|^p}. \tag{5.5}$$

Inserting

$$\boldsymbol{u} = {}^t(u(0), \cdots, u(N-1)) = {}^t(b, 1, \cdots, 1) \in \mathbf{C}^N$$

into (5.5), we obtain

$$|b|^p = (a+1)^q \qquad \Leftrightarrow \qquad b = \pm(a+1)^{q/p} = \pm(a+1)^{\frac{1}{p-1}}. \tag{5.6}$$

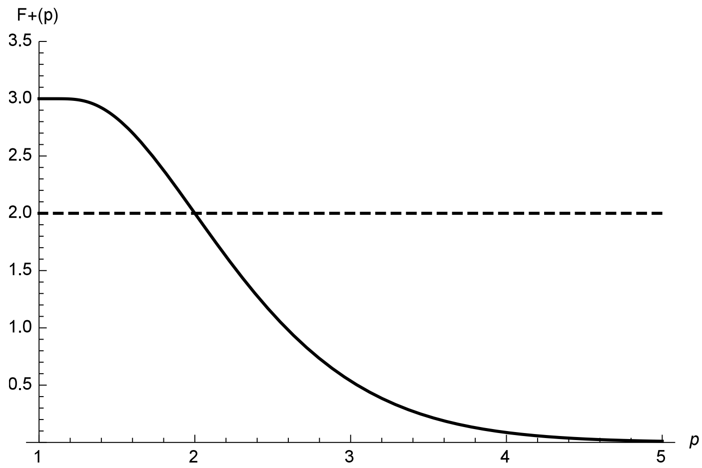

Figure 3 The graph of $F_+(p)$ for $a = 2$.

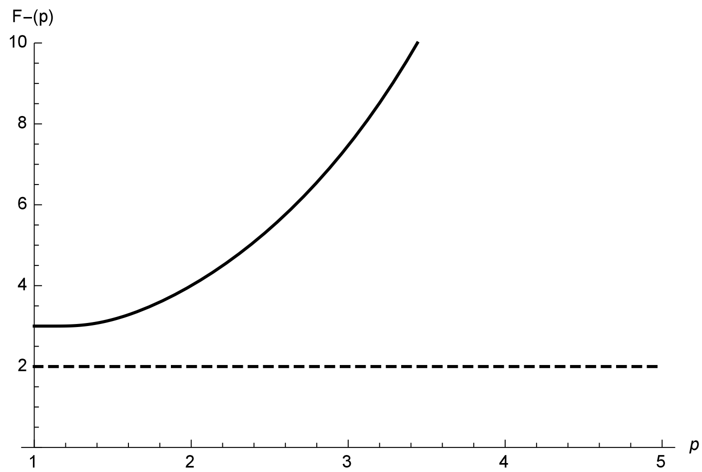

Figure 4 The graph of $F_-(p)$ for $a = 2$.

Then, since

$$|(\boldsymbol{Bu},\, \boldsymbol{BG\delta}_0)| = (N-1)|b-1|\frac{1}{a+N},$$
$$\|\,\boldsymbol{Bu}\,\|_p = (N-1)^{1/p}|b-1|, \qquad \|\,\boldsymbol{BG\delta}_0\,\|_q = (N-1)^{1/q}\frac{1}{a+N},$$

we can see that (5.3) also holds. Successively, from the third inequality of (5.2), we have

$$\frac{\| \boldsymbol{B G \delta}_0 \|_q^q}{\| \boldsymbol{B u} \|_p^p} = \frac{\| \boldsymbol{G \delta}_0 \|_q^q}{\| \boldsymbol{u} \|_p^p} \qquad \Leftrightarrow$$

$$\frac{(N-1)\dfrac{1}{(a+N)^q}}{(N-1)|b-1|^p} = \frac{\dfrac{1}{(a(a+N))^q}\left((a+1)^q + N - 1\right)}{|b|^p + N - 1} \qquad \Leftrightarrow$$

$$a^q = \frac{|b-1|^p}{|b|^p + N - 1}\left((a+1)^q + N - 1\right).$$

Inserting (5.6) into the above relation, we have

$$a = \left| \pm (a+1)^{\frac{1}{p-1}} - 1 \right|^{p-1}. \tag{5.7}$$

We can see that (5.7) holds only in the case of $p = 2$. Following Figure 3 and 4 show the behavior of

$$F_{\pm}(p) = \left| \pm (a+1)^{\frac{1}{p-1}} - 1 \right|^{p-1}.$$

To sum up the above argument, we can see that the proof of Theorem 2.1 of the present paper can not be applicable to obtain the best constant of the inequality (5.1) for the case of $p > 1$ and $p \neq 2$.

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