Quadrangulations of a Polygon with spirality

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Abstract

Given an n-sided polygon P on the plane with $n \geq 4$, a *quadrangulation* of P is a geometric plane graph such that the boundary of the outer face is P and that each finite face is quadrilateral. Clearly, P is *quadrangulatable* (i.e., admits a quadrangulation) only if n is even, but there is a non-quadrangulatable even-sided polygon. Ramaswami et al. [10] proved that every n-sided polygon P with $n \geq 4$ even admits a quadrangulation with at most $\lfloor \frac{n-2}{4} \rfloor$ Steiner points, where a *Steiner point* for P is an auxiliary point which can be put in any position in the interior of P. In this paper, introducing the notion of the *spirality* of P to control a structure of P (independent of n), we estimate the number of Steiner points to quadrangulate P.

1 Introduction

An n-sided *polygon* P is a simple cycle with n straight segments in the plane. A *triangulation* of P is a geometric plane graph with vertex set $V(P)$ such that its outer cycle coincides with P and that each finite face is triangular, where $V(P)$ is the vertex set of P. The following is a well-known result for a triangulation on a polygon, which was used in the elegant proof of the *art gallery problem* by Fisk [4].

Proposition 1. *Every polygon with at least three sides admits a triangulation.*

We turn our attention to a *quadrangulation* of a polygon P, that is, a geometric plane graph with vertex set $V(P)$ such that the outer cycle coincides with P and that each finite face is quadrilateral. Let us consider whether or not, a given polygon P is *quadrangulatable*, i.e., P admits a quadrangulation. (Figure 1 shows an example of a polygon P and its quadrangulation. We always give a "proper vertex-2-coloring of $V(P)$", that is, giving black and white alternately to the vertex sequence on the boundary of P. Since every plane quadrangulation G on P is bipartite, the 2-coloring of $V(P)$ extends to the unique 2-coloring of G.)

We immediately find a combinatorial condition on the parity of length of P, that is, if an n-sided polygon P with $n \geq 4$ admits a quadrangulation, then n must be even. Hence we always consider an *even-sided* polygon, i.e., an n-sided polygon with $n \geq 4$ even. It is easy to see that every even-sided *convex* polygon is quadrangulatable, but we find a number of non-quadrangulatable even-sided polygons, as shown in Figure 2.

For an even-sided polygon P on the plane, a *Steiner point* is an auxiliary point which can be put in any position in the interior of P. Ramaswami et al. [10] asked whether

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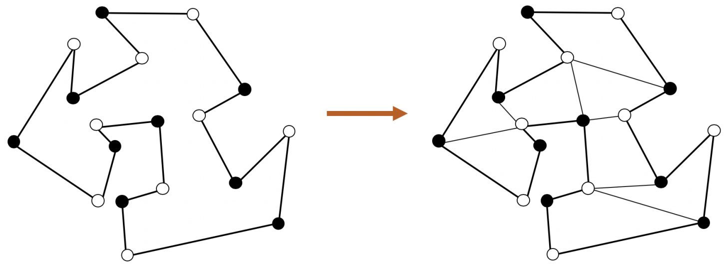

Figure 1: A polygon P and a quadrangulation of P

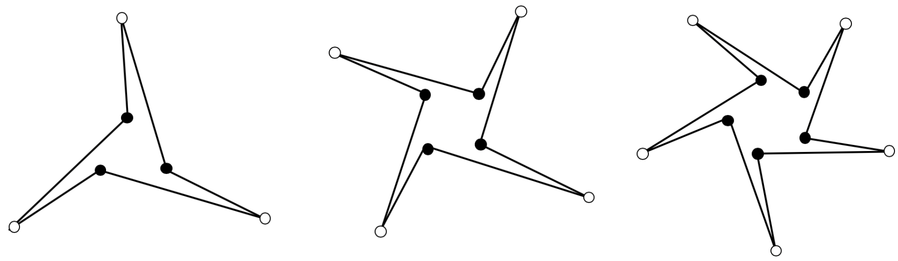

Figure 2: Non-quadrangulatable even-sided polygons

P admits a quadrangulation with a set S of Steiner points, that is, whether P admits a geometric plane graph G with vertex set $V(P) \cup S$ such that the outer cycle of G coincides with P and that each finite face is quadrilateral. (See Figure 3.) Note that no odd-sided polygon P admits a quadrangulation no matter how many Steiner points are added, since every quadrangulation is bipartite.

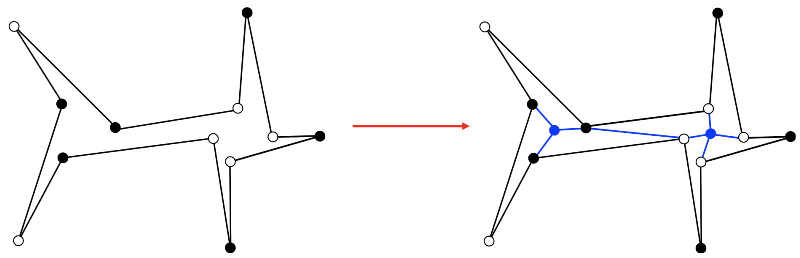

Figure 3: Non-quadrangulatable polygon P and a quadrangulation of P with two Steiner points

Ramaswami et al. [10] proved the the following result:

Theorem 2 (Ramaswami et al. [10]). *Every n-sided polygon with $n \geq 4$ even admits a quadrangulation with at most $\lfloor \frac{n-2}{4} \rfloor$ Steiner points.*

On the other hand, Nakamoto et al. [9] defined the notion of "spirality" of a polygon P, which measures how far P is from being convex in a sense. We explain this notion, as follows.

Let P be an n-sided polygon, and let $v_1 \cdots v_n$ be the sequence of the n vertices along the boundary of P, where $V(P) = \{v_1, \ldots, v_n\}$. In this case, we write $P = v_1 \cdots v_n$. A vertex v_i is *convex* (resp., *concave*) if the inner angle at v_i is less (resp., greater) than π. The interval $[v_p \cdots v_q]$ of P (possibly $p = q$) is a *spiral* if v_i is concave for $i = p, p+1, \ldots, q$ but v_{p-1} and v_{q+1} are convex, where the subscripts are taken modulo n. The *spirality* of

P, denoted by $sp(P)$, is the number of the spirals of P. For example, see Figure 4. Note that P is convex if and only if $sp(P) = 0$. Moreover, $sp(P)$ is independent of n, but we obviously have $sp(P) \leq \frac{n}{2}$, where the equality holds for the one in Figure 4(c). Therefore, we wonder if the quadrangulatability of a polygon P can be described by the spirality of P, and the following result answers this question.

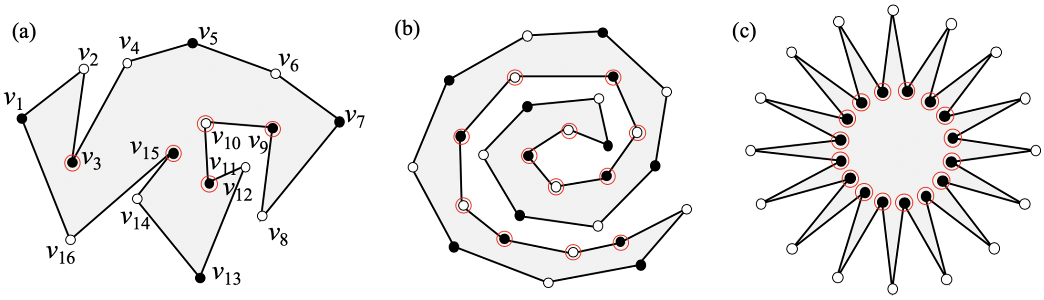

Figure 4: (a) $P = v_1 \cdots v_{16}$ with spirality 3, in which $[v_3]$, $[v_9v_{10}v_{11}]$ and $[v_{15}]$ are three spirals. (b) A polygon with spirality 1. (c) An n-sided polygon with spirality $\frac{n}{2}$. The marked vertices are concave.

Theorem 3 (Nakamoto et al. [9])**.** *Let P be an n-sided polygon P with $n \geq 4$ even. If $sp(P) \leq 1$, then P is quadrangulatable.*

The condition of the spirality in Theorem 3 is best possible, since there exists a non-quadrangulatable polygon with spirality 2, as shown in Figure 5. Moreover, we can construct a non-quadrangulatable polygon with spirality 2 which has arbitrarily large number of vertices, as in the right of Figure 5.

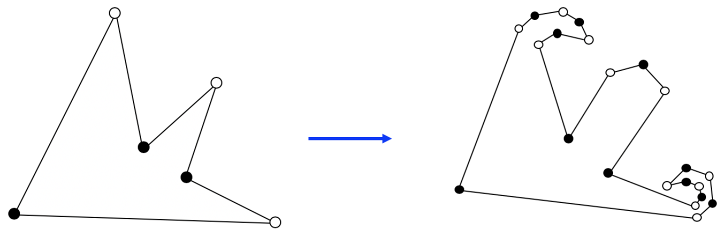

Figure 5: Non-quadrangulatable polygon with spirality 2 and its extension

When considering the quadrangulatability of an even-sided polygon P, there are two directions, in which one is to estimate the number of Steiner points to quadrangulate P with them (Theorem 2), and the other is to bound the spirality of P to be quadrangulatable (Theorem 3). Combining them, we have the following problem:

Problem 4. *Let P be an n-sided polygon with $n \geq 4$ even which has spirality k. Can we estimate the number of Steiner points to quadrangulate P by a function of k?*

Our main contribution of this paper is to prove the following, answering Problem 4:

Theorem 5. *Let P be an n-sided polygon with $n \geq 4$ even. If $sp(P) = k \geq 1$, then P admits a quadrangulation with at most $2k - 2$ Steiner points. Moreover, the estimation for the number of Steiner points is best possible for every k.*

Note that in Theorem 5, if $k = 1$, then G is quadrangulatable with no Steiner points, and hence Theorem 5 implies Theorem 3. On the other hand, Figure 6 shows polygons P with spirality k which need at least $2k - 2$ Steiner points for its quadrangulation.

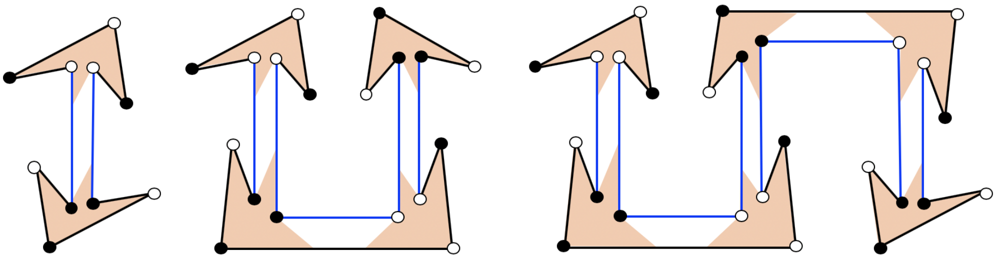

Figure 6: Polygons with spirality $k = 2, 3, 4$ requiring at least $2k - 2$ Steiner points for its quadrangulation. In each polygon, the blue parts are spirals, and the shaded parts show an area which needs at least one Steiner point. Similarly, we can construct such an example for any $k \geq 2$

Surprisingly, the examples in Figure 6 show a tightness of Theorem 2 too, though Ramaswami et al. did not give such an example [10]. Since this polygon with spirality $k \geq 2$ requiring $2k - 2$ Steiner points has $n = 8k - 6$ vertices, it is an example with n vertices requiring $\frac{n-2}{4}$ Steiner points. Hence, we also contribute to show the tightness of Theorem 2. However, Theorem 5 does not immediately imply Theorem 2, since we have only $sp(P) \leq \frac{n}{2}$ for any polygon P with n vertices, in general. (Even if we apply Theorem 5 to P, the number of Steiner points for P is at most $2k - 2 \leq 2(\frac{n}{2}) - 2 \leq n - 2$, which is much worse than Theorem 2.)

2 Related Results

In this section, we would like to mention some results related to our topic, which are geometric triangulations and quadrangulations on a point set.

Let Q be a set of points in the plane in *general position*, i.e., no three points are colinear, and let $\mathrm{Conv}(Q)$ denote the boundary of the convex hull of Q. A triangulation on Q is a geometric plane graph with vertex set Q whose outer cycle coincides with $\mathrm{Conv}(Q)$ and that each finite face is triangular. There are many results on triangulations on a point set, and edge flippings in those triangulations, which are related to Delaunay triangulations and Voronoi diagrams. For example, see [3, 5, 6].

A quadrangulation on Q can be defined similarly to a triangulation on Q. It is easy to see that if $\mathrm{Conv}(Q)$ has even length, then Q admits a quadrangulation. There are several papers on quadrangulation on a colored point set with Steiner points avoiding monochromatic edges, for example, [1, 2, 7, 8]. However, nothing is known for edge flips in those geometric quadrangulations of a point set, but we know a result for a quadrangulation of an even-sided polygon P [9], in which if $sp(P) \leq 2$, then any two quadrangulations on P can be transformed into each other by flipping edges.

3 Proof of the theorem

Let P be an even-sided polygon with a proper vertex-2-coloring. By Proposition 1, P admits a triangulation, denoted by T_P. A triangular face f of T_P is *monochromatic* if the three vertices of f have the same color. Similarly, we can define *monochromatic* edges. Note that if we fix T_P on P, then the monochromatic faces and edges are well-defined, since the 2-coloring of $V(P)$ is unique. The monochromatic faces and edges in T_P play an essential role in our proof of Theorem 5.

If T_P has no monochromatic face, then we easily get a quadrangulation of P with no Steiner points, by removing all monochromatic edges from T_P. On the other hand, if T_P has several monochromatic faces, then we have the following lemma.

Lemma 6. *For each monochromatic face f of T_P, put a single Steiner point in the interior of f. Then P is quadrangulatable with these Steiner points.*

Proof. Suppose that all vertices of a face f are white in T_P. Then we put a single black vertex in the interior of f in T_P, and join it to the three vertices of T. Do the same procedures to all monochromatic faces. Removing all monochromatic edges from the resulting triangulations of P with the set S of the added points, we get a quadrangulation of P with Steiner points S. (See Figure 7.) □

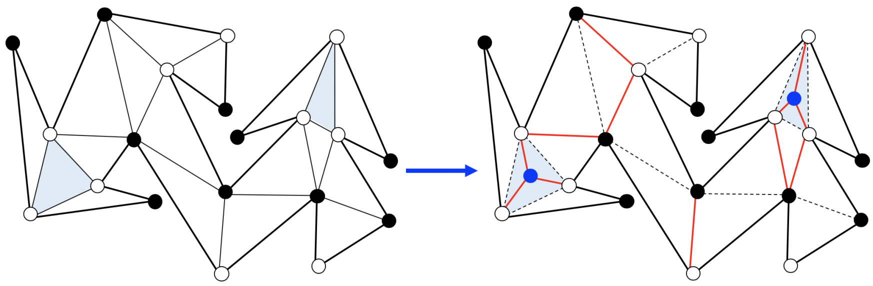

Figure 7: Construct a quadrangulation of P from T_P by putting a single Steiner point in each monochromatic face of T_P and removing all monochromatic edges of T_P

By Lemma 6, we have only to show the following lemma, in order to prove Theorem 5.

Lemma 7. *If P has spirality k, then P admits a triangulation T_P with at most $2k-2$ monochromatic faces.*

Proof. Since $sp(P) = k$, P has precisely k spirals, denoted by $A_1, \ldots, A_k$. Observe that no edge of P joins two vertices of distinct spirals. Hence, removing $A_1, \ldots, A_k$ from P, we have k disjoint intervals, denoted by $B_1, \ldots, B_k$. Choose T_P as a triangulation of P with the fewest monochromatic faces. Then we claim that no two vertices of any monochromatic face of T_P are contained in a single A_i nor B_i in the following argument.

For contradictions, suppose that T_P has a monochromatic face xyz such that two of x, y and z are contained in a single A_i or B_i for some i. We let $I[x, y]$ denote the interval of P with endpoints x and y, not containing z, and may suppose without loss of generality that $I[x, y]$ is contained in either A_i or B_i for some i.

First suppose that $I[x, y] \subset A_i$. Since x and y have the same color, they are not adjacent in P. Hence, T_P cannot have the edge xy, since every vertex in $I[x, y] - \{x, y\}$ is concave, as shown in Figure 8(a). So this case does not happen.

Secondly, suppose $I[x, y] \subset B_i$. Let $P' = I[x, y] \cup \{xz, yz\}$ be the sub-polygon of P. Observe that each inner vertex of $I[x, y]$ is convex in P' since so is it in P. Moreover, each of x and y is convex in P', since so is it in P and since its inner angle in P' is smaller than that in P. Finally, z is convex in P' since xyz forms a triangle in T_P. Consequently, P' is a convex polygon. Hence we can modify the interior of P' in T_P so that z is adjacent to all vertices in $I(x, y)$, as shown in Figure 8(b). This decreases the number of monochromatic faces by at least one, contrary to the assumption of T_P.

Now we regard T_P as a *combinatorial* plane graph, and let $\tilde{T}$ denote the maximal outer plane graph with $2k$ vertices obtained from T_P by contracting each of A_i's and B_i's into a single vertex. Observe that all monochromatic faces of T_P correspond to distinct triangular

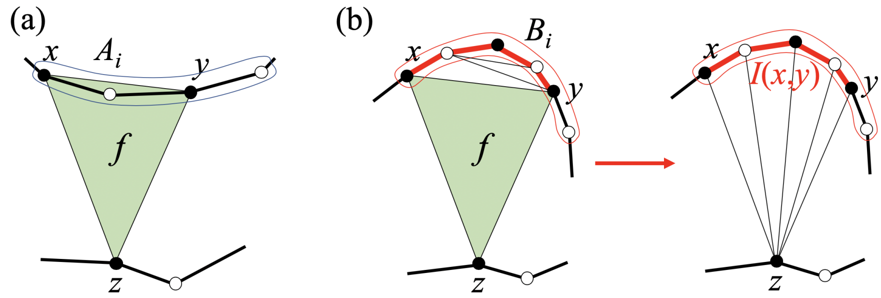

Figure 8: The cases when x and y are contained in A_i or B_i

faces of $\tilde{T}$, as shown in Figure 9, since all three vertices of any monochromatic face are contained in distinct three of $A_1, \dots, A_k, B_1, \dots, B_k$, respectively. By Euler's formula, $\tilde{T}$ has precisely $2k - 2$ triangular faces, and hence T_P has at most $2k - 2$ monochromatic faces. Thus, Lemma 7 follows. □

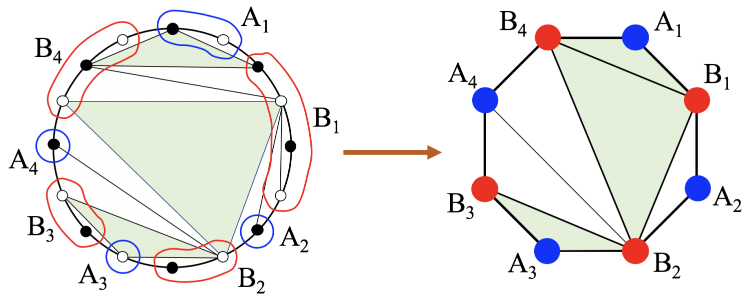

Figure 9: Combinatorial representation of T_P and the maximal outer plane graph $\tilde{T}$

Proof of Theorem 5. Let P be an n-sided polygon with $n \geq 4$ even which has spirality k. Then, by Lemma 7, P admits a triangulation T_P with at most $2k - 2$ monochromatic faces. By applying Lemma 6 to T_P, we get a quadrangulation of P with at most $2k - 2$ Steiner points. □

4 Conclusion

In this paper, we consider quadrangulatability of an even-sided polygon P. Though P is not quadrangulatable in general, we find two earlier results for this problem, in which one is to use Steiner points to get a quadrangulation of P (Theorem 2), and the other is to bound the spirality to get a quadrangulation of P (Theorem 3). This paper combines these two into Theorem 5, in which we prove that if a polygon P has spirality k, then P is quadrangulatable with at most $2k-2$ Steiner points. Our proof is short and combinatorial, and the result is best possible.

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