博士論文

On the boundary components of central streams and determining their Newton polygons

Central streamの境界成分およびそのNewton polygonの決定について

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Chapter 1

Introduction

In [12], Oort introduced the notion of *leaves* on a family of p-divisible groups. Note that p-divisible groups are often called Barsotti-Tate groups, to study the moduli space of abelian varieties in positive characteristic. Let p be a prime number. Fix an algebraically closed field k of characteristic p. Let S be a noetherian scheme over k. For a p-divisible group Y over k, in [12, 2.1], Oort defined $\mathcal{C}_Y(\mathrm{S})$ by a locally closed subset of S for a p-divisible group $\mathcal{Y}$ over S characterized by s belongs to $\mathcal{C}_Y(\mathrm{S})$ if and only if $\mathcal{Y}_s$ is isomorphic to Y over an algebraically closed field containing $k(s)$ and k; see the first paragraph of Section 2.1 for a review. We call $\mathcal{C}_Y(\mathrm{S})$ the central leaf associated to Y and $\mathcal{Y}$, if $\mathcal{Y} \to \mathrm{S}$ is a universal family over a deformation space or a moduli space.

In [12, 2.2] Oort showed that $\mathcal{C}_Y(\mathrm{S})$ is closed in an open Newton polygon stratum. We regard $\mathcal{C}_Y(\mathrm{S})$ as a locally closed subscheme of S by giving the induced reduced scheme structure. We are interested in the boundaries of leaves on the deformation space, and in the next paragraph we state this as a problem. In [12, 6.10], Oort treated this question in the polarized case, i.e., the case that p-divisible groups associated to polarized abelian varieties.

Let us formulate the problem on the boundaries of central leaves. Let X_0 be a p-divisible group over k. Put $\mathfrak{Def}(X_0) = \mathrm{Spf}(\Gamma)$ the deformation space of X_0. This deformation space is the formal scheme pro-representing the functor $\mathrm{Art}_k \to \mathrm{Sets}$ which maps R to the set of isomorphism classes of p-divisible groups X over R such that $X_k \simeq X_0$. We denote by Art_k the category of local Artinian rings with residue field k. Let $\mathfrak{X}' \to \mathrm{Spf}(\Gamma)$ be the universal p-divisible group. In [6, 2.4.4] de Jong showed that there exists an equivalence of categories between the category of p-divisible groups over $\mathrm{Spf}(\Gamma)$ and the category of the p-divisible groups over $\mathrm{Spec}(\Gamma) =: \mathrm{Def}(X_0)$. Let $\mathfrak{X}$ be the p-divisible group over $\mathrm{Def}(X_0)$ obtained from $\mathfrak{X}'$ by this equivalence. In [12, 2.7] Oort studied $\mathcal{C}_{X_0}(\mathrm{Def}(X_0))$. We are interested in $\mathcal{C}_Y(\mathrm{Def}(X))$ for $X \neq Y$ with $\mathcal{Y} = \mathfrak{X}$. Here is a basic problem:

Problem 1.1. Let Y be a p-divisible group over k. Classify p-divisible groups X over k such that $\mathcal{C}_Y(\mathrm{Def}(X)) \neq \emptyset$. Here $\mathcal{C}_Y(\mathrm{Def}(X)) \neq \emptyset$ means that X appears as a specialization of a family of p-divisible groups whose geometric fibers are isomorphic to Y.

Since the general case looks difficult, In this paper we discuss the case that the p-divisible group Y is "minimal". Oort introduced the notion of minimal p-divisible groups in [13, 1.1], and he showed in [13, 1.2] that the property: *Let X and Y be p-divisible groups over k. Suppose that X is minimal. If the kernel of the p-multiplication on X is isomorphic to that of Y, then X and Y are isomorphic.* For a Newton polygon ξ, we obtain the minimal p-divisible group $H(\xi)$. See the third and fourth paragraphs of Section 2.1 for the definitions of Newton polygons and minimal p-divisible groups.

We call $\mathcal{C}_Y(\mathrm{Def}(X_0))$ with Y a minimal p-divisible group a *central stream*. This notion is a "central" tool in the theory of foliations. For instance, it is known that the difference between central leaves and central streams comes from isogenies of p-divisible groups. Thus to study boundaries of general leaves, it is natural to start with investigating boundaries of central streams.

Let ξ be a Newton polygon. For the notation as above, we may treat the problem:

Problem 1.2. Classify p-divisible groups X over k such that $\mathcal{C}_{H(\xi)}(\mathrm{Def}(X)) \neq \emptyset$.

Let us translate this problem into the terminology of the Weyl group of GL_h. We denote by $W = W_h$ the Weyl group of GL_h. We identify this W with the symmetric group $\mathfrak{S}_h$ in the usual way. Define $J = J_c$ by $J_c = \{s_1, \ldots, s_h\} - \{s_c\}$, with simple reflections $s_i = (i, i+1)$. Put $d = h - c$. Then there exists a one-to-one correspondence between the isomorphism classes of BT_1's of rank p^h and dimension d over k and the subset ${}^JW$ of W, see Section 2.3. Let X be a p-divisible group. Let $w \in {}^JW$. We say w is the (p-kernel) type of $X[p]$ if the BT_1 $X[p]$ corresponds to w by this bijection.

In Proposition 2.1 we will show that: *Let X and Y be p-divisible groups over k with $\mathcal{C}_{H(\xi)}(\mathrm{Def}(X)) \neq \emptyset$ and $X[p] \simeq Y[p]$. Then $\mathcal{C}_{H(\xi)}(\mathrm{Def}(Y)) \neq \emptyset$.* Thanks to this proposition, Problem 1.2 is reduced to

Problem 1.3. Classify elements w of ${}^JW$ such that

($\diamond$) there exists a p-divisible group X over k such that w is the type of $X[p]$ and $\mathcal{C}_{H(\xi)}(\mathrm{Def}(X)) \neq \emptyset$.

In this paper, we treat the following problem:

Problem 1.4. Classify $w \in {}^JW$ satisfying ($\diamond$) and $\ell(w) = \ell(H(\xi)[p]) - 1$.

Theorem 1.5 and Theorem 1.6 reduce Problem 1.4 to the case that Newton polygons ξ consisting of two slopes satisfying that one slope is less than $1/2$ and the other slope is greater than $1/2$. In Section 4, we solve the problem for that case.

Before we state the main theorems, we explain the above formulations using specializations of p-divisible groups. For p-divisible groups X and Y over k, we say X is a *specialization* of Y if there exists a family of p-divisible group $\mathfrak{X} \to \mathrm{Spec}(R)$ with discrete valuation ring R in characteristic of p such that $\mathfrak{X}$ is isomorphic to Y over an algebraically closed field containing L and k, and $\mathfrak{X}_k$ is isomorphic to X over an algebraically closed field containing K and k, where L is the field of fractions of R, and $K = R/\mathfrak{m}$ is the residue field of R. Note that X is a specialization of Y if and only if $\mathcal{C}_Y(\mathrm{Def}(X)) \neq \emptyset$ holds. For a p-divisible group X, we define the *length* $\ell(X[p])$ of the p-kernel by the length of the element of the Weyl group which is the type of $X[p]$. It is known that for the p-divisible group X_0, the length $\ell(X_0[p])$ is equal to the dimension of the locally closed subscheme of $\mathrm{Def}(X_0)$ obtained by giving the induced reduced structure to the subset of $\mathrm{Def}(X_0)$ consisting of points $s \in \mathrm{Def}(X_0)$ such that $\mathfrak{X}'_s[p]$ is isomorphic to $X_0[p]$ over an algebraically closed field; see [15, 6.10] and [8, 3.1.6]. We say a specialization X of Y is *generic* if $\ell(X[p]) = \ell(Y[p]) - 1$.

Let ξ be a Newton polygon. We define $B(\xi)$ by

$$B(\xi) = \{\text{types of } X_s[p] \mid X_{\overline{\eta}} = H(\xi) \text{ and } \ell(X_s[p]) = \ell(X_{\overline{\eta}}[p]) - 1 \text{ for some } X \to \mathrm{S}\}, \tag{1.1}$$

where $\mathrm{S} = \mathrm{Spec}(R)$ with a discrete valuation ring $(R, \mathfrak{m})$, $s = \mathrm{Spec}(\kappa)$ and $\overline{\eta} = \mathrm{Spec}(\overline{K})$ with $\kappa = R/\mathfrak{m}$ and $K = \mathrm{frac}(R)$. Problem 1.4 asks us to determine the set $B(\xi)$. We call $B(\xi)$ the set of boundary components of the central stream associated to ξ. The first result is:

Theorem 1.5. Let $\xi = \sum_{i=1}^{z}(m_i, n_i)$ be a Newton polygon. Let $\xi_i = (m_i, n_i) + (m_{i+1}, n_{i+1})$ be the Newton polygon consisting of two adjacent segments for $i = 1, \ldots, z-1$. For any $w \in B(\xi_i)$, the direct sum $w_{\zeta^{(i)}} \oplus w$ is contained in $B(\xi)$, where $w_{\zeta^{(i)}}$ is the type of $H(\zeta^{(i)})$ with $\zeta^{(i)} = (m_1, n_1) + \cdots + (m_{i-1}, n_{i-1}) + (m_{i+2}, n_{i+2}) + \cdots + (m_z, n_z)$. Moreover the obtained map

$$\coprod_{i=1}^{z-1} B(\xi_i) \to B(\xi) \tag{1.2}$$

which sends $w \in B(\xi_i)$ to $w_{\zeta^{(i)}} \oplus w$ is bijective.

This theorem implies that the problem of determining boundary components of central streams is reduced to the case that the Newton polygon consists of two segments. Moreover, for the two segments case, we will show the following result:

Theorem 1.6. Let $\xi = (m_1, n_1) + (m_2, n_2)$ be a Newton polygon satisfying that $n_1/(m_1 + n_1) > n_2/(m_2 + n_2) \geq 1/2$. Put $\xi^{\mathrm{C}} = (m_1, n_1 - m_1) + (m_2, n_2 - m_2)$. Then the map sending w to $w|_{\{1,\dots,n_1+n_2\}}$ gives a bijection from $B(\xi)$ to $B(\xi^{\mathrm{C}})$.

For a Newton polygon $\xi = (m_1, n_1) + (m_2, n_2)$, we set $\xi^{\mathrm{D}} = (n_2, m_2) + (n_1, m_1)$. By the duality, it is easy to see that the map sending w to $i \mapsto l - w(l - i)$, with $l = m_1 + n_1 + m_2 + n_2 + 1$, gives a bijection from $B(\xi)$ to $B(\xi^{\mathrm{D}})$. Using repeatedly this duality and Theorem 1.6, we can reduce Problem 1.4 to the case of [5], i.e., to the case that the Newton polygon ξ consists of two slopes such that one slope is less than 1/2 and the other slope is greater than 1/2. In this paper, we treat this case in Chapter 4. There results give a complete answer to Problem 1.4.

Next, we formulate a problem on determining the Newton polygon of each boundary component in $B(\xi)$. Let ξ and ζ be Newton polygons. We write $\zeta \prec \xi$ if each point of ζ is above or on ξ with $\zeta \neq \xi$. We say $\zeta \prec \xi$ is *saturated* if there exists no Newton polygon η such that $\zeta \precneqq \eta \precneqq \xi$. We denote by w_ξ the element of the Weyl group corresponding to $H(\xi)[p]$. Our next problem is

Problem 1.7. Let ξ be a Newton polygon. We fix a generic specialization X of $H(\xi)$. Show that the existence of a Newton polygon ζ such that

(∗) $H(\zeta)$ appears as a specialization of X and $\zeta \prec \xi$ is saturated,

and determine this ζ.

Note that for the Newton polygon $\mathrm{np}(X)$ of X, since $\zeta \prec \xi$ is saturated, we have that $\mathrm{np}(X) = \zeta$, and in particular $\mathrm{np}(X) \prec \xi$ is saturated, if the above problem is affirmatively solved. see [5, Corollary 1.2].

Let us translate this problem to the terminology of the Weyl group of GL_h. We say that w' is a specialization of w, denoted by $w' \subset w$, if there exists a discrete valuation ring R of characteristic p such that there exists a finite flat commutative group scheme G over R satisfying that $G_{\overline{\kappa}}$ is a BT_1 of the type w', and $G_{\overline{L}}$ is a BT_1 of the type w, where L (resp. κ) is the fractional field of R (resp. is the residue field of R). A *generic specialization* w' of w is a specialization of w satisfying $\ell(w') = \ell(w) - 1$. For these notations, our main result is

Theorem 1.8. Let ξ be any Newton polygon. Let $w \in {}^J W$ be a generic specialization of w_ξ. Then there exists a Newton polygon ζ such that

(i) $\zeta \prec \xi$ is saturated, and

(ii) $w_\zeta \subset w$.

By Theorem 1.5, to show Theorem 1.8, the case that ξ consists of two segments is essential, see Theorem 6.6 and its proof.

For the two-slopes case, using the map given in Theorem 1.6, we have:

Theorem 1.9. Let $\xi = (m_1, n_1) + (m_2, n_2)$ be a Newton polygon satisfying that $n_1/(m_1 + n_1) > n_2/(m_2 + n_2) \geq 1/2$. Put $\xi^{\mathrm{C}} = (m_1, n_1 - m_1) + (m_2, n_2 - m_2)$. For a generic specialization $w \in B(\xi)$, let $w' \in B(\xi^{\mathrm{C}})$ be the generic specialization corresponding to w by the map of Theorem 1.6. Then a Newton polygon $\zeta = \sum(c_i, d_i)$ satisfies (i) and (ii) of Theorem 1.8 for w_ξ and w if and only if the Newton polygon $\zeta^{\mathrm{C}} = \sum(c_i, d_i - c_i)$ satisfies (i) and (ii) for $w_{\xi^{\mathrm{C}}}$ and w'.

This paper is organized as follows. In Chapter 2, we recall the notions of p-divisible groups, Newton polygons and truncated Dieudonné modules of level one, and we review the classification of BT_1's. In Chapter 3, we introduce arrowed binary sequences, and show some properties of ABS's corresponding to minimal DM_1's. We mainly use this notion to show the main results. In Chapter 4, we treat central streams corresponding to Newton polygons satisfying that the one slope is greater than 1/2 and the other is less than 1/2. We will solve Problem 1.4 for such Newton polygons ξ. Moreover, we show a key statement to solve Problem 1.7, see Proposition 4.9. In Chapter 5, we introduce Euclidean algorithm for Newton polygons. Using this algorithm, we solve Problem 1.4 and Problem 1.7 for all Newton polygons ξ consisting of two segments by reducing the problems to the case of Chapter 3. Finally, in Chapter 6, we solve the problems for all Newton polygons.

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Chapter 2

Preliminaries

In this chapter, first we recall the notions of p-divisible groups, leaves and Dieudonné modules. Next, in Section 2.3, we review the definition of truncated Barsotti-Tate groups of level one and a classification of BT_1's.

2.1 p-divisible groups and Dieudonné modules

Fix a prime number p. Let S be a scheme in characteristic p. Let h be a non-negative integer. A *p-divisible group* (Barsotti-Tate group) of height h over S is an inductive system $(G_v, i_v)_{v\geq 1}$, where G_v is a finite locally free commutative group scheme over S of order p^{vh}, and for every v, there exists the exact sequence of commutative group schemes

$$0 \longrightarrow G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{p^v} G_{v+1}, \tag{2.1}$$

with canonical inclusion i_v, i.e., $G_v \simeq G_{v+1}[p^v]$ for all v. Let $X = (G_v, i_v)$ be a p-divisible group over S. Since $G_{v+1} \simeq G_{v+2}[p^{v+1}]$, we see that G_{v+1} is killed by p^{v+1}. Hence the multiplication by $p : G_{v+1} \to G_{v+1}$ can be regarded as $p : G_{v+1} \to G_{v+1}[p^v] \simeq G_v$. Thus we get maps $j_v : G_{v+1} \to G_v$. Let G^t denote the inductive system $(D_S(G_v), D_S(j_v))_{v\geq 1}$, where $D_S(-)$ is the Cartier dual. We call this G^t the *Serre dual of* G. Moreover, let T be a scheme over S. Then we have the p-divisible group X_T over T, which is defined as $(G_v \times_{\mathrm{S}} T,\ i_v \times \mathrm{id})$. For the case T is a closed point s over S, we call the p-divisible group X_s the fiber of X over s. Let k be an algebraically closed field of characteristic p. Let $Y \to \mathrm{Spec}(k)$ be a p-divisible group, and let $\mathcal{Y} \to \mathrm{S}$ be a p-divisible group over S. In [12, 2.1] Oort defined a *leaf* by

$$\mathcal{C}_Y(\mathrm{S}) = \{s \in \mathrm{S} \mid \mathcal{Y}_s \text{ is isomorphic to } Y \text{ over an algebraically closed field}\}, \tag{2.2}$$

as a set. He showed that $\mathcal{C}_Y(\mathrm{S})$ is closed in a Newton stratum (cf. [12, 2.2]). We regard $\mathcal{C}_Y(\mathrm{S})$ as a locally closed subscheme of S by giving the induced reduced structure on it.

Let K be a perfect field of characteristic p. Let $W(K)$ denote the ring of Witt-vectors with coefficients in K. Let σ be the Frobenius over K. We denote by the same symbol σ the Frobenius over $W(K)$ if no confusion can occur. A *Dieudonné module over* K is a finite $W(K)$-module M equipped with σ-linear homomorphism $\mathrm{F} : M \to M$ and σ^{-1}-linear homomorphism $\mathrm{V} : M \to M$ satisfying that $\mathrm{F} \circ \mathrm{V}$ and $\mathrm{V} \circ \mathrm{F}$ equal the multiplication by p. For each p-divisible group X, we have the Dieudonné module $\mathbb{D}(X)$ using the covariant Dieudonné functor. The covariant Dieudonné theory says that the functor $\mathbb{D}$ induces a canonical categorical equivalence between the category of p-divisible groups over K and that of Dieudonné modules over K which are free as $W(K)$-modules. Moreover, there exists a categorical equivalence from the category of finite commutative group schemes over K to that of Dieudonné modules over K which are of finite length.

Let $\{(m_i, n_i)\}_{i=1,\dots,z}$ be a set of a finite number of pairs of coprime non-negative integers satisfying that if $i < j$, then $\lambda_i \geq \lambda_j$ with $\lambda_i = n_i/(m_i+n_i)$ for each i. A *Newton polygon* is a lower convex polygon in $\mathbb{R}^2$, which breaks on integral coordinates and consists of slopes λ_i. We write

$$\sum_{i=1}^{z} (m_i, n_i) \tag{2.3}$$

for the Newton polygon. We call each coprime pair (m_i, n_i) the *i-th segment of the Newton polygon.* For a Newton polygon $\xi = \sum_i (m_i, n_i)$, we define the p-divisible group $H(\xi)$ by

$$H(\xi) = \bigoplus_i H_{m_i,n_i}, \tag{2.4}$$

where $H_{m,n}$ is the p-divisible group over $\mathbb{F}_p$, which is of dimension n and its Serre-dual is of dimension m. Moreover the Dieudonné module $\mathbb{D}(H_{m,n})$ is described as

$$\mathbb{D}(H_{m,n}) = \bigoplus_{i=1}^{m+n} W(\mathbb{F}_p) e_i, \tag{2.5}$$

where with respect to the basis $\{e_i\}_i$, the operations F and V satisfy that $\mathrm{F}e_i = e_{i-m}$ and $\mathrm{V}e_i = e_{i-n}$ with $e_{i-(m+n)} = pe_i$. Note that $W(\mathbb{F}_p)$ is isomorphic to the ring $\mathbb{Z}_p$ of p-adic integers .

We say a p-divisible group X is *minimal* if X is isomorphic to $H(\xi)$ over an algebraically closed field for a Newton polygon ξ. For a p-divisible group X, the p-kernel $X[p]$ is obtained by the kernel of the multiplication by p. It is known that the Dieudonné module of $H(\xi)[p]$ makes a truncated Dieudonné module of level one (abbreviated as DM_1) $\mathbb{D}(H(\xi)[p])$. A DM_1 over K of height h is the triple $(N, \mathrm{F}, \mathrm{V})$ consisting of a K-vector space N of height h, a σ-linear map $\mathrm{F} : N \to N$ and a σ^{-1}-linear map $\mathrm{V} : N \to N$ satisfying that $\ker \mathrm{F} = \operatorname{im} \mathrm{V}$ and $\operatorname{im} \mathrm{F} = \ker \mathrm{V}$.

Let $\xi = \sum (m_i, n_i)$ be a Newton polygon. We denote by N_ξ the DM_1 associated to $H(\xi)[p]$. Then N_ξ is described as

$$N_\xi = \bigoplus N_{m_i,n_i}, \tag{2.6}$$

where $N_{m,n}$ is the DM_1 corresponding to the p-kernel of $H_{m,n}$. We call such DM_1 N_ξ a *minimal DM_1*.

We use the same notation as in Chapter 1. The following proposition would be well-known to the specialists, but as any good reference cannot be found, we give a proof for the reader's convenience. In the polarized case, a proof is given in [11, 12.5].

Proposition 2.1. Let ξ be a Newton polygon. Put $Y = H(\xi)$. Let X and X' be p-divisible groups over an algebraically closed field of characteristic p. If $\mathcal{C}_Y(\mathrm{Def}(X)) \neq \emptyset$ and $X[p] \simeq X'[p]$, then $\mathcal{C}_Y(\mathrm{Def}(X')) \neq \emptyset$.

Proof. Let h and c be positive integers such that $X[p]$ is the type of $w \in {}^J W$ with $W = W_h$ and $J = J_m$. Put $n = h - m$. Let F (resp. V) denote the σ-linear map (resp. σ^{-1}-linear map) of the DM_1 $\mathbb{D}(X[p]) = \mathbb{D}(X)/p\mathbb{D}(X)$ with σ the Frobenius. Take a basis $\bar{z}_{n+1}, \dots, \bar{z}_h$ of the image of V, and choose $\bar{z}_1, \dots, \bar{z}_n \in \mathbb{D}(X[p])$ so that $\bar{z}_1, \dots, \bar{z}_h$ is a basis of $\mathbb{D}(X[p])$. We choose lifts $z_1, \dots, z_h$ of $\bar{z}_1, \dots, \bar{z}_h$ to $\mathbb{D}(X)$. Then $\{z_1, \dots, z_h\}$ is a basis of $\mathbb{D}(X)$. We write

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for the display of X with respect to the basis $\{z_1, \dots, z_h\}$, where A is the $n \times n$ matrix, and D is the $(h - n) \times (h - n)$ matrix. See [10] for the construction of the display. Then for the Dieudonné module $\mathbb{D}(X)$ of X equipped with the operations F and V, we have

$$(\mathrm{F}z_1, \dots, \mathrm{F}z_h) = (z_1, \dots, z_h) \begin{pmatrix} A & pB \\ C & pD \end{pmatrix}$$

and

$$(\mathrm{V}z_1, \ldots, \mathrm{V}z_h) = (z_1, \ldots, z_h) \begin{pmatrix} p\alpha & p\beta \\ \gamma & \delta \end{pmatrix}^{\sigma^{-1}},$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is the inverse matrix of the display of X. The operations F and V on $\mathbb{D}(X[p])$ satisfy that

$$(\mathrm{F}\bar{z}_1, \ldots, \mathrm{F}\bar{z}_h) = (\bar{z}_1, \ldots, \bar{z}_h) \begin{pmatrix} \bar{A} & 0 \\ \bar{C} & 0 \end{pmatrix}$$

and

$$(\mathrm{V}\bar{z}_1, \ldots, \mathrm{V}\bar{z}_h) = (\bar{z}_1, \ldots, \bar{z}_h) \begin{pmatrix} 0 & 0 \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}^{\sigma^{-1}}.$$

For the p-divisible group $\mathfrak{X} \to \mathrm{Spec}(R)$ corresponding to $\mathrm{Spec}(R) \to \mathcal{C}_Y(\mathrm{Def}(X)) \subset \mathrm{Def}(X)$, the display of $\mathfrak{X}$ induces that

$$(\mathrm{F}\bar{z}_1, \ldots, \mathrm{F}\bar{z}_h) = (\bar{z}_1, \ldots, \bar{z}_h) \begin{pmatrix} \bar{A} + \bar{T}\bar{C} & 0 \\ \bar{C} & 0 \end{pmatrix}$$

and

$$(\mathrm{V}\bar{z}_1, \ldots, \mathrm{V}\bar{z}_h) = (\bar{z}_1, \ldots, \bar{z}_h) \begin{pmatrix} 0 & 0 \\ \bar{\gamma} & -\bar{\gamma}\bar{T} + \bar{\delta} \end{pmatrix}^{\sigma^{-1}},$$

where $\bar{T}$ is an $(h-n) \times n$ matrix on R. On the other hand, we denote by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the display of X'. Using the isomorphism from $X[p]$ to $X'[p]$, we have the basis $\bar{e}_1, \ldots, \bar{e}_h$ of $\mathbb{D}(X'[p]) = \mathbb{D}(X')/p\mathbb{D}(X')$. We have then

$$(\mathrm{F}\bar{z}_1, \ldots, \mathrm{F}\bar{z}_h) = (\bar{z}_1, \ldots, \bar{z}_h) \begin{pmatrix} \bar{a} & 0 \\ \bar{c} & 0 \end{pmatrix}$$

and

$$(\mathrm{V}\bar{z}_1, \ldots, \mathrm{V}\bar{z}_h) = (\bar{z}_1, \ldots, \bar{z}_h) \begin{pmatrix} 0 & 0 \\ \bar{\gamma}' & \bar{\delta}' \end{pmatrix}^{\sigma^{-1}}$$

for the inverse matrix

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

of the display of X'. Let $\mathcal{Y}$ be the p-divisible group having

$$\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as its display, where T is a matrix such that $T \bmod p$ equal $\bar{T}$. Then for the display of $\mathcal{Y}[p]$, we see

$$(\mathrm{F}\bar{z}_1, \ldots, \mathrm{F}\bar{z}_h) = (\bar{z}_1, \ldots, \bar{z}_h) \begin{pmatrix} \bar{a} + \bar{T}\bar{c} & 0 \\ \bar{c} & 0 \end{pmatrix}$$

and

$$(\mathrm{V}\bar{z}_1, \ldots, \mathrm{V}\bar{z}_h) = (\bar{z}_1, \ldots, \bar{z}_h) \begin{pmatrix} 0 & 0 \\ \bar{\gamma'} & -\bar{\gamma'}\bar{T} + \bar{\delta'} \end{pmatrix}^{\sigma^{-1}},$$

whence $\mathcal{Y}$ belongs to $\mathcal{C}_Y(\mathrm{Def}(X'))$. □

2.2 Specializations

Let R be a commutative ring of positive characteristic p. Let σ be the frobenius endomorphism on R defined by $\sigma(a) = a^p$.

Definition 2.2. A DM_1 over R of height h is a quintuple $\mathcal{N} = (\mathcal{N}, C, D, F, V^{-1})$, where

(1) $\mathcal{N}$ is a free R-module of rank h,

(2) C and D are submodules of $\mathcal{N}$ which are locally direct summands of $\mathcal{N}$,

(3) $F : (\mathcal{N}/C) \otimes_{R,\sigma} R \to D$ and $V^{-1} : C \otimes_{R,\sigma} R \to \mathcal{N}/D$ are R-linear isomorphisms.

Let k be an algebraically closed field of characteristic p, and let $R = k[\![t]\!]$ be the ring of formal power series over k. For an arbitrary DM_1 $\mathcal{N}$ over R, we can consider $\mathcal{N}_k := \mathcal{N} \otimes_R k$, which is a DM_1 over k. Hence we have the canonical map

$$\{\mathrm{DM}_1 \text{ over } R\} \longrightarrow \{\mathrm{DM}_1 \text{ over } k\}$$

sending $\mathcal{N}$ to $\mathcal{N}_k$. We call this the *specialization map.*

2.3 Classification of BT_1's

In this section, we work over an algebraically closed field k. Let us review the classification of truncated Barsotti-Tate groups of level one.

Fix a prime number p. Let S be a scheme of characteristic p. We denote by $\mathrm{frob}_{\mathrm{S}} : \mathrm{S} \to \mathrm{S}$ the absolute Frobenius morphism of S. Let $\pi : N \to \mathrm{S}$ be a finite commutative group scheme. We define $\pi^{(p)} : N^{(p)} \to \mathrm{S}$ to be the pull-back of $\pi : N \to \mathrm{S}$ via $\mathrm{frob}_{\mathrm{S}}$. Using the cartesian product, we obtain the map $N \to N^{(p)}$. We write this map for $\mathrm{F} = \mathrm{F}_N$. For the dual N^D of N, we have $\mathrm{F}_{N^D} : N^D \to (N^D)^{(p)} \cong (N^{(p)})^D$. We define V by the dual $\mathrm{V}_N : N^{(p)} \to N$ of F_{N^D}.

Definition 2.3. A *truncated Barsotti-Tate group of level one* (BT_1) is a commutative, finite and flat group scheme N over a scheme in characteristic p satisfying properties $[p]_N = 0$, and

$$\mathrm{im}\,(\mathrm{V} : N^{(p)} \to N) \;=\; \ker\,(\mathrm{F} : N \to N^{(p)}), \tag{2.7}$$

$$\mathrm{im}\,(\mathrm{F} : N \to N^{(p)}) \;=\; \ker\,(\mathrm{V} : N^{(p)} \to N). \tag{2.8}$$

A DM_1 appears as a Dieudonné module of a BT_1. Let $W = W_h$ be the Weyl group of the general linear group GL_h as Chapter 1. This W can be identified with the symmetric group $\mathfrak{S}_h$. Let Ω denote the standard generator of $W = \mathfrak{S}_h$. We write s_i for the simple reflection $(i, i+1)$. We define $J = J_c$ by $J_c = \Omega - \{s_c\}$. Put $d = h - c$. For the set $W_J := W_c \times W_d$, let ${}^J W$ be the set consisting of elements $w \in W_h$ such that w is the shortest element of $W_J \cdot w$, see [1, Chap. IV, Ex. §1 (3)]. Then we have

Theorem 2.4. There exists a one-to-one correspondence

$${}^J W \longleftrightarrow \{\mathrm{BT}_1\text{'s over } k \text{ of height } h \text{ of dimension } d\}/\cong. \tag{2.9}$$

Moreover, running over all d, we have

$$\bigsqcup_d {}^J W \longleftrightarrow \{0,1\}^h. \tag{2.10}$$

Kraft [7], Oort [11] and Moonen-Wedhorn [9] show the existence of a one-to-one correspondence:

$$\{0,1\}^h \longleftrightarrow \{\mathrm{DM}_1\text{'s over } k \text{ of height } h\}/\cong. \tag{2.11}$$

For $\nu \in \{0,1\}^h$, we construct the DM_1 $D(\nu)$ as follows. We write $\nu(i)$ for the i-th coordinate of ν. Set $N = ke_1 \oplus \cdots \oplus ke_h$. We define the maps F and V as follows:

$$\mathrm{F}e_i = \begin{cases} e_j, \ j = \#\{l \mid \nu(l) = 0, \ l \leq i\} & \text{for } \nu(i) = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

Let $j_1, \ldots, j_c$ be the natural numbers satisfying $\nu(j_l) = 1$ with $j_1 < \cdots < j_c$. Put $d = h - c$. Then the map V is defined by

$$\mathrm{V}e_i = \begin{cases} e_{j_l}, \ l = i - d & \text{for } i > d, \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

Therefore the DM_1 $D(\nu)$ is given by the triple $D(\nu) = (N, \mathrm{F}, \mathrm{V})$. Thus we can identify DM_1's with sequences consisting of 0 and 1.

Let us construct a bijection between ${}^JW$ and the set of isomorphism classes of DM_1's over k of height h and dimension d. For an element w of ${}^JW$, we define $\nu(j) = 0$ if and only if $w(j) > c$ for $j = 1, \ldots, h$, and we obtain the element $(\nu(1), \nu(2), \ldots, \nu(h))$ of $\{0,1\}^h$. This gives a one-to-one correspondence between ${}^JW$ and the subset of $\{0,1\}^h$ consisting of elements ν satisfying $\#\{j \mid \nu(j) = 0\} = d$.

Here, we show a lemma used for the construction of generic specializations. We define $x \in W$ by $x(i) = i + d$ if $i \leq c$ and $x(i) = i - c$ otherwise. Let θ be the map from W to itself defined by $\theta(u) = xux^{-1}$. By [14, 4.10], we have $w' \subset w$ if and only if there exists $u \in W_J$ such that $u^{-1}w'\theta(u) \leq w$ with the Bruhat order $\leq$.

Lemma 2.5. Let $w \in {}^JW$. Let w' be a specialization of w. If w' is generic, then there exist $v \in W$ and $u \in W_J$ such that

(i) $v = ws$ for a transposition s,

(ii) $\ell(v) = \ell(w) - 1$,

(iii) $w' = uv\theta(u^{-1})$.

Proof. Let $w \in {}^JW$. Let $w' \in {}^JW$ satisfying that $w' \subset w$ and $\ell(w') = \ell(w) - 1$. Choose an element u of W_J satisfying that $u^{-1}w'\theta(u) < w$. Set $v = u^{-1}w'\theta(u)$. Let us show (ii). Since w' belongs to ${}^JW$, we have $\ell(v) \geq \ell(u^{-1}w') - \ell(\theta(u)^{-1}) = \ell(u) + \ell(w') - \ell(\theta(u))$. Moreover, we have $\ell(u) + \ell(w') - \ell(\theta(u)) = \ell(w')$ since for all element u' of W_J we have $\ell(u') = \ell(\theta(u'))$ by the definition of θ. As $v < w$, we have $\ell(v) < \ell(w)$. Thus we see (ii). Let $w = s_{i_1}s_{i_2}\ldots s_{i_l}$ be a reduced expression of w with $v = s_{i_1}\ldots s_{i_{q-1}}s_{i_{q+1}}\ldots s_{i_l}$. Set $s = (s_{i_l}\ldots s_{i_{q+1}})s_{i_q}(s_{i_{q+1}}\ldots s_{i_l})$. Then s is a transposition, and this s satisfies $v = ws$. $\square$

Chapter 3

Arrowed binary sequences

In this chapter we introduce arrowed binary sequences as a generalization of classifying data ${}^{J}W$ of BT_1's. Arrowed binary sequences are a main tool to prove the main theorems. For instance, in Section 3.2, we introduce a combinatorial method to construct specializations of minimal DM_1's.

3.1 The definition of arrowed binary sequences

Definition 3.1. An *arrowed binary sequence* (we often abbreviate as ABS) is the triple (T, Δ, Π) consisting of a totally ordered finite set $T = \{t_1 < t_2 < \cdots < t_h\}$, a map $\Delta : T \to \{0, 1\}$ and a bijection $\Pi : T \to T$. For an ABS S, let $T(S)$ denote the totally ordered finite set of S. Similarly, we denote by $\Delta(S)$ (resp. $\Pi(S)$) the map from $T(S)$ to $\{0, 1\}$ (resp. the map from $T(S)$ to itself). For an ABS S, we define the *length* $\ell(S)$ of S by

$$\ell(S) = \#\{(t, t') \in T(S) \times T(S) \mid t < t' \text{ with } \Delta(S)(t) = 0 \text{ and } \Delta(S)(t') = 1\}. \tag{3.1}$$

Remark 3.2. Let $N = (N, \mathrm{F}, \mathrm{V})$ be a DM_1. We construct the arrowed binary sequence (Λ, δ, π) associated to N as follows. Let ν be the element of $\{0, 1\}^h$ corresponding to N. For a totally ordered set $\Lambda = \{t_1, \dots, t_h\}$, let $\delta : \Lambda \to \{0, 1\}$ be the map which sends t_i to the i-th coordinate of ν. Using the basis of N satisfying (2.12) and (2.13), we define the map $\pi : \Lambda \to \Lambda$ by $\pi(t_i) = t_j$, where j is uniquely determined by

$$\begin{cases} \mathrm{F}e_i = e_j & \text{if } \delta(t_i) = 0, \\ \mathrm{V}e_j = e_i & \text{otherwise.} \end{cases} \tag{3.2}$$

We say an ABS is *admissible* if it is associated to some DM_1 by the above correspondence.

Remark 3.3. For the DM_1 $N_{m,n}$ corresponding to the p-divisible group $H_{m,n}$, we get the ABS S as follows. Set $T(S) = \{t_1, \ldots, t_{m+n}\}$. The map $\Delta(S)$ is defined by $\Delta(S)(t_i) = 1$ if $i \leq m$, and $\Delta(S)(t_i) = 0$ otherwise. The map $\Pi(S)$ is defined by $\Pi(S)(t_i) = t_{i+n}$ if $i \leq m$, and $\Pi(S)(t_i) = t_{i-m}$ otherwise.

Let S be an ABS. Put $\delta = \Delta(S)$ and $\pi = \Pi(S)$. The *binary expansion* $b(t)$ of $t \in T(S)$ is the real number $b(t) = 0.b_1 b_2 \ldots$, where $b_i = \delta(\pi^{-i}(t))$.

Proposition 3.4. Let S be an admissible ABS. For elements t_i and t_j of $T(S) = \{t_1, t_2, \ldots, t_h\}$, the following holds.

(i) Suppose $\Delta(S)(t_i) = \Delta(S)(t_j)$. Then $t_i < t_j$ if and only if $\Pi(S)(t_i) < \Pi(S)(t_j)$.

(ii) Suppose $b(t_i) \neq b(t_j)$. Then $b(t_i) < b(t_j)$ if and only if $i < j$.

Proof. (i) follows from the construction of admissible ABS's. Let us see (ii). Put $\delta = \Delta(S)$ and $\pi = \Pi(S)$. By the construction of admissible ABS's, for elements t and t' of $T(S)$, if $\delta(t) = 1$ and $\delta(t') = 0$, then $\pi(t') < \pi(t)$. First, assume $b(t_i) < b(t_j)$. Then there exists a non-negative integer u such that $\delta(\pi^{-v}(t_i)) = \delta(\pi^{-v}(t_j))$ for $0 \leq v < u$ and $\delta(\pi^{-u}(t_i)) = 0$, $\delta(\pi^{-u}(t_j)) = 1$. We have then $\pi^{-u+1}(t_i) < \pi^{-u+1}(t_j)$, and the assertion follows from (i). Next, assume $i < j$. To lead a contradiction, we suppose that $b(t_j) < b(t_i)$. Then there exists a non-negative integer u such that $\delta(\pi^{-u}(t_j) = 0$ and $\delta(\pi^{-u}(t_i)) = 1$, and for non-negative integers v satisfying $v < u$, we have $\delta(\pi^{-v}(t_j) = \delta(\pi^{-v}(t_i))$. This implies that $\pi^{-u+1}(t_j) < \pi^{-u+1}(t_i)$, and this is a contradiction. $\square$

Next, in Definition 3.5, we introduce the direct sum of ABS's. The construction of the direct sum is induced from the direct sum of corresponding DM_1's.

Definition 3.5. Let S_1 and S_2 be ABS's. We define the *direct sum* $S = S_1 \oplus S_2$ of S_1 and S_2 as follows. Let $T(S) = T(S_1) \sqcup T(S_2)$ as sets. We define the map $\Delta(S) : T(S) \to \{0, 1\}$ to be $\Delta(S)|_{T(S_i)} = \Delta(S_i)$ for $i = 1, 2$. Let $\Pi(S)$ be the map from $T(S)$ to itself satisfying that $\Pi(S)|_{T(S_i)} = \Pi(S_i)$ for $i = 1, 2$. We define the order on $T(S)$ so that for elements t and t' of $T(S)$,

(i) if $b(t) \leq b(t')$, then $t < t'$;

(ii) $t < t'$ if and only if $\Pi(S)(t) < \Pi(S)(t')$ when $\Delta(S)(t) = \Delta(S)(t')$.

Notation 3.6. Let N_ξ be the minimal DM_1 of a Newton polygon $\xi = \sum_{i=1}^{z}(m_i, n_i)$. Let S be the ABS associated to N_ξ. Then S is described as $S = \bigoplus_{i=1}^{z} S_i$, where S_i is the ABS associated to the DM_1 N_{m_i,n_i}. If an element t of $T(S)$ belongs to $T(S_r)$, then we denote by t^r or τ^r this element t with $\tau = \Delta(S)(t)$. If we want to say that the element t^r is the i-th element of $T(S_r)$, we write t^r_i for the element t^r. Furthermore, we often write τ^r_i for the element t^r_i of $T(S)$ with $\tau = \Delta(S)(t^r_i)$.

For certain Newton polygons ξ, the ABS associated to N_ξ is described as follows:

Lemma 3.7. Let N_ξ be the minimal DM_1 of $\xi = (m_1, n_1) + (m_2, n_2)$ with $\lambda_2 < 1/2 < \lambda_1$. For the above notation, the sequence S associated to N_ξ is obtained by the following:

$$\underbrace{1^1_1 \cdots 1^1_{m_1}}_{m_1} \underbrace{0^1_{m_1+1} \cdots 0^1_{n_1}}_{n_1-m_1} \underbrace{1^2_1 \cdots 1^2_{n_2}}_{n_2} \underbrace{0^1_{n_1+1} \cdots 0^1_{h_1}}_{m_1} \underbrace{1^2_{n_2+1} \cdots 1^2_{m_2}}_{m_2-n_2} \underbrace{0^2_{m_2+1} \cdots 0^2_{h_2}}_{n_2}. \tag{3.3}$$

Proof. See [2], Proposition 4.20. $\square$

We denote by $\mathcal{H}'(h,d)$ the set of admissible ABS's whose corresponding DM_1's are of height h and dimension d. We have a natural bijection from ${}^JW$ to $\mathcal{H}'(h,d)$. Via this bijection, the ordering $\subset$ on ${}^JW$ defines an ordering on $\mathcal{H}'(h,d)$.

We shall give a method to construct a typical specialization of ABS's. It will turns out to correspond to specializations $w' \subset w$ with $v = ws < w$ and $w' = uv\theta(u^{-1})$, where s denotes a transposition and $u \in W_J$.

Definition 3.8. Let S be an admissible ABS with $T(S) = \{t_1 < \cdots < t_h\}$. Let i and j be natural numbers with $i < j$. We define an ABS $S^{(0)}$ as follows. We set $T(S^{(0)}) = \{t'_1 <' \cdots <' t'_h\}$ to be $t'_z = t_{f(z)}$ for $f = (i,j)$ transposition. Let $\Delta(S^{(0)}) = \Delta(S)$. For a natural number z with $1 \le z \le h$, we denote by $g(z)$ the natural number satisfying $\Pi(S)(t_z) = t_{g(z)}$. We define $\Pi(S^{(0)}) : T(S^{(0)}) \to T(S^{(0)})$ by

$$\Pi(S^{(0)})(t'_z) = \begin{cases} t'_{g(j)} & \text{if } z = i, \\ t'_{g(i)} & \text{if } z = j, \\ t'_{g(z)} & \text{otherwise.} \end{cases} \tag{3.4}$$

Thus we obtain an ABS $S^{(0)}$. We call this ABS the *small modification by* (t_i, t_j).

For an ABS S, we often describe th map $\Pi(S)$ using the arrows:

$$\begin{tikzcd} \bullet & \bullet \arrow[l, bend right, "\Pi(S)"'] \end{tikzcd}, \qquad \begin{tikzcd} \bullet \arrow[r, bend right, "\Pi(S)"'] & \bullet \end{tikzcd}, \tag{3.5}$$

where $\bullet$ are elements of $\Delta(S)$.

Example 3.9. Let $\xi = (2,7)+(3,5)$. Let S be the ABS corresponding to the DM_1 N_ξ. Then S is described as

$$S = 1^1_1\ 1^1_2\ 0^1_3\ \underline{0^1_4}\ 0^1_5\ 1^2_1\ \underline{1^2_2}\ 0^1_6\ 0^1_7\ 1^2_3\ 0^2_4\ 0^2_5\ 0^1_8\ 0^1_9\ 0^2_6\ 0^2_7\ 0^2_8\ . \tag{3.6}$$

Moreover, the small modification $S^{(0)}$ by $(0^1_4, 1^2_2)$ is described as

$$S^{(0)} = 1^1_1\ 1^1_2\ 0^1_3\ \underline{1^2_2}\ 0^1_5\ 1^2_1\ \underline{0^1_4}\ 0^1_6\ 0^1_7\ 1^2_3\ 0^2_4\ 0^2_5\ 0^1_8\ 0^1_9\ 0^2_6\ 0^2_7\ 0^2_8\ . \tag{3.7}$$

Definition 3.10. Let S be an admissible ABS. Let $S^{(0)}$ be the small modification by (t_i, t_j). Put $T(S^{(0)}) = \{t_1 < \cdots < t_h\}$. An ABS is a *full modification of* $S^{(0)}$ if

(i) $T(S') = T(S^{(0)})$ as sets,

(ii) $\Delta(S') = \Delta(S^{(0)})$,

(iii) $\Pi(S') = \Pi(S^{(0)})$, and

(iv) $<'$ is an ordering of $T(S')$ to be $t_x <' t_y \Rightarrow b(t_x) \leq b(t_y)$

for elements t_x and t_y of $T(S')$. We denote by S' a full modification obtained by $S^{(0)}$. We say a full modification S' of $S^{(0)}$ is *generic* if $\ell(S') = \ell(S) - 1$.

By construction, for a small modification $S^{(0)}$ of an ABS S, there exists at least one full modification S'. Put $T(S') = \{t'_1 <' \cdots <' t'_h\}$. Full modifications are not always unique but the sequence $\Delta(S')(t'_1), \dots, \Delta(S')(t'_h)$ is unique, see Example 3.13. Moreover, if the full modification S' of S by $(0^1_i, 1^2_j)$ is unique, then this corresponds to a specialization of w, where w is the element of ${}^JW$ corresponding to S. In Proposition 3.20 and Proposition 3.23, we will see that if full modifications are not unique, then these are not generic for an ABS S corresponding to a Newton polygon ξ.

Let us see some examples of constructing full modifications.

Example 3.11. Let us see an example of constructing a full modification. Let $\xi = (2,7)+(3,5)$, and let S be the ABS associated to N_ξ. Let S' denote the full modification of the small modification obtained by 0^1_4 and 1^2_2. Then S' is described as

$$S' = 1^1_1\ 0^1_3\ 1^1_2\ 1^2_2\ 0^1_5\ 1^2_1\ 0^1_4\ 0^1_6\ 0^1_7\ 0^2_4\ 1^2_3\ 0^2_5\ 0^1_8\ 0^1_9\ 0^2_7\ 0^2_6\ 0^2_8\ . \tag{3.8}$$

One can see that these S and S' satisfy $\ell(S') = \ell(S) - 1$, i.e., this S' is a generic full modification of S.

Example 3.12. Next, let us treat a Newton polygon consisting of three segments. Let $\xi = (2,7) + (1,2) + (3,5)$. Then the ABS S corresponding to N_ξ is

$$S = 1^1_1\ 1^1_2\ 0^1_3\ \underline{0^1_4}\ 0^1_5\ 1^2_1\ 1^3_1\ \underline{1^3_2}\ 0^1_6\ 0^1_7\ 0^2_2\ 1^3_3\ 0^3_4\ 0^3_5\ 0^1_8\ 0^1_9\ 0^2_3\ 0^3_6\ 0^3_7\ 0^3_8\ . \tag{3.9}$$

For this S, the full modification S' of the small modification obtained by exchanging 0^1_4 and 1^3_2 is

$$S' = 1^1_1\ 0^1_3\ 1^1_2\ \underline{1^3_2}\ 0^1_5\ 1^3_1\ 1^2_1\ \underline{0^1_4}\ 0^1_6\ 0^1_7\ 0^3_4\ 1^3_3\ 0^2_2\ 0^3_5\ 0^1_8\ 0^1_9\ 0^3_7\ 0^3_6\ 0^2_3\ 0^3_8\ . \tag{3.10}$$

We see that this S' is not generic.

Example 3.13. Let $\xi = (3,4) + (3,2)$. Then the ABS S corresponding to ξ is described as

$$1^1_1\ 1^1_2\ 1^1_3\ \underline{0^1_4}\ 1^2_1\ 1^2_2\ 0^1_5\ 0^1_6\ 0^1_7\ \underline{1^2_3}\ 0^2_4\ 0^2_5\ . \tag{3.11}$$

Let us consider the full modification of the small modification obtained by 0^1_4 and 1^2_3. For elements t of the small modification $T(S^{(0)})$, binary expansions $b(t)$ are obtained by

$$b(t) = \begin{cases} 0.010101\cdots & \text{if } \Delta(S)(t) = 1, \\ 0.101010\cdots & \text{otherwise.} \end{cases} \tag{3.12}$$

Thus in this case, full modifications are not unique. However the DM_1 is uniquely determined as N_ζ with $\zeta = 6(1,1)$.

Remark 3.14. Let $S \in \mathcal{H}'(h,d)$. Let S' be a full modification of the small modification $S^{(0)}$ obtained by exchanging t_i and t_j with $T(S') = \{t'_1 <' \cdots <' t'_h\}$. We denote by w the element of ${}^JW$ corresponding to S. Put $s = (i,j)$ transposition. Maps $\Pi(S)$ and $\Pi(S')$ can be regarded as elements of W. We have then $\Pi(S) = xw$. For the small modification $S^{(0)}$ with $T(S^{(0)}) = \{t^{(0)}_1 < \cdots < t^{(0)}_h\}$, we define $\varepsilon \in W$ to be $t^{(0)}_z = t'_{\varepsilon(z)}$. Since $b(t^{(0)}_z) < 0.1$ if $z \leq d$ and $b(t^{(0)}_z) > 0.1$ otherwise, ε stabilizes $\{1, 2, \ldots, d\}$. Put $v = ws$. Then $w' = uv\theta(u^{-1})$ corresponds to S' for $u = x^{-1}\varepsilon^{-1}x \in W_J$. The map $\Pi(S')$ is obtained by $\varepsilon^{-1}\Pi(S)s\varepsilon$.

After this, S denotes the ABS corresponding to a Newton polygon ξ. In Definitions 3.15 and 3.16 below, we introduce sets A_n, B_n and a method to construct full modifications S' of S combinatorially. Using these sets and the method, we can calculate the lengths of full modifications, and classify generic full modifications. For instance, in Proposition 6.2 and Corollary 6.3, using this construction, we give a necessary condition for a full modification to be generic.

Definition 3.15. Let S be the ABS of a minimal DM_1. Let $S^{(0)}$ be the small modification by $(0^r_i, 1^q_j)$. Set $\delta = \Delta(S^{(0)})$ and $\pi = \Pi(S^{(0)})$. For non-negative integers n, we write α_n for $\pi^n(0^r_i)$. We define a subset A_0 of $T(S^{(0)})$ to be

$$A_0 = \{t \in T(S^{(0)}) \mid t < \alpha_0 \text{ and } \alpha_1 < \pi(t) \text{ in } T(S^{(0)}),\ \text{with } \delta(t) = 0\} \tag{3.13}$$

endowed with the order induced from $T(S^{(0)})$. Let n be a natural number. We construct an ABS $S^{(n)}$ and a set A_n from the ABS $S^{(n-1)}$ and the set A_{n-1} as follows. Let $T(S^{(n)}) = T(S^{(n-1)})$ as sets. We define the order on $T(S^{(n)})$ so that for $t < t'$ in $S^{(n-1)}$, we have $t > t'$ if and only if $\alpha_n < t' \leq \pi(t_{\max})$ and $t = \alpha_n$ in $S^{(n-1)}$. Here $t_{\max}$ is the maximum element of A_{n-1}. We define the set A_n by

$$A_n = \{t \in T(S^{(n)}) - T(S_q) \mid t < \alpha_n \text{ and } \alpha_{n+1} < \pi(t) \text{ in } T(S^{(n)}) \text{ with } \delta(t) = \delta(\alpha_n)\} \tag{3.14}$$

endowed with the order induced from $S^{(n)}$. Thus we obtain the ABS $S^{(n)} = (T(S^{(n)}), \delta, \pi)$ and the set A_n. We call these sets $\{A_n\}$ *A-sequence* associated to S, 0_i^r and 1_j^q.

Proposition 3.20 implies that if a full modification obtained by a small modification is generic, then there exists a non-negative integer a such that $A_a = \emptyset$. Now we suppose that there exists such an integer a. Then we can define the following ABS's and sets.

Definition 3.16. For the ABS S corresponding to a minimal DM_1, let $S^{(0)}$ be the small modification by $(0_i^r, 1_j^q)$. We write δ for $\Delta(S^{(0)})$ and π for $\Pi(S^{(0)})$. Put $\beta_n = \pi^n(1_j^q)$ for non-negative integers n. Assume that there exists the minimum non-negative integer a such that $A_a = \emptyset$, we define a set B_0 by

$$B_0 = \{t \in T(S^{(a)}) \mid \beta_0 < t \text{ and } \pi(t) < \beta_1 \text{ in } T(S^{(a)}) \text{ with } \delta(t) = 1\} \tag{3.15}$$

endowed with the order induced from $T(S^{(a)})$. For the ABS $S^{(a+n-1)}$ and the set B_{n-1}, we define an ABS $S^{(a+n)}$ as follows. Let $T(S^{(a+n)}) = T(S^{(a+n-1)})$ as sets. Let $\Delta(S^{(a+n)}) = \Delta(S^{(a+n-1)})$ and $\Pi(S^{(a+n)}) = \Pi(S^{(a+n-1)})$. The ordering of $T(S^{(a+n)})$ is given so that for $t < t'$ in $S^{(a+n-1)}$, we have $t > t'$ if and only if $\pi(t_{\min}) \leq t < \beta_n$ and $t' = \beta_n$, where $t_{\min}$ is the minimum element of B_{n-1}. We define the set B_n as

$$B_n = \{t \in T(S^{(a+n)}) \mid \beta_n < t \text{ and } \pi(t) < \beta_{n+1} \text{ in } T(S^{(a+n)}) \text{ with } \delta(t) = \delta(\beta_n)\} \tag{3.16}$$

with the ordering obtained from the order on $S^{(a+n)}$. Thus we obtain the ABS $S^{(a+n)}$ and the set B_n. We call these sets $\{B_n\}$ *B-sequence* associated to S, 0_i^r and 1_j^q.

For a small modification by $(0_i^r, 1_j^q)$, if there exists non-negative integers a and b such that $A_a = \emptyset$ and $B_b = \emptyset$, then the ABS $S^{(a+b)}$ is *the* full modification by $(0_i^r, 1_j^q)$. In Proposition 3.23, we will see that if a full modification is generic, then for the above sets B_n, there exists a non-negative integer b such that $B_b = \emptyset$, i.e., a generic full modification is obtained by the ABS $S^{(a+b)}$ for some integers a and b.

Example 3.17. Let $\xi = (2,7) + (3,5)$. Let S be the ABS of ξ. The small modification $S^{(0)}$ by $(0_4^1, 1_2^2)$ constructed in Example 3.9, we have sets $A_0 = \{0_5^1\}$ and $A_1 = \emptyset$. The ABS $S^{(1)}$ is obtained by

$$S^{(1)} = 1_1^1 \ 0_3^1 \ 1_2^1 \ 1_2^2 \ 0_5^1 \ 1_1^2 \ 0_4^1 \ 0_6^1 \ 0_7^1 \ 1_3^2 \ 0_4^2 \ 0_5^2 \ 0_8^1 \ 0_9^1 \ 0_6^2 \ 0_7^2 \ 0_8^2 \,. \tag{3.17}$$

By the above $B_0 = \{1_1^2\}$. We have $B_1 = \{0_7^2\}$ with the ABS

$$S^{(2)} = 1_1^1\ 0_3^1\ 1_2^1\ 1_2^2\ 0_5^1\ 1_1^2\ 0_4^1\ 0_6^1\ 0_7^1\ 1_3^2\ 0_4^2\ 0_5^2\ 0_8^1\ 0_9^1\ 0_7^2\ 0_6^2\ 0_8^2\,. \tag{3.18}$$

Clearly $B_2 = \emptyset$. Hence we see $a = 1$ and $b = 1$. One can check that the full modification $S^{(3)}$ is equal to S' of Example 3.11.

3.2 Constructing full modifications combinatorially

In this section, using Definitions 3.15 and 3.16, we construct full modifications combinatorially.

We use the notation of Notation 3.6. Furthermore, we fix the following notation. Let S be the ABS associated to N_ξ. Let $S^{(0)}$ be the small modification by $(0_i^r, 1_j^q)$. Then we obtain arrowed binary sequences $S^{(1)}, S^{(2)}, \ldots$ and the A-sequence $A_0, A_1, \ldots$ by Definition 3.15. Put $\delta = \Delta(S^{(0)})$ and $\pi = \Pi(S^{(0)})$. We set $\alpha_n = \pi^n(0_i^r)$ and $\beta_n = \pi^n(1_j^q)$ for non-negative integers n.

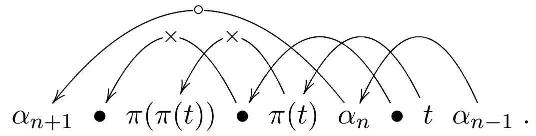
Proposition 3.18. Let n be a natural number with $n < a'$, where a' is the minimum number such that $\alpha_{a'} = \beta_0$. The set A_n is equal to

$$\{\pi(t) \mid t \in A_{n-1},\ \pi(t) \notin T(S_q) \text{ and } \delta(\pi(t)) = \delta(\alpha_n)\}. \tag{3.19}$$

Proof. Note that for elements t and t' of $T(S^{(n)})$, with $n < a'$, we have $t < t'$ and $\pi(t') < \pi(t)$ with $\delta(t) = \delta(t')$ if and only if $t \in A_n$ and $t' = \alpha_n$, or $t = \beta_0$ and $t' \in B_0' = \{t \in T(S^{(0)}) \mid \beta_0 < t \text{ and } \pi(t) < \beta_1 \text{ with } \delta(t) = 1\}$. First take an element $\pi(t)$ of the set (3.19). Let us show that this $\pi(t)$ belongs to A_n. The part of $S^{(n-1)}$ can be described as

$$\alpha_{n+1}\ \bullet\ \pi(\pi(t))\ \ \alpha_n\ \bullet\ \pi(t)\ \bullet\ t\ \ \alpha_{n-1}\,.$$

Since A_{n-1} contains t, we see $\alpha_n < \pi(t)$ in $T(S^{(n-1)})$. As $\delta(\pi(t)) = \delta(\alpha_n)$, we have $\alpha_{n+1} < \pi(\pi(t))$ in $T(S^{(n-1)})$. In the set $T(S^{(n)})$, the element α_n is located in the right side of the maximum element of $\pi(A_{n-1})$. By construction, the part of $S^{(n)}$ is described as



We have then $\pi(t) < \alpha_n$ and $\alpha_{n+1} < \pi(\pi(t))$ in $T(S^{(n)})$. Hence we see that A_n contains $\pi(t)$.

Conversely, let t' be an element of A_n. Let t be an element of $T(S^{(n)})$ such that $\pi(t) = t'$. It is enough to show that t belongs to A_{n-1}. By the definition of A_n, this t satisfies that $\pi(t) < \alpha_n$ and $\alpha_{n+1} < \pi(\pi(t))$ in $T(S^{(n)})$. As $\delta(\pi(t)) = \delta(\alpha_n)$, we have $\alpha_{n+1} < \pi(\pi(t))$ and $\alpha_n < \pi(t)$ in $T(S^{(n-1)})$. To make a contradiction, let us suppose $\alpha_{n-1} < t$ in $T(S^{(n-1)})$. Then for the maximum element $t_{\max}$ of A_{n-1}, we have $\alpha_n < \pi(t)$ in $T(S^{(n)})$ since $\alpha_n < \pi(t_{\max}) < \pi(t)$ in $T(S^{(n-1)})$. This contradicts the definition of A_n. Thus we have shown $t < \alpha_{n-1}$ in $T(S^{(n-1)})$, and it implies that t belongs to A_{n-1}. Hence this t is an element of the set (3.19). $\square$

Proposition 3.19. *A_n does not contain elements α_m for $m \leq n$.*

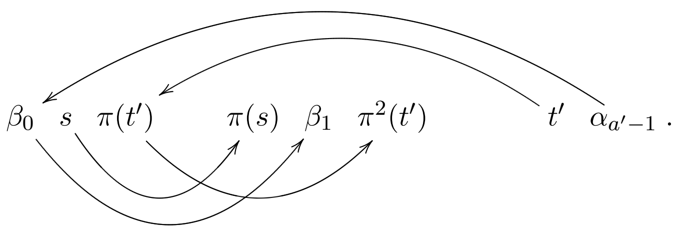
Proof. Note that for all non-negative integers n, sets A_n do not contain the inverse image of α_0, which is an the element of $T(S_q)$. Here S_q is the ABS corresponding to the DM$_1$ N_{m_q,n_q}. Let us show the assertion by induction on n. The case $n = 0$ is obvious. For a natural number n, suppose that A_n contains α_m for a non-negative integer m with $m \leq n$. By Proposition 3.18, then A_{n-1} contains α_{m-1}. This contradicts the hypothesis of induction. $\square$

Proposition 3.20. *If there exists no non-negative integer a such that $A_a = \emptyset$, then every full modification S' of the small modification $S^{(0)}$ is not generic.*

Proof. First, let us construct a full modification combinatorially. Let a' be the minimum number satisfying $\alpha_{a'} = \beta_0$. Let B'_0 be the set

$$B'_0 = \{t \in T(S^{(0)}) \mid \beta_0 < t \text{ and } \pi(t) < \beta_1 \text{ in } T(S^{(0)}) \text{ with } \delta(t) = 1\}.$$

We can describe a part of $S^{(a'-1)}$ as



Assume that $\pi(A_{a'-1})$ is contained in B'_0. Then there exists no element s of $T(S^{(a')})$ such that $s < \alpha_{a'}$ and $\alpha_{a'+1} < \pi(s)$ since $t'' < \beta_0$ and $\pi(t'') < \beta_1$ in $T(S^{(a')})$ for all $t'' \in \pi(A_{a'-1})$. Thus there exists an element t' of $A_{a'-1}$ such that $\beta_0 < \pi(t')$ and $\beta_1 < \pi(\pi(t'))$ in $T(S^{(a'-1)})$ with $\delta(\beta_0) = \delta(\pi(t'))$. Then for all elements s of B'_0, we have $s < \pi(t')$ since if $\pi(t') < s$, then $\pi(t') < s$ and $\pi(s) < \pi(\pi(t'))$ holds with $\delta(s) = \delta(\pi(t'))$. This is a contradiction. Thus there exists no element u of $T(S^{(a')})$ such that $\beta_0 < u$ and $\pi(u) < \beta_1$. Let m be a non-negative integer such that $|A_m| = |A_{m+1}| = \cdots$. Then for elements u and u' of $T(S^{(m)})$, we have $u < u'$ if $b(u) \leq b(u')$, where $b(u)$ is the binary expansion of u, see the paragraph before Proposition 3.4 for the definition of binary expansions. Hence a full modification S' of S is obtained by $(S^{(m)}, \delta, \pi)$.

Let us compare lengths of S and S'. One can see that $\ell(S) - \ell(S^{(0)}) = |A_0| + |B'_0| + 1$. Since $\ell(S^{(m)}) - \ell(S^{(0)}) \leq |A_0| - |A_m|$, we see $\ell(S') < \ell(S) - 1$. $\square$

By Proposition 3.20, we may assume that there exists a non-negative integer a such that A_a is an empty set to classify generic full modifications of arrowed binary sequences.

In Proposition 3.23, we will show that to classify generic full modifications, it suffices to consider the case there exists a non-negative integer b such that $B_b = \emptyset$. Let us see some properties of sets B_n, which is used for the proof of Proposition 3.23.

Proposition 3.21. Let n be a natural number. The set B_n is obtained by

$$B_n = \{\pi(t) \mid t \in B_{n-1} \text{ and } \delta(\pi(t)) = \delta(\beta_n)\}. \tag{3.20}$$

Proof. A proof is given in the same way as Proposition 3.18. $\square$

Proposition 3.22. B_n does not contain elements β_m for $m \leq n$.

Proof. A proof is given in the same way as Proposition 3.19. $\square$

Proposition 3.23. If there exists no non-negative integer b such that $B_b = \emptyset$, then every full modification of the small modification $S^{(0)}$ is not generic.

Proof. In this hypothesis, there exists a non-negative integer m such that $|B_m| = |B_{m+1}| = \cdots$. Then the elements of $T(S^{(a+m)})$ are ordered by these binary expansions. Thus we obtain a full modification $S' = (T(S^{(a+m)}), \delta, \pi)$.

Let us compare the lengths of S and S'. It is clear that $\ell(S) - \ell(S^{(0)}) = |A_0| + |B'_0| + 1$, where the set B'_0 is as in Proposition 3.20. Let a' be as in Proposition 3.20. If the non-negative integer a satisfies $a \geq a'$, then $|B_0| < |B'_0|$, see the proof of Proposition 3.20.

Since $\ell(S^{(a)}) - \ell(S^{(0)}) \leq |A_0|$ and $\ell(S') - \ell(S^{(a)}) \leq |B_0| - |B_m|$, we see that S' is not generic in this case. Let us see the case $a < a'$. For a natural number n such that $\delta(\alpha_n) = 1$ and A_{n-1} contains the inverse image of β_0, one can see that β_0 belongs to B_0. Let I denote the set consisting of such α_n. We have then $B_0 = B_0' \cup I$. If α_n belongs to I, then $\ell(S^{(n)}) - \ell(S^{(n-1)}) = |A_{n-1}| - |A_n| - 1$ since A_n does not contain β_0. Thus we have $\ell(S^{(a)}) - \ell(S^{(0)}) \leq |A_0| - |I|$. Since $\ell(S') - \ell(S^{(a)}) \leq |B_0| - |B_m|$, we see $\ell(S') - \ell(S) \leq |A_0| + |B_0'| - |B_m|$, and it implies that S' is not generic. $\square$

By Propositions 3.20 and 3.23, we construct the full modifications for all small modifications. Moreover, these propositions imply that, to classify generic full modifications, we may suppose that there exist non-negative integers a and b such that $A_a = \emptyset$ and $B_b = \emptyset$ for a small modification. For the ABS $S^{(a+b)}$, if elements t and t' of $T(S^{(a+b)})$ satisfy that $t < t'$ and $\delta(t) = \delta(t')$, then $\pi(t) < \pi(t')$ holds. Thus we see that by Definitions 3.15 and 3.16, for a small modification, we get a full modification S' of S by $S^{(a+b)}$. We call this ABS $S^{(a+b)}$ the full modification by $(0_i^r, 1_j^q)$.

Chapter 4

The case of 1/2-separated Newton polygons

In this chapter, we treat Newton polygons ξ satisfying the condition

- ξ consists of two segments satisfying that one slope is less than 1/2 and the other is greater than 1/2.

We call such a Newton polygon a *1/2-separated Newton polygon*, i.e., a Newton polygon $\xi = (m_1, n_1) + (m_2, n_2)$ is 1/2-separated if $n_2/(m_2 + n_2) < 1/2 < n_1/(m_1 + n_1)$. In this chapter, we solve Problem 1.4 for these 1/2-separated Newton polygons. Moreover, we shall show a key proposition (Proposition 4.9) to solve Problem 1.7 for the case that the Newton polygon ξ is 1/2-separated.

4.1 Classifying generic specializations for $1/2$-separated Newton polygons

In this section we classify generic full modifications for ABS's corresponding to 1/2-separated Newton polygons. Let ξ be a 1/2-separated Newton polygon, say $\xi = (m_1, n_1) + (m_2, n_2)$. Let S be the ABS corresponding to ξ. Let S' be a full modification obtained by the small modification by a pair $(0^1_i, 1^2_j)$ with $m_1 < i \leq n_1$ and $1 \leq j \leq m_2$. Then we obtain the A-sequence and B-sequence $A_0, \ldots, A_a$ and $B_0, \ldots, B_b$, where a (resp. b) is the smallest integer such that $A_a = \emptyset$ (resp. $B_b = \emptyset$). Note that, by Propositions 3.20 and 3.23, we may assume that there exist non-negative integers a and b such that $A_a = \emptyset$ and $B_b = \emptyset$ to classify generic full modifications of S. Moreover, we have ABS's $S^{(0)}, \ldots, S^{(a+b)}$ with $S^{(a+b)} = S'$. Theorem 4.1 below gives a classification of generic specializations of w_ξ with 1/2-separated Newton polygons ξ. We denote by the element ϕ_i (resp. ψ_j) of

$T(S^{(0)})$ the inverse image of 0_i^1 (resp. 1_j^2) by the map $\Pi(S^{(0)})$. Since $T(S^{(n)})$ for all n are the same as sets, we use the same symbol ϕ_i (resp. ψ_j) for the same element of $T(S^{(n)})$ corresponding to the element ϕ_i (resp. ψ_j) of $T(S^{(0)})$.

Theorem 4.1. Let ξ be a 1/2-separated Newton polygon. Let S be the ABS corresponding to N_ξ. A full modification S' obtained by the small modification by 0_i^1 and 1_j^2 is generic if and only if the subsets A_n of $T(S^{(n)})$ (resp. B_n of $T(S^{(a+n)})$) do not contain ϕ_i (resp. ψ_j) for all n.

By Lemma 3.7, for 0_i^1 and 1_j^2, we can consider the following three cases:

(i) $m_1 < i \leq n_1$ and $1 \leq j \leq n_2$,

(ii) $m_1 < i \leq n_1$ and $n_2 < j \leq m_2$,

(iii) $n_1 < i \leq m_1 + n_1$ and $n_2 < j \leq m_2$.

By the duality, it suffices to deal with cases (i) and (ii). We fix some notations. Put $\alpha_n = \Pi(S)^n(0_i^1)$ and $\beta_n = \Pi(S)^n(1_j^2)$. For non-negative integers n, set $e(n) = \ell(S^{(n+1)}) - \ell(S^{(n)})$. Moreover, we write $d_1(n) = |A_n| - |A_{n+1}|$ and $d_2(n) = |B_n| - |B_{n+1}|$.

Propositions 4.2 and 4.3, which compare values $e(n)$ and $d_x(n)$, are key propositions to give a criterion of generic full modifications.

Proposition 4.2. For all non-negative integers n with $n < a$, we have $e(n) \leq d_1(n)$. Moreover, the equality holds for all n if and only if there exists no non-negative integer n such that A_n contains ψ_j.

Proof. By Definition 3.15, we clearly have $e(n) \leq d_1(n)$ for all n. If for all n the sets A_n do not contains ψ_j, then $e(n) = d_1(n)$ holds for all n. Conversely, assume that A_n contains ψ_j for some n. Since $m_1 < i \leq n_1$, the inverse image of 1_j^2 is $0_{i+m_1}^1$ in $S^{(n)}$. If $\Delta(S)(\alpha_n) = 1$, then $e(n) = d_1(n) - 1$. On the other hand, if $\Delta(S)(\alpha_n) = 0$, then we have $e(n) = -1$ and $d_1(n) = 1$. This completes the proof. $\square$

Recall that the set I is the subset of B_0 consisting of elements which are of the form α_m, see the proof of Proposition 3.23.

Proposition 4.3. For all non-negative integers n with $a \leq n < a+b$, we have $e(n) \leq d_2(n)$. Moreover, for the case $1 \leq j \leq n_2$, the equality holds if and only if

(i) there exists no non-negative integer n such that B_n contains ϕ_i, and

(ii) $I = \emptyset$.

Proof. By Definition 3.16, clearly the inequality $e(n) \leq d_2(n)$ holds, and if B_n do not contain ϕ_i for all n, then $e(n) = d_2(n)$ holds. In the case $1 \leq j \leq n_2$, the inverse image of 0_i^1 is $0_{j+m_2}^2$. Assume that B_n contains $0_{j+m_2}^2$ for some n. Then $e(n) = -1$ and $d_2(n) = 1$ since $\delta(\pi(t)) = 1$ for elements t of B_n except $0_{j+m_2}^2$, where $\delta = \Delta(S^{(n)})$ and $\pi = \Pi(S^{(n)})$. Next, suppose $I \neq \emptyset$. We divide the proof into two cases depending on values of $\delta(\beta_1)$. If $\delta(\beta_1) = 1$, then as $e(0) = -|I|$ and $d_2(0) = |I|$. On the other hand, if $\delta(\beta_1) = 0$, then $e(1) = -|I|$ and $d_2(1) = |I|$. $\square$

Proof of Theorem 4.1. First, let us treat the case $m_1 < i \leq n_1$ and $1 \leq j \leq n_2$. In this case, we have $\ell(S^{(0)}) - \ell(S) = -(n_1 - i + j)$. Note that if there exists no non-negative integer n such that A_n contains ψ_j, then the set I is empty. By definition we have $\sum_{n=0}^{a-1} d_1(n) = n_1 - i$ and $\sum_{n=a}^{a+b-1} d_2(n) = j - 1$. Thus, if the subsets A_n of $T(S^{(n)})$ (resp. B_n of $T(S^{(a+n)})$) do not contain ϕ_i (resp. ψ_j) for all n, then by Proposition 4.2 and Proposition 4.3, we obtain $\ell(S') - \ell(S^{(0)}) = n_1 - i + j - 1$. Hence S' is generic. Let us consider the converse. If I is empty, then we have $\sum_{n=0}^{a-1} e(n) < n_1 - i$ or $\sum_{n=a}^{a+b-1} e(n) < j - 1$ with $\ell(S^{(0)}) - \ell(S) = n_1 - i + j$. On the other hand, if $I \neq \emptyset$, then $\sum_{n=0}^{a-1} e(n) \leq n_i - i - |I|$. Moreover, by the proof of Proposition 4.3 we have $\sum_{n=a}^{a+b-1} e(n) < j - 1$.

Next, suppose that $m_1 < i \leq n_1$ and $n_2 < j \leq m_2$. We have then $\ell(S) - \ell(S^{(0)}) = m_1 + n_1 - i + j$. In this case, $\sum_{n=0}^{a-1} e(n) < m_1 + n_1 - i$ since A_0 contains the ψ_j. Moreover, $\sum_{n=1}^{a+b-1} e(n) \leq j - 1$, and hence S' is not generic. $\square$

Example 4.4. Here let us see an example of constructing a generic full modification for a 1/2-separated Newton polygon. Let $\xi = (2,5) + (3,2)$. For the ABS S

$$S = 1_1^1\ 1_2^1\ 0_3^1\ 0_4^1\ 0_5^1\ 1_1^2\ 1_2^2\ 0_6^1\ 0_7^1\ 1_3^2\ 0_4^2\ 0_5^2\ , \tag{4.1}$$

corresponding to ξ, let us construct the full modification from the small modification by $(0_4^1, 1_2^2)$. The ABS $S^{(0)}$ is described as

$$S^{(0)} = 1_1^1\ 1_2^1\ 0_3^1\ 1_2^2\ 0_5^1\ 1_1^2\ 0_4^1\ 0_6^1\ 0_7^1\ 1_3^2\ 0_4^2\ 0_5^2\ , \tag{4.2}$$

We have the set $A_0 = \{0^1_5\}$ and the ABS $S^{(1)}$:

$$S^{(1)} = 1^1_1\ 0^1_3\ 1^1_2\ 1^2_2\ 0^1_5\ 1^2_1\ 0^1_4\ 0^1_6\ 0^1_7\ 1^2_3\ 0^2_4\ 0^2_5\ . \tag{4.3}$$

We see that A_1 is an empty set. Moreover, the set B_0 is $\{1^2_1\}$. Since B_1 is empty, the ABS $S^{(2)}$ is the full modification S'. We obtain the full modification S' by

$$S' = 1^1_1\ 0^1_3\ 1^1_2\ 1^2_2\ 0^1_5\ 1^2_1\ 0^1_4\ 0^1_6\ 0^1_7\ 0^2_4\ 1^2_3\ 0^2_5\ . \tag{4.4}$$

One can see that this full modification is generic.

Here we state some properties of generic full modifications S' of S. Note that for a small modification of S, the full modification is unique if it is generic. These properties are useful for constructing Newton polygons of generic specializations w of w_ξ. By the proof of Theorem 4.1, to study generic specializations, it suffices to deal with full modifications obtained by the small modification by 0^1_i and 1^2_j with $m_1 < i \leq n_1$ and $1 \leq j \leq n_2$. Let $S = S_1 \oplus S_2$ be the ABS associated to a 1/2-separated Newton polygon, where S_i corresponds to i-th segment of the Newton polygon. For a generic full modification, by Theorem 4.1, the sets A_n (resp. B_n) only depend on i (resp. j) since A_n (resp. B_n) are subsets of $T(S_1)$ (resp. $T(S_2)$) as sets. Thus we can define the following sets.

Definition 4.5. Set

$$G_1 = \{(0^1_i, 1^2_j) \mid S' \text{ obtained by } 0^1_i \text{ and } 1^2_j \text{ is generic with } m_1 < i \leq n_1 \text{ and } 1 \leq j \leq n_2\}.$$

By the above, we can describe this set G_1 as $G_1 = C' \times D'$, with $C' \subset T(S_1)$ and $D' \subset T(S_2)$. Put $C = C' - \{0^1_{n_1}\}$ and $D = D' - \{1^2_1\}$.

Remark 4.6. By the duality, the set

$$G_2 = \{(0^1_i, 1^2_j) \mid S' \text{ obtained by } 0^1_i \text{ and } 1^2_j \text{ is generic}, n_1 < i \leq h_1 \text{ and } n_2 < j \leq m_2\},$$

wheret $h_1 = m_1 + n_1$, can be described as $G_2 = C'' \times D''$, with $C'' \subset T(S_1)$ and $D'' \subset T(S_2)$. Moreover, the set G consisting of pairs $(0^1_i, 1^2_j)$ such that the full modifications S' obtained by 0^1_i and 1^2_j are generic is equal to $G_1 \cup G_2$.

This definition says that a full modification obtained by the small modification by 0^1_i and 1^2_j is generic if and only if 0^1_i belongs to C' and 1^2_j belongs to D'. Note that in [4], we treated the generic full modification obtained by the small modification by $0^1_{n_1}$ and 1^2_1.

Example 4.7. Let $\xi = (2,5) + (3,2)$. Let S be the ABS corresponding to ξ. Then the set G is obtained by

$$G = \{(0^1_4, 1^2_1),\ (0^1_4, 1^2_2),\ (0^1_5, 1^2_1),\ (0^1_5, 1^2_2),\ (0^1_6, 1^2_3),\ (0^1_7, 1^2_3)\}. \tag{4.5}$$

Lemma 4.8. Let n be a non-negative integer. For a generic full modification, if $d_1(n) > 0$ (resp. $d_2(n) > 0$) and A_{n+1} (resp. B_{n+1}) is not empty, then the maximum element of A_{n+1} (resp. the minimum element of B_{n+1}) is $1^1_{m_1}$ (resp. $0^2_{m_2+1}$).

Proof. For a non-negative integer n satisfying that $d_1(n) > 0$ and $A_{n+1} \neq \emptyset$, if $\Delta(S)(\alpha_n) = 1$, then $A_{n+1} = \Pi(S^{(n)})(A_n)$ and $d_1(n) = 0$. Moreover, if $\Delta(S)(\alpha_n) = \Delta(S)(\alpha_{n+1}) = 0$, then similarly $d_1(n) = 0$. For the case that $\Delta(S)(\alpha_n) = 0$ and $\Delta(S)(\alpha_{n+1}) = 1$, if $d_1(n) > 0$, then $\Pi(S^{(n)})(A_n)$ contains $1^1_{m_1}$. Hence the maximum element of A_{n+1} is $1^1_{m_1}$. In the same way we can see that if $d_2(n) > 0$ and $B_{n+1} \neq \emptyset$, then the minimum element of B_{n+1} is $0^2_{m_2+1}$. $\square$

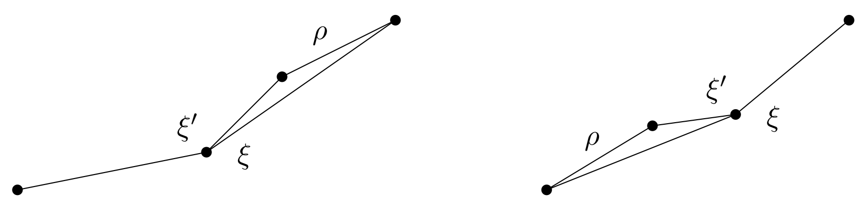
4.2 Determining the Newton polygons of generic specializations for 1/2-separated Newton polygons

In this section, we show Proposition 4.9. In Section 6.2, we will show Theorem 1.8, which is a complete answer to Problem 1.7, by induction. Proposition 4.9 is a key proposition to be applied the induction step.

Proposition 4.9. Let ξ be a 1/2-separated Newton polygon. Assume that $\xi \neq (0,1) + (1,0)$. For every element w of $B(\xi)$, there exist a generic specialization w^- of w and a segment $\rho = (c,d)$ such that

$$w^- = w' \oplus w_\rho, \tag{4.6}$$

with $w' \in B(\xi')$, where $\xi' = (m_1 - c, n_1 - d) + (m_2, n_2)$ or $\xi' = (m_1, n_1) + (m_2 - c, n_2 - d)$:



so that the area of the triangle surrounded by ξ, ξ' and ρ is one.

The next proposition (Proposition 4.10) is more concretely described in terms of ABS than Proposition 4.9. We shall prove Proposition 4.10. Let us fix some notations. Let $\xi = (m_1, n_1) + (m_2, n_2)$ be a 1/2-separated Newton polygon. Let $S = S_1 \oplus S_2$ be the ABS of ξ, and let S^- denote the generic full modification obtained by the small modification by 0^1_i and 1^2_j. By the result of Section 4.1, we may assume that $m_1 < i \leq n_1$ and $1 \leq j \leq n_2$. Constructing the full modification, we obtain the A-sequence and the B-sequence $A_0, \ldots, A_a, B_0, \ldots, B_b$ and ABS's $S^{(0)}, \ldots, S^{(a+b)}$, where $S^{(a+b)} = S'$. We write $\alpha_n = \pi^n(0^1_i)$ and $\beta_n = \pi^n(1^2_j)$, where $\pi = \Pi(S^{(0)})$. As with Section 4.1, let ϕ_i and ψ_j denote the inverse images of 0^1_i and 1^2_j by the bijection map π respectively. Using Proposition 4.9, we obtain Theorem 1.8 by induction. In the following proposition, we give a concrete construction method to obtain (4.6) from every generic full modification.

Proposition 4.10. For the 1/2-separated Newton polygon ξ satisfying that $0 < \lambda_2 < 1/2 < \lambda_1 < 1$, let S^{--} denote the ABS corresponding to a specialization w^- for a generic specialization w of w_ξ. Then this w^- satisfies (4.6) if

(i) for the case $n_1 > m_1 + 1$,

- (a) S^{--} is the full modification obtained by the small modification S^- by $0^1_{m_1+1}$ and $1^1_{m_1}$, or
- (b) S^{--} is the full modification obtained by the small modification S^- by 0^1_{i-1} and 1^2_j,

(ii) for the case $n_1 = m_1 + 1$,

- (c) S^{--} is the full modification obtained by the small modification S^- by $0^2_{m_2+1}$ and $1^2_{m_2}$,
- (d) S^{--} is the full modification obtained by the small modification S^- by 0^1_i and 1^2_{j+1}, or
- (e) S^{--} is the full modification obtained by the small modification S^- by $0^1_{m_1+n_1}$ and $1^2_{n_2+1}$.

In the cases (a) and (b), the Newton polygon ξ' of (4.6) is of the form $\xi' = (m_1 - f, n_1 - g) + (m_2, n_2)$. On the other hand, in the cases (c) and (d) we have $\xi' = (m_1, n_1) + (m_2 - f, n_2 - g)$. In particular, in the case (e) we determine the Newton polygons ρ and ξ' by $\rho = (1, 1)$ and $\xi' = (m_1 - 1, n_1 - 1) + (m_2, n_2)$.

First we show Proposition 4.10 in case (i). We use some notation of Definition 4.5. By construction, in $T(S^-)$ we have $0^1_{i-1} < 1^2_j$ since if $1^2_j < 0^1_{i-1}$ in $T(S^{(n)})$ and $0^1_{i-1} < 1^2_j$ in $T(S^{(n-1)})$ for some n, then $\alpha_n = 0^1_{i-1}$ and the set A_{n-1} contains ψ_j. This contradicts the condition of generic full modifications as shown in Theorem 4.1. To treat the case (a), we shall show that $0^1_{m_1+1} < 1^1_{m_1}$ in $T(S^-)$. To do this, we introduce

Notation 4.11. For an ABS S, let $t \in T(S)$. Put $\pi = \Pi(S)$. We often express the subset $\{t, \pi(t), \pi^2(t), \dots, \pi^n(t)\}$ of $T(S)$ as

$$t \to \pi(t) \to \pi^2(t) \to \cdots \to \pi^n(t), \tag{4.7}$$

and we call such a diagram *pass.* We often call an element of a pass a *vertex* of the pass.

Proposition 4.12. For the generic full modification S^-, we have $0^1_{m_1+1} < 1^1_{m_1}$. Moreover, there exists no non-negative integer n such that $\alpha_n = 0^1_{m_1+1}$ with $n \leq a$.

Proof. In the ABS S_1 corresponding to the first segment of ξ, binary expansions of $1^1_{m_1}$ and $0^1_{m_1+1}$ are obtained by

$$\begin{aligned} b(1^1_{m_1}) &= 0.b_1 b_2 \cdots b_{h-2} 01, && (4.8) \\ b(0^1_{m_1+1}) &= 0.b_1 b_2 \cdots b_{h-2} 10, && (4.9) \end{aligned}$$

where $h = m_1 + n_1$. In $T(S^-)$ we have two paths

$$\begin{aligned} 0^1_i &\to \cdots \to 1^1_{m_1}, && (4.10) \\ 0^1_{i+1} &\to \cdots \to 0^1_{m_1+1}. && (4.11) \end{aligned}$$

Clearly 0^1_{i+1} belongs to A_0. Moreover $\Pi(S^{(0)})^a(0^1_{i+1})$, which is equal to $0^1_{m_1+1}$, belongs to $\Pi(S^{(0)})(A_{a-1})$. By the construction of the A-sequences, we have $0^1_{m_1+1} < 1^1_{m_1}$ in $T(S^-)$. Let us see the latter statement. Assume that $\alpha_n = 0^1_{m_1+1}$ for some n. Then A_{a-1} contains α_{n-1}, and this contradicts Proposition 3.19. $\square$

Notation 4.13. For the ABS S_1, we define paths P and Q of $T(S_1)$ by

$$\begin{aligned} P &: 1^1_{m_1} \to 0^1_{m_1+n_1} \to \cdots \to 0^1_{2m_1+1}, && (4.12) \\ Q &: 0^1_{m_1+1} \to 1^1_1 \to \cdots \to 0^1_{2m_1}. && (4.13) \end{aligned}$$

These paths P and Q are useful. For instance, in the case (a) of Proposition 4.10, for the ABS S_ρ associated to the segment ρ, the set $T(S_\rho)$ is equal to P. Moreover, we have

Lemma 4.14. The set C is contained in Q.

Proof. Take $0^1_i \in C$. If 0^1_i belongs to P, then there exists a natural number n such that $\Pi(S)^n(0^1_i) = 0^1_{m_1+1}$ with $n < a$. This contradicts Proposition 4.12. $\square$

Definition 4.15. We define a set C_1 (resp. C_2) to be the subset of C' consisting of elements 0^1_i satisfying that for the generic full modification S^- obtained by 0^1_i and 1^2_j, we obtain (4.6) of Proposition 4.9 by S^{--} of (a) (resp. (b)).

Clearly if $C' = C_1 \cup C_2$, then we complete the proof of Proposition 4.10 (i). From now on, for each element of C, we not only show that the element belongs to $C_1 \cup C_2$ but also determine which of C_1 or C_2 the element belongs to. The goal is Proposition 4.22. To do this, let us see the construction of the ABS S^{--} of (a) and (b) in Proposition 4.9 concretely. Using a path, the ABS S^- obtained by 0^1_i and 1^2_j is described as

$$\begin{tikzcd}
\bullet \arrow[d] & \cdots \arrow[l] & 1^1_{m_1} \arrow[l] & 0^1_{2m_1} \arrow[l] & \cdots \arrow[l] & 0^1_i \arrow[l] & 0^2_{j+m_2} \arrow[l] & \cdots \arrow[l] \\
\bullet \arrow[r] & \cdots \arrow[r] & 0^1_{2m_1+1} \arrow[r] & 0^1_{m_1+1} \arrow[r] & \cdots \arrow[r] & 0^1_{i+m_1} \arrow[r] & 1^2_j \arrow[r] & \cdots \arrow[u, "\text{elements of } S_2"']
\end{tikzcd} \tag{4.14}$$

First let us treat the case (a) of Proposition 4.9: S^{--} is the full modification of S^- obtained by exchanging $0^1_{m_1+1}$ and $1^1_{m_1}$. By construction, the full modification obtained by $(0^1_{m_1+1}, 1^1_{m_1})$ for S^- is

$$\begin{tikzcd}
\bullet \arrow[d] & \cdots \arrow[l] & 1^1_{m_1} \arrow[l] & 0^1_{2m_1} \arrow[l, dotted, "\times" marking] \arrow[d] & \cdots \arrow[l] & 0^1_i \arrow[l] & 0^2_{j+m_2} \arrow[l] & \cdots \arrow[l] \\
\bullet \arrow[r] & \cdots \arrow[r] & 0^1_{2m_1+1} \arrow[r, dotted, "\times" marking] \arrow[u] & 0^1_{m_1+1} \arrow[r] & \cdots \arrow[r] & 0^1_{i+m_1} \arrow[r] & 1^2_j \arrow[r] & \cdots \arrow[u, "\text{elements of } S_2"']
\end{tikzcd} \tag{4.15}$$

which consists of two components. It is easy to see that the former component consists of elements of P. This component is associated to w_ρ with a segment $\rho = (f, g)$. As this component is equal to the component obtained from S_1 applying [2, Lemma 5.6] to $1^1_{m_1}$ and $0^1_{m_1+1}$, we have $fn_1 - gm_1 = 1$. Next we deal with the case (b) of Proposition 4.9, i.e., let us treat the ABS S^{--} which is the full modification of S^- obtained by $(0^1_{i-1}, 1^2_j)$. This ABS S^{--} is described as

$$\begin{tikzcd}
\bullet \arrow[d] & \cdots \arrow[l] & 0^1_{i-1} \arrow[l] & 0^1_{i-1+m_1} \arrow[l, dotted, "\times" marking] \arrow[d] & \cdots \arrow[l] & \bullet \arrow[l] \\
\bullet \arrow[r] & \cdots \arrow[r] & 0^1_{i+m_1} \arrow[r, dotted, "\times" marking] \arrow[u] & 1^2_j \arrow[r] & \cdots \arrow[r, "\text{elements of } S_2"'] & 0^1_i \arrow[u]
\end{tikzcd} \tag{4.16}$$

This S^{--} consists of two components. By construction, the latter component contains elements which are of the form $\Pi(S^-)^n(0^1_i)$ for non-negative integers n with $n \le a$. The former component is the ABS corresponding to w_ρ with a segment $\rho = (f, g)$. Since this component coincides with the ABS obtained from S_1 by applying [2, Lemma 5.6] to 0^1_{i-1} and 0^1_i, we have $fn_1 - gm_1 = 1$. Thus for cases (a) and (b), We can write $S^{--} = R_0 \oplus S_\rho$, where the latter is associated to a Newton polygon ρ of one slope. We shall show that there exists a Newton polygon ξ' such that the other component R_0 of S^{--} corresponds to a generic specialization w' of $w_{\xi'}$ satisfying that (4.6) of Proposition 4.9, i.e., for the Newton polygon ξ', we have $R_0 = R^-$ with the ABS R corresponding to $w_{\xi'}$.

Proposition 4.16. If R_0 contains no element t satisfying that $0^1_{m_1+1} < t < 1^1_{m_1}$ (resp. $0^1_{i-1} < t < 1^2_j$) in $T(S^-)$, then 0^1_i belongs to C_1 (resp. C_2).

Proof. Let S'' be the full modification of S^- obtained by exchanging $0^1_{m_1+1}$ and $1^1_{m_1}$. To see that R_0 corresponds to a specialization of $w_{\xi'}$ for some Newton polygon ξ', we consider the small modification $R_0^{(0)}$ of R_0 by $(1^2_j, 0^1_i)$. Here we define the sets

$$\mathcal{A}_0 = \{t \in T(R_0^{(0)}) \mid \alpha_0 < t \text{ and } \Pi(R_0^{(0)})(t) < \alpha_1 \text{ in } T(R_0^{(0)}) \text{ with } \Delta(R_0^{(0)})(t) = 0\} \quad (4.17)$$

and

$$\mathcal{B}_0 = \{t \in T(R_0^{(0)}) \mid \beta_0 < t \text{ and } \Pi(R_0^{(0)})(t) < \beta_1 \text{ in } T(R_0^{(0)}) \text{ with } \Delta(R_0^{(0)})(t) = 1\}, \quad (4.18)$$

where $\alpha_n = \Pi(R_0^{(0)})^n(0^1_i)$ and $\beta_n = \Pi(R_0^{(0)})^n(1^2_j)$. For ABS's $R_0^{(0)}, \dots, R_0^{(n-1)}$ and for ordered sets $\mathcal{A}_0, \dots, \mathcal{A}_{n-1}$, we construct an ABS $R_0^{(n)}$ and a set $\mathcal{A}_n$ as follows. Set $T(R_0^{(n)}) = T(R_0^{(n-1)})$ as sets. Put $\Pi(R_0^{(n)}) = \Pi(R_0^{(n-1)})$ and $\Delta(R_0^{(n)}) = \Delta(R_0^{(n-1)})$. Let us define an order on $T(R_0^{(n)})$. For $t < t'$ in $T(R_0^{(n-1)})$, we have $t > t'$ if and only if $\Pi(R_0^{(n-1)})(t_{\min}) \le t < \alpha_n$ in $T(R_0^{(n-1)})$ and $t' = \alpha_n$, where $t_{\min}$ is the minimum element of $\mathcal{A}_{n-1}$. Thus we obtain the ABS $R_0^{(n)}$. Let $\mathcal{A}_n$ be a subset of $T(R_0^{(n)})$ defined by

$$\mathcal{A}_n = \{t \mid \alpha_n < t \text{ and } \pi(t) < \alpha_{n+1} \text{ in } T(R_0^{(n)}) \text{ with } \delta(t) = \delta(\alpha_n)\}, \quad (4.19)$$

where $\pi = \Pi(R_0^{(n)})$ and $\delta = \Delta(R_0^{(n)})$. By hypothesis, we have $\mathcal{A}_n = A_n - T(S_\rho)$, whence there exists a non-negative integer a' such that $\mathcal{A}_{a'} = \emptyset$. Next, for ABS's $R_0^{(a')}, \dots, R_0^{(a'+n-1)}$ and for ordered sets $\mathcal{B}_0, \dots, \mathcal{B}_{n-1}$, to construct the ABS $R_0^{(a'+n)}$, let $T(R_0^{(a'+n)}) = T(R_0^{(a'+n-1)})$ as sets. Put $\Pi(R_0^{(a'+n)}) = \Pi(R_0^{(a'+n-1)})$ and $\Delta(R_0^{(a'+n)}) = \Delta(R_0^{(a'+n-1)})$. The ordering of $T(R_0^{(a'+n)})$ is given so that for $t < t'$ in $T(R_0^{(a'+n-1)})$, set $t > t'$ if and only if $\beta_n < t' \le \Pi(R_0^{(a'+n)})(t_{\max})$ in $T(R_0^{(a'+n-1)})$ and $t = \beta_n$, where $t_{\max}$ is the maximum element of $\mathcal{B}_{n-1}$. This ordering determines the ABS $R_0^{(a'+n)}$, and we define a subset $\mathcal{B}_n$ of $T(R_0^{(a'+n)})$ by

$$\mathcal{B}_n = \{t \mid t < \beta_n \text{ and } \beta_{n+1} < \pi(t) \text{ in } T(R_0^{(a'+n)}) \text{ with } \delta(t) = \delta(\beta_n)\}, \tag{4.20}$$

where $\pi = \Pi(R_0^{(a'+n)})$ and $\delta = \Delta(R_0^{(a'+n)})$. By hypothesis, we have $\mathcal{B}_n = B_n$ for all n. Hence we see that $\mathcal{B}_{b'}$ is empty with $b' = b$. It is easy to see that $R_0^{(a'+b')}$ is the ABS associated to $\xi' = (m_1 - f, n_1 - g) + (m_2, n_2)$, and that $\ell(R^{(a'+b')}) = \ell(R_0) + 1$. It induces that R_0 correspondts to a specialization of $w_{\xi'}$. □

We give a condition for $t \in C$ to belong to C_2:

Proposition 4.17. For $0^1_i \in C$ and $1^2_j \in D'$, there exists an element $t \in T(S^-)$ of the generic full modification satisfying $0^1_{i-1} < t < 1^2_j$ if and only if there is a non-negative integer n such that the maximum element of $\Pi(S^{(n)})(A_n)$ is 0^1_{i-1}.

Proof. For the ABS $S^{(0)}$, clearly $T(S^{(0)})$ has no element t satisfying $0^1_{i-1} < t < 1^2_j$. Hence if there exists an element t such that $0^1_{i-1} < t < 1^2_j$ in $T(S^{(m)})$ for some m, then this t is $\Pi(S^{(0)})^n(0^1_i)$ with $n \leq m$. By construction, the maximum element of $\Pi(S^{(n)})(A_n)$ is 0^1_{i-1}. □

After this we assume that C is not empty. We first introduce an ordering of C, which plays an important role to divide the set C into C_1 and C_2.

Notation 4.18. Put $c = |C|$. For $x = 1, \ldots, c$, let i_x be the natural number such that $m_1 < i_x \leq m_1 + n_1$ and $0^1_{i_x}$ is the element of C appearing in the x-th vertex in the path Q. In other words, we set

$$(\pi^{q_1}(0^1_i), \ldots, \pi^{q_c}(0^1_i)) = (0^1_{i_1}, \ldots, 0^1_{i_c}), \tag{4.21}$$

where $\pi = \Pi(S)$, for elements $\pi^{q_1}(0^1_i), \ldots, \pi^{q_c}(0^1_i)$ of C with non-negative integers $q_1 < \cdots < q_c$.

Here we give a characterization of "the first element of C" $0^1_{i_1}$.

Lemma 4.19. If there exists a minimum number x such that the element $t = \Pi(S)^x(0^1_{m_1+1})$ of Q satisfies $0^1_{m_1+1} < t < 0^1_{n_1}$ in $T(S_1)$, then $t = 0^1_{i_1}$.

Proof. To show the assertion, first let us see that the sets A_n do not contain $0^1_{m_1+1}$. Assume that A_n contains $0^1_{m_1+1}$ for some n. Then $\Pi(S^{(n)})^{n+1}(0^1_i) < 1^1_1$, and this is a contradiction.

To see that the element t belongs to C, let us construct the full modification by $(t, 1_j^2)$ with $1_j^2 \in D'$. Suppose that the set A_n contains the inverse image ψ_j of 1_j^2 for some n, and let us lead a contradiction. In $T(S_1)$, this ψ_j is the inverse image of t. Clearly ψ_j belongs to Q. Let t' be the element of $S^{(0)}$ such that $\Pi(S^{(0)})^n(t') = \psi_j$. Then this t' belongs to A_0 by Proposition 3.18. Now we treat elements τ_y^1, with $\tau = 0$ or 1, of the path Q between $0_{m_1+1}^1$ and t, i.e., elements τ_y^1 appearing in

$$0_{m_1+1}^1 \to \cdots \to \tau_y^1 \to \cdots \to t \to \cdots \to 0_{2m_1}^1. \tag{4.22}$$

By the minimality of x, these elements τ_y^1 satisfy that $y < m_1$ or $n_1 < y$. It implies that these elements do not belong to A_0 as A_0 is a subset of $\{0_{m_1+1}^1, \ldots, 0_{n_1}^1\}$. Hence t' belongs to P, and there exists a natural number m such that $\Pi(S^{(0)})^m(t') = 0_{m_1+1}^1$. This contradicts the statement of the first paragraph. $\square$

Notation 4.20. For an element 0_i^1 of C', we often write $A_{i,n}$ for sets A_n obtained by the full modification of the small modification by $(0_i^1, 1_j^2)$ to avoid confusion. Moreover, we often write a_i for the minimum integer a satisfying $A_{i,a} = \emptyset$. We put $E_i = \Pi(S^{(0)})(A_{i,a_i-1})$ for all i. This set consists of all elements t satisfying that $0_{m_1+1}^1 \le t < 1_{m_1}^1$ in $T(S^-)$.

Proposition 4.21. Put $i = n_1 - \gamma$, with $\gamma = |E_{i_1}|$. Then 0_i^1 belongs to C. Moreover $E_{i_1} = E_i$.

Proof. Since E_{i_x} is a subset of $\{0_{m_1+1}^1, \ldots, 0_{n_1}^1\}$ as sets, we have $|E_{i_x}| < n_1 - m_1$ for all elements $0_{i_x}^1$ of C. It implies that $m_1 < i < n_1$. To show that 0_i^1 belongs to C, consider the full modification of the small modification by 0_i^1 and 1_j^2. It suffices to see that there exists a natural number m such that $A_{i,0} = A_{i_1,m}$. Indeed, if there exists such a number m, then $A_{i,n} = A_{i_1,m+n}$ for all n. Put $\alpha = \pi^{m-1}(0_{i_1}^1)$ with $\pi = \Pi(S^{(0)})$. Note that this α is the inverse image of 0_i^1 in the sets $T(S^{(n)})$. By Proposition 3.19, sets $A_{i_1,m+n}$ do not contain α. Hence the sets $A_{i,n}$ do not contain the inverse image of 0_i^1, and we are done.

Let us show that existence of such m. By the definition of γ and Lemma 4.8, there exists a natural number m' such that $A_{i_1,m'} = \{1_{m_1-\gamma+1}^1, \ldots, 1_{m_1}^1\}$. We have then $A_{i_1,m'+2} = A_{i,0}$, whence we obtain the desired m by $m = m' + 2$. $\square$

Let d be the natural number such that $i_d = n_1 - \gamma$. We fix the notations of d and γ. To divide the set C' into C_1 and C_2, these numbers play an important role as we can see below.

Proposition 4.22. Let x be a natural number satisfying $1 \le x \le |C|$. Then

(1) if $x \le d$, then $0_{i_x}^1$ belongs to C_1,

(2) if $x > d$, then $0^1_{i_x}$ belongs to C_2.

If we accept this proposition, we can show Proposition 4.10 (i).

Proof of Proposition 4.10 (i). Let S^- denote the generic full modification obtained by 0^1_i and 1^2_j. By Proposition 4.22, it only remains to treat the case $i = n_1$. In this case we have $A_0 = \emptyset$, and there exists no element t of $T(S^-)$ satisfying $0^1_{i-1} < t < 1^2_j$. Hence $0^1_{n_1}$ belongs to C_2. □

To prove Proposition 4.22, let us show some properties of sets E_i, of natural numbers d and γ.

Proposition 4.23. We have the following properties:

(i) If $x < y$, then $E_{i_x} \subset E_{i_y}$,

(ii) for all non-negative integers n with $n < a_{i_d}$, we have $|A_{i_d,n}| = \gamma$,

(iii) for all x and n with $n < a_{i_x}$, we have $|A_{i_x,n}| \geq \gamma$,

(iv) $E_{i_d} \subsetneq E_{i_x}$ for all x with $x > d$,

(v) $E_{i_x} = E_{i_d}$ if and only if $x \leq d$.

Proof. (i): Put $\pi = \Pi(S^{(n)})$ for the small modification by $(0^1_{i_x}, 1^2_j)$. Then there exists a non-negative integer n such that $\pi^n(0^1_{i_x}) = 0^1_{i_y}$ with $n < a_{i_x}$. We have then $A_{i_x,n} = \{0^1_{i_y+1}, \ldots, 0^1_z\}$ with $z \leq n_1$. Clearly $A_{i_x,n}$ is a subset of $A_{i_y,0}$, and it induces that E_{i_x} is a subset of E_{i_y}.

(ii): It is obvious that $|A_n| \geq |A_{n+1}|$ for all n. By the definition of d and γ, we have $|A_{i_d,0}| = \gamma$. Moreover (i) implies that $|A_{i_d,a_{i_d}}| \geq \gamma$ and hence $|A_{i_d,0}| = \cdots = |A_{i_d,a_{i_d}}| = \gamma$.

(iii): By (i) and the definition of γ, we have $|E_{i_x}| \geq \gamma$ for all x and hence $|A_{i_x,n}| \geq \gamma$ for all n.

(iv): Fix a natural number x with $x > d$. It suffices to see that $|E_{i_x}| > \gamma$. To lead a contradiction, assume $|E_{i_x}| = \gamma$. Then we have $A_{i_x,u} = \{1^1_{m_1-\gamma+1}, \ldots, 1^1_{m_1}\}$ and $\pi^u(0^1_{i_x}) = 1^1_{m_1-\gamma}$ for some u, where $\pi = \Pi(S^{(0)})$. Since $\pi^v(0^1_{i_d}) = 0^1_{i_x}$ for some v, we have $\pi^{v+u+2}(0^1_{i_d}) = 0^1_{i_d}$ with $u + v + 2 < m_1 + n_1$. This is a contradiction.

(v): This statement follows from (i), (iv) and Proposition 4.21. □

Proposition 4.24. Let $\mathcal{L}' = \{x \in \mathbb{N} \mid 1 \leq x \leq |C|\}$. We define a set $\mathcal{L}$ by

$$\mathcal{L} = \{x \in \mathcal{L}' \mid \text{the maximum element of } A_{i_d,n} \text{ is } 0^1_{i_x-1} \text{ for some } n\}. \tag{4.23}$$

Assume that $\mathcal{L}$ is not empty. Let d' denote the maximum number of $\mathcal{L}$. We have then $\mathcal{L} = \{1, 2, \ldots, d'\}$.

Proof. Fix a natural number x with $x \leq d'$. Let us show that $0^1_{i_x - 1}$ belongs to $A_{i_d,n}$ for some n. Let u be the minimum number such that $A_{i_d,n}$ has the maximum element $0^1_{i_{d'}-1}$. Consider the path consisting of maximum elements of $A_{i_d,u'}$ with $0 \leq u' \leq u$.

$$0^1_{n_1} \to \cdots \to 0^1_{i_{d'}-1}. \tag{4.24}$$

Comparing this path to the path of Q:

$$0^1_{m_1+1} \to 1^1_{m_1} \to 0^1_{n_1+1} \to \cdots \to 0^1_{i_x} \to \cdots \to 0^1_{i_{d'}}, \tag{4.25}$$

we see that (4.24) contains $0^1_{i_x-1}$. □

Proposition 4.25. Assume that the set $\mathcal{L}$ of Proposition 4.24 is not empty. Then we have $d' \leq d$.

Proof. First, note that there exists no non-negative integer n such that the maximum element of $A_{i_d,n}$ is $1^1_{m_1-1}$. Indeed, if $A_{i_d,n}$ contains $1^1_{m_1-1}$ for some n, then the minimum element of $A_{i_d,n}$ is $1^1_{m_1-\gamma}$, and $A_{i_d,n+1}$ contains the inverse image $0^1_{i_d+m_1}$ of $0^1_{i_d}$. This is a contradiction.

Assume $d < d'$. Fix a natural number x with $d < x \leq d'$. Let us consider the generic full modification obtained by $0^1_{i_d}$ and 1^2_j. By Proposition 4.24, we obtain the path consisting of maximum elements of sets $A_{i_d,n}$ and E_{i_d}:

$$0^1_{n_1} \to \cdots \to 0^1_{i_x-1} \to \Pi(S^{(0)})(0^1_{i_x-1}) \to \cdots \to 0^1_{m_1+\gamma}. \tag{4.26}$$

Let us consider the path of $T(S_1)$

$$0^1_{i_x-1} \to \Pi(S_1)(0^1_{i_x-1}) \to \cdots \to 0^1_{m_1+\gamma+1}. \tag{4.27}$$

By the claim of the first paragraph, as the path (4.26) does not contain $1^1_{m_1-1}$, the path (4.27) does not contain $1^1_{m_1}$. Now let us consider the generic full modification $R^{(0)}, \ldots, R^{(a+b)}$ by $0^1_{i_x}$ and 1^2_j. There exists a natural number n such that $\Pi(R^{(0)})^n(0^1_{i_x}) = 0^1_{m_1+\gamma+1}$ with $n < a_{i_x}$ by (4.27). On the other hand, Proposition 4.23 (iv) implies that E_{i_x} contains $0^1_{m_1+\gamma+1}$. This contradicts Proposition 3.19. □

Proof of Proposition 4.22. Let $S^{(0)}, \ldots, S^{(a+b)}$ be the ABS's obtained by the small modification obtained by $(0^1_i, 1^2_j)$, and let $S^- = S^{(a+b)}$ be the generic full modification. Recall that the specialization S^{--} obtained by (a) or (b) of Proposition 4.10 is of the form $S^{--} = R_0 \oplus S_\rho$, where S_ρ is the ABS associated to $\rho = (f, g)$, and R_0 is a full modification of the ABS corresponding to the Newton polygon $\xi' = (m_1 - f, n_1 - g) + (m_2, n_2)$. Note that in the case (a) of Proposition 4.10, the set $T(S_\rho)$ consists of all elements of the path P. See Notation 4.13 for the definition of P and Q. First let us see the case (1). To apply Proposition 4.16, we shall show that if $x \le d$, then by the full modification by $(0^1_{m_1+1}, 1^1_{m_1})$, all elements of E_{i_x} other than $0^1_{m_1+1}$ belong to $T(S_\rho)$. By Proposition 4.23 (v), it suffices to show that all elements of E_{i_1} belong to $P = T(S_\rho)$. We divide the path Q into two paths as follows:

$$0^1_{m_1+1} \to \cdots \to 0^1_{i_1+m_1}, \quad 0^1_{i_1} \to \cdots \to 0^1_{2m_1}. \tag{4.28}$$

It follows from Proposition 3.19 that all elements $\Pi(S)^m(0^1_i)$ with $m < a_{i_1}$ of the latter component of (4.28) does not belong to E_{i_1}. The property of i_1, which is shown in Lemma 4.19, implies that all elements of the former component of (4.28) other than $0^1_{m_1+1}$ do not belong to $E_{i_1} \subset \{0^1_{m_1+1}, \dots, 0^1_{n_1}\}$. Hence we see that all elements t of S^- satisfying that $0^1_{m_1+1} < t < 1^1_{m_1}$ belong to $T(S_\rho)$.

Next, let us see the case (2). Fix an integer x with $x > d$. We consider the full modification by $0^1_{i_x-1}$ and 1^2_j for S^-. By Proposition 4.16, it suffices to show that there exists no element t with $0^1_{i_x-1} < t < 1^2_j$ in $T(S^-)$. To lead a contradiction, assume the existence of t between $0^1_{i_x-1}$ and 1^2_j. By Proposition 4.17, the maximum element of $\pi(A_{i_x,v})$ is $0^1_{i_x-1}$ for some v, where $\pi = \Pi(S^{(0)})$. Here we have the path consisting of maximum elements of $A_{i_d,0}, \dots, A_{i_d,a-1}, E_{i_d}$:

$$0^1_{n_1} \to \cdots \to 0^1_{m_1+\gamma}. \tag{4.29}$$

We define the non-negative integer m to be $A_{i_x,m} = \{1^1_{m_1-u+1}, \dots, 1^1_{m_1}\}$ with $u = |\pi(A_{i_x,v})|$. Then $m < v$, and the set $A_{i_x,m+2}$ has the maximum element $0^1_{n_1}$. We treat the path $O : 0^1_{n_1} \to \cdots \to 0^1_{i_x-1}$ consisting of maximum elements of $A_{i_x,m+2}, A_{i_x,m+3}, \dots, \pi(A_{i_x,v})$. If the path O can be regarded as a sub-path of (4.29), we complete the proof. It suffices to check that O does not contain $0^1_{m_1+\gamma}$. If $0^1_{m_1+\gamma}$ belongs to O, then $|E_{i_x}| \le \gamma$ holds, and this contradicts with Proposition 4.23 (v). Hence O is contained in (4.29), and it implies that $0^1_{i_x-1}$ is the maximum element of $A_{i_d,n}$ for some n. This contradicts the definition of d'. $\square$

It remains to show Proposition 4.10 in case (ii): $n_1 = m_1 + 1$ for the Newton polygon $\xi = (m_1, n_1) + (m_2, n_2)$. The proof of Proposition 4.10 in case (ii) is given in the same way as the proof of Proposition 4.10 in case (i). As with Proposition 4.12, in a generic full modification, $0^2_{m_2+1} < 1^2_{m_2}$ holds. In the case $n_1 = m_1 + 1$, we have $C' = \{0^1_{n_1}\}$. We now suppose that D is not empty.

Notation 4.26. We define paths P' and Q' of $T(S_2)$ as follows:

$$P' \;:\; 1^2_{m_2} \to 0^2_{m_2+n_2} \to 1^2_{n_2} \to \cdots \to 1^2_{m_2-n_2+1}, \tag{4.30}$$

$$Q' \;:\; 0^2_{m_2+1} \to 1^2_1 \to 1^2_{n_2+1} \to \cdots \to 1^2_{m_2-n_2}. \tag{4.31}$$

Clearly the set $T(S_2)$ is the disjoint union of P' and Q'.

In the proof of Proposition 4.10 (ii), the above P' and Q' play a important role as the paths P and Q do so in the proof of Proposition 4.10 (i). In the case (c) of Proposition 4.10, the ABS S_ρ corresponding to the segment $\rho = (f, g)$ consists of all the elements of Q'.

Lemma 4.27. For the above sets, D is a subset of P'. Moreover, let $j_1, \ldots, j_{|D|}$ be natural numbers such that $1^2_{j_x}$ is the x-th element of D appearing in the x-th in P', i.e., for $D = \{\pi^{q_1}(1^2_{m_2}), \ldots, \pi^{q_{|D|}}(1^2_{m_2})\}$, we have $(\pi^{q_1}(1^2_{m_2}), \ldots, \pi^{q_{|D|}}(1^2_{m_2})) = (1^2_{j_1}, \ldots, 1^2_{j_{|D|}})$ with $q_1 < \cdots < q_{|D|}$ and $\pi = \Pi(S^{(0)})$. We have then $j_1 = n_2$.

Proof. For the former part, in the same way as Proposition 4.12 we have no non-negative integer n with $n \leq b$ such that $\pi^n(1^2_j) = 1^2_{m_2}$ for every element 1^2_j of D. As with Lemma 4.14, we can see $D \subset P'$. Let us show the latter part. By the former part, if $1^2_{n_1}$ belongs to D, then we immediately obtain $j_1 = n_2$. Consider a full modification by $(0^1_i, 1^2_{n_1})$ with $0^1_i \in C'$, and assume that the set B_n contains the inverse image $0^2_{m_2+n_2}$ of 0^1_i in $S^{(a+n)}$ for some n. We have then $0^2_{m_2+n_2} < \pi^n(1^2_j)$ in $S^{(a+n-1)}$. Since $0^2_{m_2+n_2}$ is the maximum element of $T(S^{(0)})$, this is a contradiction. □

Notation 4.28. Let D_1 (resp. D_2) be the subset of D consisting of 1^2_j such that by the generic full modification S^- obtained by 0^1_i and 1^2_j, we obtain (4.6) of Proposition 4.9 by (c) (resp. (d)) of Proposition 4.10.

To show Proposition 4.10, we will show $D = D_1 \cup D_2$. To divide the set D into D_1 and D_2, we shall introduce a key element of D in Proposition 4.30.

Notation 4.29. For an element 1^2_j of D', we write $B_{j,n}$ for B_n obtained by the full modification by $(0^1_i, 1^2_j)$. For sets $B_{j,0}, \ldots, B_{j,b}$, we define $E'_j = \pi(B_{j,b-1})$ with $\pi = \Pi(S^{(0)})$. Note that this set consists of all elements t satisfying $0^2_{m_2+1} < t \leq 1^2_{m_2}$ in S^-. Moreover we denote by b_j the non-negative integer b.

Proposition 4.30. Put $j = 1 + \mu$ with $\mu = |E'_{j_1}|$. Then 1^2_j belongs to D. Let e be the natural number satisfying $j = j_e$. Then $E'_{j_1} = E'_{j_e}$.

Proof. By Lemma 4.8, we have $B_{j_1,n} = \{0^2_{m_2+1}, \dots, 0^2_{m_2+\mu}\}$ for some non-negative integer n. Then $B_{j_1,n+1} = B_{j,0}$. We can show the statement in the same way as Proposition 4.21. $\square$

This number e divides the set D into D_1 and D_2 as follows.

Proposition 4.31. Let x be a natural number with $x \leq |D|$. Then

(1) if $x \leq e$, then $1^2_{j_x}$ belongs to D_1,

(2) if $x > e$, then $1^2_{j_x}$ belongs to D_2.

Proof. First let us see the statement (1). Let S^- denote the generic full modification obtained by 0^1_i and 1^2_j. For this S^-, the ABS S^{--} of (c) or (d) of Proposition 4.10 consists of two components R_0 and S_ρ, where S_ρ is associated to a Newton polygon $\rho = (f, g)$. For these S^{--}, since it coincides with the component obtained from S_2 by applying [2, Lemma 5.6] to the adjacent $1^2_{m_2}\ 0^2_{m_2+1}$ and $1^2_j\ 1^2_{j+1}$ respectively, we have $gm_1 - fn_2 = 1$. In the same way as Proposition 4.16, we obtain the property: If there exists no element t of $T(R_0)$ satisfying that $0^2_{m_2+1} < t < 1^2_{m_2}$ (resp. $0^1_i < t < 1^2_{j+1}$) in S^-, then 1^2_j belongs to D_1 (resp. D_2). By the same way as Proposition 4.23, we have

(i) if $x < y$, then $E'_{j_x} \subset E'_{j_y}$,

(ii) for all n with $n < b_{j_e}$, we have $|B_{j_e,n}| = \mu$,

(iii) for all x and all n with $n < b_{j_x}$, we have $|B_{j_x,n}| \geq \mu$,

(iv) $E'_{j_e} \subsetneq E'_{j_x}$ holds for all x with $x > e$,

(v) $E'_{j_e} = E'_{j_x}$ if and only if $x \leq e$.

By (v), to show the statement (1), it suffices to consider the case $x = 1$. Note that in this case, $T(S_\rho)$ consists of all elements of Q'. We claim that there exists no element t of P' satisfying that $0^2_{m_2+1} < t < 1^2_{m_2}$ in S^-. Indeed, if we divide the path P' into two paths:

$$1^2_{m_2} \to 0^2_{m_2+n_2}, \quad 1^2_{j_1} \to \cdots \to 1^2_{m_2-n_2+1}, \tag{4.32}$$

clearly for $t = 1^2_{m_2}$ or $0^2_{m_2+n_2}$, these t do not satisfy $0^2_{m_2+1} < t < 1^2_{m_2}$ in S^-. Moreover, since each element of the latter component is of the form $\pi^n(1^2_{j_1})$ for some n with $n < b$, by Proposition 3.22, these elements do not belong to E'_{j_1}. Hence we see that all elements in between $0^2_{m_2+1}$ and $1^2_{m_2}$ in S^- do not belong to $T(R_0)$, i.e., if t belongs to E'_{j_1}, then this t is an element of $T(S_2) - P' = T(S_\rho)$.

Next let us show the statement (2). We define the non-negative integer e' to be the maximum number of the set

$$\mathcal{M} = \{x \in \mathcal{M}' \mid 1^2_{j_x+1} \text{ is the maximum element of } B_{j_e,n} \text{ for some } n\} \tag{4.33}$$

with $\mathcal{M}' = \{x \in \mathbb{N} \mid 1 \leq x \leq |D|\}$. By the same way as the proof of Proposition 4.25, if $\mathcal{M}$ is not empty, then $e' \leq e$. A proof is given in the same way as the proof of Proposition 4.22 (2). $\square$

Proof of Proposition 4.10 (ii). By Proposition 4.31, it remains to the case $j = 1$. If $n_2 > 1$, we have then $0^1_i < 1^2_{j+1}$, and there exists no element t of $T(S^-)$ satisfying $0^1_i < t < 1^2_{j+1}$. Hence 1^2_i belongs to D_2. Next suppose $n_2 = 1$. In this case, for S^-, we construct the full modification by $(0^1_{m_1+n_1}, 1^2_{n_2+1})$. Then for $S^{--} = R_0 \oplus S_\rho$, the latter component S_ρ is described as $1^2_1\ 0^1_{m_1+n_1}$, and R_0 is a full modification of the ABS corresponding to the Newton polygon $(m_1 - 1, n_1 - 1) + (m_2, n_2)$. Hence in this case we obtain (4.6) by (e). $\square$

Proof of Proposition 4.9. By Proposition 4.10, it remains only to show the case $\lambda_1 = 1$ or $\lambda_2 = 0$. For the case $\lambda_1 = 1$, using Proposition 4.10, we get the required w^{--}_ξ by (c) or (d). If $\lambda_2 = 0$, then we obtain w^{--}_ξ by (a) or (b). $\square$

Example 4.32. Let $\xi = (2, 5) + (3, 2)$. Let S denote the ABS corresponding to ξ. In Example 4.4, we obtain the generic full modification S^- of S obtained by $(0^1_4, 1^2_2)$. Consider the full modification by $(0^1_3, 1^2_2)$ for S^-. Then this full modification can be described as

$$1^1_1\ 0^1_3\ 1^2_2\ 1^1_2\ 0^1_4\ 0^1_6\ 0^2_4\ 1^2_3\ 0^2_5 \oplus 1^1_2\ 0^1_5\ 0^1_7\ . \tag{4.34}$$

The first component is a specialization of $N_{(1,3)+(3,2)}$, and we have $\rho = (1, 2)$. For the first component, let us consider the full modification by $(0^2_4, 1^2_3)$. We have then the specialization $N^-_{(1,3)+(2,1)} \oplus N_{(1,1)}$. Thus, constructing a specialization of $N^-_{(1,1)+(1,1)}$, we obtain the Newton polygon ζ by $\zeta = 2(1, 2) + 3(1, 1)$.

Chapter 5

The case of Newton polygons consisting of two segments

In this chapter, we treat all Newton polygons consisting of two segments, and reduce Problem 1.4 to the case that the Newton polygon ξ is 1/2-separated, i.e., we reduce the problem to the case of Chapter 4. Moreover, we show Proposition 5.9 which is a key statement to show Theorem 1.8. This proposition is a generalization of Proposition 4.9. In Section 6, we will see that it suffices to deal with Newton polygons ξ consisting of two segments to classify generic specializations of $H(\xi)$, and to determine its Newton polygons for an arbitrary ξ.

5.1 Euclidean algorithm for Newton polygons

We denote by NP the set of Newton polygons whose all segments are not the same. Let $\mathrm{NP}^{\mathrm{sep}}$ be the subset of NP consisting of Newton polygons $(m_1, n_1)+(m_2, n_2)+\cdots+(m_z, n_z)$ with $n_z/(m_z+n_z) < 1/2 < n_1/(m_1+n_1)$. In this section, we introduce *Euclidean algorithm for Newton polygons* $\Phi : \mathrm{NP} \to \mathrm{NP}^{\mathrm{sep}}$ which is used for reducing Problems 1.4 and 1.7 to the case that the Newton polygon ξ is 1/2-separated. Moreover, using this map, we will show some properties of the ABS's corresponding to minimal DM_1's, see Lemma 5.5 and Proposition 5.7.

First, we introduce two operations of Newton polygons to construct the map $\Phi : \mathrm{NP} \to \mathrm{NP}^{\mathrm{sep}}$. See Section 2.1 (2.3) for the notation of Newton polygons. For a Newton polygon $\xi = \sum_{i=1}^{z}(m_i, n_i)$, we define the Newton polygon ξ^{D} by

$$\xi^{\mathrm{D}} = \sum_{i=1}^{z}(n_{z-i+1}, m_{z-i+1}). \tag{5.1}$$

We call this ξ^{D} the *dual of* ξ. Moreover, for a Newton polygon ξ satisfying $m_i \leq n_i$ for all i, we define the Newton polygon ξ^{C} by

$$\xi^{\mathrm{C}} = \sum_{i=1}^{z} (m_i, n_i - m_i), \tag{5.2}$$

and we call this ξ^{C} the *curtailment of* ξ.

Example 5.1. Let $\xi_0 = (5,3) + (2,1) + (7,2)$. Then $\xi_0^{\mathrm{D}} = (2,7) + (1,2) + (3,5)$. For $\xi_1 = \xi_0^{\mathrm{D}}$, we have $\xi_1^{\mathrm{C}} = (2,5) + (1,1) + (3,2)$.

Example 5.2. Let $\xi = (2,5) + (3,2)$. Then the ABS S corresponding to N_ξ is described as

$$S = \ 1_1^1\ 1_2^1\ 0_3^1\ 0_4^1\ 0_5^1\ 1_1^2\ 1_2^2\ 0_6^1\ 0_7^1\ 1_3^2\ 0_4^2\ 0_5^2\ . \tag{5.3}$$

The dual ξ^{D} of ξ is $\xi^{\mathrm{D}} = (2,3) + (5,2)$. The ABS S^{D} associated to the minimal DM_1 of ξ^{D} is described as

$$S^{\mathrm{D}} = \ 1_1^1\ 1_2^1\ 0_3^1\ 1_1^2\ 1_2^2\ 0_4^1\ 0_5^1\ 1_3^2\ 1_4^2\ 1_5^2\ 0_6^2\ 0_7^2\ . \tag{5.4}$$

Using the above operations C and D, let us construct a map $\Phi : \mathrm{NP} \to \mathrm{NP}^{\mathrm{sep}}$. Proposition 5.7 and Lemma 6.1 below are properties of ABS's corresponding to minimal ABS's N_ξ for arbitrary Newton polygons ξ. Thanks to the map Φ, proofs of these claims are reduced to the case that the Newton polygons belong to $\mathrm{NP}^{\mathrm{sep}}$.

For a Newton polygon $\xi = \sum_{i=1}^{z} (m_i, n_i)$, we define the *height* of ξ by $\mathrm{ht}(\xi) = m_1 + n_1 + m_2 + n_2 + \cdots + m_z + n_z$. First, let us construct the image $\Phi(\xi)$ of a Newton polygon ξ in NP with two segments. If ξ belongs to $\mathrm{NP}^{\mathrm{sep}}$, then we define $\Phi(\xi) = \xi$. Otherwise, the Newton polygon $\xi = (m_1, n_1) + (m_2, n_2)$ satisfies that $m_i \leq n_i$ for $i = 1, 2$, or $n_i \leq m_i$ for $i = 1, 2$. For the former case, we define the image $\Phi(\xi)$ of ξ to be the image of ξ^{C} by Φ. For the latter case, we define the image $\Phi(\xi)$ of ξ to be the image of ξ^{DC} by Φ. Let us show this map Φ is well defined. It is clear that $\mathrm{ht}(\xi^{\mathrm{C}}) < \mathrm{ht}(\xi)$ and $\mathrm{ht}(\xi^{\mathrm{D}}) = \mathrm{ht}(\xi)$. If $\mathrm{ht}(\xi) = 2$, then since ξ has two types of segments, we see that $\xi = (0,1) + (1,0)$ which belongs to $\mathrm{NP}^{\mathrm{sep}}$.

For the case $z > 2$, let $\eta = (m_1, n_1) + (m_z, n_z)$. let W denote the word of C and D such that $\Phi(\eta) = \eta^{\mathrm{W}}$. For this W, we define the image $\Phi(\xi)$ of ξ by ξ^{W}. We call this map $\Phi : \mathrm{NP} \to \mathrm{NP}^{\mathrm{sep}}$ *Euclidean algorithm for Newton polygons.*

Remark 5.3. By the above construction, the Newton polygon $\Phi(\xi)$ is described as $\Phi(\xi) = \xi^{\mathrm{Q}_1\mathrm{Q}_2\cdots\mathrm{Q}_m}$, where Q_i is either the operation C or the operation D for every i. Thus by the duality and Theorem 1.6, for all Newton polygons ξ consisting of two segments, we obtain a bijection from $B(\xi)$ to $B(\Phi(\xi))$.

Example 5.4. As seen in Example 5.1, for the Newton polygon $\xi = (5,3)+(2,1)+(7,2)$, we have $\Phi(\xi) = (2,5) + (1,1) + (3,2)$.

Let S (resp. R) be the ABS of ξ (resp. ξ^{C}). Next, we describe a relation between S and R. In the following lemma, we show that the set $T(R)$ can be regarded as a subset of $T(S)$ as ordered sets. This relation is used for the proofs of Lemma 6.1 and Theorem 1.6.

Lemma 5.5. Let $\xi = (m_1, n_1)+(m_2, n_2)$ be a Newton polygon consisting of two segments with $m_i \leq n_i$ for $i = 1, 2$. Let S and R be the ABS of ξ and ξ^{C} respectively. Then $T(R)$ is contained in $T(S)$ as an ordered set. We have

$$\{t \in T(R) \mid \Delta(R)(t) = 1\} = \{t \in T(S) \mid \Delta(S)(t) = 1\} \tag{5.5}$$

and

$$T(S) - T(R) = \{\Pi(S)(t) \mid t \in T(S) \text{ with } \Delta(S)(t) = 1\}. \tag{5.6}$$

Let t be an element of $T(R)$. We also regard t as an element of $T(S)$. Then

$$\Pi(R)(t) = \begin{cases} \Pi(S)(t) & \text{if } \Delta(R)(t) = 0, \\ \Pi(S)^2(t) & \text{otherwise.} \end{cases} \tag{5.7}$$

holds.

Proof. Since $\xi^{\mathrm{C}} = (m_1, n_1 - m_1) + (m_2, n_2 - m_2)$, it is clear that $T(R)$ is a subset of $T(S)$ as sets, and there is the standard one-to-one correspondence between sets $\{t \in T(R) \mid \Delta(R)(t) = 1\}$ and $\{t \in T(S) \mid \Delta(S)(t) = 1\}$. Let us show these sets coincide as ordered sets. Take elements t and s in $T(R)$. Let t' and s' denote the elements of $T(S)$ corresponding t and s by the standard one-to-one correspondence respectively. Then for binary expansions $b(t)$ and $b(s)$, we see $b(t) < b(s)$ if and only if $b(t') < b(s')$. Thus we obtain the equality (5.5), and this induces that $T(R)$ is a subset of $T(S)$. We immediately obtain (5.6) and (5.7). □

Example 5.6 below is an example of Lemma 5.5.

Example 5.6. Let $\xi = (2,7) + (3,5)$. Let S and R be the ABS's corresponding to ξ and ξ^{C} respectively. We have then

$$S \;=\; 1^1_1\ 1^1_2\ 0^1_3\ 0^1_4\ 0^1_5\ 1^2_1\ 1^2_2\ 0^1_6\ 0^1_7\ 1^2_3\ 0^2_4\ 0^2_5\ 0^1_8\ 0^1_9\ 0^2_6\ 0^2_7\ 0^2_8\,, \tag{5.8}$$

$$R \;=\; 1^1_1\ 1^1_2\ 0^1_3\ 0^1_4\ 0^1_5\ 1^2_1\ 1^2_2\ 0^1_6\ 0^1_7\ 1^2_3\ 0^2_4\ 0^2_5\,. \tag{5.9}$$

One can check that these S and R satisfy (5.5), (5.6) and (5.7).

Proposition 5.7. Let S be the ABS of a minimal DM_1 N_ξ with $\xi = \sum_{i=1}^{z}(m_i, n_i)$. For natural numbers r and q with $r < q \le z$, we have

(i) $1^r_1 < 1^q_1$,

(ii) $0^r_{m_r+n_r} < 0^q_{m_q+n_q}$,

(iii) $0^r_{m_r+1} < 0^q_{m_q+1}$

in the set $T(S)$.

Proof. Note that (iii) follows from (i). Indeed, $0^r_{m_r+1}$ and $0^q_{m_q+1}$ are the inverse image of 1^r_1 and 1^q_1 by $\Pi(S)$ respectively.

It suffices to treat the case $z = 2$. For a Newton polygon ξ, we denote by $\mathrm{P}(\xi)$ the assertion: *The ABS associated to the minimal DM_1 N_ξ satisfies (i) and (ii).* By Proposition 3.7, if ξ satisfies that $\lambda_2 < 1/2 < \lambda_1$, then $\mathrm{P}(\xi)$ holds. To show that $\mathrm{P}(\xi)$ is true for all Newton polygons ξ consisting of two segments, we claim

(A) If $\mathrm{P}(\xi^{\mathrm{D}})$ holds, then $\mathrm{P}(\xi)$ also holds,

(B) If $m_i \le n_i$ for all i and $\mathrm{P}(\xi^{\mathrm{C}})$ holds, then $\mathrm{P}(\xi)$ also holds.

By the duality, the claim (A) is obvious. Let us show that the claim (B) follows from Lemma 5.5. Let R denote the ABS corresponding to $N_{\xi^{\mathrm{C}}}$. By Lemma 5.5, it is clear that $1^1_1 < 1^2_1$ in $T(S)$. Moreover, since $0^1_{n_1} < 0^2_{n_2}$ in $T(R)$, we see that $0^1_{m_1+n_1} < 0^2_{m_2+n_2}$, which are the inverse images of $0^1_{n_1}$ and $0^2_{n_2}$ by $\Pi(S)$, holds in $T(S)$. The assertion of the lemma follows from (A), (B) and the map $\Phi : \mathrm{NP} \to \mathrm{NP}^{\mathrm{sep}}$. $\square$

5.2 Classifying generic specializations for Newton polygons consisting of two segments

In this section, we give a proof of Theorem 1.6. This theorem is a key statement to classify all generic specializations of $H(\xi)$ with a Newton polygon ξ consisting of two segments. The notation is as Chapter 3. For a Newton polygon ξ, we have the one-to-one correspondence between $B(\xi)$ and $B(\xi^{\mathrm{D}})$, see the paragraph after Theorem 1.6. Moreover, to get a bijection between $B(\xi)$ and $B(\xi^{\mathrm{C}})$, we use Lemma 5.5.

Proof of Theorem 1.6. The assertion is paraphrased as follows: *Let S (resp. R) denote the ABS associated to ξ (resp. ξ^{C}). The map from a generic full modification S' of S obtained by the small modification by $(0^1_i, 1^2_j)$ to the generic full modification R' of R obtained by the small modification by $(0^1_i, 1^2_j)$ is bijective.* The set $T(R)$ can be regarded as a subset of $T(S)$. By Lemma 5.5, we have

$$\{(0^1, 1^2) \in T(S)^2 \mid 0^1 < 1^2 \text{ in } T(S)\} = \{(0^1, 1^2) \in T(R)^2 \mid 0^1 < 1^2 \text{ in } T(R)\}. \tag{5.10}$$

Suppose that the full modification S' of S by the small modification by $(0^1_i, 1^2_j)$ is generic. Consider the small modifications $S^{(0)}$ and $R^{(0)}$ by the same $(0^1_i, 1^2_j)$. For non-negative integers n, let $\{A_n\}$ (resp. $\{A'_n\}$) be the A-sequence obtained by Definition 3.15 for the small modification by $(0^1_i,\ 1^2_j)$ for S (resp. R). Clearly we have $A_0 = A'_0$ and $\ell(S^{(0)}) = \ell(R^{(0)})$. For a non-negative integer n, suppose that $A_n = A'_n$ and $\ell(S^{(n)}) = \ell(R^{(n)})$. If elements t of A_n satisfy that $\Delta(S)(t) = 0$, then by Lemma 5.5 we see that $A_{n+1} = A'_{n+1}$ and $\ell(S^{(n+1)}) = \ell(R^{(n+1)})$. Moreover, if elements t of A_n satisfy $\Delta(S)(t) = 1$, then it follows from Lemma 5.5 that $A_{n+2} = A'_{n+1}$ and $\ell(S^{(n+2)}) = \ell(R^{(n+1)})$. Since the full modification S' is generic, by the above, there exist non-negative integers a and a' such that $A_a = \emptyset$ and $A'_{a'} = \emptyset$. Similarly, for the B-sequences $\{B_n\}$ and $\{B'_n\}$ obtained by Definition 3.16, we have $B_0 = B'_0$. For a non-negative integer n, we suppose that $B_n = B'_n$. Similarly as above, we have

$$\begin{cases} B_{n+1} = B'_{n+1} \text{ and } \ell(S^{(a+n+1)}) = \ell(R^{(a'+n+1)}) & \text{if } \Delta(S)(t) = 0 \text{ for } t \in B_n, \\ B_{n+2} = B'_{n+1} \text{ and } \ell(S^{(a+n+2)}) = \ell(R^{(a'+n+1)}) & \text{otherwise.} \end{cases} \tag{5.11}$$

Thus we see that $\ell(S') = \ell(S) - 1$ if and only if $\ell(R') = \ell(R) - 1$. This completes the proof. □

Let ξ be a Newton polygon consisting of two segments. By Theorem 1.6, we obtain a one-to-one correspondence between $B(\xi)$ and $B(\xi^{\mathrm{C}})$. Moreover, we have a bijection from $B(\xi)$ to $B(\xi^{\mathrm{D}})$, see the paragraph after Theorem 1.6. Thus, using Euclidean algorithm for Newton polygons $\Phi : \mathrm{NP} \to \mathrm{NP}^{\mathrm{sep}}$ given in Section 5.1, to classify generic specializations of the minimal p-divisible group $H(\xi)$, we may assume that ξ satisfies $\lambda_2 < 1/2 < \lambda_1$.

Example 5.8. For the Newton polygon $\xi = (2,7) + (3,5)$ of Example 3.11, we have $\Phi(\xi) = (2,5) + (3,2)$. Let S be the ABS associated to $\Phi(\xi)$. For this ABS

$$S = \ 1^1_1 \ 1^1_2 \ 0^1_3 \ 0^1_4 \ 0^1_5 \ 1^2_1 \ 1^2_2 \ 0^1_6 \ 0^1_7 \ 1^2_3 \ 0^2_4 \ 0^2_5 \, , \tag{5.12}$$

the full modification S' obtained by the small modification by 0^1_4 and 1^2_2 is

$$S' = \ 1^1_1 \ 0^1_3 \ 1^1_2 \ 1^2_2 \ 0^1_5 \ 1^2_1 \ 0^1_4 \ 0^1_6 \ 0^1_7 \ 0^2_4 \ 1^2_3 \ 0^2_5 \, . \tag{5.13}$$

For the small modification by $(0^1_4, 1^2_2)$, we have sets $A_0 = \{0^1_5\}$, $A_1 = \emptyset$, $B_0 = \{1^2_1\}$ and $B_1 = \emptyset$. Thus we have $a = 1$ and $b = 1$. One can check that these S and S' satisfy $\ell(S') = \ell(S) - 1$. Moreover, the sets of pairs $(0^1_i, 1^2_j)$ constructing generic full modifications for ABS's corresponding to ξ and $\Phi(\xi)$ are both

$$\{(0^1_4, 1^2_1),\ (0^1_4, 1^2_2),\ (0^1_5, 1^2_1),\ (0^1_5, 1^2_2),\ (0^1_6, 1^2_3),\ (0^1_7, 1^2_3)\}. \tag{5.14}$$

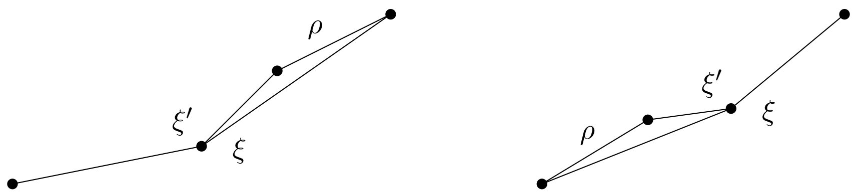
5.3 Determining Newton polygons of generic specializations for Newton polygons consisting of two segments

Theorems 1.5 and 6.6 say that it suffices to see the case that the Newton polygons of central streams consist of two segments in order to determine Newton polygons of generic specializations. From now on, we mainly treat Newton polygons consisting of two segments. Proposition 5.9 below implies that Proposition 4.9 is true for any Newton polygons ξ consisting of two segments. This proposition is a key step to prove Theorem 1.8.

Proposition 5.9. Let $\xi = (m_1, n_1) + (m_2, n_2)$ be a Newton polygon consisting of two segments. Assume that $m_2 n_1 - m_1 n_2$ is greater than 1. For every element w of $B(\xi)$, there exist a generic specialization w^- of w and a segment $\rho = (c, d)$ such that

$$w^- = w' \oplus w_\rho, \tag{5.15}$$

with $w' \in B(\xi')$, where $\xi' = (m_1 - c, n_1 - d) + (m_2, n_2)$ or $\xi' = (m_1, n_1) + (m_2 - c, n_2 - d)$:



so that the area of the triangle surrounded by ξ, ξ' and ρ is one.

Proof. For a Newton polygon $\xi = (m_1, n_1) + (m_2, n_2)$, the height $\mathrm{ht}(\xi)$ of ξ is defined by $\mathrm{ht}(\xi) = m_1 + n_1 + m_2 + n_2$. Let us prove the statement by induction on the height of ξ. If ξ is 1/2-separated, then by Proposition 4.9, we are done. It remains to see the case that ξ satisfies either

- $m_i \leq n_i$ for $i = 1, 2$, or
- $n_i \leq m_i$ for $i = 1, 2$.

If ξ satisfies the latter, then we replace ξ with ξ^{D}, and we may assume that ξ satisfies the former. Put $\eta = \xi^{\mathrm{C}}$. Take $w \in B(\xi)$. We denote by w^{C} the image of the map $B(\xi) \to B(\eta)$ obtained in Theorem 1.6. Clearly we have $\mathrm{ht}(\eta) < \mathrm{ht}(\xi)$. By the hypothesis of induction, there exist a generic specialization $(w^{\mathrm{C}})^-$ of w^{C} and a segment τ such that $(w^{\mathrm{C}})^- = v \oplus w_\tau$ with $v \in B(\eta')$, where η' is uniquely determined by η and τ so that the area of the triangle surrounded by η, η' and τ is one. Let us show that there exists a generic specialization w^- of w such that w^- consists of two components w' and w_ρ with $(w')^{\mathrm{C}} = v$ and $\rho^{\mathrm{C}} = \tau$. Let S (resp. R) denote the ABS corresponding to w (resp. w^{C}). Assume that the full modification R^- of R by the small modification by $(0^1_i, 1^2_j)$ corresponds to $(w^{\mathrm{C}})^- = v \oplus w_\tau$. Let us see that the full modification S^- of S by the small modification by $(0^1_i, 1^2_j)$ corresponds to the required w^-. By Theorem 1.6, if we write $\{s_1 < \cdots < s_h\}$ (resp. $\{s'_1 < \cdots < s'_{h'}\}$) for the ordered set $T(S)$ (resp. $T(R)$), with $h = \mathrm{ht}(\xi)$ and $h' = \mathrm{ht}(\eta)$, then $s_x = s'_x$ for $x = 1, \ldots, h'$. Moreover, if $\Delta(S)(s_x) = 1$ for an integer x, then $\Pi(S)^2(s_x) = \Pi(R)(s'_x)$. If we denote by $\{t_1 <' \cdots <' t_h\}$ and $\{t'_1 <' \cdots <' t'_{h'}\}$ the ordered sets $T(S^-)$ and $T(R^-)$ respectively, then $t_x = t'_x$ for $x = 1, \ldots, h'$. Moreover, if $\Delta(S^-)(t_x) = 1$ for an integer x, then $\Pi(S^-)^2(t_x) = \Pi(R^-)(t'_x)$. Thus we see that S^- corresponding to $w^- = w' \oplus w_\rho$, with $(w')^{\mathrm{C}} = v$ and $\rho^{\mathrm{C}} = \tau$. $\square$

Example 5.10. Let $\xi = (2,7) + (3,5)$. In Example 3.11 we obtain the full modification of S' by 0^1_4 and 1^2_2. The full modification of S' by 0^1_3 and 1^1_2 is described as

$$1^1_1\ 0^1_3\ 1^2_2\ 1^1_2\ 0^1_4\ 0^1_6\ 0^2_4\ 1^2_3\ 0^2_5\ 0^1_8\ 0^2_7\ 0^2_6\ 0^2_8 \oplus 1^1_2\ 0^1_5\ 0^1_7\ 0^1_9\ . \tag{5.16}$$

The former component corresponds to a specialization of $N_{(1,4)+(3,5)}$, and we have the Newton polygon $\rho = (1,3)$. We obtain the required Newton polygon ζ by $\zeta = 2(1,3) + 3(1,2)$. Compare with Example 4.32.

Chapter 6

Reduction to the two segments case

In this chapter, first we will prove Theorem 1.5. Thanks to this theorem, the problem of classification of generic specializations (Problem 1.4) is reduced to the case that the Newton polygon of a central stream consists of two slopes. Moreover, we will show Theorem 1.8 in Section 6.2. This theorem gives a complete answer to Proposition 1.7.

6.1 Classifying generic specializations for any Newton polygons

The main purpose of this section is to prove Theorem 1.5. Let S be the ABS corresponding to a Newton polygon ξ. Let $S^{(0)}$ denote the small modification by $(0_i^r, 1_j^q)$. Lemma 6.1 and Proposition 6.2 below imply that to classify generic full modifications, we may suppose $q = r + 1$; see Corollary 6.3. Note that the condition $q = r + 1$ implies that the r-th segment of ξ is adjacent to the q-th segment.

Lemma 6.1. Let S be the ABS associated to N_ξ with $\xi = \sum_{i=1}^{z}(m_i, n_i)$ a Newton polygon. Let 0^r and 1^q be elements of $T(S)$ satisfying that $q - r > 1$ and $0^r < 1^q$ in $T(S)$. Then there exists an element t^x of $T(S)$ such that $r < x < q$ and $0^r < t^x < 1^q$.

Proof. For a Newton polygon ξ, we write $\mathrm{Q}(\xi)$ for the assertion: *For elements 0^r and 1^q of the ABS associated to N_ξ satisfying that $q - r > 1$ and $0^r < 1^q$, there exists an element t^x such that $r < x < q$ and $0^r < t^x < 1^q$.* It suffices to treat the case $z = 3$, $r = 1$ and $q = 3$. If $\lambda_1 = \lambda_2$ (resp. $\lambda_2 = \lambda_3$) holds, then we immediately have the desired element t^x since for elements $0_i^1 < 1_j^3$, the element 0_i^2 (resp. 1_j^2) satisfies $0_i^1 < 0_i^2 < 1_j^3$ (resp. $0_i^1 < 1_j^2 < 1_j^3$).

From now on, we assume that the slopes are different from each other. Now we treat Newton polygons satisfying one of the following:

(i) $\lambda_3 < 1/2 \leq \lambda_2 < \lambda_1$,

(ii) $\lambda_3 < \lambda_2 \leq 1/2 < \lambda_1$.

By the duality, if $\mathrm{Q}(\xi)$ is true for all ξ satisfying (i), then $\mathrm{Q}(\xi)$ holds for all ξ satisfying (ii). Suppose that ξ satisfies (i). Put $h_x = m_x + n_x$ for all x. By Lemma 3.7, in the ABS corresponding to the DM_1 $N_{(m_1,n_1)+(m_3,n_3)}$, there exists no element t satisfying that $0^1_{n_1} < t < 1^3_1$ or $0^1_{h_1} < t < 1^3_{n_3+1}$. Hence it is enough to show that there exist elements t^2_x and t^2_y such that $0^1_{n_1} < t^2_x < 1^3_1$ and $0^1_{h_1} < t^2_y < 1^3_{n_3+1}$. If $\lambda_2 > 1/2$, then these elements are obtained by $t^2_x = 0^2_{n_2}$ and $t^2_y = 0^2_{h_2}$. Indeed, by Proposition 5.7 (ii), we have $0^1_{n_1} < 0^2_{n_2}$ and $0^1_{h_1} < 0^2_{h_2}$. Moreover, by the construction of the ABS of $N_{(m_2,n_2)+(m_3,n_3)}$ we have $0^2_{n_2} < 1^3_1$ and $0^2_{h_2} < 1^3_{n_3+1}$. If $\lambda_2 = 1/2$, then the required elements t^2_x and t^2_y are obtained by 1^2_1 and 0^2_2.

Now we claim that

(A) If $\mathrm{Q}(\xi^{\mathrm{D}})$ holds, then $\mathrm{Q}(\xi)$ also holds,

(B) If $m_i \leq n_i$ for all i and $\mathrm{Q}(\xi^{\mathrm{C}})$ holds, then $\mathrm{Q}(\xi)$ also holds.

If the claim (A) and (B) are true, then by Euclidean algorithm for Newton polygons $\Phi : \mathrm{NP} \to \mathrm{NP}^{\mathrm{sep}}$ defined in Section 5.1, the proposition is reduced to the case (i) or (ii), and this completes the proof. The claim (A) is obvious. Let us show (B). Let S (resp. R) denote the ABS associated to ξ (resp. ξ^{C}). We can regard $T(R)$ as a subset of $T(S)$, see Lemma 5.5. Let U (resp. V) be the subset of $T(R) \times T(R)$ (resp. $T(S) \times T(S)$) consisting of pairs $(0^1, 1^3)$ of elements of $T(R)$ (resp. $T(S)$) satisfying $0^1 < 1^3$. Again by Lemma 5.5 we have $U = V$, whence (B) holds. □

In Proposition 6.2 and Corollary 6.3 below, we give necessary conditions for full modifications to be generic. For the definition of the sets A_0 and B_0, see Definition 3.15 and Definition 3.16 respectively.

Proposition 6.2. For the small modification by $(0^r_i, 1^q_j)$, if either of the following assertions

(i) the set A_0 contains an element 0^x with $r < x$, or

(ii) the set B_0 contains an element 1^x with $x < q$,

holds, then the full modification by this small modification is not generic.

Proof. Let $S^{(0)}$ be the small modification by $(0_i^r, 1_j^q)$. Put $\pi = \Pi(S^{(0)})$. Set $\alpha_n = \pi^n(0_i^r)$ and $\beta_n = \pi^n(1_j^q)$. By Proposition 3.20 and Proposition 3.23, we may assume that there exists the full modification $S^{(a+b)}$ by the small modification by $(0_i^r, 1_j^q)$. Let B_0' be the subset of $T(S^{(0)})$ defined by

$$B_0' = \{t \in T(S^{(0)}) \mid \beta_0 < t \text{ and } \pi(t) < \beta_1 \text{ in } S^{(0)} \text{ with } \delta(t) = 1\}. \tag{6.1}$$

For this set, $\ell(S) - \ell(S^{(0)}) = |A_0| + |B_0'| + 1$. For non-negative integers n, we have $\ell(S^{(n+1)}) - \ell(S^{(n)}) \leq d(n)$, where

$$d(n) = \begin{cases} |A_n| - |A_{n+1}| & \text{if } n < a, \\ |B_n| - |B_{n+1}| & \text{if } n \geq a. \end{cases} \tag{6.2}$$

We have then $\ell(S^{(n+1)}) - \ell(S^{(n)}) \leq d(n)$ for all n. Clearly $\ell(S') - \ell(S^{(0)}) \leq |A_0| + |B_0|$ holds. First, let us see that $\ell(S') - \ell(S^{(0)}) \leq |A_0| + |B_0'|$. Let I denote the subset of B_0 consisting of elements which are of the form α_m. We have then $|B_0| \leq |B_0'| + |I|$. For a non-negative integer m, if α_{m+1} belongs to I, then A_m contains the inverse image of β_0. We have then $\ell(S^{(m+1)}) - \ell(S^{(m)}) = d(m) - 1$. Hence we see $\ell(S^{(a)}) - \ell(S^{(0)}) \leq |A_0| - |I|$. Moreover, we have $\ell(S') - \ell(S^{(a)}) = |B_0|$. Thus we get the desired inequality.

Let us see that in the case (i) the full modification is not generic. Let m be the minimum number such that the set A_m contains no element t^x with $r < x$. Fix an element t^x of A_{m-1}. Put $t = \pi(t^x)$. Now we claim that $\delta(t) = 1$ and $\delta(\alpha_m) = 0$. If $\delta(t) = 0$ and $\delta(\alpha_m) = 1$ is true, then there exists an element 1^x satisfying $\alpha_m < 1^x < t$ in $T(S^{(m-1)})$. Indeed, if $1_n^x < \alpha_m$ holds in $T(S^{(m-1)})$ for all n, then we have $1_{m_x}^x < 1_{m_r}^r$ with $r < x$. By Proposition 5.7 this is a contradiction. Thus we see that the set A_m contains the element 1^x. This contradicts the minimality of m. Hence we have $\delta(t) = 1$ and $\delta(\alpha_m) = 0$, and it implies that $\ell(S^{(m)}) - \ell(S^{(m-1)}) < d(m)$, and we have $\ell(S^{(a)}) - \ell(S^{(0)}) < |A_0| - |I|$.

Let us treat the case (ii). In the same way as the case (i), if B_0 contains an element t^x with $x < q$, then there exists a non-negative integer m such that $\ell(S^{(m)}) - \ell(S^{(m-1)}) < d(m)$. Indeed, for the minimum number m such that B_m contains no element t^x with $x < q$, fix an element t^x of B_{m-1}. Then for $t = \pi(t^x)$, we have $\delta(t) = 0$ and $\delta(\beta_m) = 1$ since if $\delta(t) = 1$ and $\delta(\beta_m) = 0$ is true, then there exists an element 0^x of $T(S^{(m-1)})$ satisfying that $t < 0^x < \beta_m$ by Proposition 5.7. It implies that B_m contains an element 0^x, and this is a contradiction. Thus we obtain $\ell(S') - \ell(S^{(a)}) < |B_0|$.

By the above, in the case (i) and (ii), we have $\ell(S') - \ell(S^{(0)}) < |A_0| + |B_0'|$, and it follows that $\ell(S') < \ell(S) - 1$. $\square$

Corollary 6.3. If $q - r \geq 2$, then the full modification is not generic.

Proof. Put $\delta = \Delta(S)$. For a small modification by $(0^r, 1^q)$, by Lemma 6.1, there exists an element t^x of $T(S)$ such that $0^r < t^x < 1^q$ and $r < x < q$. If $\delta(t^x) = 0$, then the element t^x belongs to A_0, and the assertion follows from Proposition 6.2. Let us see the case $\delta(t^x) = 1$. If the set B_0 contains t^x, then Proposition 6.2 completes the proof. Let us see the case that B_0 does not contain t^x. For the set B_0' as the proof of Proposition 6.2, this t^x belongs to B_0'. We have $a \geq a'$, where a' is the minimum number satisfying that $\alpha_{a'} = \beta_0$ since we have $|B_0| < |B_0'|$ only if $a \geq a'$. Then $|B_0| < |B_0'| + |I|$ holds, where the set I is the same as the proof of Proposition 6.2. Hence we see $\ell(S') < \ell(S) - 1$. $\square$

Example 6.4. For the ABS S of $\xi = (2,7)+(1,2)+(3,5)$, consider the small modification by $(0^1_4, 1^3_2)$. Then the ABS $S^{(0)}$ is

$$S^{(0)} = 1^1_1\ 1^1_2\ 0^1_3\ 1^3_2\ 0^1_5\ 1^2_1\ 1^3_1\ 0^1_4\ 0^1_6\ 0^1_7\ 0^2_2\ 1^3_3\ 0^3_4\ 0^3_5\ 0^1_8\ 0^1_9\ 0^2_3\ 0^3_6\ 0^3_7\ 0^3_8\ . \tag{6.3}$$

We have $A_0 = \{0^1_5\}$ and $A_1 = \emptyset$. Thus we see $a = 1$. We have the set $B_0 = \{1^2_1, 1^3_1\}$ and $B_1 = \{0^2_3, 0^3_6\}$ with the ABS $S^{(2)}$

$$S^{(2)} :\ 1^1_1\ 0^1_3\ 1^1_2\ 1^3_2\ 0^1_5\ 1^2_1\ 1^3_1\ 0^1_4\ 0^1_6\ 0^1_7\ 0^2_2\ 1^3_3\ 0^3_4\ 0^3_5\ 0^1_8\ 0^1_9\ 0^3_7\ 0^2_3\ 0^3_6\ 0^3_8\ . \tag{6.4}$$

By the ABS $S^{(2)}$, we obtain the set $B_2 = \{0^2_2\}$ and the ABS

$$S^{(3)} :\ 1^1_1\ 0^1_3\ 1^1_2\ 1^3_2\ 0^1_5\ 1^2_1\ 1^3_1\ 0^1_4\ 0^1_6\ 0^1_7\ 0^3_4\ 0^2_2\ 1^3_3\ 0^3_5\ 0^1_8\ 0^1_9\ 0^3_7\ 0^2_3\ 0^3_6\ 0^3_8\ . \tag{6.5}$$

Similarly, we obtain $B_3 = \{1^2_1\}$, $B_4 = \{0^2_3\}$ and $B_5 = \emptyset$ with the ABS's $S^{(4)}$, $S^{(5)}$ and $S^{(6)}$. Hence we have $b = 5$, and the full modification $S^{(6)}$ is equal to S' of Example 3.12.

Let $\xi = \sum_{i=1}^{z}(m_i, n_i)$ be a Newton polygon. Let S be the ABS of the DM_1 N_ξ. Recall that the ABS S is described as $S = \bigoplus_{i=1}^{z} S_i$ for ABS's S_i corresponding to the DM_1 N_{m_i,n_i}. We say a full modification S' is generic if $\ell(S') = \ell(S) - 1$. Propositions 3.20 and 3.23 imply that all generic full modifications are given by full modifications. Now let us show Theorem 1.5, which implies that to determine generic specializations of $H(\xi)$, it is enough to deal with Newton polygons consisting of two segments.

Proof of Theorem 1.5. Let S be the ABS of ξ. We will construct a bijection map

$$\bigsqcup_{i=1}^{z-1}\{\text{generic full modifications of } R_i\} \longrightarrow \{\text{generic full modifications of } S\}, \tag{6.6}$$

where R_i denotes the ABS of the two slopes Newton polygon $(m_i, n_i) + (m_{i+1}, n_{i+1})$. Since we can regard $T(R_r)$ as a subset of $T(S)$ as ordered sets, we write t_i^r the i-th element of the first component of R_r. Similarly we denote by t_j^{r+1} the j-th element of the second component of R_r. By Corollary 6.3, it suffices to show the claim: *Let r be a natural number with $r < z$. The full modification of S by the small modification by $(0_i^r, 1_j^{r+1})$ is generic if and only if the full modification of $R_r = S_r \oplus S_{r+1}$ by the small modification by the same 0_i^r and 1_j^{r+1} is generic.* Put $R = R_r$. Let R' be a generic full modification of R obtained by the small modification by $(0_i^r, 1_j^{r+1})$, and let S' be the full modification of S by the small modification by $(0_i^r, 1_j^{r+1})$. We shall show that $\ell(S) - \ell(S') = \ell(R) - \ell(R')$. For the small modification by 0_i^r and 1_j^{r+1} of S, we use the same notations as Definition 3.15 and Definition 3.16. Let E (resp. F) denote the subset of A_0 (resp. B_0) consisting of elements 0^x (resp, 1^y) with $x \neq r$ (resp. $y \neq r+1$). Then we have $\ell(S) - \ell(S^{(0)}) = \ell(R) - \ell(R^{(0)}) + |E| + |F|$. So it suffices to show that

$$\ell(S') - \ell(S^{(0)}) = \ell(R') - \ell(R^{(0)}) + |E| + |F|. \tag{6.7}$$

Let m be the minimum number such that $\alpha_m = 1_{m_r}^r$. Let $C_0, C_1, \dots$ be the A-sequence associated to R, 0_i^r and 1_j^{r+1}. Since R' is generic, there exists a non-negative integer a' such that $C_{a'} = \emptyset$. Let us show the following three claims:

(a) for every element 0^x of E, we have $x < r$,

(b) there exists no element t^x of A_m such that $x \neq r$,

(c) there exists a non-negative integer a satisfying $A_a = \emptyset$, and

$$\ell(S^{(a)}) - \ell(S^{(0)}) = \ell(R^{(a')}) - \ell(R^{(0)}) + |E|. \tag{6.8}$$

Put $q = r+1$. To simplify, set $\pi = \Pi(S^{(0)})$ and $\delta = \Delta(S^{(0)})$. To show (a), assume $r < x$ for an element 0^x of E. In $T(S^{(0)})$, we have $0^x < 0^r_i$. Thus $0^x < 1^q_j$ holds in $T(S)$ by definition. Then it is clear that $0^x_{m_x+1} < 0^q_{m_q+1}$. By Proposition 5.7, we have $x < q$. Since r is adjacent to q, this is a contradiction, and we have shown (a). To show (b), assume that an element t^x belongs to A_{m-1} with $x \neq r$. We have then $x < r$ by Proposition 3.18 and (a). Then we have $\delta(\pi(t^x)) = 0$. Indeed, if $\delta(\pi(t^x)) = 1$, then $1^r_{m_r} < 1^x_{m_x}$ holds since $\alpha_m < \pi(t^x)$ in $T(S^{(m)})$. As $x < r$, this is a contradiction. Since the values of $\delta(\alpha_m)$ and $\delta(\pi(t_x))$ are different from each other, we obtain (b). Let us show (c). Let $E_n = \{t^x \in A_n \mid x \neq r\}$ for all n. Note that the sets A_n equal $C_n \cup E_n$ for all n. Thus by (b), there exists a non-negative integer a such that A_a is empty. We shall show

$$\ell(S^{(n+1)}) - \ell(S^{(n)}) = \ell(R^{(n+1)}) - \ell(R^{(n)}) + |E_n| \tag{6.9}$$

for all n. If it is true, then by (b) we obtain the equation (6.8). Let us show (6.9). Let n be a non-negative integer, and assume that for an element t^x of E_n, the element $\pi(t^x)$ does not belong to A_{n+1}. Then $\delta(\alpha_{n+1}) = 1$ and $\delta(\pi(t^x)) = 0$. Note that by the definition of A-sequences, the values $\delta(\alpha_{n+1})$ and $\delta(\pi(t^x))$ are different from each other. To make a contradiction, suppose that $\delta(\alpha_{n+1}) = 0$ and $\delta(\pi(t^x)) = 1$. We have then $1^r_{m_r} < 1^x_{m_x}$ with $x < r$. This is a contradiction, and hence (6.9) holds.

In the same way we have the "dual" of (a), (b) and (c). Let m' be the minimum number such that $\beta_{m'} = 0^q_{m_q+1}$. Then

(d) for every element 1^y of F, we have $q < y$,

(e) there exists no element t^y of $B_{m'}$ such that $y \neq q$,

(f) we have
$$\ell(S') - \ell(S^{(a)}) = \ell(R') - \ell(R^{(a')}) + |F|. \tag{6.10}$$

By (6.8) and (6.10) we get (6.7). Thus we see that if the generic full modification R' of R by the small modification by $(0^r_i, 1^{r+1}_j)$ is generic, then the full modification S' of S by the small modification $(0^r_i, 1^{r+1}_j)$ is generic. Similarly, one can see that if the generic full modification S' of S by the small modification by $(0^r_i, 1^{r+1}_j)$ is generic, then so is the full modification R' of R by the small modification by $(0^r_i, 1^{r+1}_j)$. $\square$

Example 6.5 below is an example of Theorem 1.5.

Example 6.5. For a Newton polygon ξ, let $G(\xi)$ be the set consisting of pairs $(0^r_i, 1^q_j)$ such that the full modification obtained by 0^r_i and 1^q_j is generic. Set $\xi = (2,7) + (3,5) + (2,1)$.

We have then

$$G(\xi) = \{(0_4^1, 1_1^2),\ (0_4^1, 1_2^2),\ (0_5^1, 1_1^2),\ (0_5^1, 1_2^2),\ (0_6^1, 1_3^2),\ (0_7^1, 1_3^2),\ (0_5^2, 1_1^3),\ (0_6^2, 1_2^3),\ (0_8^2, 1_2^3))\}. \tag{6.11}$$

Let ξ_1 (resp. ξ_2) denote the Newton polygon $(2,7)+(3,5)$ (resp. $(3,5)+(2,1)$). Then we get

$$G(\xi_2) = \{(0_5^1, 1_1^2),\ (0_6^1, 1_2^2),\ (0_8^1, 1_2^2)\}, \tag{6.12}$$

and we can regard $G(\xi)$ as the disjoint union of $G(\xi_1)$ and $G(\xi_2)$. See also Example 4.7.

6.2 Determining Newton polygons of generic specializations for any Newton polygons

By the following theorem, Problem 1.7 is reduced to the case that ξ consists of two segments.

Theorem 6.6. If Theorem 1.8 is true for ξ with two segments, then Theorem 1.8 holds for any ξ.

Proof. Let $\xi = \sum_{i=1}^{z}(m_i, n_i)$ be a Newton polygon. For the Newton polygon $\xi_i = (m_i, n_i) + (m_{i+1}, n_{i+1})$ and a generic specialization w' of w_{ξ_i}, if we have a Newton polygon $\zeta' = (c_i, d_i)+(c_{i+1}, d_{i+1})$ satisfying (i) and (ii) of Theorem 1.8, then for the generic specialization w of w_ξ corresponding to w' by Theorem 1.5, we obtain required Newton polygon ζ by $\zeta = (m_1, n_1) + \cdots + (m_{i-1}, n_{i-1}) + \zeta' + (m_{i+2}, n_{i+2}) + \cdots + (m_z, n_z)$. $\square$

Thanks to Proposition 5.9, we can show Theorem 1.8.

Proof of Theorem 1.8. By Theorem 6.6, we may assume that ξ consists of two segments. Let us show the statement by induction on the value $m_2 n_1 - m_1 n_2$. If the value is one, then the only element w of $B(\xi)$ is the type of w_ζ with ζ the segment $(m_1 + m_2, n_1 + n_2)$, and $\zeta \prec \xi$ is saturated. Indeed, if $\xi = (0,1) + (1,0)$, then it is clear that the specialization of $w \in B(\xi)$ corresponds to w_ζ with $\zeta = (1,1)$. For the other Newton polygons, we can show the claim in the same way as Proposition 5.9.

Assume that the value $m_2 n_1 - m_1 n_2$ is greater than one. Let $w \in B(\xi)$. By Proposition 5.9, there exists a generic specialization w^- of w and a segment ρ such that $w^- = w' \oplus w_\rho$, with $w' \in B(\xi')$, where $\xi' = (m_1', n_1') + (m_2', n_2')$ is uniquely determined by ξ and ρ so that the area of the triangle surrounded by ξ, ξ' and ρ is one. Note that $m_2' n_1' - m_1' n_2' < m_2 n_1 - m_1 n_2$. By the hypothesis of induction, for the Newton polygon

ξ' we have a Newton polygon ζ' such that ζ' satisfies the condition (i) and (ii) of the statement for ξ'. Put $\zeta = \zeta' + \rho$. Then $\zeta \prec \xi$ is saturated, and w_ζ is a specialization of w. $\square$

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