Re-embedding structures of graphs on surfaces and related topics for graph colorings

閉曲面上のグラフの再埋蔵構造およびグラフ彩色との関連

Kengo Enami

Graduate School of Environment and Information Sciences
Yokohama National University

March 2021

Acknowledgment

I am grateful, first and foremost, to Professor Seiya Negami, who has been my supervisor since 2014. I have learned so much from working with him. He has taught me not only mathematics but also the way of mathematician's life.

I thank Professor Atsuhiro Nakamoto and Professor Kenta Ozeki for careful reading my papers and giving useful advice. I could not have come this far without their support.

I also thank Professor Kenta Noguchi and Professor Yumiko Ohno for doing joint works. Through discussion with them, I reached interesting results. Two of the papers underlying this thesis are joint work with them.

Finally, I would like to express my gratitude to my family, members of the laboratory and all mathematicians who had mathematical contact with me.

The author
Kengo Enami
March 2021

Preface

This thesis is written on the subject "Re-embedding structures of graphs on surfaces and related topics for graph colorings" and it is to be submitted to get the degree of Doctor of Science at Yokohama National University. I will present my work on topological graph theory which is based on our four papers written when I was in master's and doctor's course. In particular, we focus on "re-embedding theory".

The study of "embeddability of graphs on surfaces" is one of the most important topics in the early days of topological graph theory, which originated in Kuratowski's Theorem and led to the deep Robertson-Seymour theory on graph minors. On the other hand, since a graph may not have only one embedding on an embeddable surface, the following questions also attracted many topological graph theorists: (1) How many (distinct) embeddings on a surface does a graph have? (2) What kind of structures generate these embeddings of the graph? We call the topic about these questions *re-embedding theory*. We hope that this thesis can contribute to development of re-embedding theory.

After the introduction, the reader can find six chapters. In Chapter 2, we shall give general terminology of (topological) graph theory. The remaining chapters consist of two parts.

In the first part, we will present my work about re-embedding structures of graphs on surfaces. In Chapter 3, we shall characterize re-embedding structures of 3-connected 3-regular planar graphs on the projective-plane, the torus and the Klein bottle. These results enable us to give explicit bounds for the number of inequivalent embeddings of such a graph on each surface, and propose effective algorithms for enumerating and counting these embeddings. In Chapter 4, we shall expand the above results to strongly 2-edge-connected 2-regular diplanar digraphs. That is, we characterize re-embedding structures of strongly 2-edge-connected 2-regular diplanar digraphs on the projective-plane, the torus and the Klein bottle. Moreover, we give a simpler proof of Archdeacon, Bonnington and Mohar's theorem [7, Corollary 2.3] than the original one, which states that every strongly 2-edge-connected diplanar graph is uniquely embeddable on the sphere.

In the second part, we will present my work about graph coloring problems from the viewpoint of re-embeddings of graphs. In Chapter 5, we shall give an affirmative answers of Kündgen and Ramamurthi's conjecture [44, Conjecture 8.1]: For each positive integer k, there is a graph that has two different embeddings on the same surface whose weak chromatic numbers differ by at least k. Moreover, we prove that there is a graph having two triangulations on a surface, only one of which is weakly k-colorable if and only if $k \geq 3$. In Chapter 6, we shall give the upper bound for the difference of the facial 3-achromatic numbers between two triangular embeddings of a simple graph on a surface in terms of the genus of the surface.

Papers underlying the thesis

- K. Enami, Embeddings of 3-connected 3-regular planar graphs on surfaces of non-negative Euler characteristic, *Discrete Mathematics and Theoretical Computer Science*, Vol. **21** no. 4 (2019).
- K. Enami, Embeddings of 2-regular diplanar digraphs on surfaces, submitted to *Australasian Journal of Combinatorics.*
- K. Enami and K. Noguchi, Embeddings of a graph into a surface with different weak chromatic numbers, *Graphs and Combinatorics* (2020) Online First.
- K. Enami and Y. Ohno, Difference of facial achromatic numbers between two triangular embeddings of a graph, submitted to *Journal of Combinatorial Optimization.*

Contents

Chapter 1

Introduction

In graph theory, a *graph* $G = (V(G), E(G))$ is usually defined as a composite structure of a non-empty and finite set $V(G)$ and a family $E(G)$ of 2-element subsets in $V(G)$, possibly empty. Each element in $V(G)$ is called a *vertex* and each element in $E(G)$ is called an *edge*. To visualize this structure, we often draw a figure consisting of points and curves joining two points, which correspond to the vertices and the edges, respectively. In particular, drawing graphs on topological spaces with no pair of edges crossing is the center of attention in *topological graph theory*. In this thesis, we deal with such drawn graphs. So it is convenient that we regard a graph as a topological space with the structure of a simplicial 1-complex, not only as a combinatorial object.

A (closed) *surface* is a compact connected 2-dimensional manifold without boundary. An *embedding* of a graph G on a surface F^2 is an injective continuous map $f : G \to F^2$. Roughly speaking, an embedding of G on F^2 is a drawing of G on F^2 with no pair of edges crossing. We often call the subspace $f(G)$ in F^2 with the structure of a graph induced by f 'an embedding of G on F^2' rather than the map f. If a graph G has an embedding on a surface F^2, we say that G is *embeddable* on F^2. A *planar* graph is a graph embeddable on the sphere (or, equivalently, embeddable on the plane).

We sometime consider that a graph G is already embedded on a surface and denote its image by G itself to simplify the notation. However, if we deal with two or more embeddings of G on some surfaces, then we denote them by $f_1(G), f_2(G), \dots$ to distinguish them, or denote a first given embedding of G by G itself and another embedding of G by $f(G)$. In the latter situation, we call $f(G)$ a *re-embedding* of G.

The *faces* of a graph G embedded on a surface F^2 are the connected components of the open set $F^2 - G$. We denote by $V(F)$ the set of vertices in the boundary of a face F of G, and by $\mathcal{F}(G)$ the set of faces of G. We call G *cellular* if each face of G must be homeomorphic to an *open 2-cell*, that is, homeomorphic to the unit open disk $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$. In this thesis, we mainly focus on cellular embeddings unless we particularly mention it.

Uniqueness and re-embedding problems

Two embeddings $f_1, f_2 : G \to F^2$ are *equivalent* if there is a homeomorphism $h : F^2 \to F^2$ such that $hf_1 = f_2$, and they are *inequivalent* otherwise. We say that a graph G is *uniquely embeddable* on F^2 (up to equivalence) if any two embeddings of G on F^2 are equivalent. The following two questions are important and have attracted many topological graph theorists:

(I) (Uniqueness Problem) What kind of graphs is uniquely embeddable on a given surface?

(II) (Re-embedding Problem) If a graph has two or more inequivalent embeddings on a surface, then how many inequivalent embeddings on a fixed surface does the graph have, and what kind of structures generates inequivalent embeddings of a given graph? (We often call such a structure the *re-embedding structure*.)

We call the topic about these problems *Re-embedding Theory*. (This topic is also called the *flexibility of embeddings* in some papers.) The following theorem is the most classical and famous result.

Theorem 1.1 (Whitney [74,75])**.** *Every* 3*-connected planar graph is uniquely embeddable on the sphere.*

This theorem was obtained as a corollary of the statement that one of any two embeddings of a 2-connected planar graph on the sphere can be obtained from the other by a sequence of simple local re-embeddings, called *Whitney's* 2*-flipping* (see Chapter 2 for more details).

With regard to the uniqueness of a graph G embedded on a non-spherical surface F^2 and the upper bound for the number of inequivalent embeddings of G on F^2, "face-width" plays an effective role. The *face-width* $\mathbf{fw}(G)$, which is also called the *representativity*, of G is defined by

$$\mathbf{fw}(G) = \min\{|G \cap \ell| : \ell \text{ is a noncontractible simple closed curve on } F^2\}.$$

Robertson and Vitray [66] proved that every 3-connected graph G embedded on an orientable surface of genus g with $\mathbf{fw}(G) \geq 2g+3$ is uniquely embeddable on the surface, and G is a minimum genus embedding. Mohar [49] and Seymour and Thomas [69] improved this to any surface and $\mathbf{fw}(G) = \Omega(\log(g)/\log(\log(g)))$. Robertson, Zha and Zhao [67] proved that every 3-connected toroidal graph G except for a small number of graphs with

$\mathbf{fw}(G) \geq 4$ is uniquely embeddable on the torus. Mohar and Robertson [51] proved that for any surface F^2, there is an integer $N(F^2)$ such that every 3-connected graph G has at most $N(F^2)$ inequivalent embeddings $f(G)$ on F^2 with $\mathbf{fw}(f(G)) \geq 3$. In these results, the assumption of the 3-connectivity is necessary to avoid inequivalent embeddings obtained by Whitney's 2-flipping.

Not only the face-width but also the connectivity has a strong relation to the number of inequivalent embeddings on surfaces with lower genera, which can be expected from Whitney's theorem. Negami [57] proved that every 6-connected toroidal graph except for three graphs is uniquely embeddable on the torus. Kitakubo and Negami [42] and Suzuki [71] studied the number of inequivalent embeddings of 5- and 4-connected non-planar graphs on the projective-plane, respectively. Recently, Maharry et al. [46] constructed re-embedding structures of non-planar graphs on the projective-plane completely and pointed out some mistakes in the past studies. In these papers, they analyzed re-embedding structures of "non-planar" graphs.

On the other hand, Mohar, Robertson and Vitray [53] and Mohar and Robertson [52] showed that 2-connected "planar" graphs embedded on non-spherical surfaces have special re-embedding structures, called "patch structures", while they have not given specific structures on each surface except for the projective-plane and not mentioned the number of inequivalent embeddings on any surface.

In Chapter 3, we shall construct the complete list of re-embedding structures of a planar graph G embedded on the projective-plane, the torus or the Klein bottle when G is 3-connected and 3-regular. In this argument, 3-connectivity is essential in order for us to avoid Whitney's 2-flippings, and when a 3-connected planar graph G is 3-regular, we can describe re-embedding structures of G on each surface completely. These re-embedding structures lead to the following results.

Theorem 1.2. *There exists a one-to-one correspondence between inequivalent embeddings of a* 3*-connected* 3*-regular planar graph on the projective-plane and subgraphs of the dual graph of the graph embedded on the sphere isomorphic to* K_2 *or* K_4*.*

Theorem 1.3. *There exists a one-to-one correspondence between inequivalent embeddings of a* 3*-connected* 3*-regular planar graph on the torus and subgraphs of the dual graph of the graph embedded on the sphere isomorphic to* $K_{2,2,2}$*,* $K_{2,2m}$ *or* $K_{1,1,2m-1}$ *for some positive integer* m*.*

Theorem 1.4. *There exists a one-to-one correspondence between inequivalent embeddings of a* 3*-connected* 3*-regular planar graph on the Klein bottle and subgraphs of the dual graph of the graph embedded on the sphere isomorphic to* $K_{2,2m-1}$ *or* $K_{1,1,2m}$ *for some positive integer* m*, or one of the six graphs* A_1 *to* A_6 *shown in Fig.* 1.1*.*

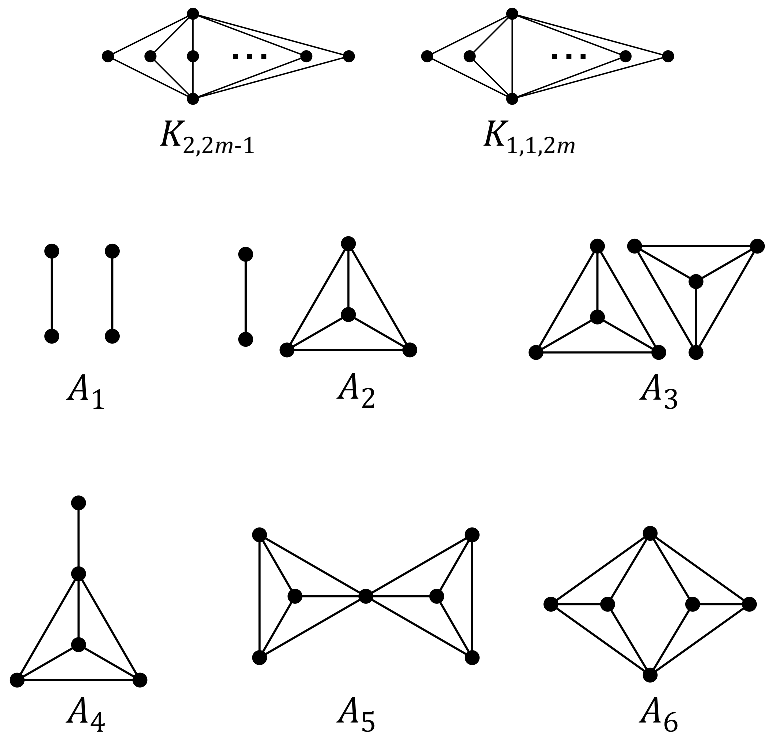

Figure 1.1: The eight graphs

Based on these theorems, we will give explicit bounds for the number of inequivalent embeddings of a 3-connected 3-regular planar graph G on each of the projective-plane, the torus and the Klein bottle. In addition, we will propose effective algorithms for enumerating and counting these embeddings. In particular, even though G may have exponentially many inequivalent embeddings on the torus and the Klein bottle, we can calculate the total number of such embeddings in polynomial time.

A *digraph* or *directed graph* D is defined as a composite structure of a non-empty and finite set $V(D)$, each of whose element is called a *vertex*, and a set $E(D)$ of ordered pairs of distinct vertices of D, called *arcs* or *directed edges*. When we present a digraph as a figure, the "direction" of each arc is indicated by an arrowhead. The *underlying graph* of a digraph D is an undirected graph obtained from D by replacing each arc with an undirected edge.

A digraph is *Eulerian* if each vertex has the same indegree and outdegree. An *embedding* of an Eulerian digraph D on a surface F^2 is defined as a 2-cell embedding of its underlying graph on F^2 with a property that each face is bounded by a directed closed walk. Hence, in- and out-edges alternate in the rotation around each vertex of an embedded digraph. An Eulerian digraph D is *diplanar* if D is embeddable on the sphere. This type of embedding was previously studied in [2, 6, 7, 9, 10, 16, 34, 40].

For an integer $k \geq 1$, a digraph D is *k-regular* if each vertex of D has both indegree and outdegree k, and is *strongly k-edge-connected* if for any $X \subseteq E(D)$ with $|X| < k$, $D - X$ is *strongly connected*, that is, there is a directed path from any vertex to any other vertex. Archdeacon et al. [6] proved an analogue of Whitney's theorem:

Theorem 1.5 (Archdeacon et al. [6]). *Every strongly 2-edge-connected 2-regular diplanar digraph is uniquely embeddable on the sphere.*

As with Whitney's theorem, Theorem 1.5 was obtained as a corollary of the stronger result that one of any two embeddings of a connected 2-regular diplanar digraph can be obtained from the other by a sequence of "directed" Whitney flips. Moreover, in [7], they proved an analogue of *Tutte's peripheral cycles theorem*, that is, every edge of a 3-connected graph is contained in at least two induced cycles, each of whose removal result in a connected graph. This analogue of Tutte's peripheral cycles theorem is also a stronger result of Theorem 1.5. Note that Theorem 1.5 does not hold arbitrary Eulerian digraphs. That is, there are infinitely many strongly 2-edge-connected Eulerian diplanar digraphs having at least two embeddings on the sphere.

In Chapter 4, we shall indicate the close relationship between an embedding of a 3-connected 3-regular graph on a surface and one of a strongly 2-edge-connected 2-regular digraph on the surface, which enables us to give a simple proof of Theorem 1.5. Moreover, we shall focus on embeddings of diplanar digraphs on non-spherical surfaces and extend the above result about 3-connected 3-regular planar graphs to strongly 2-edge-connected 2-regular diplanar graphs. That is, we shall characterize re-embedding structures of strongly 2-edge-connected 2-regular diplanar digraph on the projective-plane, the torus and the Klein bottle. In addition to this, we evaluate the number of such embeddings. In particular, we shall prove the following two theorems.

An undirected graph is *cyclically k-edge-connected* if there is no set of at most $k-1$ edges such that the graph obtained by deleting these edges has at least two components having a cycle.

Theorem 1.6. *Every strongly 2-edge-connected 2-regular diplanar digraph with n vertices has exactly $2n$ inequivalent embeddings on the projective-plane.*

Theorem 1.7. *If the underlying graph of a strongly 2-edge-connected 2-regular digraph D with n vertices is cyclically 5-edge-connected, then D has exactly n inequivalent embeddings on the torus and $n(2n-1)$ inequivalent embeddings on the Klein bottle. Moreover, in the case on the torus, the converse holds.*

Graph colorings with re-embedding structures

Hereafter, we will consider "multigraphs", which is a generalization of the notion of graphs. A multigraph G is a composite structures of a non-empty and finite vertex set $V(G)$ and a multiset $E(G)$ of 2-element subsets in $V(G)$, called an edge set. Then we allow two or more edges joining the same pair of vertices, which are called *multiple edges* or *parallel edges*. After Chapter 5, we use the term *graph* in this generalized sense, and call graphs without multiple edges *simple graphs*. (Some authors allowed *loops*, that is, edges with only one end, in multigraphs. However, throughout this thesis, we consider only graphs without loops.) A *digon* of a graph embedded on a surface is a face bounded by two multiple edges.

A (vertex) *k-coloring* of a graph G is an assignment $c : V(G) \to \{1, 2, \dots, k\}$ (we regard the set $\{1, 2, \dots, k\}$ as the set of k-colors). A coloring c of a graph G is *proper* if $c(u) \neq c(v)$ for any two adjacent vertices u and v of G. Throughout this thesis, a k-coloring c of G is not necessarily *proper* unless we mention it. The *chromatic number* $\chi(G)$ of a graph G is the minimum number k such that G has a proper k-coloring.

Historically the first problem in coloring of graphs on surfaces is "Four Color Problem", which was posed in 1852 by Francis Guthrie. It was answered affirmatively by Appel and Haken [3] in 1976 (see also [4, 5]), and the result is known as "Four Color Theorem":

Theorem 1.8 (Four Color Theorem)**.** *Every planar graph has a proper* 4*-coloring.*

Four Color Theorem implies that for any planar graph G, $\chi(G) \leq 4$. This bound is best possible as the complete graph of order 4 is embeddable on the sphere. On the other hand, in 1890, Heawood [36] proved that for any non-spherical surface F^2 and any graph G embedded on F^2, $\chi(G) \leq \lfloor (7 + \sqrt{49 - 24e(F^2)})/2 \rfloor$, where $e(F^2)$ is the Euler characteristic of F^2, and conjectured this bound is best possible. In a series of papers, partial case of the conjecture were solved, and finally in 1968, Ringel and Young [64] solved Heawood's conjecture completely. This result is called "Map Color Theorem" (see [65] for the history of Map Color Theorem).

Theorem 1.9 (Map Color Theorem)**.** *Let F^2 be a non-spherical surface and $\chi(F^2)$ be the maximum of $\chi(G)$ taken over all graphs G embedded on F^2. Then $\chi(F^2) = 6$ if F^2 is the Klein bottle, and $\chi(F^2) = \lfloor (7 + \sqrt{49 - 24e(F^2)})/2 \rfloor$ otherwise.*

Four Color Theorem and Map Color Theorem are the cornerstones for developing topological graph theory. To solve these theorems, various ideas and techniques for graphs embedded on surfaces were invented, and even after these theorems were proved, a variety of graph coloring problems have been introduced and studied. Recently, there are a large number of papers which study colorings of graphs embedded on surfaces where suitable constraints on colors appear around faces. Such colorings are called *facially-constrained colorings*. In particular, facially-constrained colorings of *plane graphs*, that is, planar graphs already embedded on the sphere, were overviewed by Czap and Jendrol' [20].

In this thesis, we shall study some facially-constrained colorings from the following viewpoint: The possibility of a facially-constrained coloring depends on the embedding in general. That is, even if a graph G embedded on a surface F^2 has a certain facially-constrained coloring, a re-embedding $f(G)$ of G may not have this coloring. Actually, such situations were pointed out in various papers dealt with facially-constrained colorings.

In this thesis, we mainly focus on two facially-constrained colorings. First, in Chapter 5, we deal with a "weak coloring". A *weak coloring* of a graph G embedded on a surface is a coloring of G such that no face is *monochromatic*, that is, all vertices on its boundary have the same color. The *weak chromatic number* of G, denoted by $\chi_w(G)$, is the minimum integer k such that G has a weak k-coloring. Kündgen and Ramamurthi [44] studied weak colorings of graphs embedded on surfaces from various viewpoints and raised the following conjecture.

Conjecture 1.10 (Kündgen and Ramamurthi [44, Conjecture 8.1])**.** *For each positive integer k, there is a graph that has two different embeddings on the same surface whose weak chromatic numbers differ by at least k.*

We shall answer this conjecture affirmatively in two ways. First, we construct two embeddings of a simple graph on the same surface such that one of them has a weak 2-coloring but the other has arbitrarily large weak chromatic number. Note that these are far from minimum genus embeddings, while we secondly construct two "triangulations" obtained from the same non-simple graph on the same surface of Euler genus g, whose weak chromatic numbers differ from $\Omega(\sqrt[3]{g})$. A *triangulation* is a graph embedded on a surface so that each face is bounded by a cycle of length 3. (Note that a triangulation is a minimum genus embedding if we avoid digons.) This order is best possible in some sense:

Dvořák, Král' and Škrekovski [24] proved the following theorem.

Theorem 1.11 (Dvořák, Král' and Škrekovski [24])**.** *For an integer $r \geq 3$, every graph G embedded on a surface of Euler genus $g \geq 5$ has a coloring using at most $3\lceil \sqrt[r]{(g-2)/2} \rceil$ colors and avoiding a monochromatic face of size at least r.*

This theorem implies that the weak chromatic number of a graph embedded on a surface of Euler genus g without digons is $O(\sqrt[3]{g})$. Thus, for two embeddings of the same graph on the same surface of Euler genus g, the difference of these weak chromatic numbers is also $O(\sqrt[3]{g})$. Our second construction attains this order.

Moreover, our second construction implies that for many positive integers k, the weak k-colorability of triangulations depends on the embedding. Actually, we shall show that the weak 2-colorability of triangulations does not depend on the embedding, while for any positive integer $k \geq 3$, the weak k-colorability of triangulations depends on the embedding.

In Chapter 6, we shall focus on a "facial complete coloring", which is an expansion of a "complete coloring". A *complete k-coloring* of a graph G is a proper k-coloring such that each pair of k-colors appears on at least one edge of G. The *achromatic number* of G is the maximum integer k such that G has a complete k-coloring. This notion was introduced by Harary and Hedetniemi [35], and has been extensively studied (see [39] for its survey). Recently, Matsumoto and Ohno [48] introduced a facial complete coloring as an extended notion of a complete coloring. A k-coloring, which is not necessarily proper, of a graph G embedded on a surface is *facially t-complete* if for any t-element subset X of the k colors, there is a face F of G such that $X \subseteq c(V(F))$. The maximum integer k such that G has a facial t-complete k-coloring is the *facial t-achromatic number* of G, denoted by $\psi_t(G)$. It seems to be natural to consider facial t-complete colorings for graphs embedded on a surface so that each face is bounded by a cycle of length t. Actually, in this thesis, we focus on facial 3-complete colorings of triangulations on a surface, and prove the following theorem:

Theorem 1.12. *Let G be a simple graph which has two triangulations $f_1(G)$ and $f_2(G)$ on a surface F^2, and let g be the Euler genus of F^2. If F^2 is orientable, then*

$$|\psi_3(f_1(G)) - \psi_3(f_2(G))| \leq \begin{cases} 9g/2 & (g \leq 2) \\ 27g/2 - 27 & (otherwise). \end{cases}$$

If F^2 is non-orientable, then

$$|\psi_3(f_1(G)) - \psi_3(f_2(G))| \leq \begin{cases} 3g & (g = 1) \\ 21g - 27 & (otherwise). \end{cases}$$

Note that we can easily construct a triangulation on each surface so that its facial 3-achromatic number is an arbitrarily large, while Theorem 1.12 implies that the difference of the facial 3-achromatic numbers between two triangulations $f_1(G)$ and $f_2(G)$ on a given surface, which is obtained from the same graph G, can be bounded by a constant.

On the other hand, the upper bounds in Theorem 1.12 do not seem to be sharp. Unfortunately, we have no construction of a graph which has two triangulations on a surface whose facial 3-achromatic numbers differ. So one may suspect that $\psi_t(f_1(G)) = \psi_t(f_2(G))$ whenever a simple graph G has two triangulations $f_1(G)$ and $f_2(G)$ on a surface. However, we do not believe that. Actually, we shall show the non-simple graphs having two triangulations on a surface whose facial 3-achromatic numbers differ (the definition of the facial complete coloring can be extended to multigraphs naturally). Hence, we hope that there exist such graphs for simple graphs.

Chapter 2

Foundations

In this chapter we define some basic terminology of (topological) graph theory, which are used throughout this thesis. We refer to [15, 23, 29, 50].

2.1 Graphs

A *graph* $G = (V(G), E(G))$ is usually defined as a composite structure of a non-empty and finite set $V(G)$ and a family $E(G)$ of 2-element subsets in $V(G)$, possibly empty. Each element in $V(G)$ is called a *vertex* and each element in $E(G)$ is called an *edge*. An edge $\{u, v\}$ is often written as uv or vu. The *order* of a graph G is the number of vertices of G, denoted by $|G|$. (We will introduce some other notions of graphs in the last of this section.)

To visualize the structure of a graph, we often draw a figure consisting of some points and curves joining two points, which correspond to the vertices and the edges of the graph, respectively. For example, Fig. 2.1 represents the graph $G = (V(G), E(G))$ with $V(G) = \{1, 2, 3, 4, 5\}$ and $E(G) = \{12, 13, 14, 23, 25, 45\}$.

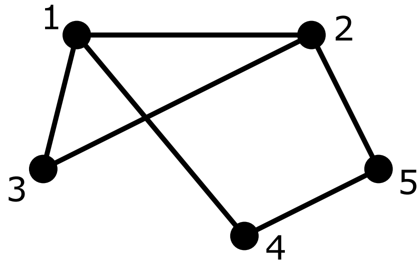

Figure 2.1: The graph G.

If two vertices u and v of a graph G are joined by an edge e, that is, $e = uv$, then they are called *neighbors* of each other, and *endvertices* or *ends* of e. In this situation, we say that they are and *adjacent* to each other, and u (or v) is *incident* with e.

The set of neighbors of v is called the *(open) neighborhood* of v, denoted by $N_G(v)$. The *degree* of v is the number of edges incident with v, denoted by $\deg_G(v)$, that is, $\deg_G(G) = |NG(v)|$. A graph is *r-regular* if the degrees of all vertices of G is r. The following lemma is one of the most fundamental and important statement in graph theory, which is well-known as *Handshaking Lemma*.

Lemma 2.1 (Handshaking Lemma). *For every graph G, we have*

$$\Sigma_{v \in v(G)} \deg_G(v) = 2|E(G)|.$$

This lemma implies that every graph has an even number of vertices with odd degree.

Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs with $|V(G_1)| = |V(G_2)|$. A bijection $\phi : V(G_1) \to V(G_2)$ is called an *isomorphism* from G_1 to G_2 if for any two vertices u and v of G_1, uv is an edge of G_1 if and only if $\phi(u)\phi(v)$ is an edge of G_2. If there is an isomorphism from G_1 to G_2, then they are called *isomorphic* to each other. An isomorphism from G_1 to itself is called an *automorphism*. We do not usually distinguish between isomorphic graphs, and write $G_1 = G_2$.

Subgraphs

For two graphs G and H, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is called a *subgraph* of G, and we say that G has H (as a subgraph). A subgraph H of G is called *spanning* if $V(H) = V(G)$. An edge not in H but with both ends in H is called *chord* of H. The subgraph H of a graph G is *induced* if H has no chord.

Let S be a subset of the vertex set of a graph G. If a subgraph of G whose vertex set is S contains all edges $e = uv$ of G with $u, v \in S$, then we say that it is a subgraph *induced* by S, and denote it by $G[S]$. Let G be a graph and H be a subgraph of G. An *edge-induced subgraph* of a graph G consists of a subset $X \subseteq E(G)$ and the endvertices of all edges in X, denoted by $G[X]$. In this situation, we say that this subgraph *induced* by X.

For a subset S of the vertex set of a graph G, we denote by $G - S$ the subgraph of G obtained by deleting all vertices in S and all edges whose endvertices contains a vertex in S. If $S = \{v\}$, we simply denote it by $G - v$. Similarly, for a subset X of $E(G)$, we denote by $G - X$ the subgraph of G obtained by deleting all edges in X. If If $S = \{e\}$, we simply denote it by $G - e$.

For example, Fig. 2.2 presents the graph G and two subgraphs H_1 and H_2 of G. The subgraph H_1 is not an induced subgraph of G since it has a chord fg, while H_2 is a induced subgraph of G, which is obtained by deleting two vertices c and g, that is, $H_2 = G - \{c, g\}$.

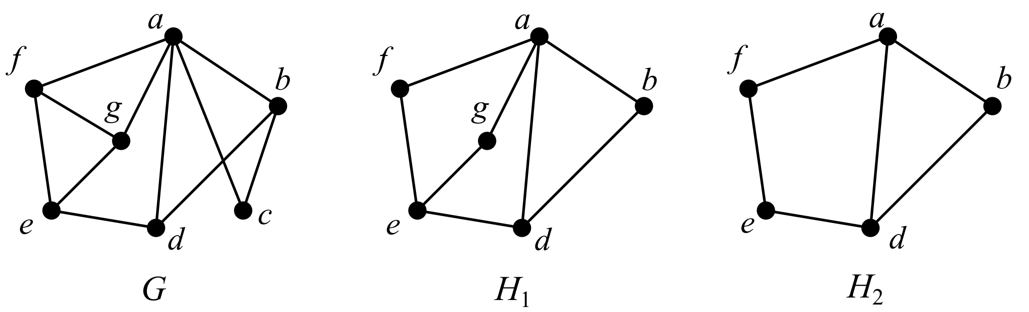

Figure 2.2: The graph G and two subgraphs H_1 and H_2 of G.

Paths and cycles

Let G be a graph. A *walk* is a sequence of vertices and edges $W = v_1, e_1, v_2, e_2, \ldots e_{n-1}, v_n$ such that $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1$. We sometime write the walk as $W = v_1, v_2, \ldots, v_n$ or $W = e_1, e_2, \ldots, e_{n-1}$. If $u = v_1$ and $v = v_n$, then W is called *u-v walk*. If $v_1 = v_n$ then the walk W is called *closed*. The *length* of W is the number of edges in W. A non-closed walk (resp. closed walk) in G in which no vertex is repeated is called a *path* (resp. *cycle*). A path and a cycle in G can be regarded as a subgraph of G. If a graph is isomorphic to a path or cycle of order n, then it is denoted by P_n or C_n, respectively (see Fig. 2.3 for example). We often call a cycle of order n a *n-cycle*.

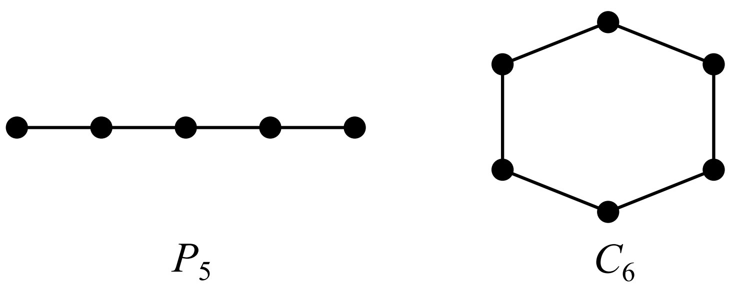

Figure 2.3: A path P_5 and a cycle C_6.

Complete graphs and miltipartite graphs

If all vertices of a graph are pairwise adjacent, then it is called *complete*. A complete graph of order n is denoted by K_n.

For an integer $k \geq 2$, a graph is *k-partite* if there is a partition of $V(G)$ into k subsets $V_1, V_2, \ldots, V_k$, called *partite sets*, such that V_i induces a subgraph having no edges for $1 \leq i \leq k$. A 2-partite graph is also called a *bipartite graph*. For a k-partite graph with partite sets $V_1, V_2, \ldots, V_k$, if any two vertices belonging to different partite sets are adjacent, then the graph is called *complete k-partite graph*, denoted by $K_{n_1,n_2,\ldots,n_k}$, where $n_i = |V_i|$ for $1 \leq i \leq k$. Fig. 2.4 presents three graphs K_5, $K_{3,3}$ and $K_{2,2,2}$.

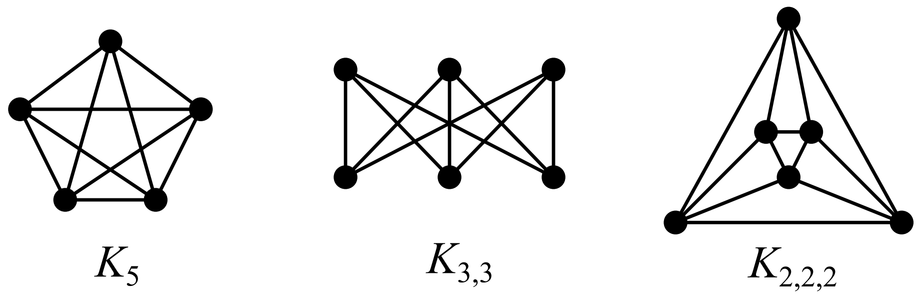

Figure 2.4: Three graphs K_5, $K_{3,3}$ and $K_{2,2,2}$.

Independent set and matching

Let G be a graph. A subset $S \subseteq V(G)$ is *independent* if no two vertices in S are adjacent. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of the largest independent set, that is, the size of a *maximum independent set* of G. A subset $X \subseteq E(G)$ is *matching* if any two edges have no common endvertex. The *matching number* of G, denoted by $\mu(G)$, is the cardinality of the largest matching, that is, the size of a *maximum matching* of G. A matching M of G is *perfect* if $2|M| = |G|$.

Connectivity

A graph G is called *connected* if for any two vertices u and v of G, there is a u-v path in G, and called *disconnected* otherwise. A *component* of a graph G is a connected induced subgraph H such that there is no path in G between a vertex of H and one of the rest of G, that is, a *maximal* connected subgraph of G.

A subset S of $V(G)$ is called a *vertex-cut set* or simply *cut set*, if $G - S$ is disconnected. If $S = \{v\}$, then we call v a *cut vertex*. Let X be a subset of $E(G)$. X is called an *edge-cut* *set* or *edge-cut*, if $G - X$ is disconnected. If $X = \{e\}$, then we call e a *cut edge* or a *bridge*.

A graph G is called *k-connected* if G has at least $k+1$ vertices, and for any subset S of $V(G)$ with at least $k-1$ vertices, $G - S$ is connected. The *connectivity* of G is the maximum number k such that G is k-connected.

A graph G is *cyclically k-edge connected* if there is no set of at most $k-1$ edges such that the graph obtained by deleting these edges has at least two components having a cycle.

H-bridge

An *H-bridge* is a subgraph of a graph G induced by a chord of H, or a component of $G - V(H)$ together with all edges joining it to H. In an H-bridge, a vertex belongs to $V(H)$ is called a *vertex of attachment*. Note that any two H-bridges are edge-disjoint and meet only the common vertices of attachment.

In Fig. 2.5 to the left, a subgraph H of a graph G is represented by the four edges drawn by bold lines with these ends, that is, H is induced by the four bold edges. There are three H-bridges in G, denoted by B_1, B_2 and B_3, as shown in Fig. 2.5.

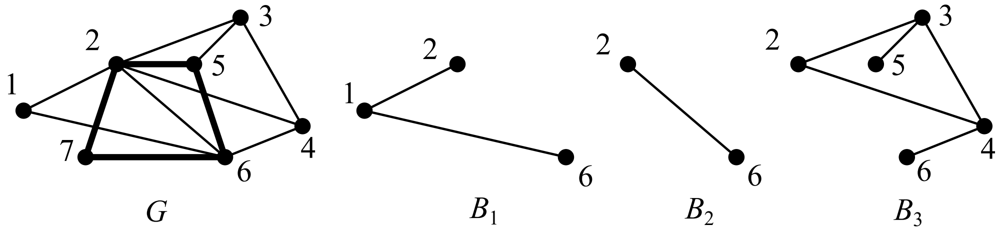

Figure 2.5: There are three H-bridges in G.

Other notions of graphs

In this thesis, we sometime mention some other notions of graphs.

A *multigraph* $G = (V(G), E(G))$ is defined as a composite structure of a non-empty and finite vertex set $V(G)$ and a multiset $E(G)$ of 2-element subsets in $V(G)$, called an edge set. That is, we allow two or more edges joining the same pair of vertices, which are called *multiple edges* or *parallel edges*. For example, Fig. 2.6 presents a multigraph G, where pairs (b, c) and (f, d) of vertices are joined by three and two multiple edges, respectively.

In Chapters 3 and 4, we will not consider multigraphs, while after Chapter 5, we use the term *graph* in this generalized sense, and call graphs without multiple edges *simple*

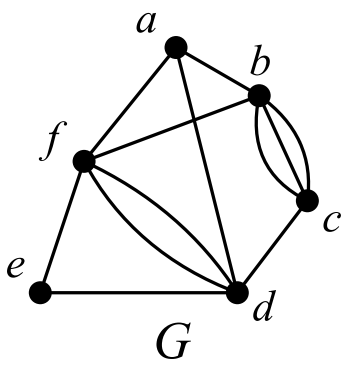

Figure 2.6: The multigraph G.

graphs. (Some authors allowed *loops*, that is, edges with only one end, in multigraphs. However, throughout this thesis, we consider only graphs without loops.)

A *digraph* $D = (V(D), E(D))$ or *directed graph* is defined as a composite structure of a non-empty and finite vertex set $V(D)$ and a family $E(D)$ of ordered pairs of distinct vertices of D, called *arcs* or *directed edges*. An arc (u, v) is often written as uv. The arc uv is said to be *directed from* u to v. The *underlying graph* of a digraph D is an undirected graph obtained from D by replacing each arc with an undirected edge. When we present a digraph as a figure, the "direction" of each arc is indicated by an arrowhead. For example, Fig. 2.7 represents the graph $D = (V(D), E(G))$ with $V(G) = \{1, 2, 3, 4, 5\}$ and $E(G) = \{12, 14, 21, 23, 25, 34, 42, 51\}$.

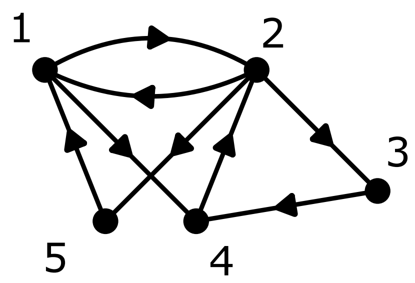

Figure 2.7: The digraph D.

For a digraph D, the *indegree* and *outdegree* of a vertex v are the numbers of arcs directed to v and that of arcs directed from v, respectively. A digraph is *Eulerian* if each vertex has the same indegree and outdegree, and *k-regular* if each vertex has both indegree and outdegree k.

A *directed walk* is a sequence of vertices and arcs $W = v_1, e_1, v_2, e_2, \dots e_{n-1}, v_n$ such that $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1$. Each of a *directed closed walk*, a *directed path* and a *directed cycle* can be defined similarly as in the undirected case. A digraph D is called *strongly connected* if there is a directed path from any vertex to any other vertex, and *strongly k-edge-connected* if for any $X \subseteq E(D)$ with $|X| \leq k-1$, $D - X$ is strongly connected.

A *hypergraph* $H = (V(H), \mathcal{E}(H))$ is defined as a composite structure of a non-empty and finite vertex set $V(D)$ and a family $\mathcal{E}(H)$ of non-empty subsets of $V(H)$, called *hyper-edge*. A hypergraph H is called *k-uniform* if each hyper-edge has k vertices.

2.2 Graphs on surfaces

A (closed) *surface* is a compact connected 2-dimensional manifold without boundary. By the classification theorem of surfaces, any surface is homeomorphic to either an *orientable surface* of genus $g \geq 0$, denoted by S_g, or a *non-orientable surface* of genus $k \geq 1$, denoted by N_k. The *Euler genus* of S_g is $2g$ and that of N_k is k. The surface S_0, S_1, N_1 and N_2 are called the *sphere*, the *torus*, the *projective-plane* and the *Klein bottle*.

An *embedding* of a graph G on a surface F^2 is an injective continuous map $f : G \to F^2$. Roughly speaking, an embedding of G on F^2 is a drawing of G on F^2 with no pair of edges crossing. We often call the subspace $f(G)$ in F^2 with the structure of a graph induced by f "an embedding of G on F^2" rather than the map f. If a graph G has an embedding on a surface F^2, we say that G is *embeddable* on F^2. A *planar* graph is a graph embeddable on the sphere (or, equivalently, embeddable on the plane). A *plane* graph is a planar graph already embedded on the sphere.

We sometime consider that a graph G is already embedded on a surface and denote its image by G itself to simplify the notation. However, if we deal with two or more embeddings of G on some surfaces, then we denote them by $f_1(G), f_2(G), \ldots$ to distinguish them, or denote a first given embedding of G by G itself and another embedding of G by $f(G)$. In the latter situation, we call $f(G)$ a *re-embedding* of G.

The *faces* of a graph G embedded on a surface F^2 are the connected components of the open set $F^2 - G$. We denote by $V(F)$ the set of vertices in the boundary of a face F of G, and by $\mathcal{F}(G)$ the set of faces of G. We call G *cellular* if each face of G must be homeomorphic to an *open 2-cell*, that is, homeomorphic to the unit open disk $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$. In this thesis, we mainly focus on cellular embeddings unless we particularly mention it. For a graph G embedded on a surface, a closed walk in G is *facial* if it bounds a face of G.

A *triangulation* on a surface is a graph embedded on a surface so that each facial closed walk has length 3. If a graph G is embeddable on a surface F^2 as a triangulation, we say that G has the triangulation on F^2. As with a triangulation, a *quadrangulation* on a surface F^2 is defined as a graph embedded on F^2 so that each facial closed walk has length 4.

Two embeddings $f_1, f_2 : G \to F^2$ are *equivalent* if there is a homeomorphism $h : F^2 \to F^2$ such that $hf_1 = f_2$, and they are *inequivalent* otherwise. We say that a graph G is *uniquely embeddable* on F^2 (up to equivalence) if any two embeddings of G on F^2 are equivalent.

The following theorem is one of the most important and famous results in topological graph theory, called the *Whitney's theorem*.

Theorem 2.2 (Whitney [74,75]). *Every 3-connected planar graph is uniquely embeddable on the sphere.*

This theorem was obtained as a corollary of the statement that one of any two embeddings of a 2-connected planar graph on the sphere can be obtained from the other by a sequence of simple local re-embeddings, called "Whitney's 2-flipping".

Let G be a graph embedded on a surface F^2, and $\mathcal{D}$ be a disk whose boundary meets G only at two vertices u and v. Suppose that there is at least one vertex into $\mathcal{D}$. Then we can obtain a re-embedding $f(G)$ of G on F^2 by applying an orientation-reversing automorphism of $\mathcal{D}$ which fixes u and v on the boundary of $\mathcal{D}$ (see Fig. 2.8 for example). Such an operation is called *Whitney's 2-flipping*. See [50, Sections 2 and 5] for the strict definition of Whitney's 2-flipping.

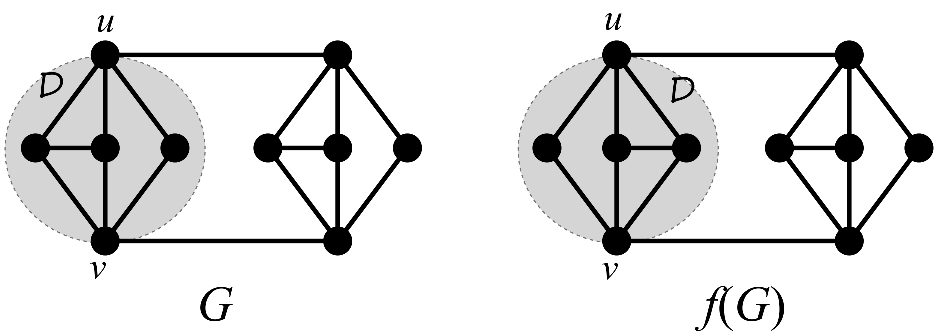

Figure 2.8: Whitney's 2-flipping D for a plane graph G.

In addition, Theorem 2.2 is also obtained as a corollary of the *Tutte's peripheral cycle theorem*. We say that a cycle C in a graph G is *peripheral* if there is exactly one C-bridge in G.

Theorem 2.3 (Tutte [73]). *Every edge in a 3-connected graph is contained in at least two peripheral cycles.*

Rotation systems and embedding schemes

We now introduce two combinatorial ways of describing embeddings of a graph; "rotation systems" and "embedding schemes". A general description on the rotation system and the embedding scheme can be found in [50].

Suppose that a connected graph G is embedded on an orientable surface. A *rotation* ρ_v around a vertex v of G is a cyclic permutation of edges incident with a vertex v such that $\rho_v(e)$ is the successor of e in the clockwise ordering around v. A *rotation system* for the embedded graph G is the collection of ρ_v, denoted by $\rho = \{\rho_v : v \in V(G)\}$. It is well-known that every embedding of a connected graph on an orientable surface is uniquely determined up to equivalence by its rotation system. Moreover, there are no rotation systems representing this embedding other than this rotation system and its inverse.

On the other hand, in order to include embeddings on non-orientable surfaces, we have to add the following concept. Let $f(G)$ be another embedding of G on a surface, which is not necessarily orientable. There are two possible cyclic ordering of edges incident with each vertex v of $f(G)$. Choose one of them and denote it by ρ_v. A *signature* of $E(G)$ is a map outputting 1 or -1 from each edge of G, denoted by λ, such that for an edge $e = uv$ with its endvertices u and v, $\lambda(e) = 1$ if a sub-walk induced by the three edges $\rho_u(e)$, e and $\rho_v^{-1}(e)$ is included in a facial walk, otherwise $\lambda(e) = -1$. It can be shown that this definition of the signature λ is consistent, that is, $\lambda(uv) = \lambda(vu)$ for every edge uv. The pair (ρ, λ), where $\rho = \{\rho_v : v \in V(G)\}$ is obtained by the above procedure, is called an *embedding scheme* for $f(G)$. An embedding scheme determines exactly one embedding of G. Unfortunately, an embedding scheme representing a given embedding of G is not uniquely determined, unlike rotation systems for the orientable case. For an embedded graph G associated with a given embedding scheme (ρ, λ), an edge e with $\lambda(e) = -1$ is called *twisted*. If there are no twisted edges then G is embedded on an orientable surface obtained by the rotation system ρ.

Cycles in embedded graphs

Let G be a graph embedded on a surface F^2. A cycle C of G can be regarded as a *simple closed curve* in F^2. The cycle C is *contractible* if it bounds a 2-cell region, and is *separating* if it separates F^2 into two parts. We say that C is 2-*sided* if it divides its annular neighborhood into two parts, and is 1-*sided* otherwise. Note that if C is contractible then it is separating and if C is 1-sided then it is non-separating.

For two disjoint cycles C_1 and C_2 of G, cut the surface F^2 along them. When one of the component in the resulting surface is an annulus with boundary components C_1 and C_2, we say that C_1 and C_2 are *homotopic*.

2.3 Facially-constrained colorings

Let G be a graph. A (vertex) *k-coloring* of G is a map $c : V(G) \to \{1, 2, \ldots, k\}$. A k-coloring c of G is *proper* if $c(u) \neq c(v)$ for any two adjacent vertices u and v of G. We say that G is *k-colorable* if G has a proper k-coloring. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k such that G is k-colorable.

The following two theorems are the most important and famous results about coloring of graphs on surfaces.

Theorem 2.4 (Four Color Theorem)**.** *Every planar graph has a proper* 4*-coloring.*

Theorem 2.5 (Map Color Theorem)**.** *Let F^2 be a non-spherical surface and $\chi(F^2)$ be the maximum of $\chi(G)$ taken over all graphs G embedded on F^2. Then $\chi(F^2) = 6$ if F^2 is the Klein bottle, and $\chi(F^2) = \lfloor (7 + \sqrt{49 - 24e(F^2)})/2 \rfloor$ otherwise.*

Recently, there are a large number of papers which study colorings of graphs embedded on surfaces where suitable constraints on colors appear around faces. Such colorings are called *facially-constrained colorings*. In particular, facially-constrained colorings of plane graph were overviewed by Czap and jendrol' [20]. Many facially-constrained colorings can be translated into colorings of some kind of hypergraphs, called "face-hypergraphs". The *face-hypergraph* $\mathcal{H}(G)$ of a graph G embedded on a surface is the hypergraph with vertex-set $V(G)$ and edge-set $\{V(F) : F \in \mathcal{F}(G)\}$, whose concept was introduced in [44].

We now introduce some facially-constrained colorings: rainbow coloring, antirainbow coloring, weak coloring, polychromatic coloring, facial complete coloring. For a vertex-colored graph G embedded on a surface, a face F is *rainbow* if any two distinct vertices in $V(F)$ has disjoint colors, and is *monochromatic* if all vertices in $V(F)$ has the same color.

Rainbow coloring

Let G be a graph embedded on a surface. A *rainbow coloring* (or originally *cyclic coloring*) of G is a coloring of G so that each face is rainbow. The minimum integer k such that G has a rainbow k-coloring is the *rainbowness* of G, denoted by $\mathrm{rb}(G)$. It is clear that the rainbowness of G is larger than or equal to the maximum $|V(F)|$ taken over all faces F in G. This concept was introduced by Ore and Plummer [60], and has been extensively studied; see for example [11–13, 26, 27, 55, 62, 68]. Recently, rainbow coloring of plane graphs and related topics were overviewed by Czap and Jendrol' [21].

Antirainbow coloring

An *antirainbow coloring* (or a *valid coloring*) of G is a coloring of G so that no face is rainbow. The maximum integer k such that G has a surjective antirainbow k-coloring is the *antirainbowness* of G, denoted by $\mathrm{arb}(G)$. This type of coloring was introduced by Ramamurthi and West [63] and Negami [58] independently. (See also [8,59] for some acts of taking the initiative.) The topics of antirainbow coloring of plane graph was studied in [25, 41, 63], and that of triangulations on surfaces was studied in [54, 58]. (Note that in [54,58], the invariant "looseness" $\xi(G)$ of a triangulation G on a surface was introduced and studied, which is corresponds to the value $\mathrm{arb}(G) + 2$.)

Weak coloring

A *weak coloring* of G is a coloring of G such that no face is monochromatic. The minimum integer k such that G has a weak k-coloring is the *weak chromatic number* of G, denoted by $\chi_w(G)$. Note that a weak coloring of an embedded graph corresponds to a proper coloring of its face-hypergraph. Weak colorings of graphs on surfaces have been studied in various contexts; see for example [14, 24, 44, 61].

Polychromatic coloring

A *polychromatic k-coloring* of G is a k-coloring of G so that all k colors appear in the boundary of each face. Note that a weak k-coloring and a polychromatic k-coloring are equivalent if and only if $k = 2$. This concept was introduced by Alon et al. [1], and has been extensively studied; see for example [1, 37, 38, 43, 45, 72]. A polychromatic coloring of plane graph is related to *art gallery problems* (see [19, 28] for details of art gallery problems).

Facial complete coloring

A *complete k-coloring* of a graph G is a proper k-coloring such that each pair of k-colors appears on at least one edge of G. The *achromatic number* of G is the maximum integer k such that G has a complete k-coloring. This notion was introduced by Harary and Hedetniemi [35], and has been extensively studied (see [39] for its survey).

Recently, Matsumoto and Ohno [48] introduced a new facially-constrained coloring, called the "facial complete coloring", which is an expansion of the complete coloring. A k-coloring, which is not necessarily proper, of a graph G embedded on a surface is *facially t-complete* if for any t-element subset X of the k colors, there is a face F of G such that $X \subseteq c(V(F))$. The maximum integer k such that G has a facial t-complete k-coloring is the *facial t-achromatic number* of G, denoted by $\psi_t(G)$. It seems to be natural to consider facial t-complete colorings for graphs embedded on a surface so that each face is bounded by a cycle of length t.

From the viewpoint of re-embeddings of graphs

We should notice that the possibility of a facially-constrained coloring of a graph embedded on a surface depends on the embedding. That is, even if a graph G embedded on a surface has a certain facially-constrained coloring, a re-embedding $f(G)$ of G may not have this coloring. In Chapters 5 and 6, we focus on weak colorings and facial complete colorings from this point of view, respectively. We now introduce some results and observations about rainbow colorings, antirainbow colorings.

Let G be the graph consisting of $m \geq 3$ cycles of length 3 with one common vertex, which has two embeddings $f_1(G)$ and $f_2(G)$ on the sphere as shown in Fig. 2.9. Then G has $2m+1$ vertices.

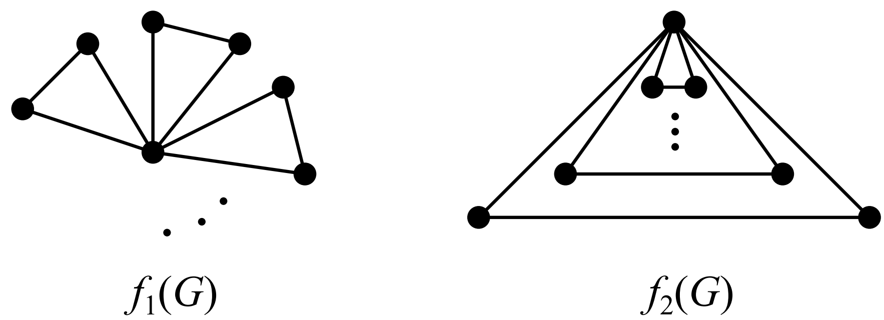

Figure 2.9: Two embeddings of G on the sphere.

As there is a face incident with all vertices in $f_1(G)$, we have $\mathrm{rb}(f_1(G)) = |V(G)| = 2m+1$. It is also easy to see that $\mathrm{rb}(f_2(G)) = 5$. Hence, the difference between $\mathrm{rb}(f_1(G))$ and $\mathrm{rb}(f_2(G))$ is $2m-4$. It implies that the rainbowness of a graph embedded on a surface depends on the embedding. Moreover, such a difference can be arbitrarily large. On the other hand, it is easy to see that $\mathrm{rb}(G) = \chi(G)$ for every triangulation on a surface, where $\chi(G)$ is the chromatic number of G. This implies that the rainbowness of a triangulation does not depend on the embedding.

Ramamurthi and West [63] observed that for the above two embeddings $f_1(G)$ and $f_2(G)$ of G on the sphere, $\mathrm{arb}(f_1(G)) = m+1$ and $\mathrm{arb}(f_2(G)) = \lceil 3m/2 \rceil$. Then the difference of these antirainbownesses is $\lceil m/2 - 1 \rceil$, and hence the antirainbowness of a graph embedded on a surface also depends on the embedding. Ramamurthi and West [63] conjectured this difference is the maximum difference for two embeddings of a graph on the sphere, that is, for every planar graph G of order n, there is no pair of embeddings of G on the sphere whose antirainbownesses differ from at least $\lfloor (n-2)/4 \rfloor$.

Arocha, Bracho and Neumann-Lara [8] studied the antirainbow 3-colorability of triangulations obtained from complete graphs, which they called the *tightness*. They proved that the complete graph of order 30 has both of a tight triangulation and an untight one on the same surface. This implies that the antirainbowness of triangulations depends on the embedding, in contrast to the rainbowness of triangulations. As the generalization of their work, Negami [58] introduced the *looseness* of a triangulation G on a surface, which corresponds to $\mathrm{arb}(G)+2$. He proved that for any graph having two triangulations $f_1(G)$ and $f_2(G)$ on a surface F^2,

$$|\mathrm{arb}(f_1(G)) - \mathrm{arb}(f_2(G))| \leq 2\lfloor g(F^2)/2 \rfloor,$$

where $g(F^2)$ is the Euler genus of F^2.

Chapter 3

3-Regular Planar Graphs on Surfaces

In this chapter, we focus on embeddings of a 3-connected 3-regular planar graph on the projective-plane, the torus or the Klein bottle. Our main results in this chapter are the followings.

Theorem 3.1. *There exists a one-to-one correspondence between inequivalent embeddings of a* 3*-connected* 3*-regular planar graph on the projective-plane and subgraphs of the dual graph of the graph embedded on the sphere isomorphic to* K_2 *or* K_4.

Theorem 3.2. *There exists a one-to-one correspondence between inequivalent embeddings of a* 3*-connected* 3*-regular planar graph on the torus and subgraphs of the dual graph of the graph embedded on the sphere isomorphic to* $K_{2,2,2}$, $K_{2,2m}$ *or* $K_{1,1,2m-1}$ *for some positive integer* m.

Theorem 3.3. *There exists a one-to-one correspondence between inequivalent embeddings of a* 3*-connected* 3*-regular planar graph on the Klein bottle and subgraphs of the dual graph of the graph embedded on the sphere isomorphic to* $K_{2,2m-1}$ *or* $K_{1,1,2m}$ *for some positive integer* m*, or one of the six graphs* A_1 *to* A_6 *shown in Fig.* 3.1.

Throughout this chapter, let G be a 3-connected 3-regular planar graph. In addition, we assume that G is already embedded on the sphere with its rotation system $\rho = \{\rho_v : v \in V(G)\}$. (By Whitney's theorem, G is uniquely embeddable on the sphere.)

3.1 Re-embeddings of 3-regular planar graphs

We now consider another embedding of G. Since G is 3-regular, there are only two possible rotations around each vertex of G, and one of them is the inverse of the other. This implies that for any embedding $f(G)$ of G on any surface, we can choose ρ_v as the local rotation around each vertex v. Thus, $f(G)$ can be determined by an embedding

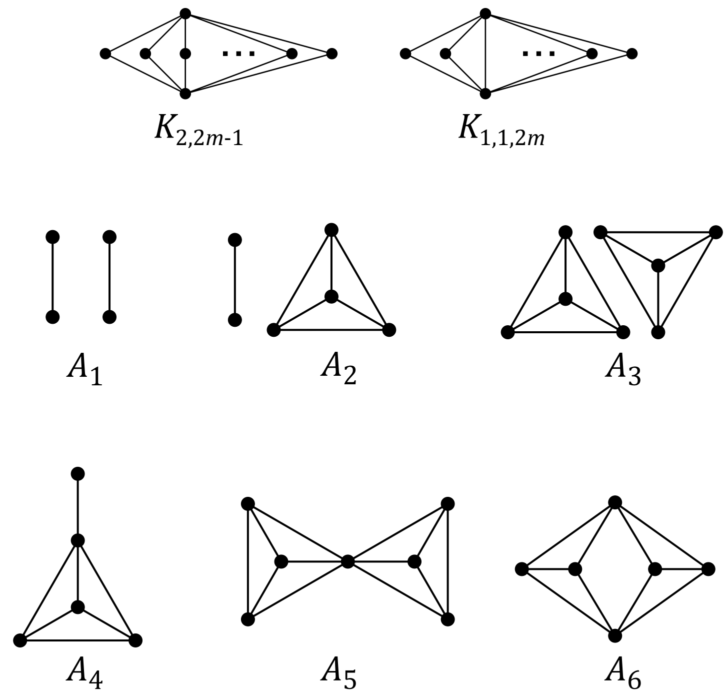

Figure 3.1: The eight graphs

scheme (ρ, λ) with a suitable signature λ. We denote the set of twisted edges associated with this embedding scheme by X. In this situation, we regard this embedding as a re-embedding of G obtained by twisting all edges of X and denote it by $f_X(G)$. In addition, let F_X^2 be the surface where $f_X(G)$ is embedded.

3.2 Facial cycles in planar graphs

Choose two distinct subsets X_1 and X_2 of $E(G)$. Then, there is an edge e belonging to only one of either X_1 or X_2. We may assume that $e \in X_2$. It is easy to check that every facial walk of a 3-connected planar graph embedded on the sphere is a cycle. Thus, there are exactly two facial cycles containing e of G, denoted by C and C'. Let e_1 and e_2 be the edges of C' adjacent to e. Fig. 3.2 presents local neighborhoods around $f_{X_1}(e)$ and $f_{X_2}(e)$. Note that $f_{X_1}(C)$ and $f_{X_2}(C)$ are drawn by bold lines in Fig. 3.2.

For the walk $W = e_1 e e_2$ of G, $f_{X_1}(W)$ constructs a consecutive part of a facial walk in $f_{X_1}(G)$ on $F_{X_1}^2$ but $f_{X_2}(W)$ are not so on $F_{X_2}^2$. Thus, $F_{X_1}^2 \neq F_{X_2}^2$, or $f_{X_1}(G)$ and $f_{X_2}(G)$ are not equivalent. It implies that the choice of a subset X of $E(G)$ uniquely induces the re-embedding $f_X(G)$ of G up to equivalence. Moreover, the total number of inequivalent embeddings of G is $2^{|E(G)|}$, and F_X^2 is homeomorphic to the sphere if and only if X is empty.

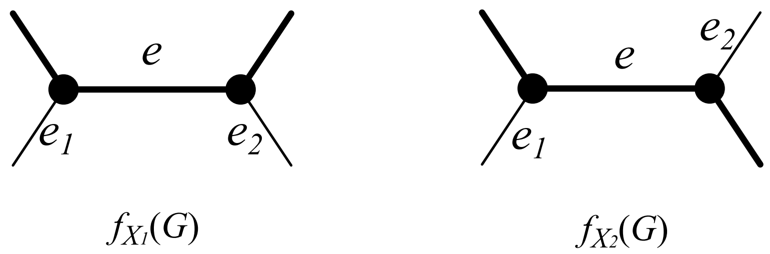

Figure 3.2: The neighborhoods around e in $f_{X_1}(G)$ and $f_{X_2}(G)$

In the situation shown in the right of Fig. 3.2, we say that two cycles $f_{X_2}(C)$ and $f_{X_2}(C')$ *cross* along an edge e. Since G is 3-connected and planar, there are no edges and vertices contained in both C and C' other than e and its endvertices. Thus, $f_{X_2}(C)$ and $f_{X_2}(C')$ cross exactly once. Note that any two cycles in $f_X(G)$ for a given subset X of $E(G)$ do not cross at a single vertex since G is 3-regular.

It has been known that for a facial cycle C of a 3-connected planar graph G, there is only one C-bridge in G (e.g., see [50, p.39–40]).

Lemma 3.4. *Let C be a facial cycle in G and let $f_X(G)$ be a re-embedding of G with a given subset X of $E(G)$. Then, $f_X(C)$ is a non-separating cycle on F_X^2 if and only if $f_X(C)$ has a twisted edge.*

Proof. It is easy to see that if $f_X(C)$ has no twisted edges then it is facial in $f_X(G)$ and hence it separates F_X^2 into two regions.

Suppose that $f_X(C)$ has a twisted edge $f_X(e)$, that is, an edge e of G is in X. Let C' be the other facial cycle of G containing e. As shown in the right of Fig. 3.2, $f_X(C)$ and $f_X(C')$ cross along e and hence two edges of $f_X(C')$ adjacent to $f_X(e)$ are located separately in opposite sides of $f_X(C)$. However, both of these edges are contained in the unique $f_X(C)$-bridge. This implies that $f_X(C)$ does not separate F_X^2. □

3.3 Characterizations of re-embedding structures

In this section, we shall characterize the structures of $f_X(G)$ when F_X^2 is homeomorphic to the projective-plane, the torus or the Klein bottle, and show the following theorems.

Theorem 3.5. *A* 3*-connected* 3*-regular graph embedded on the projective-plane is planar if and only if it has one of the two structures* ($P1$) *and* ($P2$) *shown in Fig.* 3.3.

Theorem 3.6. *A* 3*-connected* 3*-regular graph embedded on the torus is planar if and only if it has one of the two structures* ($T1$), ($T2$) *and* ($T3$) *shown in Fig.* 3.4.

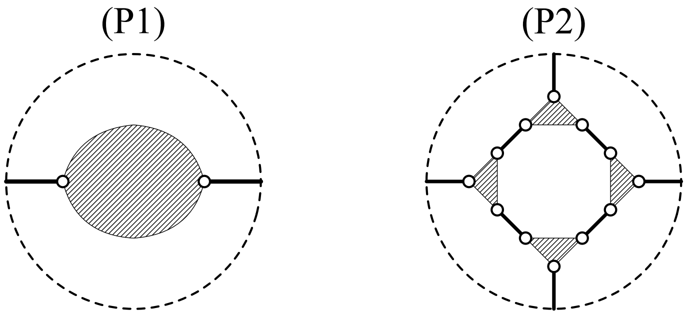

Figure 3.3: Re-embedding structures on the projective-plane

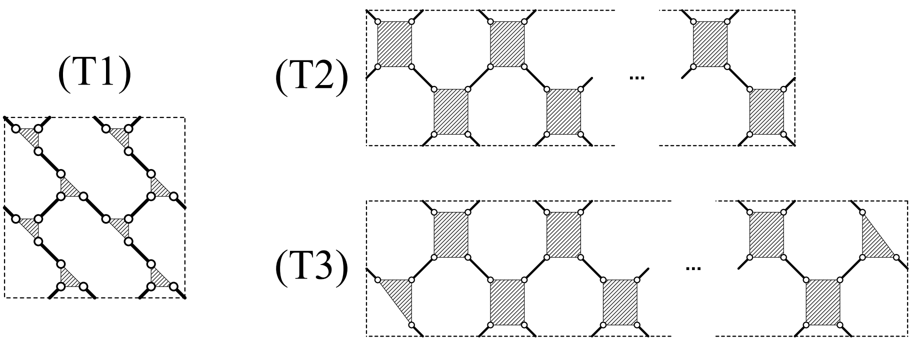

Figure 3.4: Re-embedding structures on the torus

Theorem 3.7. *A 3-connected 3-regular graph embedded on the Klein bottle is planar if and only if it has one of the eight structures* $(K1)$ *to* $(K8)$ *shown in Fig.* 3.5.

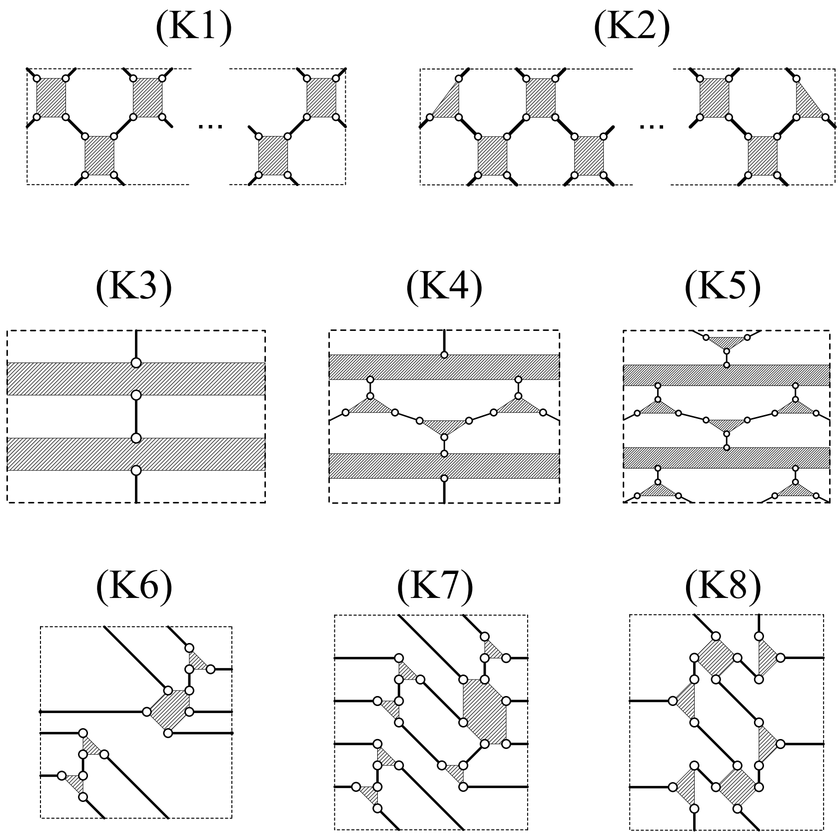

Figure 3.5: Re-embedding structures on the Klein bottle

In Fig. 3.3, each pair of antipodal points on the dashed circle should be identified to recover the projective-plane. Similarly, in Fig. 3.4, to recover the torus, both pairs of opposite sides of dashed rectangle should be identified in the same direction, and in Fig. 3.5, to recover the Klein bottle, the top and bottom sides of the dashed rectangle should be identified in the same direction while the left and right sides should be identified in the opposite direction. In these figures, each of shaded areas corresponds to a component of the graph obtained form the original graph by deleting all edges drawn by bold lines. Some vertices on the boundary of such an area may not be different from each other, that is, the edges drawn by bold lines may not be disjoint. We omit a series of shaded rectangles from $(T2)$, $(T3)$, $(K1)$ and $(K2)$. Both $(T2)$ and $(T3)$ have an even number of shaded rectangles ($(T3)$ may have no shaded rectangle), while both $(K1)$ and $(K2)$ have an odd number of shaded rectangles.

In [53], the re-embedding structure of 2-connected planar graphs on the projective-plane was analyzed in detail (see Theorem 3.2 in [53]), while we focus on 3-connected 3-

regular planar graphs. Then, our re-embedding structure on the projective-plane, shown in Theorem 3.5, is a special case in [53], and follows from it. However, our proof is very simple and important for us to understand other Theorems (e.g. Theorem 3.1). We thus provide a full proof of Theorem 3.5. Moreover, we also construct the re-embedding structures on the torus and the Klein bottle, which is not characterized completely in [52, 53]. One may think that the case of the Klein bottle can be easily obtained from the case of projective-plane, but this is not true. Some structures in Theorem 3.7 (e.g. $(K3)$, $(K4)$ and $(K5)$) can be regarded as simple combinations of the structure in Theorem 3.5, but some are not (e.g. $(K1)$ and $(K2)$).

Let H_X be the subgraph of the dual of G (embedded on the sphere) induced by all edges dual to edges of the given subset X of $E(G)$. Then, there is a vertex of H_X located in the inside of each face of G whose facial cycle has an edge in X. We shall specify what H_X is isomorphic to when F^2_X is homeomorphic to the projective-plane, the torus or the Klein bottle, which are essential ideas to prove not only Theorems 3.5, 3.6 and 3.7 but also Theorems 3.1, 3.2 and 3.3.

First of all, we give a simple condition of H_X when F^2_X is homeomorphic to a non-orientable surface. It has been known that an embedding scheme defines an embedding of a given graph on non-orientable surface if and only if there is a cycle containing an odd number of twisted edges (see [50, p.24–25]). It implies the following lemma.

Lemma 3.8. *For a given subset X of $E(G)$, F^2_X is non-orientable if and only if there is a vertex of odd degree in H_X.*

Proof. If there is a vertex of odd degree in H_X then the facial cycle, denoted by C, corresponding to the vertex contains an odd number of edges in X. Thus, $f_X(C)$ contains an odd number of twisted edges.

Suppose that F^2_X is non-orientable. Then, there is a cycle in $f_X(G)$ containing an odd number of twisted edges, that is, there is a cycle in G containing an odd number of edges in X, denoted by C'. Since C' separates the sphere into two regions, the edges dual to $X \cap E(C')$ form an edge-cut of H_X, whose cardinality is odd. Thus, there is a vertex of odd degree in H_X by the handshaking lemma. □

3.3.1 On the projective-plane

Lemma 3.9. *For a given subset X of $E(G)$, F^2_X is homeomorphic to the projective-plane if and only if H_X is isomorphic to K_2 or K_4.*

Proof. Suppose that F^2_X is homeomorphic to the projective-plane. Any two non-separating simple closed curves on the projective-plane cross at least once.

Let C and C' be any two facial cycles in G each of which has an edge of X. By Lemma 3.4, $f_X(C)$ and $f_X(C')$ are non-separating cycles on F_X^2 and cross at most once. Thus, $f_X(C)$ and $f_X(C')$ cross exactly once and hence C and C' have exactly one common edge in X. It implies that any two vertices in H_X are adjacent to each other, that is, H_X must be a complete graph. Since H_X is planar and induced by edges, H_X must be isomorphic to K_2, K_3 or K_4. However, H_X is not isomorphic to K_3 by Lemma 3.8.

If H_X is isomorphic to K_2 or K_4 then G must have one of the structures shown in Fig. 3.6 (H_X is drawn by squares and dashed lines). Note that K_4 is uniquely embeddable on the sphere and hence the structure is determined uniquely. In Fig. 3.6, we represent edges in X by bold lines and each component of $G-X$ by shaded area together with some vertices, each of which is an end vertex of an edge in X, on its boundary. Note that X does not have to be a matching, that is, two edges in X may have a common end vertex.

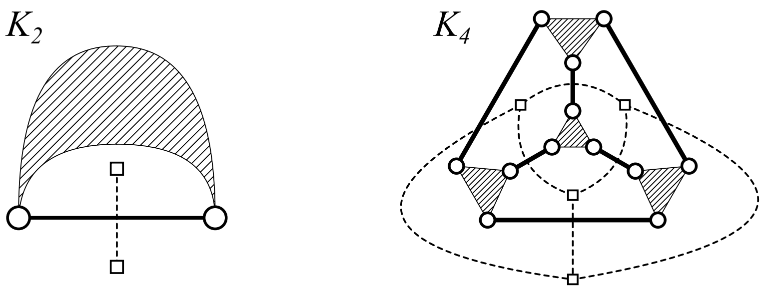

Figure 3.6: Two structures of G with H_X

In the situation shown in Fig. 3.6, by twisting all edges of X, we obtain the re-embedding $f_X(G)$ into the projective-plane shown in Fig. 3.3. □

Proof of Theorem 3.5. Let G be a 3-connected 3-regular planar graph. Any embedding of G on a non-spherical surface can be represented by $f_X(G)$ with a suitable non-empty subset X of $E(G)$. By lemma 3.9, if $f_X(G)$ is a re-embedding of G into the projective-plane then it has one of the structures shown in Fig. 3.3.

Conversely, it is easy to see that if a 3-connected 3-regular graph has one of the structures shown in Fig. 3.3, then it can be embedded on the sphere so that it has one of the structures shown in Fig. 3.6. □

3.3.2 On the torus

Lemma 3.10. *For a given subset X of $E(G)$, F_X^2 is homeomorphic to the torus if and only if H_X is isomorphic to $K_{2,2,2}$, $K_{2,2m}$, or $K_{1,1,2m-1}$ for some positive integer m.*

Proof. Suppose that F_X^2 is homeomorphic to the torus. For two simple closed curves crossing at most once on the torus, they cross if and only if they are not homotopic.

Let C and C' be any two facial cycles in G each of which has an edge of X. By Lemma 3.4, $f_X(C)$ and $f_X(C')$ are non-separating cycles on F_X^2 and cross at most once. Then, $f_X(C)$ and $f_X(C')$ are homotopic if and only if they do not cross, that is, C and C' have no common edge in X, and hence two vertices in H_X corresponding to them are not adjacent. It implies that H_X must be a complete multipartite graph and each partite set corresponds to a non-null homotopy class on the torus.

It is easy to check that any planar complete miltipartite graph is isomorphic to one of the 7 graphs $K_{1,1,1,2}$, $K_{1,1,1,1} = K_4$, $K_{2,2,2}, K_{1,2,2}$, $K_{1,1,n}$, $K_{2,n}$ and $K_{1,n}$ for some natural number n. By Lemma 3.8, any vertex of H_X has even degree. Then, as H_X is planar, H_X is isomorphic to $K_{2,2,2}$, or $K_{2,2m}$ or $K_{1,1,2m-1}$ for some positive integer m.

Conversely, if H_X is isomorphic to $K_{2,2,2}, K_{2,2m}$ or $K_{1,1,2m-1}$, then G must have one of the structure shown in Fig. 3.7. Note that all of $K_{2,2,2}, K_{2,2m}$ and $K_{1,1,2m-1}$ is uniquely embeddable on the sphere if we neglect the labels of their vertices.

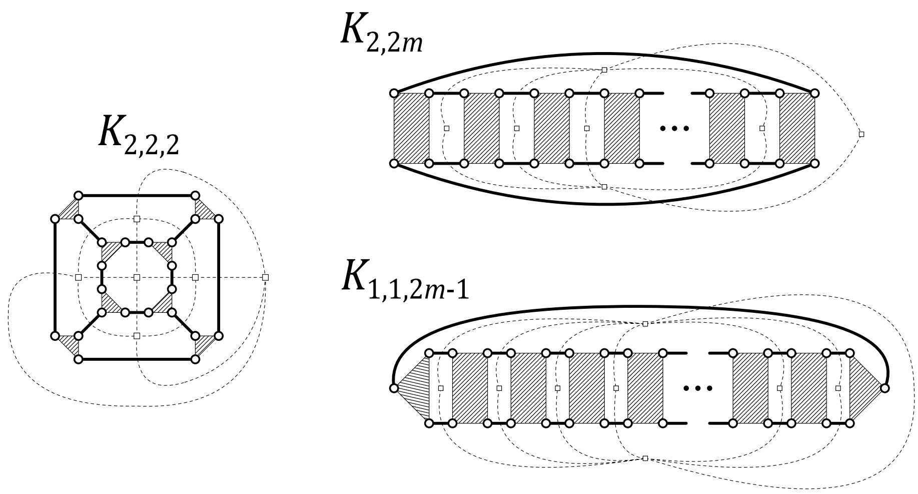

Figure 3.7: Three structures of G with H_X

In the situation shown in Fig. 3.7, by twisting all edges of X, we obtain the re-embedding $f_X(G)$ into the torus shown in Fig. 3.4.

□

Proof of Theorem 3.6. Like Theorem 3.5, this theorem follows immediately from the key lemma; Lemma 3.10. □

3.3.3 On the Klein bottle

Lemma 3.11. *For a given subset X of $E(G)$, F_X^2 is homeomorphic to the Klein bottle if and only if H_X is isomorphic to $K_{2,2m-1}$ or $K_{1,1,2m}$ for some positive integer m, or one of the six graphs A_1 to A_6 shown in Fig.* 3.1.

Proof. Suppose that F_X^2 is homeomorphic to the Klein bottle. There are exactly two mutually disjoint non-separating simple closed 1-sided curves and exactly one non-separating 2-sided curve on the Klein bottle up to homotopy.

Let C and C' be any two facial cycles in G each of which has an edge of X. Then, $f_X(C)$ and $f_X(C')$ are non-separating cycles on F_X^2 and cross at most once. We first assume that both $f_X(C)$ and $f_X(C')$ are 1-sided. Then, $f_X(C)$ and $f_X(C')$ cross if and only if they are homotopic. Second, we assume that one of $f_X(C)$ and $f_X(C')$ is 1-sided and the other is 2-sided. Then, they cross. Third, we assume that both $f_X(C)$ and $f_X(C')$ are 2-sided. Then they are homotopic and hence do not cross.

The vertex of H_X corresponding to C has odd degree if and only if $f_X(C)$ is 1-sided. Thus, the facts mentioned in the last paragraph imply that H_X has the following conditions. (1) The vertices of odd degree in H_X induce a graph having at most two components each of which is isomorphic to a complete graph. (2) Any vertex of even degree and any vertex of odd degree are adjacent. (3) The vertices of even degree in H_X are independent, that is, any pair of such vertices are not adjacent.

Let V_{odd} (resp. V_{even}) be the set of vertices of odd (resp. even) degree in H_X. Since V_{even} is a independent set and any vertex of V_{even} is adjacent to each vertex of V_{odd}, $|V_{odd}|$ is even.

Case 1: V_{odd} induces a complete graph K_m. As H_X is planar, $m = 2$ or 4.

Subcase 1a: $m = 2$. It is easy to see that $|V_{even}|$ is even. Then, H_X is isomorphic to $K_{1,1,2k}$ with some non-negative integer k. However, if H_X is isomorphic to $K_{1,1,0} = K_2$ then F_X^2 is homeomorphic to the projective-plane by Lemma 3.9. Then, $k \geq 1$.

Subcase 1b: $m = 4$. If there is at least one vertex in V_{even} then H_X is not planar since it contains K_5 as a subgraph. Moreover, if V_{even} is empty, then H_X is isomorphic to K_4 and hence F_X^2 is homeomorphic to the projective-plane by Lemma 3.9. Therefore, $m \neq 4$.

Case 2: V_{odd} induces two disjoint complete graphs K_m and K_n. Then, we have $m + n = 2, 4, 6, 8$.

Subcase 2a: $m + n = 2$, that is, $m = n = 1$. In this situation, H_X is isomorphic to $K_{2,2k-1}$ with some positive integer k.

Subcase 2b: $m + n = 4$, that is, $m = n = 2$ or $m = 1, n = 3$. Suppose that $m = n = 2$. If $|V_{even}| \geq 3$ then H_X is not planar since it contains $K_{3,3}$ as a subgraph. If $|V_{even}| = 1$ then each vertex of H_X has even degree, which contradicts Lemma 3.8. If $|V_{even}| = 0$ or 2 then H_X corresponds to A_1 or A_6, respectively.

Suppose that $m = 1$ and $n = 3$. If $|V_{even}| \geq 3$ then H_X is not planar since it contains $K_{3,3}$ as a subgraph. If $|V_{even}| = 0, 2$ then each vertex of H_X has even degree, which contradicts Lemma 3.8. If $|V_{even}| = 1$ then H_X corresponds to A_4.

Subcase 2c: $m + n = 6$, that is, $m = n = 3$ or $m = 2, n = 4$. Suppose that $m = n = 3$. If $|V_{even}| \geq 3$ then H_X is not planar since it contains $K_{3,3}$ as a subgraph. If $|V_{even}| = 0, 2$ then each vertex of H_X has even degree, which contradicts Lemma 3.8. If $|V_{even}| = 1$ then H_X corresponds to A_5.

Suppose that $m = 2, n = 4$. If $|V_{even}| \geq 1$ then H_X is not planar since it contains K_5 as a subgraph. If $|V_{even}| = 0$ then H_X corresponds to A_2.

Subcase 2d: $m + n = 8$, that is, $m = n = 4$. If $|V_{even}| \geq 1$ then H_X is not planar since it contains K_5 as a subgraph. If $|V_{even}| = 0$ then H_X corresponds to A_3.

According to the above results, H_X is isomorphic to $K_{2,2m-1}$ or $K_{1,1,2m}$ for some positive integer m, or H_X is isomorphic to one of the six graphs A_1 to A_6.

Conversely, if H_X is isomorphic to $K_{2,2m-1}$ or $K_{1,1,2m}$ for some positive integer m, or H_X is isomorphic to one of the six graphs A_1 to A_6, then G must have one of the structure shown in Fig. 3.8. Note that all graphs shown in Fig. 3.1 are uniquely embeddable on the sphere if we neglect the labels of their vertices.

In the situation shown in Fig. 3.8, by twisting all edges of X, we obtain the re-embedding $f_X(G)$ into the Klein bottle shown in Fig. 3.5. □

Proof of Theorem 3.7. Like Theorem 3.5, this theorem follows immediately from the key lemma; Lemma 3.11. □

3.3.4 Proof of Theorems

Theorems 3.1, 3.2 and 3.3 immediately follow from Lemmas 3.9, 3.10 and 3.11, respectively.

Proof of Theorems 3.1, 3.2 and 3.3. Let G be a 3-connected 3-regular planar graph. Any embedding of G on any surface is equivalent to an embedding $f_X(G)$ associated with a suitable subset X of $E(G)$. Moreover, such X is unique. Thus, Lemmas 3.9, 3.10 and 3.11 imply Theorems3.1, 3.2 and 3.3, respectively. □

3.4 Inequivalent embeddings

In this section, we first give explicit bounds for the number of inequivalent embeddings of G on each of the projective-plane, the torus and the Klein bottle. After that, we propose algorithms for enumerating and counting these embeddings.

3.4.1 The number of inequivalent embeddings

Based on Theorems 3.1, 3.2 and 3.3, we show the following three results.

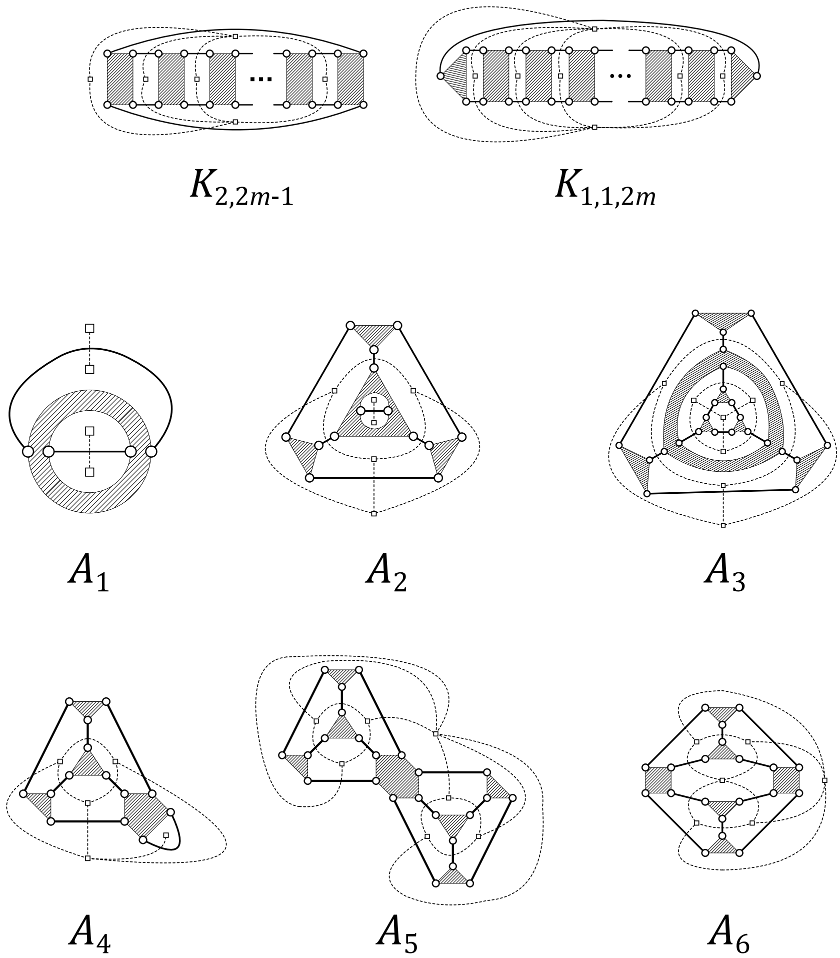

Figure 3.8: Eight structures of G with H_X

Theorem 3.12. *A* 3*-connected* 3*-regular planar graph with* n *vertices has at least* $\frac{3}{2}n$ *and at most* $2n - 1$ *inequivalent embeddings on the projective-plane.*

Theorem 3.13. *A* 3*-connected* 3*-regular planar graph with* $n \geq 5$ *vertices has at least* $\frac{5}{2}n$ *inequivalent embeddings on the torus.*

Theorem 3.14. *A* 3*-connected* 3*-regular planar graph with* n *vertices has at least* $\frac{3}{8}n(3n+2)$ *inequivalent embeddings on the Klein bottle.*

Before we prove these theorems, we consider a situation where the dual of G embedded on the sphere has many subgraphs isomorphic to K_4, which is useful for showing the upper bound of Theorem 3.12 and characterizing graphs attaining this bound.

A graph embedded on the sphere is 3-connected and 3-regular if and only if the dual is a triangulation on the sphere. For a triangulation T, a 3-*vertex addition* is an operation of adding a vertex into a face Δ of T and joining the new vertex to the vertices on the boundary of Δ.

Lemma 3.15. *Every triangulation T on the sphere has at most $(|V(T)|-3)$ subgraphs isomorphic to K_4. In particular, T attains the upper bound if and only if T is obtained from K_4 embedded on the sphere by a sequence of 3-vertex additions.*

Proof. The proof is by induction on the number of vertices.

If $|V(T)| = 4$ then T is K_4 itself and hence the result clearly holds. Thus, we assume $|V(T)| \geq 5$.

If T has no separating cycle of order 3 then T has no subgraph isomorphic to K_4. Thus, we may assume that T has a separating cycle C of order 3, which separates the sphere into two regions, denoted by R_1 and R_2. Let T_1 (resp. T_2) be the subgraph of T induced by the vertices lying on R_1 (resp. R_2) with its boundary. Then, both of T_1 and T_2 is also a triangulation on the sphere. Note that $T_1 \cap T_2 = C$ and $T_1 \cup T_2 = G$. For any vertices $x \in V(T_1) \setminus V(C)$ and $y \in V(T_2) \setminus V(C)$, there is no edge whose endvertices are x and y, and hence there are no subgraphs of T isomorphic to K_4 having both x and y. Thus, the number of subgraphs of T isomorphic to K_4 is at most

$$(|V(T_1)|-3) + (|V(T_2)|-3) = (|V(T)|+3) - 6 = |V(T)| - 3.$$

Next, we characterize triangulations attaining this upper bounds. Let $\tilde{T}$ be a triangulation on the sphere obtained from T by one operation of a 3-vertex addition and $\tilde{v}$ be the additional vertex of $\tilde{T}$. There is exactly one subgraph of $\tilde{T}$ isomorphic to K_4 including $\tilde{v}$. If T is obtained from K_4 by a sequence of 3-vertex addition then T has has exactly $|V(T)|-3$ subgraphs isomorphic to K_4, and hence $\tilde{T}$ has exactly $|V(T)|-2 = |V(\tilde{T})|-3$ subgraphs isomorphic to K_4.

Conversely, suppose that T has exactly $|V(T)|-3$ subgraphs isomorphic to K_4. We may assume that $|V(T)| \geq 5$ and T has a separating cycle C of order 3. Then, T_1 and T_2, which are defined in the same way as above, must have exactly $|V(T_1)|-3$ and $|V(T_2)|-3$ subgraphs isomorphic to K_4, respectively, and hence both are obtained from K_4 by a sequence of 3-vertex additions.

Let T' be a triangulation on the sphere obtained from K_4 by a sequence of 3-vertex addition but not K_4. It is easy to check that any two vertices of degree 3 are not adjacent in T'. Thus, an operation of a 3-vertex addition from T' will not decrease the number of vertices of order 3, and hence T' has at least two vertices of degree 3.

The above facts imply that we can obtain K_4 from T_2 by deleting a vertex of degree 3 without deleting the vertices on C. By applying these operations to T and deleting the last vertex from R_2, we have just obtained T_1. Therefore, T is also obtained from K_4 by a sequence of 3-vertex additions. □

Proof of Theorem 3.12. Let G be a 3-connected 3-regular planar graph embedded on the sphere with n vertices and G^* be its dual. Choose an edge e of G and put $X = \{e\}$.

Then, H_X is isomorphic to K_2 and hence $f_X(G)$ is embedded on the projective-plane by Lemma 3.9. It implies that G has at least $|E(G)|$ inequivalent embeddings on the projective-plane. Since G is 3-regular, we have $|E(G)| = \frac{3}{2}n$.

By Lemma 3.15, there are at most $|V(G^*)| - 3$ subgraphs of G^* isomorphic to K_4. By Euler's formula, $|V(G^*)| = (|V(G)| + 4)/2 = \frac{n+4}{2}$ and hence G^* has at most $\left(\frac{n+4}{2} - 3\right)$ subgraphs isomorphic to K_4. Thus, by Theorem 3.1, G has at most $\frac{3}{2}n + \left(\frac{n}{2} - 1\right) = 2n - 1$ inequivalent embeddings on the projective-plane. $\square$

Not only 3-connected 3-regular planar graphs, any 2-connected graph G has $|E(G)|$ inequivalent embeddings on the projective-plane by twisting each edge of G. Then, the assumptions on 3-connectivity and 3-regularity are not necessary in the lower bound in Theorem 3.12. However, these assumptions are clearly necessary in the upper bound in Theorem 3.12. In fact, we can easily construct non-3-connected or non-3-regular 2-connected planar graphs having exponentially many inequivalent embeddings on the projective-plane.

Proof of Theorem 3.13. Let G be a 3-connected 3-regular planar graph embedded on the sphere with at least 5 vertices and G^* be its dual. Every face of G^* is bounded by a cycle of order 3 ($= K_{1,1,1}$) and every edge of G^* forms a chord of a cycle of order 4 ($= K_{2,2}$) since it is incident with just two triangle faces. If G is not isomorphic to K_4, then there are no other chords in this cycle. Thus, G^* has at least $|V(G)|$ cycles of order 3 and at least $|E(G)|$ cycles of order 4 as subgraphs. As G is 3-regular, $|E(G)| = 3n/2$. Then, by Theorem 3.2, G has at least $n + 3n/2 = 5n/2$ inequivalent embeddings on the torus. $\square$

Note that a 3-connected 3-regular planar graph with at most 4 vertices must be isomorphic to K_4, which has exactly $7 \leq 5 \cdot 4/2$ inequivalent embeddings on the torus.

Proof of Theorem 3.14. Let G be a 3-connected 3-regular planar graph embedded on the sphere with n vertices and G^* be its dual. Choose two distinct edges e_1 and e_2 of G and put $X = \{e_1, e_2\}$. Then, H_X is isomorphic to $K_{2,1}$ or A_1. By Lemma 3.11, $f_X(G)$ is embedded on the Klein bottle.

In addition, we try to find subgraphs isomorphic to $K_{1,1,2}$ in G^*. Since G^* is a triangulation on the sphere, every edge e^* is incident with just two triangle faces. The five edges bounding these faces induce subgraphs of G^* isomorphic to $K_{1,1,2}$. Then, G^* has $|E(G^*)|$ subgraphs isomorphic to $K_{1,1,2}$.

These results imply that, by Theorem 3.3, G has at least $\binom{|E(G)|}{2} + |E(G^*)| = \frac{3}{8}n(3n+2)$ inequivalent embeddings on the Klein bottle. $\square$

3.4.2 Examples

First, we characterize the graphs attaining the lower bound of Theorem 3.12. By Theorem 3.12, the following clearly holds.

Corollary 3.16. *A* 3*-connected* 3*-regular planar graph G *with* n *vertices has exactly* $\frac{3}{2}n$ *inequivalent embeddings on the projective-plane if and only if the dual of* G *embedded on the sphere has no subgraph isomorphic to* K_4*.* □

By this corollary, we show the following two families of graphs attaining the lower bound of Theorem 3.12.

Proposition 3.17. *A* 3*-connected* 3*-regular planar graph* G *with* $n \geq 5$ *vertices has exactly* $\frac{3}{2}n$ *inequivalent embeddings on the projective-plane if* G *is bipartite or cyclically* 4*-edge-connected.*

Proof. We only have to show that the dual G^* of G embedded on the sphere has no subgraph isomorphic to K_4.

If G is bipartite then degree of each vertex of G^* is even, that is, G^* is a *even* triangulation. It is well-known that every even triangulation on the sphere is (vertex) 3-colorable and hence has no subgraph isomorphic to K_4.

If G is cyclically 4-edge-connected and $n \geq 5$, then G^* has no separating cycle of order 3 and hence has no subgraph isomorphic to K_4. □

Second, we characterize the graphs attaining the upper bound of Theorem 3.12. Towards this goal, we introduce a transforming operation of G, which corresponds to a 3-vertex addition in the dual of G.

Let v be a vertex of G, and u_1, u_2 and u_3 be vertices adjacent to v. A *truncation* of a vertex v in G is an operation of replacing a small part around v with a cycle of order 3 shown in Fig. 3.9; delete v and add new vertices v_1, v_2 and v_3 together with six edges $u_1v_1, u_2v_2, u_3v_3, v_1v_2, v_2v_3$ and v_3v_1.

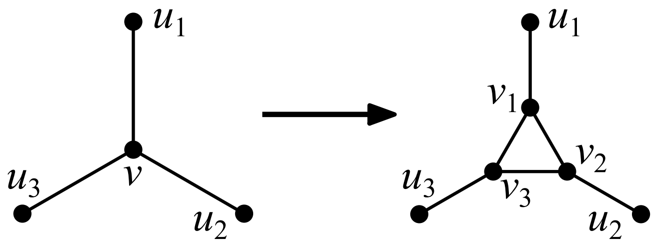

Figure 3.9: Truncation of a vertex

The resulting graph, denoted by G', is also 3-connected, 3-regular and planar. The dual $(G')^*$ is obtained from the dual G^* of G by a 3-vertex addition. Then, the following clearly holds by Lemma 3.15.

Corollary 3.18. *A 3-connected 3-regular planar graph G with n vertices has exactly $2n-1$ inequivalent embeddings on the projective-plane if and only if G is obtained from K_4 by a sequence of truncations.*

Third, we provided graphs attaining the lower bounds of Theorems 3.13 and 3.14.

Corollary 3.19. *A 3-connected 3-regular planar graph G with $n \geq 5$ vertices has exactly $\frac{5}{2}n$ inequivalent embeddings on the torus if and only if G is cyclically 5-edge-connected.*

Corollary 3.20. *A 3-connected 3-regular planar graph G with n vertices has exactly $\frac{3}{8}n(3n+2)$ inequivalent embeddings on the Klein bottle if G is cyclically 5-edge-connected.*

Proof of Corollaries 3.19 and 3.20. Suppose that G is cyclically 5-edge-connected. In each of the six structures shown in Fig. 3.7 and Fig. 3.8 corresponding to $K_{2,2,2}, A_2, A_3, A_4, A_5$ and A_6, we can easily find a set of at most four edges (drawn by bold lines) such that the graph obtained from G by deleting these edges has at least two components having a cycle. (The shaded annular area in A_2 must have a cycle.) Then, G has none of these six structures and hence G^* has no subgraph isomorphic to one of the six graphs $K_{2,2,2}, A_2, A_3, A_4, A_5$ and A_6.

Suppose that G^* has a subgraph isomorphic to $K_{2,n}$ or $K_{1,1,n}$ with $n \geq 3$. Then, G has one of the four structures shown in Fig. 3.7 and Fig. 3.8 corresponding $K_{2,2m}, K_{1,1,2m-1}, K_{2,2m-1}$ and $K_{1,1,2m}$. In both case, G has at least three shaded areas, and since G is cyclically 5-edge-connected, all shaded areas except for at most one have no cycle. Hence, there are two consecutive shaded areas having no cycle in G, one of which is rectangle. These shaded areas together with edges joining them form one of the three subgraphs shown in Fig. 3.10.

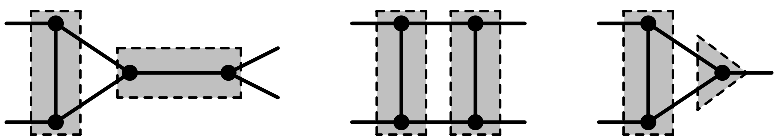

Figure 3.10: Subgraphs formed by two consecutive shaded areas

These subgraphs have a cycle and can be separated from G by deleting at most four edges. This implies that G has at most one more shaded rectangle and that it contains no cycle. Hence it must be one of the three graphs shown in Fig. 3.11. These graphs have

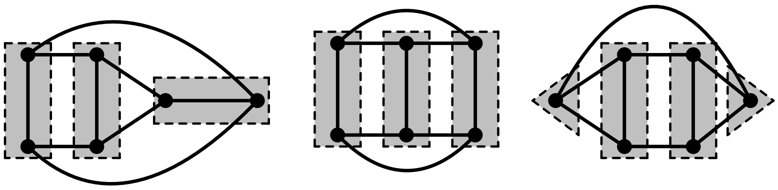

Figure 3.11: Three graphs

distinct structures but each graph is the same as the others. However, this graph is not cyclically 5-edge-connected, a contradiction.

Therefore, G^* has no subgraph isomorphic to $K_{2,n}$ or $K_{1,1,n}$ with $n \geq 3$, and hence G has only $\frac{5}{2}n$ inequivalent embeddings mentioned in the proof of Theorem 3.13 on the torus, and only $\frac{3}{8}n(3n+2)$ inequivalent embeddings mentioned in the proof of Theorem 3.14 on the Klein bottle.

Suppose G is not cyclically 5-edge-connected. Since G is 3-connected, there are three or four edges of G whose removal results in a disconnected graph having exactly two components, both of which contain a cycle. Let X be such edges. Then, H_X is isomorphic to $K_{1,1,1} = K_3$ or $K_{2,2}$, and hence $f_X(G)$ is embedded on the torus. However, this re-embedding conforms to none of re-embeddings mentioned in the proof of Theorem 3.13. Thus, the number of inequivalent embeddings of G on the torus is more than $\frac{5n}{2}$. □

A graph attaining the lower bound of Theorem 3.14 is not necessarily cyclically 5-edge-connected. For example, the following graph shown in Fig. 3.12 is such a graph. We can construct infinitely many such graphs but we omit this here.

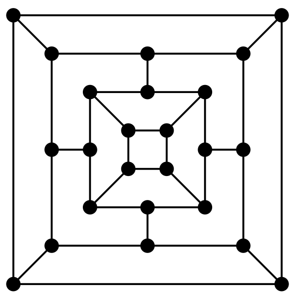

Figure 3.12: A graph attaining the lower bound of Theorem 3.14

Finally, we show graphs having exponentially many inequivalent embeddings on the torus and the Klein bottle.

Proposition 3.21. *For a 3-connected 3-regular planar graph G with n vertices, if the dual G^* of G embedded on the sphere has a subgraph isomorphic to $K_{1,1,m}$ with a positive* *integer m then G has at least $2^m - 1$ inequivalent embeddings on each of the torus and the Klein bottle.*

Proof. For the complete tripartite graph $K_{1,1,m}$ with partite sets $V_1 = \{u\}, V_2 = \{v\}$ and $V_3 = \{a_1, a_2, \ldots, a_m\}$, let S be any non-empty subset of V_3. A subgraph induced by $V_1 \cup V_2 \cup S$ is isomorphic to $K_{1,1,|S|}$. Deleting an edge uv from this subgraph, we obtain a subgraph isomorphic to $K_{2,|S|}$.

It implies that if G^* has a subgraph isomorphic to $K_{1,1,m}$, then we can find $2(2^m - 1)$ subgraphs in G^* isomorphic to $K_{1,1,k}$ or $K_{2,k}$ for some positive integer k. Moreover, in these subgraphs, the number of subgraphs isomorphic to $K_{1,1,k}$ is the same as the one isomorphic to $K_{2,k}$ for any $1 \leq k \leq m$. Then, by Theorems 3.2 and 3.3, G has at least $2^m - 1$ inequivalent embeddings on each of the torus and the Klein bottle. □

3.4.3 Algorithms

By Theorem 3.12, the number of inequivalent embeddings of G on the projective-plane is $O(n)$ with respect to the number n of vertices of G. In fact, we can easily enumerate these embeddings in polynomial-time with respect to n. Note that we regard an enumeration of embedding schemes of a graph as one of the embeddings of the graph.

Theorem 3.22. *There is a polynomial time algorithm for enumerating inequivalent embeddings of a 3-connected 3-regular planar graph on the projective-plane.*

Proof. Let G be a 3-connected 3-regular planar graph. The embedding of G on the sphere, its dual G^* and another embedding $f_X(G)$ with a given subset X of $E(G)$ can be obtained in polynomial time. Then, we only have to find subgraphs isomorphic to K_2 or K_4 in G^* by Lemma 3.9, which can be done in polynomial time. □

On the other hand, there are 3-connected 3-regular planar graphs having exponentially many inequivalent embeddings on the torus and the Klein bottle by Proposition 3.21. Then, we cannot enumerate inequivalent embeddings of such a graph on the torus or the Klein bottle in polynomial time. However, we shall give a "polynomial delay" algorithm for enumerating them. An enumeration algorithm is said to be *polynomial delay* if the maximum computation time between two consecutive outputs is polynomial in the input size.

For the complete miltipartite graphs $K_{2,m+2}$ and $K_{1,1,m+1}$ with any positive integer m, there are exactly two vertices whose degree is not two. We call them *apex vertices.*

Theorem 3.23. *There is a polynomial delay algorithm for enumerating inequivalent embeddings of a 3-connected 3-regular planar graph on each of the torus and the Klein bottle.*

Proof. Let G be a 3-connected 3-regular planar graph. We enumerate inequivalent embeddings of G on the torus and the Klein bottle simultaneously. Like the proof of Theorem 3.22, we only have to find subgraphs isomorphic to $K_{2,m+1}$ or $K_{1,1,m}$ for any positive integer m, or one of the seven graphs $A_1, \cdots, A_6$ and $K_{2,2,2}$. (Every time we find such a subgraph, output a embedding corresponding to it)

First, we find subgraphs isomorphic to one of the nine graphs $A_1, \cdots, A_6$, $K_{1,1,1} = K_3, K_{2,2}$ and $K_{2,2,2}$ in G^*, which can be done in polynomial time. Second, for a pair of vertices u and v of G^*, enumerate vertices adjacent to both of them. Third, enumerate subgraphs isomorphic to $K_{2,m+2}$ or $K_{1,1,m+1}$ whose apex vertices are u and v. In such a subgraph, all non-apex vertices are already enumerated in the second step. Thus, the third step can be done in polynomial delay time.

To repeat the second and third step for any pair of vertices, in the end, we have just enumerated all subgraphs isomorphic to $K_{2,m+2}$ or $K_{1,1,m+1}$ in G^*. □

We can calculate the total number of inequivalent embeddings of G on each of the projective-plane, the torus and the Klein bottle in polynomial time by a simple improvement of the above algorithms.

Corollary 3.24. *There is a polynomial time algorithm for counting the number of inequivalent embeddings of a* 3*-connected* 3*-regular planar graph on each of the projective-plane, the torus and the Klein bottle.*

Proof. The projective-planar case clearly holds. We only have to count all the embeddings enumerated in the algorithm of Theorem 3.22. Thus, we may consider embeddings on the torus and the Klein bottle.

Let G be a 3-connected 3-regular planar graph. On the basis of the algorithm in Theorem 3.23, we shall count subgraphs of G^* isomorphic to $K_{2,2,2}, K_{2,2m}$ or $K_{1,1,2m-1}$, which correspond to embeddings on the torus, and count subgraphs isomorphic to $K_{2,2m-1}$ or $K_{1,1,2m}$, or one of the six graphs A_1 to A_6, which correspond to embeddings on the Klein bottle.

Like the proof of Theorem 3.23, we first count the number of subgraphs of G^* isomorphic to one of the three graphs $K_{2,2,2}, K_{2,2}$ and $K_{1,1,1} = K_3$, and denote it by N_T. Similarly, we count the number of subgraphs isomorphic to one of the six graphs A_1 to A_6, and denote it by N_K.

Second, for the pair of vertices u and v in the proof of Theorem 3.23, assume that exactly k vertices are adjacent to both u and v. Let $f_T(u,v)$ (resp. $f_K(u,v)$) be the number of subgraphs isomorphic to $K_{2,2m+2}$ or $K_{1,1,2m+1}$ (resp. $K_{2,2m-1}$ or $K_{1,1,2m}$) for any positive integer m whose apex vertices are u and v. If u and v are adjacent to each other in G^*, then we have

$$f_T(u,v) = \sum_{i=3}^{k} \binom{k}{i} = 2^k - \frac{k(k-1)}{2} - k - 1 = 2^k - \frac{k^2+k+2}{2},$$

$$f_K(u,v) = \sum_{i=1}^{k} \binom{k}{i} = 2^k - 1.$$

Otherwise,

$$f_T(u,v) = \sum_{i=2}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2i} = 2^{k-1} - \frac{k(k-1)}{2} - 1 = 2^{k-1} - \frac{k^2-k+2}{2},$$

$$f_K(u,v) = \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{2i-1} = 2^{k-1}.$$

Add the sum of $f_T(u,v)$ (resp. $f_K(u,v)$) taken over all pairs of vertices u, v to N_T (resp. N_K). These are the total numbers of inequivalent embeddings of G on the torus and the Klein bottle. Thus, we can obtain this number in polynomial time. □

3.5 Remarks

In this chapter, we have shown the re-embedding structures of a 3-connected 3-regular planar graph G on the projective-plane, the torus and the Klein bottle. These structures enable us to count inequivalent embeddings of G on each surface easily. These results allow the computation and the study of the *genus distributions* of a large family of graphs. We denote the number of inequivalent embeddings of a graph G on the orientable surface of genus k (resp. the non-orientable surface of genus h) by $g_G(k)$ (resp. $\tilde{g}_G(h)$). The genus distribution (resp. non-orientable genus distribution) of G is defined as the sequence $g_G(0), g_G(1), g_G(2), \ldots$ (resp. $\tilde{g}_G(0), \tilde{g}_G(1), \tilde{g}_G(2), \ldots$). The topic of genus distributions was introduced by Gross and Furst [30] and studied in various papers; see for example [17, 18, 30–33]. Whether the genus distribution of every graph is log-concave is an interesting problem conjectured in [31], and still remains to be solved. From the genus distribution's point of view, we give explicit bounds for $g_G(1), \tilde{g}_G(1)$ and $\tilde{g}_G(2)$ of a 3-connected 3-regular planar graph G and algorithms for calculating them.

In order to extend our result to surfaces with higher genera, we should show the complete lists of re-embedding structures of G on these surfaces like Theorems 3.5, 3.6 and 3.7. However, we think that there are a large number of re-embedding types even on an orientable surface with genus 2 or a non-orientable surface with genus 3. Then, it seems to be difficult to give such complete lists without additional assumptions.

Chapter 4

2-Regular Diplanar Digraphs on Surfaces

In this Chapter, we focus on embeddings of strongly 2-edge-connected 2-regular "diplanar" digraphs on surfaces.

An *embedding* of an Eulerian digraph D on a surface F^2 is defined as a 2-cell embedding of its underlying graph on F^2 with a property that each face is bounded by a directed closed walk. Hence, in- and out-edges alternate in the rotation around each vertex of an embedded digraph. An Eulerian digraph D is *diplanar* if D has an embedding on the sphere (or the plane).

In Section 4.1, we indicate the close relationship between an embedding of a 3-connected graph on a surface and one of a strongly 2-edge-connected digraph on the surface, which enables us to give a simple proof of Theorem 1.5, which we call *the directed version of Whitney Theorem* or simply *Directed Whitney Theorem*.

Moreover, we focus on embeddings of diplanar digraphs on non-spherical surfaces. In Chapter 3, we completely characterized structures of embeddings of 3-connected 3-regular planar graphs on the projective-plane, the torus and the Klein bottle, which are useful for enumerating such embeddings and counting its total number. In Section 4.2, we extend the above result to embeddings of digraphs, that is, we characterize structures of embeddings of strongly 2-edge-connected 2-regular diplanar digraph on the projective-plane, the torus and the Klein bottle. In addition to this, we evaluate the number of such embeddings in Section 4.3.

4.1 Simple proof of Directed Whitney Theorem

For an embedded graph G associated with a given embedding scheme, we call an operation of replacing the signatures of some edges with these inverses *twisting* these edges, and the embedding associated with the resulting embedding scheme the *re-embedding* of G obtained by twisting these edges.

Now we introduce a transforming operation of a 4-regular graph. Let G be a 4-regular graph embedded on a surface, and v be a vertex of G adjacent to u_1, u_2, u_3 and u_4 such that the rotation around v corresponds this order. A *truncation* of a vertex v is an operation of replacing a small part around v with a facial cycle of order 4, called a *truncated cycle*, shown in Fig. 4.1; delete v and add four vertices v_1, v_2, v_3 and v_4 together with edges v_iu_i and v_iv_{i+1} with indices taken modulo 4.

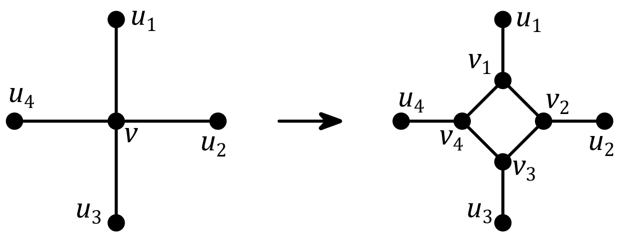

Figure 4.1: Truncation of a vertex

The *truncated graph* of a 4-regular graph G embedded on a surface is the embedded graph obtained from G by truncating all vertices, denoted by $tr(G)$. We call the edges of $tr(G)$ contained in a truncated cycle *truncated edges* and the others *original edges.* It is clear that $tr(G)$ is 3-regular and each vertex is incident with two truncated edges and one original edge. Note that the truncating operation depends on the rotation around a vertex. That is, if two embeddings $f_1(G)$ and $f_2(G)$ of G on a surface are inequivalent then the truncated graphs $tr(f_1(G))$ and $tr(f_2(G))$ may not be isomorphic to each other. However, we do not have to consider such situation when G is the underlying graph of a connected 2-regular digraph D.

Lemma 4.1. *For any two embeddings $f_1(D)$ and $f_2(D)$ of a connected 2-regular digraph D embedded on a surface with its underlying graph G, the truncated graphs $tr(f_1(G))$ and $tr(f_2(G))$ are isomorphic to each other.*

Proof. Since G is the underlying graph of D, which is 2-regular, there are only two possible rotations around each vertex of G, and one of them is the inverse of the other. This implies that the sets of truncated cycles of $tr(f_1(G))$ and $tr(f_2(G))$ are same, and hence $tr(f_1(G))$ and $tr(f_2(G))$ are isomorphic to each other. □

Lemma 4.2. *If a connected 2-regular digraph D embedded on a surface is strongly 2-edge-connected then the truncated graph of the underlying graph of D is 3-connected.*

Proof. It is easy to see that if D is strongly 2-edge-connected then the underlying graph G is 2-connected and 4-edge-connected. Suppose that the truncated graph $tr(G)$ of G is not 3-connected. Since $tr(G)$ is 3-regular, there are two edges of $tr(G)$ forming an edge-cut. If one of them is truncated then the other must be contained in the same truncated cycle. It implies that the vertex of G corresponding to this truncated cycle is a cut vertex, which contradicts the 2-connectivity of G. Hence, both edges are not truncated. However, it implies that they form an edge-cut of G, which contradicts the 4-edge-connectivity of G. □

Using Lemmas 4.1 and 4.2, we can prove Theorem 1.5 easily.

Proof of Theorem 1.5. Let D be a strongly 2-edge-connected 2-regular diplanar digraph with its underlying graph G, and $f_1(D)$ and $f_2(D)$ be two embeddings of D on the sphere. By Lemma 4.1, the two truncated graphs $tr(f_1(G))$ and $tr(f_2(G))$ of $f_1(G)$ and $f_2(G)$, respectively, are isomorphic. Moreover, by Lemma 4.2, they are 3-connected and hence equivalent to each other (by Whitney's theorem). This implies that $f_1(D)$ and $f_2(D)$ are equivalent. □

4.2 Embeddings on non-spherical surfaces

In this section, we expand Theorems 3.5, 3.6 and 3.7 to embeddings of strongly 2-edge-connected 2-regular diplanar digraphs:

Theorem 4.3. *A strongly 2-edge-connected 2-regular digraph embedded on the projective-plane is diplanar if and only if it has the structure shown in Fig. 4.2.*

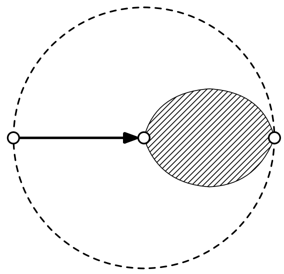

Figure 4.2: Directed embedding structures on the projective-plane

Theorem 4.4. *A strongly 2-edge-connected 2-regular digraph embedded on the torus is diplanar if and only if it has the structure shown in Fig. 4.3.*

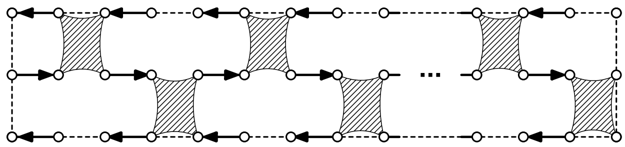

Figure 4.3: Directed embedding structures on the torus

Theorem 4.5. *A strongly* 2*-edge-connected* 2*-regular digraph embedded on the Klein bottle is diplanar if and only if it has one of the two structures shown in Fig.* 4.4*.*

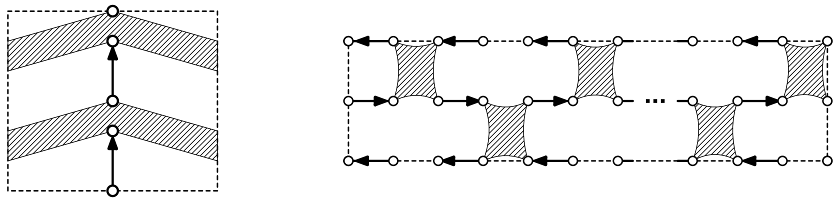

Figure 4.4: Directed embedding structures on the Klein bottle

As with Figures 3.3, 3.4 and 3.5, in these figures, each of shaded areas corresponds to a component of the digraph obtained form the original digraph by deleting all arcs drawn by bold arrows.

Proof of Theorems 4.3, 4.4 and 4.5. If an embedding of a strongly 2-edge-connected 2-regular digraph on the projective-plane, the torus or the Klein bottle has one of the structures shown in Fig. 4.2, Fig. 4.3 and Fig. 4.4, then it has one of the embeddings on the sphere shown in Fig. 4.5. Hence, we only have to show that any embedding of a strongly 2-edge-connected 2-regular diplanar digraph D on the projective-plane, the torus or the Klein bottle must have one of the structures shown in Fig. 4.2, Fig. 4.3 and Fig. 4.4.

Suppose that D is already embedded on the projective-plane, the torus or the Klein bottle with an embedding of the underlying graph G. By Lemma 4.2, the truncated graph $tr(G)$ of G is 3-connected. Moreover, we now show that every edge-cut of order 3 in $tr(G)$ is *trivial*, that is, its edges have the common end-vertex. Suppose that $tr(G)$ has a non-trivial edge-cut of order 3 (for a contradiction). If an edge in the edge-cut is truncated then either of the others is contained in the same truncated cycle. Since the edge-cut is non-trivial, in this situation, we can find a vertex-cut of order at most two in $tr(G)$, which contradicts the 3-connectivity of $tr(G)$. Thus, each of them is original one. However, this implies that G has an edge cut of order 3, which contradicts the strong 2-edge-connectivity of D. Therefore, every edge-cut of order 3 in $tr(G)$ is trivial.

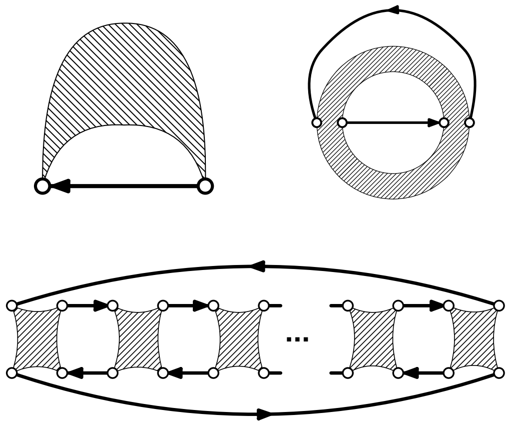

Figure 4.5: Three structures of an embedding on the sphere

Since $tr(G)$ is 3-connected, 3-regular and planar, by Theorems 3.5, 3.6 and 3.7, $tr(G)$ has one of the structures shown in Fig. 3.3, Fig. 3.4 and Fig. 3.5. In the structure, if $tr(G)$ has a shaded triangle then the three edges incident with this triangle form an edge-cut. Thus, this triangle corresponds to only one vertex, denoted by v, and hence the truncated cycle incident with v, denoted by C, bounds an empty area in the structure. This implies that this cycle does not facial in the re-embedding of $tr(G)$ on the sphere. By Theorem 1.5, D is uniquely embeddable on the sphere, and we denote this embedding by $f(D)$. By Lemma 4.1, the truncated graph $tr(f(G))$ of $f(G)$ and $tr(G)$ are isomorphic and hence C is also facial cycle of $tr(f(G))$. However, the re-embedding of $tr(G)$ on the sphere is just $tr(f(G))$, a contradiction.

Therefore, $tr(G)$ has no shaded triangle, that is, the structure is one of the four $(P1)$, $(T2)$, $(K1)$ and $(K3)$. From the above argument, an edge contained in a truncated cycle appear in a shaded area. Thus, G has the same structure, and hence D has one of the structures shown in Fig. 4.2, Fig. 4.3 and Fig. 4.4. □

4.3 The number of embeddings

For an embedding of a digraph D on a surface, the operation of twisting some edges of D can be defined as with the case of embeddings of (undirected) graphs. This operation holds the property that in- and out-edges alternate in the rotation at each vertex of D. That is, the resulting mapping of D on a surface is a re-embedding of D. Actually, for a digraph D embedded on the sphere with one of the structure shown in Fig. 4.5, we obtain the re-embedding of D with one of the structure shown in Fig. 4.2, Fig. 4.3 and Fig. 4.4 by twisting all edges which are not contained in shaded area. It can be shown that for two distinct edge-sets X_1 and X_2, the two embeddings of D obtained from the original embedding of D by twisting the edges in X_1 and X_2 are inequivalent or mapped on distinct surfaces.

Proposition 4.6. *Every connected* 2*-regular diplanar digraph D with n vertices has at least* $2n$ *inequivalent embeddings on the projective-plane and n inequivalent embeddings on the torus. If D is strongly* 2*-edge-connected, then D has at least* $n(2n-1)$ *inequivalent embeddings on the Klein bottle.*

Proof. Let D be a 2-regular diplanar digraph with n vertices, and suppose that D is now embedded on the sphere.

Twisting an edge of D, we obtain a re-embedding of D on the projective-plane, which has the structure shown in Fig. 4.2. Twisting the four edges incident with a vertex of D, we obtain a re-embedding of D on the torus, which has the structure shown in Fig. 4.3 when there are exactly two shaded rectangles and one of them represents only one vertex. Choose two edges of D and twisting them, we obtain a re-embedding of D on the Klein bottle, which has one of the two structures shown in the left of Fig. 4.4 or the right of Fig. 4.4 when there are exactly one shaded rectangles. Therefore, we can give at least $|E(D)| = 2n$ inequivalent embeddings of D on the projective-plane, and $|V(G)| = n$ inequivalent embeddings of D on the torus.

In the case of Klein bottle, since D is strongly 2-edge-connected, the re-embedding of D obtained by twisting any two edges is embedded on non-spherical surfaces. That is, this operation is not a directed Whitney flip. Actually, if we choose two edges contained in the same facial directed walk, then the re-embedding is embedded on the Klein bottle and has the structure shown in the left of Fig. 4.4. Otherwise, it has the structure shown in the right of Fig. 4.4. Thus, we can give at least $\binom{|E(G)|}{2} = n(2n-1)$ inequivalent embeddings of D on the Klein bottle. □

By Theorem 4.3, there are no embeddings of a strongly 2-edge-connected 2-regular diplanar digraph on the projective-plane other than them in Proposition 4.6.

Corollary 4.7. *Every strongly* 2*-edge-connected* 2*-regular diplanar digraph with n vertices has exactly* $2n$ *inequivalent embeddings on the projective-plane.*

Next, we show a family of digraphs attaining the lower bounds in Proposition 4.6 on the torus and the Klein bottle. An undirected graph is *cyclically k-edge-connected* if there is no set of at most $k-1$ edges such that the graph obtained by deleting these edges has at least two components having a cycle.

Corollary 4.8. *If the underlying graph of a strongly* 2*-edge-connected* 2*-regular digraph with n vertices is cyclically* 5*-edge-connected, then it has exactly n inequivalent embeddings* *on the torus and $n(2n-1)$ inequivalent embeddings on the Klein bottle. Moreover, in the case on the torus, the converse holds.*

Proof. Let D be a a strongly 2-edge-connected 2-regular digraph D with n vertices. Suppose that the underlying graph G of D is cyclically 5-edge-connected, and D has more than n inequivalent embeddings on the torus or $n(2n-1)$ inequivalent embeddings on the Klein bottle. Thus, there is an embedding of D on the torus or the Klein bottle which is not counted in Proposition 4.6. This embedding has the structure shown in Fig. 4.3 or the right of Fig. 4.4 having at least two shaded rectangle, each of which does not represent just one vertex, that is, has a cycle. In this situation, we can find four edges of G such that the graph obtained by deleting these edges has exactly two components having cycles, a contradiction.

If G is not cyclically 5-edge-connected then there are four edges such that the graph obtained by deleting these edges has two components having a cycle. Twisting these edges, we obtain a re-embedding of D on the torus, which has the structure shown in Fig. 4.3 when there are exactly two shaded rectangles. This embedding is is not counted in Proposition 4.6. $\square$

The underlying graph of a digraph attaining the lower bounds in Proposition 4.6 on the Klein bottle is not necessarily cyclically 5-edge-connected. For example, Fig. 4.6 presents the underlying graph of such a digraph D.

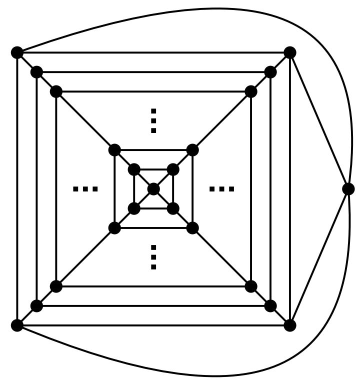

Figure 4.6: The underlying graph of D

In addition to Corollary 4.8, we can give a polynomial-time algorithm for counting the number of inequivalent embeddings of a given strongly 2-edge-connected 2-regular diplanar digraph on the torus or the Klein bottle and a polynomial-delay algorithm for enumerating them to imitate the algorithm in Theorems 3.22 and 3.2, but we omit details here.

Chapter 5

Kündgen and Ramamurth's Conjecture

Hereafter, we use the term *graph* in the generalized sense, that is, we focus on multiple graphs and call graphs without multiple edges *simple* graphs.

In this Chapter, we give affirmative answers of Conjecture 1.10 in two ways. First, we construct two embeddings of a simple graph on the same surface such that one of them has a weak 2-coloring but the other has arbitrarily large weak chromatic number in Section 5.1. Note that these are far from minimum genus embeddings, while we secondly construct two triangulations obtained from the same non-simple graph on the same surface of Euler genus g in Section 5.2.

Moreover, in Section 5.3, we prove that there is a graph having two triangulations on a surface, only one of which is weakly k-colorable if and only if $k \geq 3$.

5.1 Two embeddings of a simple graph

In this section, we prove the following theorem to give an affirmative answer of Conjecture 1.10.

Theorem 5.1. *For each positive integer $n \geq 3$, there is a simple graph G embedded on a surface F^2 with $\chi_w(G) \geq n$ such that G has another embedding $f(G)$ on F^2 which has a weak 2-coloring.*

Proof. To construct desired embeddings, we shall prepare two simple graphs, both of which have a large number of vertices.

De Brandes, Phelps and Rödl [22] showed that for each positive integer $k \geq 3$, there are k-chromatic *Steiner Triple Systems*, that is, 3-uniform hypergraphs such that every pair of vertices appears in exactly one edge, with $O(k^2 \log k)$ vertices. Kündgen and

Ramamurthi [44, Theorem 8.2] constructed an embedded complete graph whose face-hypergraph has such a Steiner Triple System as a subhypergraph (see also [44, Theorem 7.1]). Let K be such an embedded graph with k fixed to $n+1$. Hence $\chi_w(K) \geq n+1$ and K has a triangular face, denoted by F.

Let u_1, u_2 and u_3 be the vertices bounding F. Although there are such embeddings on both of orientable and non-orientable surfaces, we now assume that K is embedded on an orientable surface, and denote its genus by g_0. Let $f_1(K)$ be a maximum (orientable) genus embedding of K and g be the maximum genus of K. That is, $f_1(K)$ is an embedding of K on S_g, and K has no (cellular) embedding on any orientable surface of genus at least $g+1$. (see [76] for details of maximum genus embeddings).

Second, we let T be a simple triangulation on the sphere. We assume that $|V(T)|$ is so large that T has an embedding $f_2(T)$ on S_{g_1}, where $g_1 = g - g_0$, such that there is a triangle face F' in T whose boundary cycle also bounds a face of $f_2(T)$, denoted by $f_2(F')$. Let v_1, v_2 and v_3 be the vertices bounding F'.

Now we construct an embedding of a simple graph in which a weak coloring requires many colors. For two embedded graphs K and $f_2(T)$, paste the two faces F and $f_2(F')$ so that u_i is identified with v_i for $1 \leq i \leq 3$ (this operation corresponds to the *connected sum* of S_{g_0} and S_{g_1}). Thus, we obtain a simple graph embedded on S_g, denoted by G. That is, G has a 3-cycle which separates S_g into two surfaces S_{g_0} and S_{g_1} where K and $f_2(T)$ are embedded, respectively. Since all faces of K other than F are still faces in G, we have $\chi_w(G) \geq n$.

We next construct another embedding of G on S_g which has a weak 2-coloring. Since K is a complete graph, $f_1(K)$ has at most two faces (see [76]). Thus, it is easy to see that it has a weak 2-coloring c such that the second color is assigned to only one vertex, say, w. By the symmetry of a complete graph, we may assume that w corresponds to u_1, and the path $P = u_1u_2u_3$ forms a consecutive part of a walk bounding a face of $f_1(K)$. Add a second edge e joining u_1 and u_3 in $f_1(K)$ which yields a triangle face bounded by u_1, u_2 and u_3. (Then the resulting graph has two edges joining u_1 and u_3.) We denote this triangle face by F''. For this embedded graph and T, paste the two faces F'' and F' so that u_i is identified with v_i for $1 \leq i \leq 3$. After that, we remove the edge e. Then we obtain another embedding $f(G)$ of G on S_g. Since every planar graph is weakly 2-colorable (see [44, Theorem 2.1]), one can easily deduce that T has a weak 2-coloring c' such that $c'(v_1) = 2$ and $c'(v_2) = c'(v_3) = 1$. Therefore, two 2-colorings c and c' construct a weak 2-coloring of $f(G)$. $\square$

Note that we can also construct desired embeddings on non-orientable surfaces by using a maximum non-orientable genus embedding of K instead of $f_1(K)$ in the proof of Theorem 5.1.

5.2 Two triangulations obtained from a multiple graph

In this section, we define a "complete" triangulation G on a surface F^2 and construct another embedding of G which also triangulates F^2.

We denote the complete graph of order n by K_n. Moreover, for a positive integer $m \geq 2$, we denote the graph obtained from K_n by replacing each edge with m multiple edges by K_n^m.

A triangulation G on a surface is *complete* if its face-hypergraph is isomorphic to a complete 3-uniform hypergraph, that is, G has exactly $\binom{|V(G)|}{3}$ faces and there is a face bounded by each triple of vertices. Kündgen and Ramamurthi [44] introduced this notion, and gave a criterion for the existence of complete triangulations.

Theorem 5.2 (Kündgen and Ramamurthi [44, Theorem 6.1])**.** *There is a complete triangulation of order n on a surface if and only if n is even and at least 4.*

Suppose that a triangulation G of order $2m \geq 4$ is complete. Each edge is incident with exactly two faces, and each pair of vertices must be bounded by $2m-2$ faces. Thus, G is isomorphic to K_{2m}^{m-1} and embedded on a surface of Euler genus $(m-1)(m-2)(2m+3)/3$. Actually, Kündgen and Ramamurthi [44] constructed a complete triangulation of order $2m$ on an orientable surface. Moreover, they remarked in [44, Remark 6.2] that we can also obtain complete triangulations on non-orientable surfaces of the same Euler genus by simple replacements.

It is easy to check that for a complete triangulation G of order $2m$, we have $\chi_w(G) = m$ by the pigeon-hole principle. Our goal is to construct another triangulation by G whose weak chromatic number is smaller than G.

Let T be a simple triangulation on a surface, and G be a triangulation on a surface, which may have multiple edges, with the same vertex set as T. We call G a *T-face-hypergraph-isomorphic* triangulation (a T-FHI triangulation, for short) if the edge set of $\mathcal{H}(G)$ coincides with that of $\mathcal{H}(T)$ by ignoring the multiplicity of the edge sets. For example, suppose that graphs $T = K_4$ and $G = K_4^2$ are embedded on the sphere and the torus as a triangulation, respectively. Note that both graphs are uniquely embeddable on the surfaces (see [57]). Then it is easy to check that G is a T-FHI triangulation.

By Ringel's Map Color Theorem [65], for each positive integer m, the complete graph K_{12m} of order $12m$ can be embedded on the orientable surface $S_{g'}$, where $g' = (4m-1)(3m-1)$, as a triangulation, denoted by T. We shall show that the graph $G = K_{12m}^{6m-1}$ has a T-FHI triangulation on an orientable surface.

Theorem 5.3. *Let T be any triangulation on $S_{g'}$ isomorphic to K_{12m}, where $g' = (4m-1)(3m-1)$. Then the graph $G = K_{12m}^{6m-1}$ has a T-FHI triangulation on S_g, where $g = (6m-1)(3m-1)(4m+1)$.*

To prove this theorem, we introduce a transformation of an embedded graph. Let G be a graph embedded on a surface F^2 and u_1v_1 and u_2v_2 be two disjoint edges of G. We define the *digon pasting* for u_1v_1 and u_2v_2 as follows:

(1) Add a second edge in parallel with each of u_1v_1 and u_2v_2 so that parallel edges form digons (Fig. 5.1).

(2) Cut away each digon from F^2. Then the resulting surface has two boundary components. Next, paste these boundaries so that u_1 and v_1 are identified with u_2 and v_2, respectively (and so that the resulting surface is orientable if F^2 is orientable; see Fig. 2). Note that this operation can be regarded as *adding a handle* (see [50, p.80]).

Let G' be the resulting graph. Then G' is embedded on S_{g+1} if $F^2 = S_g$, and N_{k+2} if $F^2 = N_k$. All faces in G are still faces in G' and there are no new faces in G'.

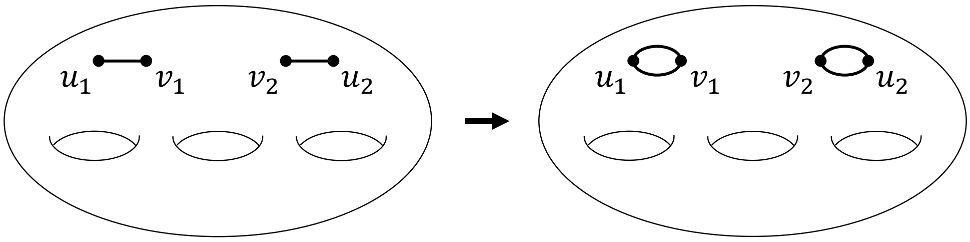

Figure 5.1: The digon pasting (1)

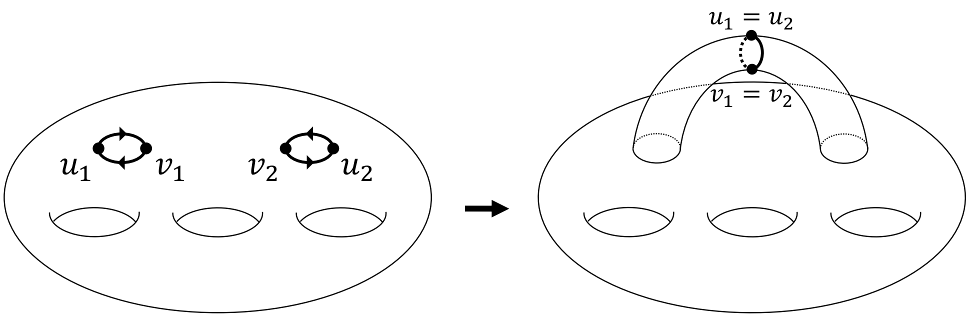

Figure 5.2: The digon pasting (2)

Proof of Theorem 5.3. Let $v_1, v_2, \ldots, v_{12m}$ be the vertices of T. Prepare $6m-1$ copies of $S_{g'}$ triangulated by T. We denote these surfaces and triangulations by $F_1^2, F_2^2, \ldots, F_{6m-1}^2$ and $T_1, T_2, \ldots, T_{6m-1}$, respectively, where T_i is embedded on F_i^2 for $1 \le i \le 6m-1$.

Moreover, we let the vertices of T_i be $v_1^i, v_2^i, \dots, v_{12m}^i$ so that $v_j^i \in V(T_i)$ corresponds to $v_j \in V(T)$ for $1 \leq i \leq 6m-1$ and $1 \leq j \leq 12m$.

We perform the digon pasting for two edges $v_1^1 v_2^1$ and $v_1^2 v_2^2$ (the digon pasting can be naturally extended to disconnected surfaces, and the second step of the digon pasting corresponds to the operation of the connected sum of surfaces). We denote the joint 2-cycle by C_J. After that, perform the digon pasting again for $v_1^3 v_2^3$ and an edge contained in C_J. Repeat these operations until all v_1^i's and v_2^i's are identified. Then we have repeated digon pastings $6m-2$ times so far, and obtained the connected orientable surface of genus $(6m-1)g'$. Moreover, for any positive integer $2 \leq l \leq 6m$, perform such a series of operations for edges $v_{2l-1}^i v_{2l}^i$'s. Thus, all vertices corresponding to a vertex $v_j \in V(T)$ are identified for $1 \leq j \leq 12m$. After that, we re-label the identified vertex corresponding to v_j as v_j itself.

Let G be the resulting embedded graph. All edges and faces of T_i's still exist in G (but labels are changed), and G has $12m$ vertices. That is, G is a triangulation and isomorphic to K_{12m}^{6m-1}. We have repeated digon pastings $6m(6m-2)$ times in total, but the first $6m-2$ times corresponds to the operations of the connected sum. Thus, the resulting surface is orientable and its genus is

$$(6m-1)g' + (6m-1)(6m-2) = (6m-1)(3m-1)(4m+1).$$

For each face F of T, G has $6m-1$ faces bounded by the same vertices as F, and there are no other faces in G. Therefore, G is a T-FHI triangulation. □

Remark 5.4. In Theorem 5.3, for a triangulation $T = K_{12m}$ on an orientable surface, we obtain a T-FHI triangulation $G = K_{12m}^{6m-1}$ on an orientable surface. By Ringel's Map Color Theorem [65], a complete graph K_n of order n can be embedded on an orientable surface (resp. a non-orientable surface) as a triangulation if and only if $n \equiv 0, 3, 4, 7 \pmod{12}$ (resp. $n \equiv 0, 1, 3, 4 \pmod 6$). Then for any triangulation T on an orientable surface (resp. T' on a non-orientable surface) which is isomorphic to K_{12m+4} (resp. K_{6m} or K_{6m+4}), we can construct a T-FHI triangulation on an orientable surface (resp. T'-FHI triangulation on a non-orientable surface) in the same way as Theorem 5.3, which is isomorphic to a complete triangulation.

For a simple triangulation T and a T-FHI triangulation G, it is clear that $\chi_w(T) = \chi_w(G)$.

Lemma 5.5. *For any positive integer m, let T be any triangulation on a surface isomorphic to K_{12m}. Then T has a weak $4m$-coloring.*

Proof. Since T is a complete graph of order at least five, we can find a non-facial 3-cycle C_1 in T. Moreover, $T - V(C_1)$ is also a complete graph of order at least five and hence T

has another non-facial 3-cycle C_2 so that C_1 and C_2 are disjoint. Repeating this operation, we can find $4m-1$ mutually disjoint and non-facial 3-cycles $C_1, C_2, \ldots, C_{4m-1}$ in T. Let u_1, u_2 and u_3 be the three vertices not contained in any of these cycles, and let C_{4m} be the cycle induced by these three vertices.

We assign color i to the vertices of C_i for $1 \leq i \leq 4m$. If C_{4m} is not facial, then no facial cycle is monochromatic, and hence this $4m$-coloring is a weak coloring of T. So we may assume that C_{4m} is facial.

Since T is a triangulation, there is exactly one facial cycle containing the edge u_1u_2 other than C_{4m}. We may assume that this cycle contains a vertex in C_1. Then u_1, u_2 and any vertex in C_i for $2 \leq i \leq 4-1$ induce a non-facial cycle. Let v_1, v_2 and v_3 be the vertices in C_2. It is easy to see that there is at least one edge of C_2, say, v_1v_2, not belonging to any facial cycle containing u_3. Thus, the cycle $C = u_3v_1v_2$ is not facial. Since the cycle $C' = v_3u_1u_2$ is not facial, if we interchange the colors of u_3 and v_3, then there is no monochromatic face. Hence the resulting coloring is a weak $4m$-coloring of T. □

Based on Theorem 5.3 and Lemma 5.5, we shall show the following theorem.

Theorem 5.6. *For any positive number m, put $g = (6m-1)(6m-2)(12m+3)/6$. Then K_{12m}^{6m-1} has two triangulations on S_g whose weak chromatic numbers differ by at least $2m$.*

Proof. It follows from Theorem 5.2 that K_{12m}^{6m-1} can be embedded on S_g as a complete triangulation, denoted by G. Then we have $\chi_w(G) = 6m$. On the other hand, by Theorem 5.3, K_{12m}^{6m-1} can also be embedded on S_g as a T-FHI triangulation, where T is a complete graph of order $12m$ embedded on an orientable surface as a triangulation. We denote this embedding by $f(G)$. Since $\chi_w(f(G)) = \chi_w(T)$, it follows from Lemma 5.5 that $\chi_w(f(G)) \leq 4m$. Hence, $\chi_w(G) - \chi_w(f(G)) \geq 2m$. □

This theorem implies that for any non-negative integer g, there is a graph having two triangulations on the same surface of Euler genus at least g, whose weak chromatic numbers differ from $\Omega(\sqrt[3]{g})$. Theorem 1.11 implies that for two embeddings of the same graph on the same surface of Euler genus g, the difference of these weak chromatic numbers is $O(\sqrt[3]{g})$. Our construction attains this order.

5.3 Weak k-colorability of triangulations

Theorem 5.6 implies that the weak colorability of triangulations depends on the embedding. On the other hand, we will show that the weak 2-colorability of triangulations does not depend on the embedding.

A *polychromatic k-coloring* of a graph embedded on a surface is a k-coloring so that all k colors appear in the boundary of each face. Note that a weak k-coloring and a polychromatic k-coloring are equivalent if and only if $k = 2$. As with a triangulation, a *quadrangulation* on a surface F^2 is defined as a graph embedded on F^2 so that each facial closed walk has length 4. Nakamoto, Noguchi and Ozeki [56] noticed that a triangulation G on a surface has a polychromatic 2-coloring if and only if G has a spanning bipartite quadrangulation. If G has a spanning bipartite quadrangulation then its bipartition induces a polychromatic 2-coloring of G. On the other hand, if G has a polychromatic 2-coloring then the subgraph obtained from G by deleting the monochromatic edges is a spanning bipartite quadrangulation. Using this result, we prove the following theorem.

Theorem 5.7. *For two triangulations T and T' on the same surface obtained from the same graph, if T has a polychromatic 2-coloring then this coloring is also a polychromatic 2-coloring of T'.*

Proof. Suppose that T has a polychromatic 2-coloring c. Then T has the spanning bipartite subgraph H whose bipartition is associated with c, that is, H is a spanning bipartite quadrangulation of T. If the embedding of H in T' had a digon, then T' would have at least one vertex in this digon, which contradicts the fact that H is a spanning subgraph of T'. Thus, H is also a spanning bipartite quadrangulation of T'. This implies that c is also a polychromatic 2-coloring of T'. □

This theorem implies that the polychromatic 2-colorability, or the weak 2-colorability, of triangulations on a surface does not depend on the embedding, while Theorem 5.6 implies that for many positive integers k, the weak k-colorability of triangulations depends on the embedding. Actually, we shall show that it is true for any $k \geq 3$.

Theorem 5.8. *For any positive integer $k \geq 3$, there are graphs having two triangulations on a surface, only one of which has a weak k-coloring.*

Proof. For any positive integer m, the graph K_{12m}^{6m-1} has a complete triangulation, denoted by G, on an orientable surface. Since $\chi_w(G) = 6m$, G is not weakly $(6m-1)$-colorable. By Theorem 5.6, G has another triangulation on the same surface which is weakly $4m$-colorable. Therefore, Theorem 5.8 holds for $k = 4, 5$ and $k \geq 8$.

We now consider the case for $k = 3$. The rotation system in [65, p.82] represents a triangulation $T = K_{12}$ on S_6, whose vertices are the element of $\mathbb{Z}_{12}$. It is easy to check that the three sets $\{0, 1, 6, 7\}$, $\{2, 3, 8, 9\}$ and $\{4, 5, 10, 11\}$ construct the color classes of a weak 3-coloring. By Theorem 5.3, the graph K_{12}^5 has a T-FHI triangulation on an orientable surface, which is weakly 3-colorable. On the other hands, it follows from Theorem 5.2 that K_{12}^5 has a complete triangulation on the orientable surface, whose weak chromatic number is 6.

We next consider the case for $k = 6, 7$. Sun [70] constructed a current graph which generates a triangulation $T' = K_{24}$ on S_{35}, whose vertices are the elements of $\mathbb{Z}_{24}$. It is easy to check that there is no face in T' bounded by three of the four vertices $4i$, $4i+1$, $4i+2$ and $4i+3$ for each integer $0 \leq i \leq 5$. Thus, assigning color $i+1$ to the four vertices $4i$, $4i+1$, $4i+2$ and $4i+3$, we can obtain a weak 6-coloring of T'. By Theorem 5.3, the graph K_{24}^{11} has a T'-FHI triangulation on an orientable surface, which is weakly 6-colorable. On the other hands, it follows from Theorem 5.2 that K_{24}^{11} has a complete triangulation on the orientable surface, whose weak chromatic number is 12. □

5.4 Remarks

In this chapter, to give an affirmative answer of Conjecture 1.10, we constructed two embeddings of a simple graph in Section 5.1. However, these embeddings are far from minimum genus embeddings. On the other hand, in Section 5.2, we constructed two triangulations obtained from the same multiple graph. Thus, Conjecture 1.10 remains open for simple triangulations (or minimum genus embeddings of simple graphs). Moreover, Theorem 5.7 holds even if triangulations are simple, while we do not know whether Theorem 5.8 is true for simple triangulations.

Chapter 6

Facial Achromatic Number of Triangulations on Surfaces

In this chapter, we focus on facial 3-complete coloring of triangulations on surfaces and the following theorem is our main result.

Theorem 6.1. *Let G be a graph which has two triangulations $f_1(G)$ and $f_2(G)$ on a surface F^2, and let g be the Euler genus of F^2. If F^2 is orientable, then*

$$|\psi_3(f_1(G)) - \psi_3(f_2(G))| \leq \begin{cases} 9g/2 & (g \leq 2) \\ 27g/2 - 27 & (otherwise). \end{cases}$$

If F^2 is non-orientable, then

$$|\psi_3(f_1(G)) - \psi_3(f_2(G))| \leq \begin{cases} 3g & (g = 1) \\ 21g - 27 & (otherwise). \end{cases}$$

6.1 Cycles in a triangulation

To prove Theorem 6.1, we prepare some lemmas.

Lemma 6.2. *Let G be a triangulation on a surface, and $C_1, C_2, \ldots, C_k$ be vertex-disjoint facial cycles of G. If there is no chord in the union $H = C_1 \cup C_2 \cup \cdots \cup C_k$, then there is only one H-bridge in G.*

Proof. Let $C = uvw$ be a facial cycle of G bounded by three vertices u, v and w. Suppose that C is not contained in H. Since H consists of vertex-disjoint cycles and has no chord, C meets at most one cycle of H. Suppose that C meets C_1 at a vertex, say u, and $v, w \notin V(C_i)$ for any $1 \leq i \leq k$. If v and w belongs to different H-bridges in G, then the edge vw joins these H-bridges, a contradiction. Hence, v and w belongs to the same H-bridge in G. It implies that all vertices and edges around C_i belongs to one H-bridge in G. Suppose that C meets none of $C_1, C_2, \ldots, C_k$. Then it is clear that u, v and w belong to the same H-bridge in G. Therefore, there is only one H-bridge in G. □

Lemma 6.3. *Let G be a graph which has two triangulations $f_1(G)$ and $f_2(G)$ on a surface, and C be a 3-cycle of G. If $f_1(C)$ is facial in $f_1(G)$ but $f_2(C)$ is not facial in $f_2(G)$, then $f_2(C)$ is non-contractible in $f_2(G)$.*

Proof. Suppose to that $f_2(C)$ is contractible in $f_2(G)$. Since $f_2(C)$ is not facial in $f_2(G)$, it separates $f_2(G)$ into two components. On the other hand, since $f_1(C)$ is facial in $f_1(G)$, it follows from Lemma 6.2 that G has only one C-bridge in G, a contradiction. □

We introduce two lemmas about sets of pairwise non-homotopic cycles. The second lemma closely follows from the proof of [47, Proposition 3.7], which corresponds to the first one. However, to keep the thesis self-contained, we give its proof.

Lemma 6.4 (Malnič and Mohar [47])**.** *Let G be a graph embedded on a surface F^2, and let g be the Euler genus of F^2. Let Γ be a set of pairwise disjoint, non-contractible and pairwise non-homotopic cycles of G. If F^2 is orientable, then*

$$|\Gamma| \leq \begin{cases} g/2 & (g \leq 2) \\ 3g/2 - 3 & (\textit{otherwise}). \end{cases}$$

If F^2 is non-orientable, then

$$|\Gamma| \leq \begin{cases} g & (g \leq 1) \\ 3g - 3 & (\textit{otherwise}). \end{cases}$$

Lemma 6.5. *Let G be a graph embedded on a non-orientable surface F^2 of Euler genus g. let Γ_1 (resp. Γ_2) be a set of pairwise disjoint, non-contractible and pairwise non-homotopic 1-sided (resp. 2-sided) cycles of G. Then $|\Gamma_1| \leq g$ and*

$$|\Gamma_2| \leq \begin{cases} 0 & (g = 1) \\ 2g - 3 & (\textit{otherwise}). \end{cases}$$

Proof. It is easy to see that this lemma holds for $g \leq 2$. Hence, we may assume that $g \geq 3$. Moreover, we may assume that Γ_1 is maximal, that is there is no 1-sided cycle in G disjoint from Γ_1. Cutting F^2 along the cycles in Γ_1, we obtain a connected surface, denoted by $\tilde{F}^2$, which has $|\Gamma_1|$ boundary components. Thus, $\chi(\tilde{F}^2) \leq 2 - |\Gamma_1|$. Since $\chi(\tilde{F}^2) = \chi(F^2) = 2 - g$, we have $|\Gamma_1| \leq g$.

We may also assume that Γ_2 is maximal, that is, all 2-sided cycles in G disjoint from Γ_2 is contractible or homotopic to some element of Γ_2. Cut F^2 along the cycles in Γ_2. Then F^2 is separated into some connected surfaces, denoted by $F_1^2, F_2^2, \ldots, F_k^2$. Note that they are all compact and with non-empty boundary. We denote by $b(\partial F_i^2)$ the number of boundary components of F_i^2 for $1 \leq i \leq k$. Since each cycle in Γ_2 gives rise to two boundary components, we have $\sum_{i=1}^k b(\partial F_i^2) = 2|\Gamma_2|$.

Let $\hat{F}_1^2, \hat{F}_2^2, \ldots, \hat{F}_k^2$ be the surfaces obtained from $F_1^2, F_2^2, \ldots, F_k^2$ by pasting a disk to each boundary component. By the maximality of Γ_2, $\hat{F}_i^2$ is the sphere or the projective-plane for $1 \leq i \leq k$. We denote by n_s and n_p the numbers of the spheres and the projective-planes among $\hat{F}_i^2$'s, respectively. Then we have $n_p \leq g$ and $\sum_{i=1}^k \chi(\hat{F}_i^2) = 2n_s + n_p$.

Now we shall show that if $\hat{F}_i^2$ is the sphere, then $b(\partial F_i^2) \geq 3$. If $b(\partial F_i^2) = 1$, then F_i^2 is a closed disk, that is, the cycle bounding F_i^2 is contractible in F^2, a contradiction. Suppose that $b(\partial F_i^2) = 2$. Then F_i^2 is an annulus. If two cycles of Γ_2 corresponding to the boundary components F_i^2 are the same, then F^2 must be the Klein bottle, a contradiction. Thus, these two cycles are different from each other. However, in this situation, they are homotopic in F^2, a contradiction. Therefore, we may assume that $b(\partial F_i^2) \geq 3$. It implies that we have $3n_s + n_p \leq 2|\Gamma_2|$.

Since $\chi(F^2)$ is equal to the sum of all F_i^2's, we have

$$\begin{aligned}
\chi(F^2) = \sum_{i=1}^k \chi(F_i^2) = \sum_{i=1}^k \chi(\hat{F}_i^2) - \sum_{i=1}^k b(\partial F_i^2) &= 2n_s + n_p - 2|\Gamma_2| \\
&= \frac{2}{3}(3n_s + n_p - 2|\Gamma_2|) + \frac{1}{3}n_p - \frac{2}{3}|\Gamma_2| \\
&\leq \frac{1}{3}g - \frac{2}{3}|\Gamma_2|.
\end{aligned}$$

Since $\chi(F^2) = 2 - g$, we have $|\Gamma_2| \leq 2g - 3$. □

6.2 Proof of the main theorem

Proof of Theorem 6.1. Suppose that $\psi_3(f_1(G)) = k$ and $\psi_3(f_2(G)) < k$. Let $c : V(G) \to \{1, 2, \ldots, k\}$ be a facial 3-complete k-coloring of $f_1(G)$. Then, every triple of k-colors appears in some face of $f_1(G)$. On the other hand, some triples do not appear in the faces of $f_2(G)$. Let $\mathcal{T}$ be a set of triples in k colors such that any triple in $\mathcal{T}$ does not appear in the faces of $f_2(G)$, and for any pair of triples T and T' in $\mathcal{T}$, $T \cap T' = \emptyset$. Moreover, we choose $\mathcal{T}$ so that $|\mathcal{T}|$ is as large as possible. Let $T_1, T_2, \ldots, T_m$ be the triples in $\mathcal{T}$, and so $|\mathcal{T}| = m$. By the maximality of $\mathcal{T}$, we can choose $k - 3m$ colors so that every triple in these colors appear in some face of $f_2(G)$. It implies that $f_2(G)$ has a facial 3-complete $(\max\{3, k - 3m\})$-coloring. Then, $|\psi_3(f_1(G)) - \psi_3(f_2(G))| \leq 3m$.

Let $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ be a set of facial cycles in $f_1(G)$ such that $c(V(C_i)) = T_i$ for $1 \le i \le m$. Since every C_i is not facial in $f_2(G)$, it follows from Lemma 6.3 that every C_i is non-contractible in $f_2(G)$.

Claim 1. *There are at most three pairwise homotopic cycles of $\mathcal{C}$ in $f_2(G)$.*

Proof. Suppose that C_1, C_2, C_3 and C_4 are pairwise homotopic in $f_2(G)$, and appear on the annulus bounded by C_1 and C_4 in this order. Thus, the union $C_2 \cup C_4$ separates C_1 from C_3, and hence there are no chords of $C_1 \cup C_3$. Similarly, $C_1 \cup C_3$ also separates C_2 from C_4. It implies that there are at least two $C_1 \cup C_3$-bridges in G. On the other hand, since both of C_1 and C_3 are facial in $f_1(G)$ and $C_1 \cup C_3$ has no chord, it follows from Lemma 6.2 that there is only one $C_1 \cup C_3$-bridge in G, a contradiction. Therefore, there are at most three pairwise homotopic cycles of $\mathcal{C}$ in $f_2(G)$. $\square$

Now we shall give the upper bound for $|\mathcal{T}| = m$, which induces the upper bound for $|\psi_3(f_1(G)) - \psi_3(f_2(G))|$. We first consider the case when the surface F^2 is homeomorphic to one of the sphere, the projective-plane, and the torus. Suppose that F^2 is the sphere. All cycles in G is contractible, and hence $\mathcal{C} = \emptyset$. Actually, it follows Lemma 6.3 that $f_1(G)$ and $f_2(G)$ are essentially equivalent embeddings. (In general, Whitney [74] showed that every 3-connected planar graph has essentially unique embedding in the sphere.) Suppose that F^2 is the projective-plane. There is no pair of disjoint non-contractible cycles in $f_2(G)$, and hence $m \le 1$. Suppose that F^2 is the torus. All non-contractible and pairwise disjoint cycles in G are pairwise homotopic. Then, all cycles in $\mathcal{C}$ are pairwise homotopic by Lemma 6.4, and hence it follows from Claim 1 that $m \le 3$.

Second, suppose that F^2 is an orientable surface of genus at least two. If $m > 9g - 9$, then there are at least four pairwise homotopic cycles in $\mathcal{C}$ by Lemma 6.4, which contradicts Claim 1. Hence, we have $m \le 9g - 9$. Finally, suppose that F^2 is a non-orientable surface of genus at least two. If $m > 7g - 9$, then there are at least $6g - 8$ 2-sided cycles in $\mathcal{C}$, and hence some four of them are pairwise homotopic by Lemma 6.5, which contradicts Claim 1. Therefore, in any case, the desired inequality holds. $\square$

6.3 Facial complete colorings of multigraphs

In this section, we consider graphs which may have multiple edges. In Chapter 5, we constructed two triangulations $f_1(G)$ and $f_2(G)$ obtained from the graph $G = K_{12m}^{6m-1}$ on a surface for any positive integer m, whose weak chromatic numbers differ by at least $2m$. We now show that the facial 3-achromatic numbers of these triangulations also differ.

For details of constructions of $f_1(G)$ and $f_2(G)$, see Chapter 5. The face-hypergraph $\mathcal{H}(f_1(G))$ of $f_1(G)$ is isomorphic to a complete 3-uniform hypergraph. That is, the triangulation $f_1(G)$ is complete. Then it is easy to see that $\psi_3(f_1(G)) = |V(G)| = 12m$.

Let T be a triangulation on a surface obtained from K_{12m} The edge-set of $\mathcal{H}(f_2(G))$ coincides with that of $\mathcal{H}(T)$ by ignoring the multiplicity of the edge-sets. It implies that $\psi_3(f_2(G)) = \psi_3(T)$. Suppose that T is facially 3-complete k-colorable. Then, T must have at least $\binom{k}{3}$ faces, and hence we obtain the following inequality:

$$\begin{aligned} |\mathcal{F}(T)| = 4m(12m-1) &\geq k(k-1)(k-2)/6 \\ 288m^2 - 24m &\geq (k-2)^3 \\ \sqrt[3]{288}\, m^{2/3} &\geq k-2 \\ 7m+2 &\geq k. \end{aligned}$$

Then, $\psi_3(f_2(G)) \leq 7m+2$ (this bound might be loose), and hence we have

$$\psi_3(f_1(G)) - \psi_3(f_2(G)) \geq 5m-2.$$

Since G is isomorphic to K_{12m}^{6m-1}, both of two triangulations $f_1(G)$ and $f_2(G)$ are embedded on a surface of Euler genus $(m-1)(m-2)(2m+3)/3$. It implies that for any non-negative integer g, there is a graph having two triangulations on a surface of Euler genus at least g, whose facial 3-achromatic numbers differ from $\Omega(\sqrt[3]{g})$.

Bibliography

[1] N. Alon, R. Berke, K. Buchin, M. Buchin, P. Csorba, S. Shannigrahi, B. Speckmann and P. Zumstein, Polychromatic colorings of plane graphs, *SCG' 08 : Proceedings of the Twenty-Fourth Annual Symposium on Computational Geometry, ACM* (2008), 338–345.

[2] L. D. Andersen, A. Bouchet and B. Jackson, Orthogonal A-trails of 4-regular graphs embedded in surfaces of low genus, *J. Combin. Theory Ser. B*, **66** (1996), 232–246.

[3] K. Appel and W. Haken, Every planar map is four colorable, *Bull. Amer. Math. Soc.*, **82** (1976) 711—712.

[4] K. Appel and W. Haken, Every planar map is four colorable, part I: Discharging, *Illinois J. Math.*, **21** (1977) 4429–490.

[5] K. Appel and W. Haken, Every planar map is four colorable, part II: Reducibility, *Bull. Amer. Math. Soc.*, **21** (1977) 491—567.

[6] D. Archdeacon, M. DeVos, S. Hannie and B. Mohar, Whitney's theorem for 2-regular planar digraphs, *Australas. J. Combin.*, **67** (2017), 159–165.

[7] D. Archdeacon, C. P. Bonnington and B. Mohar, Embedding quartic Eulerian digraphs on the plane, *Australas. J. Combin.*, **67** (2017), 364–377.

[8] J. L. Arocha, J. Bracho and V. Neumann-Lara, Tight and untight triangulations of surfaces by complete graphs, *J. Combin. Theory Ser. B*, **63** (1995),185–199.

[9] C. P. Bonnington, M. Conder, M. Morton and P. McKenna, Embedding digraphs on orientable surfaces, *J. Combin. Theory Ser. B*, **85**(2002), 1–20.

[10] C. P. Bonnington, N.Hartsfield and J. Širáň, Obstructions to directed embeddings of Eulerian digraphs in the plane, *European J. Combin.*, **25** (2004), 877–891.

[11] O. V. Borodin, Solution of the Ringel problem on vertex-face coloring of planar graphs and coloring of 1-planar graphs, *Metody Diskret. Analiz.* **41** (1984), 12–26 (in Russian).

[12] O. Borodin, A new proof of the 6 color theorem, *J. Graph Theory* **19** (1995), 507–521.

[13] O. V. Borodin, D. P. Sanders and Y. Zhao, On cyclic colorings and their generalizations, *Discrete Math.* **203** (1999), 23–40.

[14] M. I. Burstein, An upper bound for the chromatic number of hypergraphs, *Sakharth. SSR Mecn. Akad. Moambe* **75** (1974), 37–40 (In Russian).

[15] G. Chartrand, L. Lesniak and P. Zhang, Graphs & Digraphs, Fifth Edition, Chapman & Hall/CRC, 2010.

[16] Y. Chen, J. L. Gross and X. Hu, Enumeration of digraph embedding, *European J. Combin.*, **36** (2014), 660–678.

[17] Y. Chen and T. Wang, Recursive formulas for embedding distributions of cubic outerplanar graphs, *Australas. J. Combin.*, **68** (2017), 131–146.

[18] Y. Chen, J. L. Gross and T. Mansour, On the genus distributions of wheels and of related graphs, *Discrete math.*, **341** (2018), 934–945.

[19] V. Chvátal, A combinatorial theorem in plane geometry, *J. Combin. Ser. B*, **18** (1975), 39–41.

[20] J. Czap and S. Jendrol', Facially-constrained colorings of plane graphs: a survey, *Discrete Math.*, **340** (2017),2691–2703.

[21] J. Czap and S. Jendrol', A survey on the cyclic coloring and its relaxations, *Discuss. Math.*, **41** (2021),5–38.

[22] M. De Brandes, K. T. Phelps and V. Rödl, Coloring Steiner Triple Systems, *SIAM J. Algebraic Discrete Methods*, **3**(1982),241–249.

[23] R. Diestel, Graph Theory, Fourth edition, Springer, 2010.

[24] Z. Dvořák, D. Král' and R. Škrekovski, Coloring face hypergraphs on surfaces, *European J. Combin.*, **26** (2005),95–110.

[25] Z. Dvořák, D. Král' and R. Škrekovski, Non-rainbow colorings of 3-, 4- and 5-connected plane graphs, *J. Graph Theory* **63** (2010), 129–145.

[26] H. Enomoto and M. Horňák, A general upper bound for the cyclic chromatic number of 3-connected plane graphs, *J. Graph Theory* **62** (2009), 1–25.

[27] H. Enomoto, M. Horňák and S. Jendrol', Cyclic chromatic number of 3-connected plane graphs, *SIAM J. Discrete Math.* **14** (2001), 121–137.

[28] S. Fisk, A short proof of Chvátal's Watchman theorem, *J. Combin. Theory Ser. B* **24** (1978), 374.

[29] J. L. Gross and T. W. Tucker, Topological Graph Theory, Wiley-Interscience, 1987.

[30] J. L. Gross and M. L. Furst, Hierarchy for imbedding-distribution invariants of a graph, *J. Graph Theory*, **11** (1987), 205–220.

[31] J. L. Gross, D. P. Robbins and T. W. Tucker, Genus distributions for bouquets of circles, *J. Comb. Theory Ser. B*, **47** (1989), 292–306.

[32] J. L. Gross, T. Mansour, T. W. Tucker and D. G. L. Wang, Combinatorial conjectures that imply local log-concavity of graph genus polynomials, *European J. Combin.*, **52** (2016), 207–222.

[33] J. L. Gross, T. Mansour and T. W. Tucker, Partial duality for ribbon graphs, I: Distributions, *European J. Combin.*, **86** (2020), 103084.

[34] R. Hao, Y. Liu, T. Zhang and L. Xu, The genus distributions of 4-regular digraphs, *Australas. J. Combin.*, **43** (2009), 79–90.

[35] F. Harary and S. Hedetniemi, The achromatic number of a graph, *J. Combin. Theory*, **8** (1970),154–161.

[36] P. J. Heawood, Map Colour Theorem, *Quart. J. Math.*, **24** (1890),332–338.

[37] E. Horev, M.J. Katz, R. Krakovski and A. Nakamoto, Polychromatic 4-coloring of cubic bipartite plane graphs, *Discrete Math.* **312** (2012), 715–719.

[38] E. Horev and R. Krakovski, Polychromatic colorings of bounded degree plane graphs, *J. Graph Theory* **60** (2009), 269–283.

[39] F. Hughes and G. MacGilivray, The achromatic number of a graphs A survey and some new results, *Bull. Inst. Combin. Appl.*, **19** (1997),27–56.

[40] T. Johonson, *Eulerian Digraph Immersion*, PhD Thesis, Princeton University, 2002.

[41] V. Jungić, D. Král' and R. Škrekovski, Colorings of plane graphs with no rainbow faces, *Combinatorica*, **26** (2006), 169–182.

[42] S. Kitakubo and S. Negami, Re-embedding structures of 5-connected projective-planar graphs, *Discrete Math.*, **244** (2002),211–221.

[43] M. Kobayashi, A. Nakamoto and T. Yamaguchi, Polychromatic 4-coloring of cubic even embeddings on the projective plane, *Discrete Math.* **313** (2013), 2423–2431.

[44] A. Kündgen and R. Ramamurthi, Coloring face-hypergraphs of graphs on surfaces, *J. Combin. Theory Ser. B*, **85** (2002),307–337.

[45] L. Lovász, Combinatorial Problems and Exercises, AMS Chelsea Publishing, 2007.

[46] J. Maharry, N. Robertson, V. Sivaraman and D. Slilaty, Flexibility of projective-planar embeddings, *J. Comb. Theory Ser. B*, **122** (2017),241–300.

[47] A. Malnič and B. Mohar, Generating locally cyclic triangulations of surfaces, *J. Combin. Theory Ser. B*, **56** (1992),147–164.

[48] N. Matsumoto and Y. Ohno, Facial achromatic number of triangulations on the sphere, *Discrete Math.*, **343** (2020),#111651.

[49] B. Mohar, Uniqueness and minimality of large face-width embeddings of graphs, *Combinatorica*, **15** (1995),541–556.

[50] B. Mohar and C. Thomassen, Graphs on Surfaces, The Johns Hopskins University Press,2001.

[51] B. Mohar and N. Robertson, Flexibility of polyhedral embeddings of graphs in surfaces, *J. Comb. Theory Ser. B*, **83** (2001),38–57.

[52] B. Mohar and N. Robertson, Planar graphs on nonplanar surfaces, *J. Comb. Theory Ser. B*, **68** (1996),87–111.

[53] B. Mohar, N. Robertson and R. P. Vitray, Planar graphs on the projective plane, *Discrete Math.*, **149** (1996),141–157.

[54] A. Nakamoto, S. Negami, K. Ohba and Y. Suzuki, Looseness and independence number of triangulations on closed surfaces, *Discuss. Math. Graph Theory* **36** (2016), 545–554.

[55] A. Nakamoto, K. Noguchi and K. Ozeki, Cyclic 4-colorings of graphs on surfaces, *J. Graph Theory* **82** (2016), 265–278.

[56] A. Nakamoto, K. Noguchi and K. Ozeki, Spanning bipartite quadrangulations of even triangulations, *J. Graph Theory*, **90** (2019),267–287.

[57] S. Negami, Uniqueness and faithfulness of embedding of toroidal graphs, *Discrete Math.*, **44** (1983),161–180.

[58] S. Negami, Looseness ranges of triangulations on closed surface, *Discrete Math.*, **303** (2005),167–174.

[59] S. Negami and T. Midorikawa, Loosely-tightness of triangulations of closed surface, *Sci. rep. Yokohama Nat. Univ. Sect. I Math Phys Chem*, **43** (1996),25–41.

[60] O. Ore and M.D. Plummer, Cyclic coloration of plane graphs, in: W. T. Tutte(Ed.), *Recent Progress in Combinatorics* (Proceedings of the Third Waterloo Conference on Combinatorics), May 1968, Academic Press, (1969), 287–293.

[61] J. G. Penaud, Une propriété de bicoloration des hypergraphes planaires, *Cahiers Centre Études Rech. Opér* **17** (1975), 345–349.

[62] M. D. Plummer and B. Toft, Cyclic coloration of 3-polytopes, *J. Graph Theory* **11** (1987), 507–515.

[63] R. Ramamurthi and D. B. West, Maximum face-constrained colorings of plane graphs, *Discrete Math.*, **274** (2004),233–240.

[64] G. Ringel and J. W. T. Youngs, Solution of the Heawood map-coloring problem, *Proc. Nat. Acad. Sci. U.S.A.*, **60** (1968), 438–445.

[65] G. Ringel, Map Color Theorem, *Springer Science Business Media vol. 209*, (1974).

[66] N. Robertson and R. P. Vitray, Representativity of surface embeddings, In: *Paths, Flows and VLSI-Layout*, B. Korte, L. Lovász, H. J. Prömel and A. Schrijver, eds. Springer-Verlag Berlin Heidelberg,1990.

[67] N. Robertson, X. Zha and Y. Zhao, On the flexibility of toroidal embeddings, *J. Comb. Theory Ser. B*, **98** (2008),43–61.

[68] D. P. Sanders and Y. Zhao, A new bound on the cyclic chromatic number, *J. Combin. Theory Ser. B* **83** (2001), 102–111.

[69] P. D. Seymour and R. Thomas, Uniqueness of highly representative surface embeddings, *J. Graph Theory*, **23** (1996),337–349.

[70] T. Sun, A simple construction for orientable triangular embeddings of the complete graphs on 12s vertices, *Discrete Math.*, **342** (2019),1147–1151.

[71] Y. Suzuki, Re-embedding structures of 4-connected projective-planar graphs, *J. Graph Theory*, **68** (2011),213–228.

[72] M. T. Tsai and D. B. West, A new proof of 3-colorability of Eulerian triangulations, *Ars Math. Contemp.* **4** (2011), 73–77.

[73] W. T. Tutte, How to draw a graph, *Proc. London Math. Soc.* **13** (1963), 743–767.

[74] H. Whitney, Congruent Graphs and the Connectivity of Graphs, *Amer. J. Math.*, **54** (1932),150-168.

[75] H. Whitney, 2-isomophic graphs, *Amer. J. Math.*, **55** (1933),245-254.

[76] N. H. Xuong, How to determine the maximum genus of a graph, *J. Combin. Theory Ser. B*, **26**(1979),217–225.