Connected subgraphs with certain properties in dense graphs

Shunichi Maezawa
Graduate School of Environment and Information Sciences
Yokohama National University
79-7 Tokiwadai, Hodogaya-ku, Yokohama 240-8501 JAPAN

Preface

Graph Theory is an area of mathematics whose origins lie as far back as in 18th century with the solution of the Köningsberg Bridge problem by the mathematician Leonhard Euler. Since then, the subject has developed into an area with numerous interesting problems and applications in many diverse fields.

In this doctor's thesis, we show some new results on spanning subgraphs having some specified properties. We mainly deal with the problems on spanning subgraphs which are generalizations of Hamilton path problems. A research of Hamilton cycles (resp. paths) is one of major topics in graph theory. A Hamilton cycle (resp. path) in a graph is a cycle (resp. path) passing through all the vertices of the graph. In 1960, Ore [48] gave sufficient conditions for graphs to have a Hamilton cycle and a Hamitlon path. This result is one of the cornerstones of graph theory. Since a Hamilton cycle and a Hamilton path can be regarded as a spanning subgraph with some specified properties, Ore's theorem has been generalized to those of spanning subgraphs with some properties.

This thesis consists of four chapters. In Chapter 1, we give basic definitions, notations and terminologies which are needed for reading this thesis. Moreover we introduce some results of a Hamilton cycle and a Hamilton path which motivate our results.

In Chapter 2, we show some results of the existence on spanning subgraphs with constrains on the degree.

In Chapter 3, we show some results of the existence on spanning trees with some specified properties, which are generalized concepts of Hamilton paths.

In Chapter 4, we show a Fan-type condition for bipartite graphs to have long paths. As a consequence of the result, we completely determine the bipartite Ramsey numbers with respect to a path and a bistar.

I am very grateful to many people who have contributed to this thesis.

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Papers underlying the thesis

- M. Furuya, S. Maezawa, R. Matsubara, H. Matsuda, S. Tsuchiya, and T. Yashima, Degree sum conditions for the existence of spanning k-trees in star-free graphs, *Discuss. Math. Graph Theory*, to appear, (2019).

- S. Maezawa, R. Matsubara, and H. Matsuda, Degree conditions for graphs to have spanning trees with few branch vertices and leaves, *Graphs Combin.* **35**, (2019) 231–238.

- M. Furuya, S. Maezawa, and K. Ozeki, Long paths in bipartite graphs and path-bistar bipartite ramsey number, *Graphs Combin.* **36**, (2020) 167–176.

- S. Maezawa and K. Ozeki, A forbidden pair for connected graphs to have spanning k-trees, submitted to *J. Graph Theory.*

- S. Maezawa, M. Tsugaki, and T. Yashima, Closure and spanning trees with bounded total excess, submitted to *Graphs Combin.*

- S. Maezawa, R. Matsubara, and H. Matsuda, A Fan-type condition for graphs to be k-leaf-connected, submitted to *Discrete Math.*

Contents

Introduction

A *graph* G consists of a *vertex set*, denoted by $V(G)$ and an *edge set*, denoted by $E(G)$. Each edge joins two vertices, which are not necessarily distinct. For a graph G and $v \in V(G)$, the number of edges incident with v is called the *degree* of v in G and is denoted by $\deg_G(v)$. For two vertices x and y of a connected graph G, the *distance* between x and y in G is the length of a shortest path connecting x and y in G and is denoted by $\mathrm{dist}_G(x, y)$. For a nonempty vertex subset X of $V(G)$, the *subgraph of G induced by X* is defined as the subgraph of G whose vertex set is X and whose edge set consists of the edges of G joining vertices of X. The subgraph of G induced by X is denoted by $G[X]$. A subgraph H is called an *induced subgraph* of G if there exists a nonempty vertex subset X of $V(G)$ such that $H = G[X]$. For given graphs G and H, if there exists a bijection $f : V(G) \to V(H)$ such that $f(x)$ and $f(y)$ are adjacent in H if and only if x and y are adjacent in G, then G and H are *isomorphic*. For a given graph H, a graph G is said to be *H-free* if G contains no induced subgraph isomorphic to H.

A *Hamilton cycle* (resp. *path*) of a graph G is a cycle (resp. path) passing through all vertices of G. A graph G is called *Hamilton-connected*, if for any two vertices x and y of G, there is a Hamilton path of G connecting x and y. A research of the Hamiltonialy is one of major topics in graph theory. Since the problem of determining whether a given graph has a Hamilton cycle (resp. path) is NP-complete [28], we have studied a sufficient condition for graphs to have a Hamilton cycle (resp. path). The problem of determining whether a given graph is Hamilton-connected is also NP-complete [19]. We focus on degree conditions and forbidden subgraph conditions for graphs to have a Hamilton cycle (resp. path) and to be Hamilton-connected. A degree condition is to guarantee that each vertex has an enough large degree and a forbidden subgraph condition is to guarantee that a graph has no induced subgraph isomorphic to some given graphs. Let $\alpha(G)$ be the maximum cardinality of an independent set of a graph G. For a positive integer k, and a graph G, we define

$$\sigma_k(G) = \min\left\{ \sum_{x \in S} \deg_G(x) : S \text{ is an independent set of } G \text{ with } |S| = k \right\}$$

if $\alpha(G) \geq k$, and $\sigma_k(G) = \infty$ if $\alpha(G) < k$. In 1960, Ore gave a sufficient condition for graphs to have a Hamilton cycle (resp. path) [48] and to be Hamilton-connected [49]. These results are cornerstones of graph theory.

Theorem 0.1 (Ore [48, 49]) *Let G be a graph with order at least three. Suppose that $\sigma_2(G) \geq |G| + s$ with $s \in \{-1, 0, 1\}$.*

(i) *If $s = -1$, then G has a Hamilton path.*

(ii) *If $s = 0$, then G has a Hamilton cycle.*

(iii) *If $s = 1$, then G is Hamilton-connected.*

A degree sum condition on $\sigma_2(G)$ is so-called an Ore-type condition.

In 1984, Fan [25] gave a degree condition for graphs to have a Hamilton cycle (resp. path), which is weaker than the condition of Theorem 0.1. This degree condition is so-called a Fan-type degree condition. Benhocine and Wojda showed a Fan-type condition for graphs to be Hamilton-connected.

Theorem 0.2 (Fan [25], Benhocine and Wojda [5]) *Let $s \in \{-1, 0, 1\}$ and let G be a graph. Suppose that*

$$\max\{\deg_G(x), \deg_G(y)\} \geq \frac{|G| + s}{2}$$

for any two vertices $x, y \in V(G)$ with $\mathrm{dist}_G(x, y) = 2$.

(i) *If G is connected and $s = -1$, then G has a Hamilton path.*

(ii) *If G is 2-connected and $s = 0$, then G has a Hamilton cycle.*

(iii) *If G is 3-connected and $s = 1$, then G is Hamilton-connected.*

Liu, Tian, and Wu in 1986 and independently, Broersma in 1988, showed that we can relax the degree condition of Theorem 0.1 (i) by restricting graphs to be $K_{1,3}$-free, where $K_{n,m}$ is a complete bipartite graph with a size of one partite set n and the size of the other partite set m.

Theorem 0.3 (Liu, Tian, and Wu [38], Broersma [8]) *Let G be a connected $K_{1,3}$-free graph. If*

$$\sigma_3(G) \geq |G| - 2,$$

then G has a Hamilton path.

Faudree and Gould characterized the forbidden pairs for connected graphs to have a Hamilton path. The graph $N(p, q, r)$ is one obtained from the triangle xyz by joining p isolated vertices to x, q isolated vertices to y, r isolated vertices to z (Fig. 1). We denote a path with n vertices by P_n.

Theorem 0.4 (Faudree and Gould [26]) *Let H_1 and H_2 be connected graphs with $H_1, H_2 \neq P_1, P_2, P_3$. Then, every connected $\{H_1, H_2\}$-free graph has a Hamilton path if and only if H_1 is $K_{1,3}$ and H_2 is one of the graph $N(p, q, r)$ for $0 \leq p, q, r \leq 1$ or P_4.*

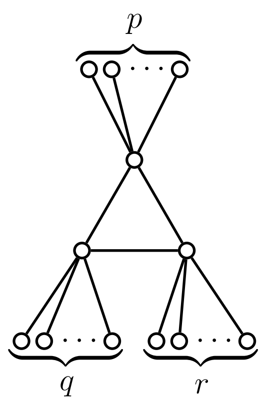


Figure 1: The graph $N(p,q,r)$

In this thesis, we deal with some extended concepts of a Hamilton path. We can regard that a Hamilton path is a spanning subgraph with maximum degree at most two. In Chapter 2, we deal with some spanning subgraphs with bounded maximum degree. For an integer $k \geq 2$, a *k-tree* T is defined as a tree with maximum degree at most k. If a k-tree T spans a graph G, then T is called a *spanning k-tree* of G. Since a spanning 2-tree is a Hamilton path, a spanning k-tree is an extended concept of a Hamilton path.

Caro, Krasikov, and Roditty in 1985 and independently, Jackson and Wormald in 1990, obtained the following result, which guarantees the existence of a spanning k-tree in connected $K_{1,k}$-free graphs.

Theorem 0.5 (Caro, Krasikov, and Roditty [11], Jackson and Wormald [32]) *For an integer $k \geq 3$, every connected $K_{1,k}$-free graph contains a spanning k-tree.*

In Chapter 2.2, we focus on a sharp condition that guarantees the existence of a spanning k-tree in connected $K_{1,k+1}$-free graphs and give a degree sum condition as follows.

Theorem 0.6 *Let k be an integer with $k \geq 2$. If a connected $K_{1,k+1}$-free graph G satisfies*

$$\sigma_{3k-3}(G) \geq |G| - 2,$$

then G has a spanning k-tree.

The degree sum condition of Theorem 0.6 is sharp in the sense we cannot replace the lower bound of $\sigma_{3k-3}(G)$ with $|G| - 3$.

In 2010, Ota and Sugiyama gave a forbidden subgraph condition for a graph to have a spanning k-tree.

Theorem 0.7 (Ota and Sugiyama [50]) *Let $k \geq 2$ be an integer. If G is a connected $\left\{K_{1,k+1}, N\left(k-1, k-1, \left\lfloor \frac{k-1}{2} \right\rfloor\right), N(k-1, k-2, k-2)\right\}$-free graph, then G has a spanning k-tree.*

However, it was not known whether the conditions of being $N\left(k-1, k-1, \left\lfloor\frac{k-1}{2}\right\rfloor\right)$-free and $N(k-1, k-2, k-2)$-free in Theorem 0.7 are sharp. They posed the following conjecture.

Conjecture 0.8 (Ota and Sugiyama [50]) *Let $k \geq 2$ be an integer. If G is a connected $\left\{K_{1,k+1}, N\left(k-1, k-1, \left\lceil\frac{k-1}{2}\right\rceil\right)\right\}$-free graph, then G has a spanning k-tree.*

We show that Conjecture 0.8 is true in Chapter 2.3.

In 1976, Bondy and Chvátal introduced a closure concept in [7]. The following result is a stronger than Theorem 0.1 (ii).

Theorem 0.9 (Bondy and Chvátal [7]) *Let G be a graph. If u and v are nonadjacent vertices with $\deg_G(u) + \deg_G(v) \geq |G|$, then G has a Hamilton cycle if and only if $G + uv$ has a Hamilton cycle.*

In [42], Matsubara et al. considered a closure concept for spanning k-trees. For a vertex subset S of a graph G, and a positive integer k with $k \leq |S|$, let

$$\Delta_k(S; G) = \max\left\{\sum_{x \in X} \deg_G(x) : X \text{ is a subset of } S \text{ with } |X| = k\right\}.$$

Theorem 0.10 (Matsubara, Tsugaki and Yamashita [42]) *Let $k \geq 2$ be an integer, and let G be a connected graph. Let u and v be two nonadjacent vertices of G. If $\Delta_k(S; G) \geq |G| - 1$ for every independent set S in G of order $k+1$ such that $\{u, v\} \subseteq S$, then G has a spanning k-tree if and only if $G + uv$ has a spanning k-tree.*

On the other hand, a tree is called a *k-ended tree* if the number of its leaves is at most k. In [9], Broersma and Tuinstra considered a closure concept for spanning k-ended trees.

Theorem 0.11 (Broersma and Tuinstra [9]) *Let $k \geq 2$ be an integer, and let G be a connected graph. Let u and v be two nonadjacent vertices of G. If $\deg_G(u) + \deg_G(v) \geq |G| - 1$, then G has a spanning k-ended tree if and only if $G + uv$ has a spanning k-ended tree.*

Let $\alpha \geq 0$ and $k \geq 2$ be integers. For a graph G, the *total k-excess* of G is defined as $\mathrm{te}(G; k) = \sum_{v \in V(G)} \max\{\deg_G(v) - k, 0\}$. We propose a new closure concept for a spanning tree with bounded total k-excess. This concept was introduced by Enomoto, Onishi and Ota in [24], and we can see some results concerning it in [27, 47, 51]. Note that for a tree T, $\mathrm{te}(T; k) = 0$ if and only if T is a k-tree, and $\mathrm{te}(T; 2) \leq k - 2$ if and only if T is a k-ended tree. In this thesis, we generalize Theorems 0.10 and 0.11 as follows.

Theorem 0.12 *Let $\alpha \geq 0$ and $k \geq 2$ be integers, and let G be a connected graph. Let u and v be two nonadjacent vertices of G. If $\Delta_k(S; G) \geq |G| - 1$ for every independent set S in G of order $k+1$ such that $\{u, v\} \subseteq S$, then G has a spanning tree T with $te(T; k) \leq \alpha$ if and only if $G + uv$ has a spanning tree T' with $te(T'; k) \leq \alpha$.*

The lower bound of $\Delta_k(S;G)$ in Theorem 0.12 is sharp. Let $\alpha \geq 0$ and $k \geq 2$ be integers, and let G be a connected graph. In [27], Fujisawa et al. showed that if $\alpha(G) \leq k + \alpha$, then G has a spanning tree T with $\mathrm{te}(T;k) \leq \alpha$. Moreover, they showed the upper bound of $\alpha(G)$ is sharp. Therefore, it is natural to consider the following problem, which corresponds to an improvement of Theorem 0.12.

Problem 0.13 *Let $\alpha \geq 0$ and $k \geq 2$ be integers, and let G be a connected graph. Let u and v be two nonadjacent vertices of G. If $\Delta_k(S;G) \geq |G| - 1$ for every independent set S in G of order $k + \alpha + 1$ such that $\{u, v\} \subseteq S$, then G has a spanning tree T with te$(T;k) \leq \alpha$ if and only if $G + uv$ has a spanning tree T' with te$(T';k) \leq \alpha$.*

However, Problem 0.13 is not true for $\alpha > 0$. Therefore, we change the condition on S so that S contains at least one of u and v, and prove the following theorem in Chapter 2.4.

Theorem 0.14 *Let $\alpha \geq 0$ and $k \geq 2$ be integers, and let G be a connected graph. Let u and v be two non-adjacent vertices of G. If $\Delta_k(S;G) \geq |G| - \alpha - 1$ for every independent set S in G of order $k + \alpha + 1$ such that $S \cap \{u, v\} \neq \emptyset$, then G has a spanning tree T with te$(T;k) \leq \alpha$ if and only if $G + uv$ has a spanning tree T' with te$(T';k) \leq \alpha$.*

The lower bound of $\Delta_k(S;G)$ in Theorem 0.14 is sharp.

In Chapter 3, we deal with spanning trees with certain properties, which are extensions of properties of a Hamilton path.

A *branch vertex* of a tree is a vertex of degree strictly greater than two. For a tree T, let $L(T)$ denote the set of leaves of T and let $B(T)$ denote the set of branch vertices of T. The following two results motivate our results in Chapter 3.2. Theorem 0.15 gives an Ore-type condition for a graph to have a spanning k-ended tree.

Theorem 0.15 (Broersma and Tuinstra [8]) *Let $k \geq 2$ be an integer and let G be a connected graph. If G satisfies $\deg_G(u) + \deg_G(v) \geq |G| - k + 1$ for every pair of two nonadjacent vertices $u, v \in V(G)$, then G has a spanning k-ended tree.*

The following theorem is stronger than Theorem 0.15 although it assumes the same condition as Theorem 0.15.

Theorem 0.16 (Nikoghosyan [46], Saito and Sano [54]) *Let $k \geq 2$ be an integer. If a connected graph G satisfies $\deg_G(x) + \deg_G(y) \geq |G| - k + 1$ for every two nonadjacent vertices $x, y \in V(G)$, then G has a spanning tree T with $|L(T)| + |B(T)| \leq k + 1$.*

We show two degree conditions for graphs to have spanning trees with bounded total number of branch vertices and leaves.

Theorem 0.17 *Let $k \geq 2$ be an integer. Suppose that a connected graph G satisfies*

$$\max\{\deg_G(x), \deg_G(y)\} \geq \frac{|G| - k + 1}{2}$$

for every two nonadjacent vertices $x, y \in V(G)$. Then G has a spanning tree T with $|L(T)| + |B(T)| \leq k + 1$.

Theorem 0.18 *Let $k \geq 2$ be an integer. Let G be a 2-connected graph. Suppose that*

$$\max\{\deg_G(x), \deg_G(y)\} \geq \frac{|G| - k + 1}{2}$$

for every two vertices $x, y \in V(G)$ with $\mathrm{dist}_G(x, y) = 2$. Then G has a spanning tree T with $|L(T)| + |B(T)| \leq k + 1$.

The lower bounds $(|G| - k + 1)/2$ in Theorems 0.17 and 0.18 are sharp. Moreover, we cannot replace the assumption of being 2-connected in Theorem 0.18 with that of being connected.

For $k \geq 2$, a graph G is said to be *k-leaf-connected* if $|G| > k$ and for each subset S of $V(G)$ with $|S| = k$, G has a spanning tree T with precisely S as the set of leaves of T. By the definition, it is easy to see that the property of being "2-leaf-connected" is equivalent to the property of being "Hamilton-connected." Hence the property is a general concept of Hamilton-connected. The following result motivates our result in Chapter 3.3. Theorem 0.19 is a fundamental result, which gives an Ore-type condition for graphs to be k-leaf-connected.

Theorem 0.19 (Egawa, Matsuda, Yamashita, and Yoshimoto [23]) *Let $k \geq 2$ be an integer and let G be a $(k + 1)$-connected graph. Suppose that*

$$\deg_G(x) + \deg_G(y) \geq |G| + 1$$

for any two nonajacent vertices x, $y \in V(G)$. Then G is k-leaf-connected.

Note that the condition of being $(k + 1)$-connected is a necessary condition for graphs to be k-leaf-connected. In fact if G has a cut set with size at most k, then there is no spanning tree with precisely the cut set as the set of leaves of the tree. We give a Fan-type condition for graphs to be k-leaf-connected.

Theorem 0.20 *Let $k \geq 2$ be an integer. Suppose that G is a $(k + 1)$-connected graph and that*

$$\max\{\deg_G(u), \deg_G(v)\} \geq \frac{|G| + 1}{2}$$

for any vertices u and v in G with $\mathrm{dist}_G(u, v) = 2$. Then G is k-leaf-connected.

The lower bound in Theorem 0.20 is sharp.

In 1963, Moon and Moser obtained a degree condition for bipartite graphs to have a Hamiton cycle (resp. path). For a bipartite graph G with bipartition (A, B), we define

$$\sigma_{1,1}(G) = \min\Big\{ \deg_G(x) + \deg_G(y) : x \in A,\ y \in B, xy \notin E(G)\Big\}$$

if G is not a complete bipartite graph, and $\sigma_{1,1}(G) = \infty$ if G is a complete bipartite graph.

Theorem 0.21 (Moon and Moser [44]) *Let G be a connected bipartite graph with bipartition (A, B).*

(i) *If $|A| \le |B| \le |A| + 1$ and $\sigma_2(G) \ge |B|$, then G has a Hamilton path.*

(ii) *If $|A| = |B| = n \ge 2$ and $\sigma_{1,1}(G) \ge n + 1$, then G has a Hamilton cycle.*

Note that the conditions $|A| \le |B| \le |A| + 1$ and $|A| = |B| \ge 2$ are necessary conditions for bipartite graphs to have a Hamilton path and a Hamilton cycle, respectively. To find a long path in graphs is one of generalizations of finding a Hamilton path. Inspired by Theorem 0.21 (i), we study a Fan-type condition for long paths in bipartite graphs. The following is one of our main results.

Theorem 0.22 *Let m and n be positive integers with $n \ge m$. Let G be a bipartite graph having partite sets X_1 and X_2 with $|X_1| = |X_2| = n$. If*

(D1) $\max\{\deg_G(x_1), \deg_G(x_2)\} \ge m$ *or*

(D2) $\min\{\deg_G(x_1), \deg_G(x_2)\} \ge \frac{n+1}{2}$

for all vertices $x_1 \in X_1$ and $x_2 \in X_2$ with $x_1x_2 \notin E(G)$, then G contains a path P with $|V(P)| \ge 2m$.

If all vertices $x_1 \in X_1$ and $x_2 \in X_2$ satisfy (D2), then G has a Hamilton path by Theorem 0.21. Hence the condition (D1) is essential in Theorem 0.22. The lower bound of (D1) is sharp. As a consequence of our main result, we completely determine the bipartite Ramsey numbers $b(P_s, B_{t_1,t_2})$, where B_{t_1,t_2} is the graph obtained from a t_1-star and a t_2-star by joining their centers.

Theorem 0.23 *Let s, t_1 and t_2 be integers with $s \ge 2$ and $t_1 \ge t_2 \ge 0$. Then the following hold.*

(i) *If $t_1 = t_2$, then $b(P_s, B_{t_1,t_2}) = \lfloor \frac{s-1}{2} \rfloor + t_1 + 1$.*

(ii) *Assume that $t_1 > t_2$.*

(ii-a) *If* $t_1 \geq \lfloor \frac{s-1}{2} \rfloor$, *then*

$$b(P_s, B_{t_1,t_2}) = \begin{cases} \lfloor \frac{s-1}{2} \rfloor + t_1 + 1 & (s \textit{ is even, or } s \textit{ is odd and } t_1 \equiv 0 \ (\textit{mod } \frac{s-1}{2})) \\ \lfloor \frac{s-1}{2} \rfloor + t_1 & (\textit{otherwise}). \end{cases}$$

(ii-b) *If* $t_1 < \lfloor \frac{s-1}{2} \rfloor$, *then*

$$b(P_s, B_{t_1,t_2}) = \begin{cases} 2t_1 + 1 & (2t_1 - t_2 \geq \lfloor \frac{s-1}{2} \rfloor) \\ \lfloor \frac{s-1}{2} \rfloor + t_2 + 1 & (\textit{otherwise}). \end{cases}$$

This thesis consists of four chapters as follows: In Chapter 1, we give basic definitions, notations, and terminologies which are needed for reading this thesis. Moreover we introduce some results of Hamiltonicity which motivate our results. In Chapter 2, we show some results of the existence of spanning subgraphs with constrains on the degree and prove Theorems 0.6, 0.12, 0.14, and show that Conjecture 0.8 is true. In Chapter 3, we show some results of the existence of spanning trees with certain properties, which are extensions of properties of a Hamiton path and prove Theorems 0.17, 0.18, and 0.20. In Chapter 4, we show a Fan-type condition for bipartite graphs to have logn paths. As a consequence of the result, we completely determine the bipartite Ramsey numbers with respect to a path and a bistar. We prove Theorems 0.22 and 0.23 in Chapter 4.

Chapter 1

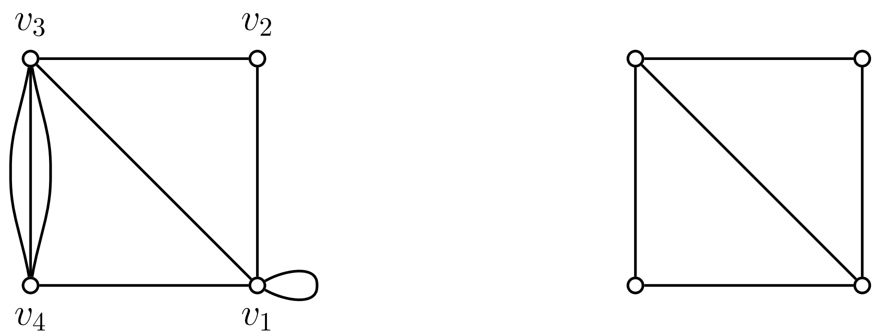
Preliminary

1.1 Graphs

A *graph* G consists of a *vertex set*, denoted by $V(G)$ and an *edge set*, denoted by $E(G)$. Each edge joins two vertices, which are not necessarily distinct. An edge joining two vertices x and y is denoted by xy or yx. An edge joining a vertex to itself is called a *loop*. Two or more edges which join a same pair of distinct two vertices are called *multiple edges*.

A graph that may have loops and multiple edges is called a *general graph*. A graph G having neither loops nor multiple edges is called a *simple graph*. In this thesis, a simple graph is called simply a graph.

The number of vertices of a graph G is called the *order* of G and is denoted by $|G|$. The number of edges of a graph G is called the *size* of G.



G G'

Figure 1.1: A general graph G and a simple graph G'

The graph in Fig. 1.1 satisfies that $V(G) = \{v_1, v_2, v_3, v_4\}$, $E(G) = \{v_1v_1, v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_3v_4, v_3v_4, v_3v_4\}$, $|G| = 4$, and the size of G is equal to 8. For the graph G in Fig. 1.1, v_1v_1 is a loop and the edges joining v_3 and v_4 are multiple edges.

If $e = xy$ is an edge of G, then x and y are *adjacent* in G and e is *incident* with x

and y. For a graph G and $v \in V(G)$, the number of edges incident with v is called the *degree* of v in G and is denoted by $\deg_G(v)$. The largest degree among the vertices of G is called the *maximum degree* in G and is denoted by $\Delta(G)$. Similarly, the smallest degree among the vertices of G is called the *minimum degree* in G and is denoted by $\delta(G)$. For example, the graph G' in Fig.1.1 satisfies $\Delta(G') = 3$ and $\delta(G') = 2$. For a graph G, the set of vertices adjacent to a vertex v in G is called the *neighborhood* of v and is denoted by $N_G(v)$.

A vertex with degree zero is called an *isolated vertex*. We denote by $i(G)$ the number of isolated vertices in G.

Theorem 1.1 (Handshaking lemma) *Let G be a graph. The sum of degree of all the vertices in G is equal to twice the size of G, that is,*

$$\sum_{v \in V(G)} \deg_G(v) = 2|E(G)|.$$

Proof. Since each edge is incident to exactly two vertices, summing the degrees of all the vertices of the graph G, each edge is counted twice. Hence this lemma holds. ∎

A *complete graph* is a graph in which every pair of two distinct vertices are adjacent and it is denoted by K_n, where n is the order of the graph.

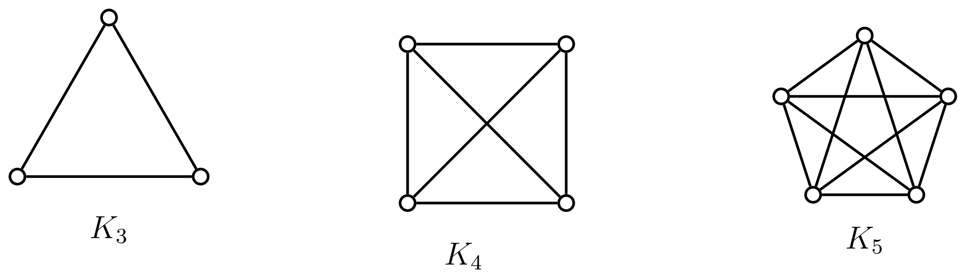


Figure 1.2: Complete graphs of order three, four, and five, respectively.

For an integer $n \geq 1$, a *path* P_n is a graph consisting of n vertices $v_1, \ldots, v_n$ and $n-1$ edges $v_i v_{i+1}$ for each $1 \leq i \leq n-1$. A *cycle* C_n is obtained from P_n by joining the two vertices with degree one in P_n.

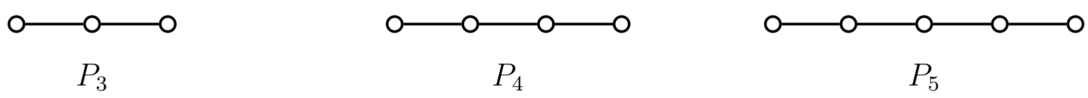


Figure 1.3: Paths

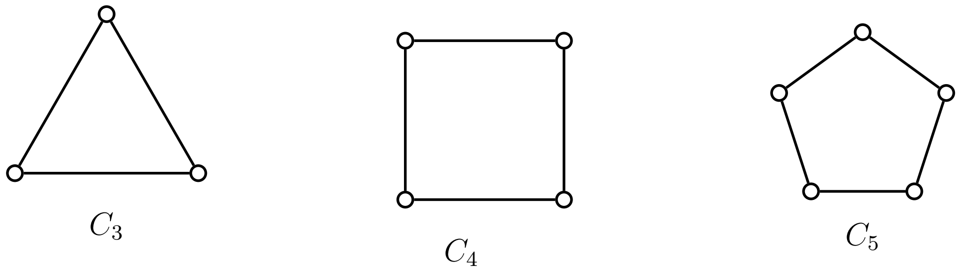


Figure 1.4: Cycles

A graph G is called a *bipartite graph* if $V(G)$ consists of two disjoint subsets A and B with $A \cup B = V(G)$ and every edge of G joins a vertex of A to a vertex of B. The two disjoint subsets A and B of $V(G)$ is called *partite sets* of G. A bipartite graph G with partite sets A and B is called a *complete bipartite graph* if any vertex of A is adjacent to all the vertices of B. If $|A| = m$ and $|B| = n$, then the complete bipartite graph G with partite sets A and B is denoted by $K_{m,n}$. For a positive integer n, the complete bipartite graph $K_{1,n}$ is called a *star*. In particular, $K_{1,3}$ is sometimes called a *claw*.

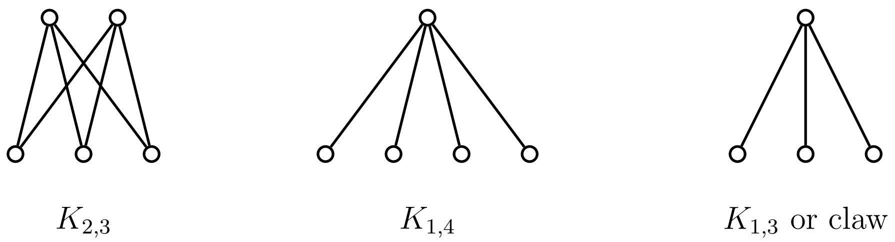


Figure 1.5: Complete bipartite graphs.

1.2 Subgraphs

A graph H is called a *subgraph* of a graph G if the vertex set of H is a subset of the vertex set of G and the edge set of H is a subset of the edge set of G. A *spanning subgraph* of G is a subgraph of G containing all the vertices of G.

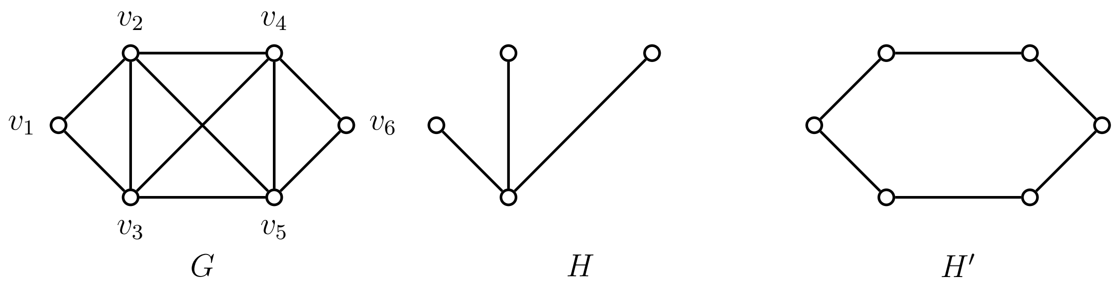


Figure 1.6: H is a subgraph of G and H' is a spanning subgraph of G.

For a nonempty vertex subset X of $V(G)$, the *subgraph of* G *induced by* X is defined as the subgraph of G whose vertex set is X and whose edge set consists of the edges of G joining vertices of X. The subgraph of G induced by X is denoted by $G[X]$. A subgraph H is called an *induced subgraph* of G if there exists a nonempty vertex subset X of $V(G)$ such that $H = G[X]$.

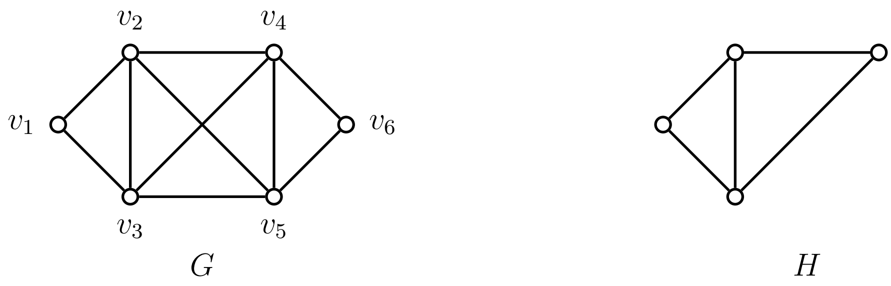


Figure 1.7: H is a subgraph of G induced by $X = \{v_1, v_2, v_3, v_4\}$.

For a vertex v in a graph G, the subgraph $G-v$ is obtained by deleting v and the edges incident with v from G. In other words, $G - v$ is the induced subgraph $G[V(G) \setminus \{v\}]$. For an edge e in a graph G, the subgraph $G-e$ is obtained by deleting e from G. In other words, $G - e$ is the spanning subgraph of G with the edge set $E(G) \setminus \{e\}$. For a proper vertex subset X of $V(G)$, the subgraph $G - X$ is the induced subgraph $G[V(G) \setminus X]$. For an edge subset Y of $E(G)$, the subgraph $G - Y$ is a spanning subgraph with the edge set $E(G) \setminus Y$. For nonadjacent two vertices x and y in a graph G, the graph $G + xy$ is obtained from G by adding the edge xy.

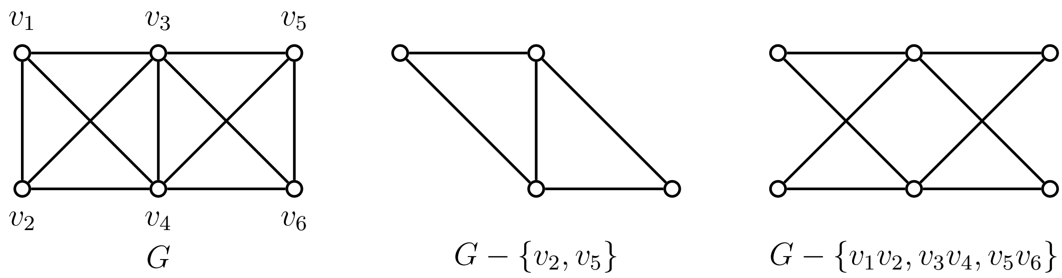


Figure 1.8: Some subgraphs of G.

For given graphs G and H, if there exists a bijection $f : V(G) \to V(H)$ such that $f(x)$ and $f(y)$ are adjacent in H if and only if x and y are adjacent in G, then G and H are *isomorphic*. For a given graph H, a graph G is said to be H-*free* if G contains no induced subgraph isomorphic to H.

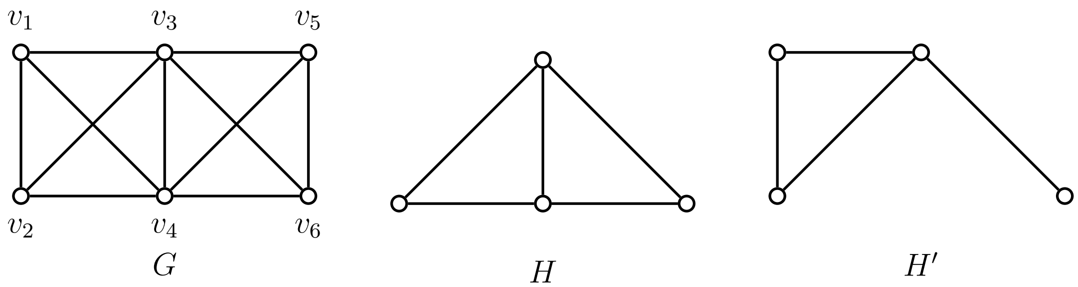


Figure 1.9: A $K_{1,3}$-free graph G and induced subgraphs H and H' of G.

The graph G in Fig.1.9 is $K_{1,3}$-free. In fact, each induced subgraph of G with four vertices is not isomorphic to $K_{1,3}$. For example, H is the subgraph of G induced by $\{v_2, v_3, v_4, v_6\}$ and H' is the subgraph of G induced by $\{v_1, v_2, v_3, v_6\}$. Then neither H nor H' are isomorphic to $K_{1,3}$.

For two graphs G and H, the *union* $G \cup H$ is the graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. The *join* $G + H$ is the graph obtained from $G \cup H$ by adding all the edges joining a vertex of G to a vertex of H. Let $k \geq 2$ be an integer. For a graph G which consists of k disjoint copies of a graph H, we write $G = kH$.

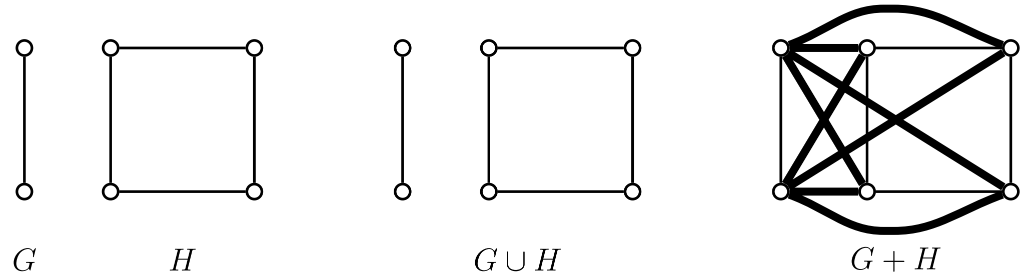


Figure 1.10: The union $G \cup H$ and the join $G + H$. The thick edges in $G + H$ are the additional edges joining a vertex of G to a vertex of H.

1.3 Paths and cycles

A *walk* in a graph G is a sequence of vertices and edges

$$v_0, e_0, v_1, \ldots, v_{i-1}, e_{i-1}, v_i, \ldots, v_{m-1}, e_{m-1}, v_m$$

such that the edge e_{i-1} is incident with the two vertices v_{i-1} and v_i for each $1 \leq i \leq m$. For the above walk, the vertices v_0 and v_m is called *end-vertices* of the walk. The *length* of a walk is the number of edges. A *trail* is a walk such that all edges are distinct. A *path* is a walk such that every vertex are distinct. For a path with end-vertices x and y in a graph G, we say that the path *connects* x and y in G. A walk whose end-vertices are the same is a *closed walk*. A closed walk with order at least four whose vertices are distinct except for the end-vertices is a *cycle*. A cycle of an even order is called an *even cycle*. A cycle of an odd order is called an *odd cycle*.

Figure 1.11: (1) A sequence $v_1v_3v_9v_{10}v_5v_7v_8v_6$ is a path. (2) A sequence $v_1v_3v_5v_7v_8v_6v_4v_2v_1$ is a cycle.

1.4 Connectivity and distance

A graph G is said to be *connected* if for any distinct two vertices are connected by a path in G. If a graph G is not connected, then G is said to be *disconnected*. For a connected graph G, a vertex v in $V(G)$ is called a *cut vertex* if $G - v$ is disconnected and an edge e in $E(G)$ is called a *cut edge* or *bridge* if $G - e$ is disconnected.

Figure 1.12: For a connected graph G, a vertex v is a cut vertex in G.

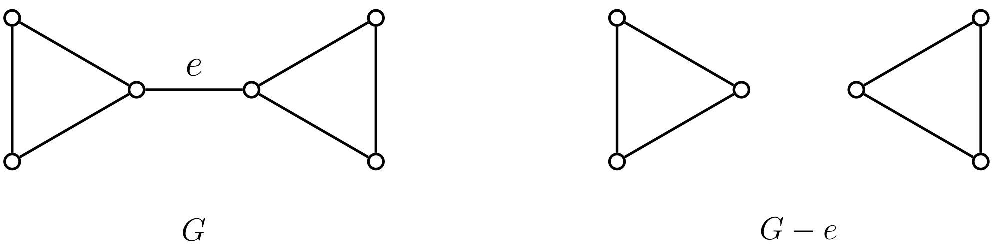


Figure 1.13: For a connected graph G, an edge e is a cut edge in G.

A maximal connected subgraph of a graph G is called a *component* of G. The number of components of G is denoted by $\omega(G)$. For example, the graph G in Fig.1.12 satisfies $\omega(G - v) = 3$ and the graph G in Fig.1.13 satisfies $\omega(G - e) = 2$.

For an integer $k \geq 1$, a connected graph G is called *k-connected* if $|G| > k$ and $G - X$ is connected for every $X \subseteq V(G)$ with $|X| \leq k - 1$. Note that if G is k-connected, then $\delta(G) \geq k$.

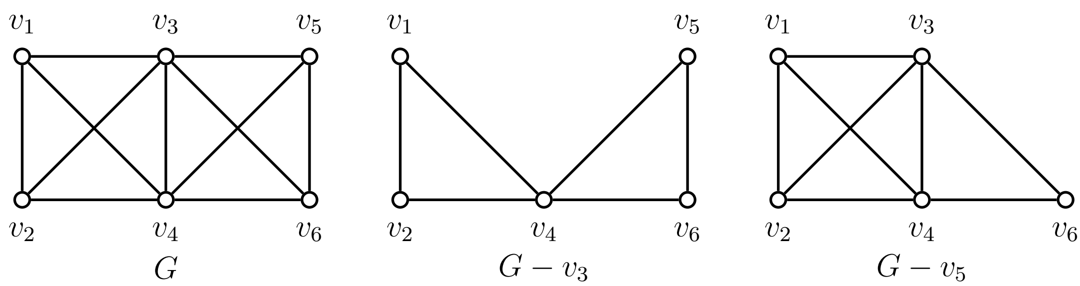


Figure 1.14: A 2-connected graph G.

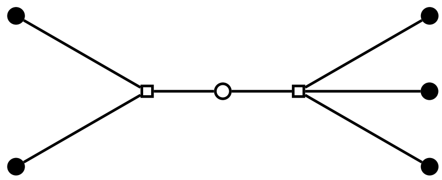
In Fig.1.14, a graph G is 2-connected. In fact, for each vertex v of G, $G - v$ is connected. Fig.1.14 shows that $G - v_3$ and $G - v_5$ are connected.

For two vertices x and y of a connected graph G, the *distance* between x and y in G is the length of a shortest path connecting x and y in G and is denoted by $\text{dist}_G(x, y)$. For example, the graph G in Fig.1.14 satisfies $\text{dist}_G(v_1, v_2) = 1$ and $\text{dist}_G(v_1, v_6) = 2$.

1.5 Trees

A connected graph having no cycle is called a *tree*. A spanning subgraph T of a graph G is called a *spanning tree* of G if T is a tree. A *leaf* of a tree is a vertex of degree one and a *branch vertex* of a tree is a vertex of degree strictly greater than two. For a tree T, let

$$L(T) = \{x \in V(T) \mid x \text{ is a leaf of } T\} \textit{ and}$$
$$B(T) = \{x \in V(T) \mid x \text{ is a branch vertex of } T\}.$$



T

Figure 1.15: A graph T is a tree, black vertices are the leaves of T and square vertices are the branch vertices of T.

G A spanning tree of G

Figure 1.16: The subgraph of G consisting of all the vertices of G and thick edges is a spanning tree of G.

Theorem 1.2 *The following properties are equivalent for a graph T:*

(i) *T is a tree,*

(ii) *T is connected and every edge of T is a cut edge,*

(iii) *any two vertices of T are connected by the unique path in T, and*

(iv) *T has no cycle and for any two vertices x, y of T, $T + xy$ has the unique cycle.*

Proof. (i)⇒(ii) Let T be a tree. Suppose that there exists an edge $e = xy$ of T such that $T - e$ is connected. Then there exists a path P in $T - e$ connecting x and y. Then $P + e$ is a subgraph of T and $P + e$ contains a cycle. This is a contradiction.
(ii)⇒(iii) Let T be a connected graph such that every edge of T is a cut edge. Suppose that there exist two paths P and Q in T connecting two vertices x and y of T. Then $P \cup Q$ is a subgraph of T and $P \cup Q$ contains a cycle C. Any edge e contained in C is not a cut edge of T. This is a contradiction.
(iii)⇒(iv) Let T be a graph such that any two vertices of T are connected by the unique path in T. Since any two vertices x, y of T are connected by the unique path P in T, $T + xy$ contains the unique cycle $P + xy$. Suppose that T has a cycle. Then for two vertices x and y in a cycle of T, there are at least two paths in T connecting x and y. This is a contradiction
(iv)⇒(i) Let T be a graph having no cycle. We prove the following statement: "if T is not connected, then $T + xy$ has no cycle for some two vertices x and y of T." Suppose that T is not a connected graph. Then T has at least two components. Let x be a vertex of T and let y be a vertex of T not contained in the component of T containing x. Then $T + xy$ contains no cycle. ∎

Theorem 1.3 *If T is a tree of order n and size m, then $m = n - 1$.*

Proof. We proceed by induction on the size of a tree. There is only one tree of size 0 and its order is 1. Thus this theorem holds for a tree of size 0. Assume that the order of every tree of size $m - 1 \geq 0$ is m. Let T be a tree of order n and size m and let e be an edge of T. By Theorem 1.2 (ii), e is a cut edge of T. Hence $T - e$ has two components T_1 and T_2. Then both T_1 and T_2 have no cycle, i.e. both T_1 and T_2 are trees. By the induction hypothesis, $|E(T_1)| = |V(T_1)| - 1$ and $|E(T_2)| = |V(T_2)| - 1$. Hence we obtain $m = |E(T_1)| + |E(T_2)| + |\{e\}| = |V(T_1)| + |V(T_2)| - 1 = n - 1$. ∎

For two distinct vertices x and y of a tree T, $P_T(x, y)$ denotes the unique path in T connecting x and y.

Given a tree T, we often regard T as a *rooted tree* in which all the edges are directed away from a specified vertex of T. Such a specified vertex of T is called a *root* of T. Let T be a rooted tree with root v. The *out-neighborhood* of x, denoted by $N^+_{T,v}(x)$, is the set of vertices adjacent from x in the rooted tree with respect to (T, v). The *in-neighborhood vertex* of x, denoted by $n^-_{T,v}(x)$, is a vertex such that $n^-_{T,v}(x) \in N_T(x) \setminus N^+_{T,v}(x)$. Note that if $x \neq r$, then $n^-_{T,v}(x)$ is unique. If there is no ambiguity, we write $N^+_T(x)$ for $N^+_{T,v}(x)$ and $n^-_T(x)$ for $n^-_{T,v}(x)$ and we use the following definitions. For a subset $X \subseteq V(T)$, X^- denotes the set of vertices adjacent to a vertex in X and for a vertex $v \in V(T)$, v^- denotes the unique vertex adjancet to v. For a subset $Y \subseteq V(T)$, Y^+ denotes the set of vertices adjacent from a vertex in Y.

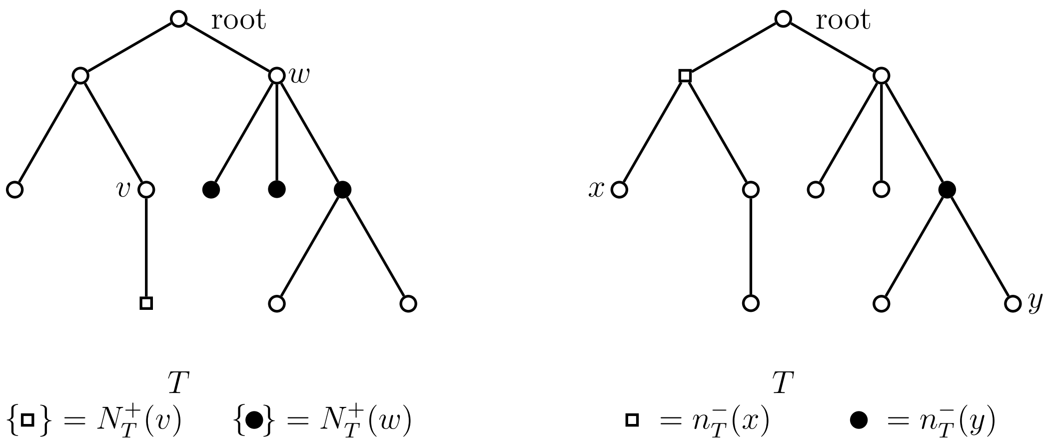


Figure 1.17: A rooted tree T.

1.6 Hamiltonian properties

In this section, we introduce some Hamitonian properties and results which give sufficient conditions for graphs to satisfy Hamiltonian properties. A cycle (resp. path) in a graph G is called a *Hamilton cycle* (resp. *path*) of G if it contains all the vertices of G. A graph G is called *Hamilton-connected*, if for any two vertices x and y of G, there is a Hamilton path of G connecting x and y. Since the problem of determining whether a given graph G has a Hamilton cycle (resp. path) is NP-complete [28], we have studied sufficient conditions for graphs to have a Hamilton cycle (resp. path). The problem of determining whether a given graph G is Hamilton-connected is also NP-complete [19]. In this section, we introduce degree conditions and forbidden subgraph conditions which motivate our results.

1.6.1 Degree conditions

For a graph G, a subset X of $V(G)$ is *independent* if no two vertices in X are adjacent in G. For a graph G, the *independence number* of G is the maximum number of vertices in an independent set of $V(G)$ and the independence number of G is denoted by $\alpha(G)$.

For a graph G, where k is an integer with $k \geq 2$, define

$$\sigma_k(G) = \min_{S \subseteq V(G)} \left\{ \sum_{x \in S} \deg_G(x) \;\middle|\; S \text{ is an independent set of } k \text{ vertices} \right\}$$

if $\alpha(G) \geq k$, and $\sigma_k(G) := \infty$ if $\alpha(G) < k$. In 1960, Ore gave a sufficient condition for graphs to have a Hamilton cycle (path) [48] and to be Hamilton-connected [49]. These results are cornerstones of graph theory.

Theorem 1.4 (Ore [48, 49]) *Let G be a connected graph with order at least three. Suppose that $\sigma_2(G) \geq |G| + s$ with $s \in \{-1, 0, 1\}$.*

(i) *If $s = -1$, then G has a Hamilton path.*

(ii) *If $s = 0$, then G has a Hamilton cycle.*

(iii) *If $s = 1$, then G is Hamilton-connected.*

Note that the degree conditions of Theorem 1.4 are best possible in the sense we cannot replace $|G| + s$ by $|G| + s - 1$.

In 1976, Bondy and Chvátal introduced a closure concept in [7]. An *s-closure* $CL_s(G)$ of a graph is recursively joining pairs of nonadjacent vertices such that the degree sum of these vertices is at least s, until no such pair remains.

Theorem 1.5 (Bondy and Chvátal [7]) *Let G be a graph. If u and v are nonadjacent vertices with $\deg_G(u) + \deg_G(v) \geq |G|$, then G has a Hamilton cycle if and only if $G + uv$ has a Hamilton cycle.*

We can obtain Theorem 1.4 (ii) by Theorem 1.5 as follows. If a graph G satisfies the condition of Theorem 1.4 (ii), then $CL_{|G|}(G)$ is a complete graph. It is easy to see that a complete graph with order at least three has a Hamilton cycle. By Theorem 1.5, G has a Hamilton cycle.

In 1984, Fan [25] gave a degree condition for graphs to have a Hamilton cycle (resp. path) which is weaker than the condition of Theorem 1.4. This degree condition is so-called a Fan-type degree condition. Benhocine and Wojda gave a Fan-type condition for graphs to be Hamilton-connected.

Theorem 1.6 (Fan [25], Benhocine and Wojda [5]) *Let $s \in \{-1, 0, 1\}$ and let G be a graph. Suppose that*

$$\max\{\deg_G(x), \deg_G(y)\} \geq \frac{|G| + s}{2}$$

for any two vertices $x, y \in V(G)$ with $\mathrm{dist}_G(x, y) = 2$.

(i) *If G is connected and $s = -1$, then G has a Hamilton path.*

(ii) *If G is 2-connected and $s = 0$, then G has a Hamilton cycle.*

(iii) *If G is 3-connected and $s = 1$, then G is Hamilton-connected.*

The conditions of Theorem 1.6 are best possible.

Liu, Tian, and Wu in 1986 and independently, Broersma in 1988, showed that we could relax the degree condition of Theorem 1.4 (i) by restricting graphs to be $K_{1,3}$-free.

Theorem 1.7 (Liu, Tian, and Wu [38], Broersma [8]) *Let G be a connected $K_{1,3}$-free graph. If*

$$\sigma_3(G) \geq |G| - 2,$$

then G has a Hamilton path.

1.6.2 Forbidden subgraph conditions

Faudree and Gould characterized the forbidden pairs for connected graphs to have a Hamilton path. The graph $N(p,q,r)$ is one obtained from the triangle xyz by joining p isolated vertices to x, q isolated vertices to y, r isolated vertices to z (Fig. 1.18).

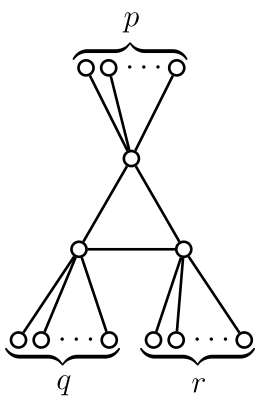


Figure 1.18: The graph $N(p,q,r)$

Theorem 1.8 (Faudree and Gould [26]) *Let H_1 and H_2 be connected graphs with $H_1, H_2 \neq P_1, P_2, P_3$. Then, every connected $\{H_1, H_2\}$-free graph has a Hamilton path if and only if H_1 is $K_{1,3}$ and H_2 is one of the graph $N(p,q,r)$ for $0 \leq p,q,r \leq 1$ or P_4.*

Note that the "if" part of Theorem 1.8 was obtained by Duffus, Jacobson, and Gould [21].

Chapter 2

Connected degree factors

In this chapter, we focus on a spanning subgraph with constrains on the degree. Such a spanning subgraph is called a *connected degree factor*.

2.1 A spanning k-tree

For an integer $k \geq 2$, T is a *k-tree* if the maximum degree of T is at most k. For a graph G, T is a *spanning k-tree* of G if T is a k-tree with $V(T) = V(G)$. Note that a Hamiltonian path of a graph G with the maximum degree two. Hence a Hamiltonian path is a spanning 2-tree. The following result is a natural extention of Theorem 1.4.

Theorem 2.1 (Win [56]) *Let $k \geq 2$ be an integer and let G be a connected graph. If*

$$\sigma_k(G) \geq |G| - 1,$$

then G has a spanning k-tree.

Proof. Let G be a graph satisfying all the conditions of Theorem 2.1, but has no spanning k-tree. The case $k = 2$ follows from Theorem 1.4. Thus we consider the case $k \geq 3$. Let T be a maximal k-tree of G. Since G is connected and T is not a spanning tree, there exists a vertex v not contained in T and adjacent to a vertex w in $V(T)$.

Claim 2.1.1 $\deg_T(w) = k$.

Proof. Suppose that $\deg_T(w) \neq k$. Since T is a k-tree, $\deg_T(w) < k$. Then $T' := T + wv$ is a k-tree with $|V(T')| > |V(T)|$. This contradicts the maximality of T. Hence $\deg_T(w) = k$. ∎

Let $D_1, D_2, \ldots, D_k$ denote the components of $T - \{w\}$. For each $i = 1, 2, \ldots, k$, let u_i be the vertex of D_i adjacent to w in T and let x_i be a leaf of T contained in D_i.

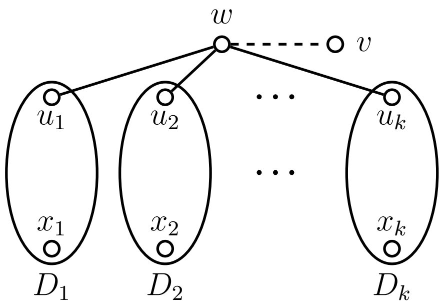


Fig. 2.1.1 A maximal k-tree T of G.

Claim 2.1.2 $\{x_1, x_2, \dots, x_k\}$ *has no vertex adjacent to a vertex not contained in* T.

Proof. Suppose that x_i is adjacent to a vertex y not contained in T for some $i = 1, 2, \dots, k$. Then $T' := T + x_i y$ is a k-tree of G with $|V(T')| > |V(T)|$. This contradicts the maximality of T. ∎

Claim 2.1.3 $\{x_1, x_2, \dots, x_k\}$ *is an independent set of* G.

Proof. Suppose that there exist two distinct vertices x_i and x_j in $\{x_1, x_2, \dots, x_k\}$ such that x_i and x_j are adjacent in G. Then $T' := T + x_i x_j + vw - u_i w$ is a k-tree of G with $|V(T')| > |V(T)|$. This contradicts the maximality of T. ∎

Let t be an integer with $1 \le t \le k$. Choose a vertex x_a from $\{x_1, x_2, \dots, x_k\} \setminus \{x_t\}$ such that

$$|N_G(x_a) \cap V(D_t)| = \max_{i \neq t} |N_G(x_i) \cap V(D_t)|.$$

Claim 2.1.4 *For every* $z \in N_G(x_a) \cap V(D_t)$, $\deg_T(z) = k$.

Proof. Suppose that there exists a vertex $z \in N_G(x_a) \cap V(D_t)$ such that $\deg_T(z) \neq k$. Since T is a k-tree, $\deg_T(z) < k$. Then $T' := T + x_a z + vw - u_a w$ is a k-tree of G with $|V(T')| > |V(T)|$. This contradicts the maximality of T. ∎

We regard D_t as a rooted tree with root x_t.

Claim 2.1.5 *For every* $z \in N_G(x_a) \cap V(D_t)$, $N^+_{D_t}(z) \cap N_G(x_t) = \emptyset$.

Proof. Suppose that there exists a vertex $z \in N_G(x_a) \cap V(D_t)$ such that there exists $z^+ \in N^+_{D_t}(z) \cap N_G(x_t)$. Then $T' := T + x_t z^+ + x_a z + vw - zz^+ - u_a w$ is a k-tree of G with $|V(T')| > |V(T)|$. This contradicts the maximality of T. ∎

Claim 2.1.6 *The vertex u_t is not in $N_G(x_a)$.*

Proof. Suppose that $u_t x_a \in E(G)$. Then $T' := T + u_t x_a + vw - wx_t$ is a k-tree of G with $|V(T')| > |V(T)|$, a contradiction. ∎

By Claims 2.1.4, 2.1.5, 2.1.6, and the choice of x_a, we obtain

$$\begin{aligned}|V(D_t)| &\geq |N_G(x_t) \cap V(D_t)| + \sum_{z \in N_G(x_a) \cap V(D_t)} |N^+_{D_t}(z)| + |\{x_t\}| \\ &= |N_G(x_t) \cap V(D_t)| + (k-1)|N_G(x_a) \cap V(D_t)| + 1 \\ &\geq |N_G(x_t) \cap V(D_t)| + \sum_{1 \leq i \neq t \leq k} |N_G(x_i) \cap V(D_t)| + 1 \\ &= \sum_{i=1}^{k} |N_G(x_i) \cap V(D_t)| + 1.\end{aligned}$$

It follows from the above inequality that

$$\begin{aligned}\sum_{i=1}^{k} \deg_G(x_i) &\leq \sum_{i=1}^{k} \left(\sum_{j=1}^{k} |N_G(x_i) \cap V(D_j)| + |\{w\}| \right) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{k} |N_G(x_i) \cap V(D_j)| + k \\ &= \sum_{j=1}^{k} \sum_{i=1}^{k} |N_G(x_i) \cap V(D_j)| + k \\ &\leq \sum_{j=1}^{k} (|D_j| - 1) + k \\ &= |T| - 1 \leq |G| - 2.\end{aligned}$$

On the other hand, $\sum_{i=1}^{k} \deg_G(x_i) \geq \sigma_k(G) = |G| - 1$. This is a contradiction. ∎

By restricting graphs to be star-free, Caro, Krasikov, and Roditty in 1985 and independently, Jackson and Wormald in 1990, obtained the following result, which guarantees the existence of a spanning k-tree.

Theorem 2.2 (Caro, Krasikov, and Roditty [11], Jackson and Wormald [32]) *For an integer $k \geq 2$, every connected $K_{1,k}$-free graph contains a spanning k-tree.*

2.2 Degree sum condition for the existence of spanning k-trees in star-free graphs

In this section, we show the degree sum condition for graphs having no $K_{1,k+1}$ as an induced subgraph to have a spanning k-tree.

Theorem 2.2 is best possible in the sense that there exist infinitely many connected $K_{1,k+1}$-free graphs which have no spanning k-tree. Thus some additional conditions are needed for connected $K_{1,k+1}$-free graphs to have a spanning k-tree. The purpose of this section is to give a degree sum condition for connected $K_{1,k+1}$-free graphs to have a spanning k-tree. Our main result is the following.

Theorem 2.3 *Let k be an integer with $k \geq 2$. If a connected $K_{1,k+1}$-free graph G satisfies*

$$\sigma_{3k-3}(G) \geq |G| - 2,$$

then G has a spanning k-tree.

Theorem 2.3 gives a generalization of Theorem 1.7. By Theorem 2.3, we also obtain an upper bound on the independence number $\alpha(G)$ for $K_{1,k+1}$-free graphs to have a spanning k-tree.

Corollary 2.4 *Let k be an integer with $k \geq 2$. If a connected $K_{1,k+1}$-free graph G satisfies*

$$\alpha(G) \leq 3k - 4,$$

then G has a spanning k-tree.

The degree sum condition of Theorem 2.3 is sharp as shown in the next subsection and the example also shows the sharpness of the independence number in Corollary 2.4.

2.2.1 Sharpness of Theorem 2.3

We show that the lower bounds of $\sigma_{3k-3}(G)$ in Theorem 2.3 and the independence number in Corollary 2.4 are best possible. In fact, we give the following example:

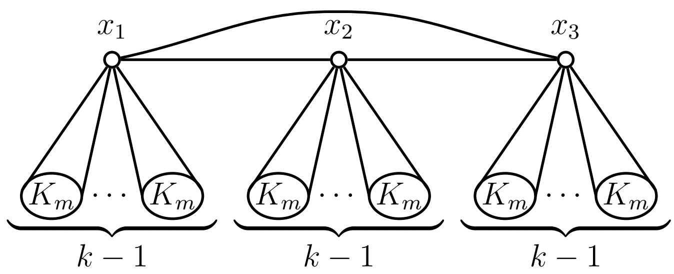


Figure 2.1: An infinite family of connected $K_{1,k+1}$-free graphs G having no spanning k-tree and satisfying $\sigma_{3k-3}(G) = |G| - 3$

Let $k \geq 2$ and $m \geq 1$ be integers. Let T be a triangle with $V(T) = \{x_1, x_2, x_3\}$. For each $i = 1, 2, 3$, define a graph H_i as $k-1$ disjoint copies of K_m. The graph G is obtained by joining x_i and all the vertices in $V(H_i)$ for each $i = 1, 2, 3$. Then G has no induced subgraph isomorphic to $K_{1,k+1}$ and $|G| = 3m(k-1)+3$. Since $\alpha(H_1 \cup H_2 \cup H_3) = 3k-3$, we can choose $3k-3$ independent vertices one by one from each complete graph K_m. Then $\sigma_{3k-3}(G) = 3m(k-1) = |G|-3$. For any spanning tree T of G, one of the three vertices x_1, x_2 and x_3 must have degree more than k in T. Hence G has no spanning k-tree, and thus the lower bounds of $\sigma_{3k-3}(G)$ in Theorem 2.3 and the independence number in Corollary 2.4 are sharp.

Note that the graphs in Figure 2.1 show that $K_{1,k}$-freeness in Theorem 2.2 cannot be replaced by $K_{1,k+1}$-freeness.

2.2.2 Proof of Theorem 2.3

Let k be an integer with $k \geq 2$, and let G be a connected $K_{1,k+1}$-free graph satisfying $\sigma_{3k-3}(G) \geq |G|-2$. The case $k=2$ follows from Theorem 1.7. Thus we consider the case when $k \geq 3$. Let T be a maximal k-tree of G. Suppose that T is not a spanning tree of G. Then G has a vertex u_0 not contained in T which is adjacent to a vertex v in $V(T)$.

Claim 2.2.1 $\deg_T(v) = k$.

Proof. Suppose that $\deg_T(v) \neq k$. Since T is a k-tree, $\deg_T(v) < k$. Then $T + vu_0$ is a k-tree of order $|V(T)|+1$. This contradicts the maximality of T. Hence $\deg_T(v) = k$. ∎

Let $S_1, S_2, \ldots, S_k$ denote the components of $T - v$. For each $1 \leq i \leq k$, let s_i be the vertex of S_i which is adjacent to v in T. Note that $\deg_{S_i}(s_i) \leq k-1$ for each i.

Claim 2.2.2 *For each $1 \leq i \leq k$, u_0 is nonadjacent to s_i in G.*

Proof. Suppose that $u_0 s_i \in E(G)$ for some $1 \leq i \leq k$. Then $T + vu_0 + u_0 s_i - vs_i$ is a k-tree of order $|V(T)|+1$, which contradicts the maximality of T. ∎

Since v is a common neighbor of u_0, s_1, s_2, …, s_k in G, by the $K_{1,k+1}$-freeness of G and Claim 2.2.2, s_i and s_j are adjacent in G for some $1 \leq i < j \leq k$. Without loss of generality, we may assume that $s_{k-1}s_k \in E(G) \setminus E(T)$.

Claim 2.2.3 $\deg_T(s_{k-1}) = \deg_T(s_k) = k$.

Proof. By symmetry, it suffices to show that $\deg_T(s_k) = k$. If $\deg_T(s_k) \neq k$, then $\deg_T(s_k) < k$ since T is a k-tree, and hence $T + s_{k-1}s_k + u_0 v - vs_k$ is a k-tree of order $|V(T)|+1$. This contradicts the maximality of T. ∎

As seen in Figure 2.2, we redefine $T_i = S_i$ and $t_i = s_i$ for each $1 \leq i \leq k-2$ and let $T_{k-1}, \ldots, T_{2k-3}$ and $T_{2k-2}, \ldots, T_{3k-4}$ be the components of $S_{k-1} - s_{k-1}$ and $S_k - s_k$, respectively. Let t_{k-1}, …, t_{2k-3} (resp. t_{2k-2}, …, t_{3k-4}) denote the vertices of $T_{k-1}, \ldots, T_{2k-3}$ (resp. $T_{2k-2}, \ldots, T_{3k-4}$) which are adjacent to s_{k-1} (resp. s_k) in T.

Since $T_1, T_2, \dots, T_{3k-4}$ are vertex-disjoint k-trees, we can choose a leaf $u_i \in V(T_i)$ of T for each $1 \le i \le 3k-4$. By the maximality of T and $\deg_T(u_i) = 1$, $N_G(u_i) \subseteq V(T)$ for each $1 \le i \le 3k-4$.

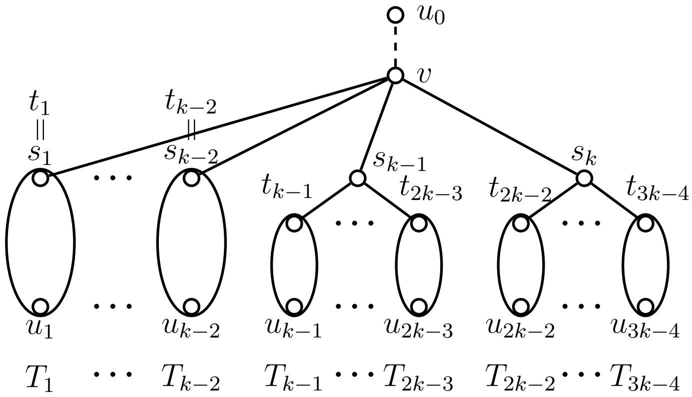


Figure 2.2: A maximal k-tree T

Claim 2.2.4 *The set $\{u_0, u_1, \dots, u_{3k-4}\}$ is an independent set of G.*

Proof. For $1 \le i \le 3k-4$, since $N_G(u_i) \subseteq V(T)$, we have $u_0u_i \notin E(G)$. Suppose that $u_iu_j \in E(G)$ for some $1 \le i < j \le 3k-4$. Consider the following tree T_A;

$$T_A := \begin{cases} T + u_iu_j + u_0v - vt_i & \text{if } 1 \le i \le k-2 \\ T + u_iu_j + u_0v + s_{k-1}s_k - vs_k - s_{k-1}t_i & \text{if } k-1 \le i \le 2k-3 \\ T + u_iu_j + u_0v + s_{k-1}s_k - vs_{k-1} - s_kt_i & \text{if } 2k-2 \le i \le 3k-4. \end{cases}$$

Then T_A is a k-tree of order $|V(T)| + 1$, which contradicts the maximality of T. Hence the claim holds. ∎

For each $1 \le i \le 3k-4$, define

$$W_i = \left(\bigcup_{0 \le j \le 3k-4, j \ne i} N_G(u_j) \right) \cap V(T_i).$$

Claim 2.2.5 *For each $1 \le i \le 3k-4$, $t_i \notin W_i$.*

Proof. If $t_i \in W_i$ for some $1 \le i \le 3k-4$, then t_i is adjacent to a leaf u_j of T_j with $j \ne i$ or to the vertex u_0. Consider the following tree T_B;

$$T_B := \begin{cases} T + t_iu_j + u_0v - vt_i & \text{if } 1 \le i \le k-2 \\ T + t_iu_j + s_{k-1}s_k + u_0v - s_{k-1}t_i - vs_k & \text{if } k-1 \le i \le 2k-3 \\ T + t_iu_j + s_{k-1}s_k + u_0v - s_kt_i - vs_{k-1} & \text{if } 2k-2 \le i \le 3k-4. \end{cases}$$

Then T_B is a k-tree of order $|V(T)| + 1$, which contradicts the maximality of T. Consequently, $t_i \notin W_i$ for each $1 \le i \le 3k-4$. ∎

Claim 2.2.6 *For each $1 \le i \le 3k-4$, any vertex $w \in W_i$ satisfies the following three statements:*

(i) $\deg_T(w) = k$;

(ii) *no vertex u_j with $1 \le j \le 3k-4$ is adjacent to any vertex of $N^+_{T_i,u_i}(w)$ in G; and*

(iii) $|(N_G(w) \cap \{u_0, u_1, u_2, \ldots, u_{3k-4}\}) \setminus \{u_i\}| \le k-1$.

Proof. (i) Suppose that $\deg_T(w) \ne k$ for some $w \in W_i$ with $1 \le i \le 3k-4$. Since T is a k-tree, $\deg_T(w) < k$. By the definition of W_i, w is adjacent to a vertex u_j with $j \ne i$ in G (possibly, $j = 0$). Consider the following tree T_C;

$$T_C := \begin{cases} T + u_j w + u_0 v - v t_i & \text{if } 1 \le i \le k-2 \\ T + u_j w + s_{k-1}s_k + u_0 v - s_{k-1}t_i - v s_k & \text{if } k-1 \le i \le 2k-3 \\ T + u_j w + s_{k-1}s_k + u_0 v - s_k t_i - v s_{k-1} & \text{if } 2k-2 \le i \le 3k-4. \end{cases}$$

Then T_C is a k-tree of order $|V(T)| + 1$, which contradicts the maximality of T. Hence $\deg_T(w) = k$ as desired.

(ii) Suppose that for some $1 \le j \le 3k-4$, u_j is adjacent to a vertex $w^+ \in N^+_{T_i,u_i}(w)$ in G. By the definition of W_i, w is adjacent to a leaf u_ℓ with $\ell \ne i$ or to the vertex u_0. Note that $w \ne t_i$ by Claim 2.2.5. Consider the following k-tree T_D;

$$T_D := \begin{cases} T + u_\ell w + u_j w^+ + u_0 v - v t_i - w w^+ & \text{if } 1 \le i \le k-2 \\ T + u_\ell w + u_j w^+ + s_{k-1}s_k + u_0 v - v s_k - s_{k-1}t_i - w w^+ & \text{if } k-1 \le i \le 2k-3 \\ T + u_\ell w + u_j w^+ + s_{k-1}s_k + u_0 v - v s_{k-1} - s_k t_i - w w^+ & \text{if } 2k-2 \le i \le 3k-4. \end{cases}$$

Then T_D is a k-tree of order $|V(T)| + 1$. This contradicts the maximality of T.

(iii) To the contrary, assume that $|(N_G(w) \cap \{u_0, u_1, u_2, \ldots, u_{3k-4}\}) \setminus \{u_i\}| \ge k$. Since $\deg_T(w) = k \ge 3$ by Claim 2.2.6 (i), a vertex $w_1 \in N^+_{T_i,u_i}(w)$ exists. Note that w_1 is different from any u_j with $j \ne i$ because $w_1 \in V(T_i)$ and $(\{u_0, u_1, u_2, \ldots, u_{3k-4}\} \setminus \{u_i\}) \cap V(T_i) = \emptyset$. Then w_1 and k vertices in $(N_G(w) \cap \{u_0, u_1, u_2, \ldots, u_{3k-4}\}) \setminus \{u_i\}$ are all neighbors of w in G. Moreover, Claims 2.2.4 and 2.2.6 (ii) assart that w_1 and k vertices in $(N_G(w) \cap \{u_0, u_1, u_2, \ldots, u_{3k-4}\}) \setminus \{u_i\}$ are independent in G. This contradicts the assumption that G is $K_{1,k+1}$-free. Hence $|(N_G(w) \cap \{u_0, u_1, u_2, \ldots, u_{3k-4}\}) \setminus \{u_i\}| \le k-1$.
∎

Claim 2.2.7 *We have $|N_G(s_i) \cap \{u_0, u_1, \ldots, u_{3k-4}\}| \le k-1$ for each $i = k-1$ and k.*

Proof. We first prove that

$$N_G(s_k) \cap \{u_0, u_1, \ldots, u_{3k-4}\} \subseteq \{u_{2k-2}, \ldots, u_{3k-4}\}.$$

By Claim 2.2.2, $s_k u_0 \notin E(G)$. If $s_k u_i \in E(G)$ for some $i = 1, 2, \ldots, 2k-3$, then $T + s_k u_i + u_0 v - v s_k$ is a k-tree of order $|V(T)| + 1$. This contradicts the maximality of T. Hence $N_G(s_k) \cap \{u_0, u_1, \ldots, u_{3k-4}\} \subseteq \{u_{2k-2}, \ldots, u_{3k-4}\}$ as desired. This implies that $|N_G(s_k) \cap \{u_0, u_1, \ldots, u_{3k-4}\}| \leq k-1$. By symmetry, applying the preceding argument, we obtain the claim for the case when $i = k-1$. ∎

Claim 2.2.8 $|N_G(v) \cap \{u_0, u_1, \ldots, u_{3k-4}\}| \leq k-1$.

Proof. We show that $N_G(v) \cap \{u_0, u_1, \ldots, u_{3k-4}\} \subseteq \{u_0, \ldots, u_{k-2}\}$. Suppose that $vu_i \in E(G)$ for some $k-1 \leq i \leq 3k-4$. Then $T + u_i v + s_{k-1}s_k + u_0 v - s_{k-1}v - s_k v$ is a k-tree of order $|V(T)| + 1$. This contradicts the maximality of T. Hence $N_G(v) \cap \{u_0, u_1, \ldots, u_{3k-4}\} \subseteq \{u_0, \ldots, u_{k-2}\}$. Thus $|N_G(v) \cap \{u_0, u_1, \ldots, u_{3k-4}\}| \leq k-1$. ∎

By Claim 2.2.6 (i), $|N^+_{T_i,u_i}(w)| = k-1$ for any $w \in W_i$ with $1 \leq i \leq 3k-4$. It follows from Claim 2.2.6 (ii) that

$$\begin{aligned} |N_G(u_i) \cap V(T_i)| &\leq |V(T_i)| - (k-1)|W_i| - |\{u_i\}| \\ &= |V(T_i)| - (k-1)|W_i| - 1. \end{aligned} \tag{2.1}$$

For each $0 \leq j \leq 3k-4$ with $j \neq i$, Claim 2.2.6 (iii) asserts that

$$\sum_{\substack{0 \leq j \leq 3k-4 \\ j \neq i}} |N_G(u_j) \cap V(T_i)| \leq (k-1)|W_i|. \tag{2.2}$$

By (2.1) and (2.2), we obtain

$$\sum_{0 \leq j \leq 3k-4} |N_G(u_j) \cap V(T_i)| \leq |V(T_i)| - 1.$$

Hence we obtain

$$\begin{aligned} \sum_{1 \leq i \leq 3k-4} \sum_{0 \leq j \leq 3k-4} |N_G(u_j) \cap V(T_i)| &\leq \sum_{1 \leq i \leq 3k-4} (|V(T_i)| - 1) \\ &\leq |T| - |\{s_{k-1}, s_k, v\}| - (3k-4) = |T| - 3k + 1. \end{aligned} \tag{2.3}$$

By (2.3), Claims 2.2.7 and 2.2.8,

$$\begin{aligned} \sum_{0 \leq i \leq 3k-4} \deg_G(u_i) &\leq |T| - 3k + 1 + (k-1)|\{s_{k-1}, s_k, v\}| + |N_{G-V(T)}(u_0)| \\ &\leq |T| - 2 + |G| - |T| - |\{u_0\}| = |G| - 3. \end{aligned}$$

This contradicts the degree sum condition of Theorem 2.3 and hence the proof of Theorem 2.3 is completed. ∎

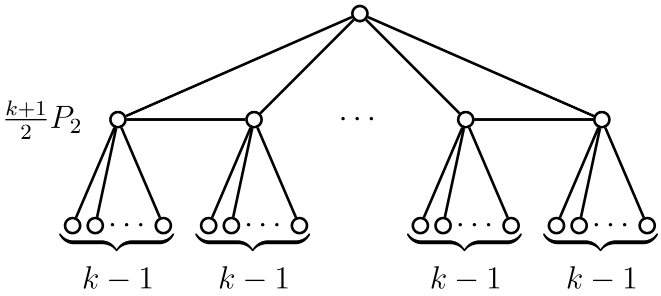


Figure 2.3: The graph G (k is an odd integer.)

2.3 A forbidden pair for connected graphs to have spanning k-trees

In this section, we show the forbidden pair for connected graphs to have a spanning k-tree. The result gives a positive answer to the conjecture posed by Ota and Sugiyama in 2010.

If a graph G has a vertex v such that $G-v$ has at least $k+1$ components, then G does not have a spanning k-tree. In order to forbid such a situation, it is natural to consider connected $K_{1,k+1}$-free graphs for the existence of a spanning k-tree. Ota and Sugiyama obtained a forbidden subgraph condition for a graph to have a spanning k-tree.

Theorem 2.5 (Ota and Sugiyama [50]) *Let $k \geq 2$ be an integer. If G is a connected $\left\{K_{1,k+1}, N\left(k-1, k-1, \left\lfloor \frac{k-1}{2} \right\rfloor\right), N(k-1, k-2, k-2)\right\}$-free graph, then G has a spanning k-tree.*

However, it was not known whether the conditions of being $N\left(k-1, k-1, \left\lfloor \frac{k-1}{2} \right\rfloor\right)$-free and $N(k-1, k-2, k-2)$-free in Theorem 2.5 are sharp. They posed the following conjecture.

Conjecture 2.6 (Ota and Sugiyama [50]) *Let $k \geq 2$ be an integer. If G is a connected $\left\{K_{1,k+1}, N\left(k-1, k-1, \left\lceil \frac{k-1}{2} \right\rceil\right)\right\}$-free graph, then G has a spanning k-tree.*

They showed that if Conjecture 2.6 is true, then it is stronger than Theorem 2.5 and the condition is sharp in the sense that we cannot replace $N\left(k-1, k-1, \left\lceil \frac{k-1}{2} \right\rceil\right)$-free by $N\left(k-1, k-1, \left\lceil \frac{k+1}{2} \right\rceil\right)$-free. The graphs G and G' in Fig. 2.3 and 2.4, respectively, are not $N\left(k-1, k-1, \left\lceil \frac{k-1}{2} \right\rceil\right)$-free and are $N\left(k-1, k-1, \left\lceil \frac{k+1}{2} \right\rceil\right)$-free but these graphs have no spanning k-tree. Hence the conditions of Conjecture 2.6 are sharp. In this thesis, we prove Conjecture 2.6.

Theorem 2.7 *Let $k \geq 2$ be an integer. If G is a connected $\left\{K_{1,k+1}, N\left(k-1, k-1, \left\lceil \frac{k-1}{2} \right\rceil\right)\right\}$-free graph, then G has a spanning k-tree.*

In order to show Theorem 2.7, we prove a technical but stronger result. We will explain that in the next section.

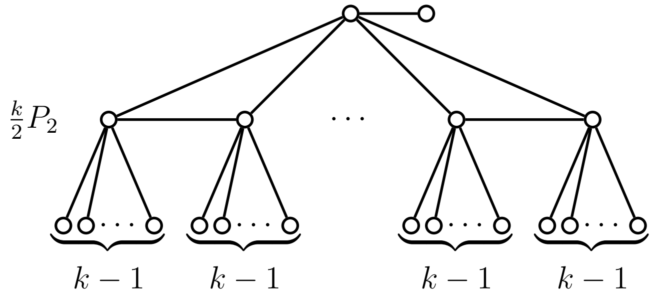


Figure 2.4: The graph G' (k is an even integer.)

2.3.1 Techniques for the proof of Theorem 2.7

In order to show Theorem 2.5, Ota and Sugiyama proved the following stronger statement for the inductive argument.

Theorem 2.8 (Ota and Sugiyama [50]) *Let $k \geq 2$ be an integer. Suppose that G is a connected $\left\{K_{1,k+1}, N\left(k-1, k-1, \lfloor \frac{k-1}{2} \rfloor\right), N(k-1,k-2,k-2)\right\}$-free graph and u is a vertex of G such that the number of components in $G-u$ is at most $k-1$. Then G has a spanning k-tree T such that $\deg_T(u) \leq k-1$.*

Since every graph has a vertex that is not a cut-vertex, Theorem 2.8 implies Theorem 2.5. They showed that each of the conditions of being $K_{1,k+1}$-free, $N\left(k-1, k-1, \lfloor \frac{k-1}{2} \rfloor\right)$-free, and $N(k-1,k-1,k-2)$)-free are necessary for the conclusion of Theorem 2.8. So, in order to show Conjecture 2.6, it is impossible to replace the condition of Theorem 2.8 with $\left\{K_{1,k+1}, N\left(k-1, k-1, \lceil \frac{k-1}{2} \rceil\right)\right\}$-free graphs.

We introduce some definitions and show our result that is stronger than Theorems 2.7 and 2.8. Let $k \geq 2$ be an integer. Let G be a graph. For a vertex u of G, a *u-bridge* of G is a subgraph of G induced by the edges in a component of $G-u$ and all edges from that component to u. Let $\mathcal{H}(G,u)$ be the set of u-bridges of G. Note that for each u-bridge H of G, H is connected and u is not a cut-vertex of H. We recursively define functions g^k : $\{(G,v)$: G is a connected graph and $v \in V(G)$ such that $G-v$ is connected$\} \to \{1,2\}$ and $f_G^k : V(G) \to \{0,1,2,\ldots\}$ as follows.

$$g^k(G,v) = \begin{cases} 1 & \text{if either } f_{G-v}(x) \leq k-1 \text{ for some } x \in N_G(v) \text{ or } |N_G(v)| = 1, \\ 2 & \text{otherwise.} \end{cases}$$

$$f_G^k(u) = \begin{cases} 0 & \text{if } G \text{ consists of only } u, \\ \sum_{H \in \mathcal{H}(G,u)} g^k(H,u) & \text{otherwise.} \end{cases}$$

Now, we are ready to state our technical theorem.

Theorem 2.9 *Let $k \geq 3$ be an integer, G a connected $\left\{K_{1,k+1}, N\left(k-1, k-1, \lceil \frac{k-1}{2} \rceil\right)\right\}$-free graph, and let u be a vertex of G. Then G has a spanning k-tree T such that $\deg_T(u) \leq f_G^k(u)$.*

Theorem 2.7 is a direct corollary of Theorem 2.9. We prove Theorem 2.9 by the induction on $|V(G)|$. Since we never change the value of k as in Theorem 2.9 in the rest of Section 2.3, for convenience, we will write $g(\cdot)$ and $f_G(\cdot)$ instead of $g^k(\cdot)$ and $f_G^k(\cdot)$.

We briefly explain our idea to improve the argument by Ota and Sugiyama [50]. As in Theorem 2.8, they considered the number of components in $G-u$ for a specified vertex u in a graph G and the existence of a spanning k-tree T with $\deg_T(u) \leq k-1$. This was succeeded to show Theorem 2.5. However, counting the number of components in $G-u$ was not enough to reach a proof of Conjecture 2.6. In this paper, we focus on not only counting the number of components in $G-u$ but also the detailed structure of each component by the functions g and f_G. In fact, if a u-bridge H of G, which is obtained by a component of $G-u$ together with u, has certain conditions, then H is counted as 2 in $g(H,u)$, and requires two edges from $H-u$ to u in the desired spanning k-tree in Theorem 2.9.

This idea appears also in the proof of Theorem 2.9. We show by induction several properties of a vertex v and a v-bridge C distinguishing the following three types;

- $g(C,v) = 2$ and $\alpha(C[N_C(v)]) = 1$, see Lemma 2.13 and Claims 2.3.2 and 2.3.8,
- $g(C,v) = 2$ and $\alpha(C[N_C(v)]) \geq 2$, see Claim 2.3.7,
- $g(C,v) = 1$ see Claims 2.3.4 and 2.3.6.

Those are crucial ideas to prove Theorem 2.7.

2.3.2 Preliminary

In this section, we show some lemmas which are used in the proof of Theorem 2.9. In 1972, Chvátal and Erdős obtained the following result, which gives a sufficient condition for graphs to have a Hamilton cycle.

Theorem 2.10 (Chvátal and Erdős [18]) *Let G be a k-connected graph. If $\alpha(G) \leq k$, then G has a Hamilton cycle.*

Using Theorem 2.10, we show the first lemma.

Lemma 2.11 *Let G be a connected graph. If $\alpha(G) = 2$, then there exist nonadjacent vertices v_1 and v_2 such that there exists a Hamilton path P_i with end v_i for each $i = 1, 2$.*

Proof. If G is 2-connected, then G has a Hamilton cycle by Theorem 2.10 and so this lemma holds. We may assume that G has a cut-vertex w. Let H_1 and H_2 be components of $G-w$. Since $\alpha(G) = 2$, H_1 and H_2 are complete graphs. If w is adjacent to all vertices of H_1 and H_2, then there exists a Hamilton path connecting a vertex in H_1 and a vertex in H_2, and this lemma holds. We may assume that there exists a vertex x of H_2 not adjacent to w. Since $\alpha(G) = 2$, $|V(H_2)| \geq 2$ and w is adjacent to all vertices of H_1. Then there exists a Hamilton path connecting x and a vertex of H_1 and this lemma holds. ∎

Next, we show some properties of the functions g and f in a connected $\{K_{1,k+1}, N(k-1, k-1, \lceil \frac{k-1}{2} \rceil)\}$-free graphs.

Lemma 2.12 *Let v be a vertex of a $K_{1,k+1}$-free graph G. Suppose that $\alpha(C[N_C(v)]) \geq 2$ for each v-bridge C of G with $g(C, v) = 2$. There exists an independent set S of $N_G(v)$ such that $|S| = f_G(v)$ and $|S \cap V(C')| = g(C', v)$ for each v-bridge C' of G. Moreover, $f_G(v) \leq k$.*

Proof. We take a vertex adjacent to v from each v-bridge C of G with $g(C, v) = 1$. Since $\alpha(C'[N_{C'}(v)]) \geq 2$ for each v-bridge C' of G with $g(C', v) = 2$, we can take nonadjacent two vertices from $N_{C'}(v)$. The set of taken vertices is an independent set with desired property. Since G is $K_{1,k+1}$-free, $f_G(v) \leq k$. ∎

We often use the following fact, which is obtained in a similar way to the proof of Lemma 2.12. For a vertex v of a graph G, if $f_G(v) = \ell$, then the number of v-bridges of G is at least $\lceil \frac{\ell}{2} \rceil$, since $g(C, v) \leq 2$ for each v-bridge C of G.

Lemma 2.13 *Let $k \geq 3$ be an integer. Let v be a vertex of a connected $\{K_{1,k+1}, N\left(k-1, k-1, \lceil \frac{k-1}{2} \rceil\right)\}$-free graph G such that $G - v$ is connected. If $g(G, v) = 2$ and $\alpha(G[N_G(v)]) = 1$, then there exist two vertices w_1 and w_2 in $N_G(v)$ such that there exists an independent set of $N_{G-v}(w_i) \setminus V(C_i)$ with size $k-1$ for each $i = 1, 2$, where C_i is the unique w_i-bridge of $G - v$ containing $N_G(v)$. In particular, G has an induced subgraph N isomorphic to $N(k-1, k-1, 0)$ such that* $\deg_N(v) = 2$.

Proof. We prove this lemma by induction on $|V(G)|$. Since $\alpha(G[N_G(v)]) = 1$, for each vertex w in $N_G(v)$, there is exactly one w-bridge of $G - v$ containing $N_G(v)$. Moreover, $f_{G-v}(w) \geq k$ for each vertex w in $N_G(v)$ since $g(G, v) = 2$.

Claim 2.3.1 *Let w be a vertex in $N_G(v)$ and let C_w be the w-bridge of $G - v$ containing $N_G(v)$. Suppose that $g(C_w, w) = 1$. Then there exists an independent set of $N_{G-v}(w) \setminus V(C_w)$ with size $k-1$.*

Proof. Suppose that $\alpha(C'_w[N_{C'_w}(w)]) \geq 2$ for each w-bridge C'_w of $G - v$ with $g(C'_w, w) = 2$. It follows from Lemma 2.12 that there exists an independent set of $N_{G-v}(w) \setminus V(C_w)$ with size $f_{G-v}(w) - g(C_w, w) \geq k-1$. Hence we may assume that there exists a w-bridge C'_w of $G - v$ with $g(C'_w, w) = 2$ and $\alpha(C'_w[N_{C'_w}(w)]) = 1$. By the induction hypothesis, C'_w contains an induced subgraph N_w isomorphic to $N(k-1, k-1, 0)$ such that $\deg_{N_w}(w) = 2$. Since $g(C_w, w) = 1$, the number of w-bridges of $G - v$ except for C_w is at least $\lceil (k-1)/2 \rceil$ and so there exists an independent set S_w of $N_{G-v}(w) \setminus (V(C_w) \cup V(C'_w))$ with size $\lceil (k-3)/2 \rceil$. Then $G[V(N_w) \cup S_w \cup \{v\}]$ is isomorphic to $N\left(k-1, k-1, \lceil \frac{k-1}{2} \rceil\right)$. This is a contradiction. ∎

Then we are ready to prove Lemma 2.13. Suppose first that $|N_G(v)| = 2$. Let w_1 and w_2 be two vertices in $N_G(v)$. Since $g(G, v) = 2$, we have $f_{G-v}(w_i) \geq k$ for each $i = 1, 2$. For each $i = 1, 2$, let C_i be the w_i-bridge of $G - v$ containing $N_G(v)$. Suppose that $g(C_1, w_1) = 2$. Then $f_{C_1-w_1}(w_2) \geq k$. Suppose that there exists no w_2-bridge C_2' of $C_1 - w_1$ such that $g(C_2', w_2) = 2$ and $\alpha(C_2'[N_{C_2'}(w_2)]) = 1$. By Lemma 2.12, there exists an independent set S of $N_{C_1-w_1}(w_2)$ with size k. Then $G[S \cup \{v, w_2\}]$ is isomorphic to $K_{1,k+1}$. This is a contradiction. Hence there exists a w_2-bridge C_2' of $C_1 - w_1$ with $g(C_2', w_2) = 2$ such that $\alpha(C_2'[N_{C_2'}(w_2)]) = 1$. By the induction hypothesis, C_2' contains an induced subgraph N isomorphic to $N(k-1, k-1, 0)$ such that $\deg_N(w_2) = 2$. Since $f_{C_1-w_1}(w_2) - g(C_2', w_2) \geq k - 2$, the number of w_2-bridges of $C_1 - w_1$ except for C_2' is at least $\lceil (k-2)/2 \rceil$ and so there exists an independent set S' of $N_{C_1-w_1}(w_2) \setminus V(C_2')$ with size $\lceil (k-2)/2 \rceil$. Then $G[V(N) \cup S' \cup \{v\}]$ is isomorphic to $N\left(k-1, k-1, \lceil \frac{k}{2} \rceil\right)$. This is a contradiction. Hence $g(C_1, w_1) = g(C_2, w_2) = 1$ by the symmetry. By Claim 2.3.1, there exists an independent set of $N_{G-v}(w_i) \setminus V(C_i)$ with size $k-1$ for each $i = 1, 2$, and we are done.

Suppose next that $|N_G(v)| \geq 3$. Let x_1, x_2, x_3 be three vertices in $N_G(v)$. Let C_i be the x_i-bridge of $G - v$ containing $N_C(v)$. Suppose that $g(C_i, x_i) = 1$ for each $i = 1, 2, 3$. By Claim 2.3.1, there exists an independent set S_i of $N_{G-v}(x_i) \setminus V(C_i)$ with size $k-1$. Then $G[S_1 \cup S_2 \cup S_2 \cup \{x_1, x_2, x_3\}]$ is isomorphic to $N(k-1, k-1, k-1)$. This is a contradiction. Hence $g(C_i, x_i) = 2$ for some $i = 1, 2, 3$. Without loss of generality, we may assume that $g(C_1, x_1) = 2$. Since $N_G(v) \cap V(C_1 - x_1) \subseteq N_{C_1}(x_1)$, we have $f_{C_1-x_1}(y) \geq k$ for each vertex y in $N_G(v) \cap V(C_1 - x_1)$. Moreover, $|N_G(v) \cap V(C_1 - x_1)| \geq 2$. Hence $g(G[V(C_1 - x_1) \cup \{v\}], v) = 2$. Then $|V(G[V(C_1 - x_1) \cup \{v\}])| < |V(G)|$. By the induction hypothesis, $G[V(C_1 - x_1) \cup \{v\}]$ has desired two vertices and this lemma holds. ∎

2.3.3 Proof of Theorem 2.9

We prove Theorem 2.9 by induction on $|V(G)|$.

Claim 2.3.2 *Let C be a connected induced subgraph of G and v be a vertex of C. Suppose that there exists a v-bridge D of C such that $g(D, v) = 2$ and $\alpha(D[N_D(v)]) = 1$. Then k is an even integer and each v-bridge D' of C satisfies $g(D', v) = 2$ and $\alpha(D'[N_{D'}(v)]) = 1$. Moreover, $f_C(v) = k$.*

Proof. By Lemma 2.13, C has an induced subgraph N isomorphic to $N(k-1, k-1, 0)$ such that $\deg_N(v) = 2$. Suppose that there exists a v-bridge D of C with $g(D, v) = 1$. Then the number of v-bridges of C except for D is at least $\lceil (k-3)/2 \rceil + 1 = \lceil (k-1)/2 \rceil$. Hence there exists an independent set S contained in $N_C(v) \setminus V(D)$ with size $\lceil (k-1)/2 \rceil$.

Then $C[V(N) \cup S]$ is isomorphic to $\left(k-1, k-1, \lceil \frac{k-1}{2} \rceil\right)$. This is a contradiction. Hence for each v-bridge D of C, we have $g(D, v) = 2$.

Suppose that there exists a v-bridge D' of C such that $\alpha(D'[N_{D'}(v)]) \geq 2$. Then there exists an independent set S' of $N_C(v) \setminus V(D)$ with size $\lceil (k-4)/2 \rceil + 2 = \lceil k/2 \rceil$. Then $C[V(N) \cup S']$ is isomorphic to $N\left(k-1, k-1, \lceil \frac{k}{2} \rceil\right)$. This is a contradiction and hence $\alpha(D'[N_{D'}(v)]) = 1$ for each v-bridge D' of C.

Suppose that $f_C(v) \geq k+1$. Then there exists an independent set S'' of $N_C(v) \setminus V(D)$ with size $\lceil k - 1/2 \rceil$. Then $C[V(N) \cup S'']$ is isomorphic to $N\left(k-1, k-1, \lceil \frac{k-1}{2} \rceil\right)$, a contradiction. Hence $f_C(v) = k$. Since $f_C(v) = k$ and $g(D, v) = 2$ for each v-bridge D of G, we have k is an even integer. ∎

For a connected induced subgraph C of G, a vertex $w \in V(C)$ with $f_C(w) \geq k$ is called a *clique-vertex* in C, if $\alpha(C_w[N_{C_w}(w)]) = 1$ and $g(C_w, w) = 2$ for each w-bridge C_w of C.

Claim 2.3.3 *Let v be a vertex of G with $f_C(v) \geq k$ for a connected induced subgraph C of G. Then either one of the following holds and $f_C(v) = k$.*

(i) *k is an even integer and v is a clique-vertex in C.*

(ii) *There exists a maximum independent set S of $N_C(v)$ such that $|S \cap V(D)| = g(D, v)$ for each v-bridge D of C. This implies that there exists no v-bridge D' of C such that $|S \cap V(D')| > g(D', v)$.*

Proof. if there exists a v-bridge D of C such that $g(D, v) = 2$ and $\alpha(D[N_D(v)]) = 1$, then it follows from Claim 2.3.2 that k is an even integer, v is a clique-vertex in C, and $f_C(v) = k$. Thus, (i) holds. On the other hand, if there exists no v-bridge D of C such that $g(D, v) = 2$ and $\alpha(D[N_D(v)]) = 1$, then it follows from Lemma 2.12 that there exists an independent set S of $N_C(v)$ such that $|S \cap V(D)| = g(D, v)$ for each v-bridge D of C and $f_C(v) = k$. Since G is $K_{1,k+1}$-free, S is maximum, and (ii) holds. ∎

Claim 2.3.4 *Let C be a connected induced subgraph of G and v be a vertex of C. If D is a v-bridge of C such that $g(D, v) = 1$, then there exists a vertex w in $N_D(v)$ such that $f_{D-v}(w) \leq k-1$.*

Proof. If $|N_D(v)| \geq 2$, then this claim holds by the definition of g. We assume that $|N_D(v)| = 1$. Let w be the unique vertex in $N_D(v)$. Suppose that $f_{D-v}(w) \geq k$. If w satisfies Claim 2.3.3 (ii), then $G[S \cup \{v, w\}]$ contains an induced subgraph isomorphic to $K_{1,k+1}$, a contradiction, where S is an independent set of $N_{D-v}(w)$ satisfying the condition of Claim 2.3.3 (ii). Thus, we may assume that w satisfies Claim 2.3.3 (i). Let D' be a w-bridge of $D - v$. By Lemma 2.13, D' has an induced subgraph N isomorphic to $N(k-1, k-1, 0)$ such that $\deg_N(w) = 2$. Since the number of w-bridges of $D - v$

except for D' is $(k-2)/2$, there exists an independent set S of $N_{D-v}(w) \setminus V(D')$ with size $(k-2)/2$. Then $G[V(N) \cup S \cup \{v\}]$ is isomorphic to $N\left(k-1, k-1, \frac{k}{2}\right)$. This is a contradiction. ∎

Claim 2.3.5 *Let C be a connected induced subgraph of G and let v be a vertex of C such that $C-v$ is connected and $g(C,v)=2$. Let w be a vertex in $N_C(v)$ such that only one w-bridge of $C-v$ contains $N_C(v)$, and let H_w be the union of w-bridges of $C-v$ not containing $N_C(v) \setminus \{w\}$. If $g(C-V(H_w), v) = 2$, then $f_{H_w}(w) \le k-2$.*

Proof. Since $g(C,v)=2$, we have $f_{C-v}(w)=k$. Let C_w be the w-bridge of $C-v$ containing $N_C(v)$. Note that $C_w - w = C - V(H_w) - v$. If $g(C_w, w) = 2$, then $f_{H_w}(w) = f_{C-v}(w) - g(C_w, w) = k-2$. We may assume that $g(C_w, w) = 1$. Then w is not a clique-vertex in $C-v$ by the definition of a clique-vertex. By Claim 2.3.3, there exists an independent set of $N_{H_w}(w)$ with size $k-1$. Since G is $K_{1,k+1}$-free, $N_{C_w}(w) \subseteq N_{C-V(H_w)}(v)$. By Claim 2.3.4, there exists a vertex $x \in N_{C_w}(w)$ with $f_{C_w-w}(x) \le k-1$. Since $N_{C_w}(w) \subseteq N_{C-V(H_w)}(v)$, we have $g(C-V(H_w), v) = 1$. This contradicts the assumption of this claim. ∎

Claim 2.3.6 *Let C be a u-bridge of G such that $g(C,u)=1$. Then C has a spanning k-tree T such that $\deg_T(u)=1$.*

Proof. By Claim 2.3.4, there exists a vertex v in $N_C(u)$ such that $f_{C-u}(v) \le k-1$. By the induction hypothesis, $C-u$ has a spanning k-tree T such that $\deg_T(v) \le k-1$. Then $T+uv$ is a desired spanning k-tree. ∎

Claim 2.3.7 *Let C be a u-bridge of G such that $g(C,u)=2$ and $\alpha(C[N_C(u)])=2$. Then C has a spanning k-tree T such that $\deg_T(u) \le 2$.*

Proof. Let v be a vertex in $N_C(u)$ such that only one v-bridge of $C-u$ contains $N_C(u)$, and let D be such a v-bridge. The vertex v satisfies either (i) or (ii) in Claim 2.3.3 for $C-u$ and we prove this claim dividing into two cases.

Case 2.3.1 *The vertex v satisfies (i) in Claim 2.3.3.*

Note that k is an even integer and v is a clique-vertex in $C-u$. It follows from the definition of a clique-vertex that each v-bridge C_v of $C-u$ satisfies $g(C_v, v)=2$ and $\alpha(C_v[N_{C_v}(v)])=1$ and so v and C_v satisfy the assumption of Lemma 2.13. We show that $N_D(v) \subseteq N_G(u)$. Suppose that there exists a vertex v' in $N_D(v)$ not adjacent to u. Since $k \ge 4$, there exists a v-bridge D' of $C-u$ except for D. By Lemma 2.13, D' has an induced subgraph N isomorphic to $N(k-1,k-1,0)$ such that $\deg_N(v)=2$. Since the number of v-bridges of $C-u$ except for D and D' is $(k-4)/2$, there exists an independent set S of $N_{C-u}(v) \setminus (V(D) \cup V(D'))$ with size $(k-4)/2$. Then $G[V(N) \cup S \cup \{v', u\}]$ is isomorphic to $N\left(k-1, k-1, \frac{k}{2}\right)$. This is a contradiction. Hence $N_D(v) \subseteq N_G(u)$.

By the induction hypothesis, $C-u$ has a spanning k-tree T_v. Let w be a vertex in $V(D)$ such that $vw \in E(T_v)$. Then $T_v + uv + uw - vw$ is a desired spanning k-tree.

Case 2.3.2 *The vertex v satisfies (ii) in Claim 2.3.3.*

Suppose that $g(D, v) = 1$. By Claim 2.3.3 (ii), there exists an independent set of $N_{C-u}(v) \setminus V(D)$ with size $k-1$. Since G is $K_{1,k+1}$-free, $N_D(v) \subseteq N_G(u)$. By the induction hypothesis, $C-u$ has a spanning k-tree T_v. Let w be a vertex in $V(D)$ such that $vw \in E(T_v)$. Then $T_v + uv + uw - vw$ is a desired spanning k-tree.

Hence we assume that $g(D, v) = 2$. By Claim 2.3.3 (ii), $\alpha(D[N_D(v)]) = 2$. Since G is $K_{1,k+1}$-free, for any nonadjacent two vertices in $N_D(v)$, u is adjacent to one of the two vertices. Let $P = w_1 w_2 \dots w_m$ be a path in $D[N_D(v)]$ such that

- w_1 is adjacent to u,
- if $D[N_D(v)]$ is connected, then P is a Hamilton path of $D[N_D(v)]$, and
- if $D[N_D(v)]$ is not connected, then P is a Hamilton path of one of the components of $D[N_D(v)]$ such that all vertices in $N_D(v) - \{w_1, \dots, w_i\}$ are contained in a same component of $D - \{v, w_1, \dots, w_i\}$ for each $1 \leq i \leq m$.

By Lemma 2.11, such a path P exists in the case that $D[N_D(v)]$ is connected. Suppose that $D[N_D(v)]$ is not connected. Since $\alpha(D[N_D(v)]) = 2$, $D[N_D(v)]$ consists of two components both of which are cliques. Since u is adjacent to at least one of any nonadjacent two vertices, all vertices in one component of $D[N_D(v)]$ are neighbors of u and let A be such a component. Since $D - v$ is connected, there exists a vertex w in A with a path from w to $N_D(v) - A$ in $D - v$ disjoint from $A - \{w\}$. Then, any path in A ending w satisfies the desired condition for P. In this case, we have $w_m = w$.

Let $D_0 = D$. For each $1 \leq i \leq m$, we define the graphs H_i and D_i such that

- H_i is the union of w_i-bridges of D_{i-1} not containing $\{v, w_{i+1}, \dots, w_m\}$ and
- $D_i = D_{i-1} - V(H_i)$.

Note that if $D[N_D(v)]$ is connected, then $V(D_m) = \{v\}$ otherwise, $D_m[N_{D_m}(v)]$ is a clique. By the choice of P, $D_i - v$ is connected for each $1 \leq i \leq m$.

We claim that there exists an integer i such that $g(D_i, v) = 1$. If $D[N_D(v)]$ is connected, then since $|N_{D_{m-1}}(v)| = |\{w_m\}| = 1$, it follows from the definition of g that $g(D_{m-1}, v) = 1$. We assume that $D[N_D(v)]$ is not connected, and claim that $g(D_m, v) = 1$. Suppose that $g(D_m, v) = 2$. Since $\alpha(D_m[N_{D_m}(v)]) = 1$, it follows from Lemma 2.13 that D_m has an induced subgraph N isomorphic to $N(k-1, k-1, 0)$ such that $\deg_N(v) = 2$.

Since Claim 2.3.3 (ii) holds, there exists an independent set S of $N_{C-u}(v) \setminus V(D)$ with size $k-2$. Then $G[V(N) \cup S]$ is isomorphic to $N(k-1, k-1, k-2)$, a contradiction. Thus, in either case, there exists an integer i such that $g(D_i, v) = 1$.

Let

$$t = \min_{1 \le i \le m} \{i : g(D_i, v) = 1\}.$$

By Claim 2.3.5, replacing C, w, and H_w with D_i, w_i, and H_i, respectively for $1 \le i \le t-1$, we have $f_{H_i}(w_i) \le k-2$. By the induction hypothesis, H_i has a spanning k-tree T_i such that $\deg_{T_i}(w_i) \le k-2$ for each $1 \le i \le t-1$ and H_t has a spanning k-tree T_t such that $\deg_{T_t}(w_t) \le k-1$. Since $g(D_t, v) = 1$, it follows from the induction hypothesis that D_t has a spanning k-tree T_v such that $\deg_{T_v}(v) = 1$. Let H_v be the union of v-bridges of $C-u$ except for D. Since $f_{H_v}(v) = f_{C-u}(v) - g(D, v) = k-2$, it follows from the induction hypothesis that H_v has a spanning k-tree T'_v such that $\deg_{T'_v}(v) \le k-2$. Then $T_1 \cup T_2 \cup \cdots \cup T_t \cup T_v \cup T'_v \cup w_1Pw_t + uw_1 + uv$ is a desired spanning k-tree, where w_1Pw_t is the path in P from w_1 to w_t. ∎

Claim 2.3.8 *Let C be a u-bridge of G such that $g(C, u) = 2$ and $\alpha(C[N_C(u)]) = 1$. Then C has a spanning k-tree T such that $\deg_T(u) = 2$.*

Proof. By Lemma 2.13, there exists a vertex v in $N_C(u)$ such that there exists an independent set of $N_{C-u}(v) \setminus V(C_v)$ with size $k-1$, where C_v is the v-bridge of $C-u$ containing $N_C(u)$. Since G is $K_{1,k+1}$-free, $N_{C_v}(v) \subseteq N_G(u)$. By the induction hypothesis, $C-u$ has a spanning k-tree T_v. Let w be a vertex in $V(C_v)$ such that $vw \in E(T_v)$. Then $T_v + uv + uw - vw$ is desired spanning k-tree. ∎

By Claims 2.3.6, 2.3.7, and 2.3.8, each u-bridge C of G has a spanning tree T_C such that $\deg_{T_C}(u) \le g(C, u)$. Let $T = \bigcup_{C \in \mathcal{H}(G,u)} T_C$. Then T is a spanning tree of G such that $\deg_T(u) \le f_G(u)$. Therefore this theorem holds. ∎

2.4 Closure and spanning trees with bounded total excess

For a vertex subset S of G, and a positive integer k with $k \le |S|$, let

$$\Delta_k(S; G) = \max\Big\{\sum_{x \in X} \deg_G(x) : X \text{ is a subset of } S \text{ with } |X| = k\Big\}.$$

Bondy and Chvátal introduced a closure concept in [7]. They showed that it plays an important role for the existence of cycles, and some other subgraphs in graphs. We refer the reader to the survey [10] on several closure concepts. In [42], Matsubara et al. considered a closure concept for spanning k-trees.

Theorem 2.14 (Matsubara, Tsugaki and Yamashita [42]) *Let $k \geq 2$ be an integer, and let G be a connected graph. Let u and v be two non-adjacent vertices of G. If $\Delta_k(S;G) \geq |G|-1$ for every independent set S in G of order $k+1$ such that $\{u,v\} \subseteq S$, then G has a spanning k-tree if and only if $G+uv$ has a spanning k-tree.*

On the other hand, a tree is called a *k-ended tree* if the number of its leaves is at most k. In [9], Broersma and Tuinstra considered a closure concept for spanning k-ended trees.

Theorem 2.15 (Broersma and Tuinstra [9]) *Let $k \geq 2$ be an integer, and let G be a connected graph. Let u and v be two nonadjacent vertices of G. If $\deg_G(u) + \deg_G(v) \geq |G|-1$, then G has a spanning k-ended tree if and only if $G+uv$ has a spanning k-ended tree.*

Let G be a graph. The *total k-excess* of G is defined as

$$\mathrm{te}(G;k) = \sum_{v \in V(G)} \max\{\deg_G(v) - k, 0\}.$$

This concept was introduced by Enomoto, Onishi and Ota in [24], and we can see some of results concerning it in [27], [47] and [51]. Note that for a tree T, $\mathrm{te}(T;k) = 0$ if and only if T is a k-tree, and $\mathrm{te}(T;2) \leq k-2$ if and only if T is a k-ended tree. We generalize Theorems 2.14 and 2.15 as follows.

Theorem 2.16 *Let $\alpha \geq 0$ and $k \geq 2$ be integers, and let G be a connected graph. Let u and v be two nonadjacent vertices of G. If $\Delta_k(S;G) \geq |G|-1$ for every independent set S in G of order $k+1$ such that $\{u,v\} \subseteq S$, then G has a spanning tree T such that $te(T;k) \leq \alpha$ if and only if $G+uv$ has a spanning tree T' such that $te(T';k) \leq \alpha$.*

The degree sum condition of Theorem 2.16 is best possible. Let $\alpha \geq 0$ and $k \geq 2$ be integers, and let V_1 and V_2 be disjoint vertex sets such that $|V_1| = k+\alpha-1$, $|V_2| = k-1$. Let u, v and w be distinct vertices not contained in $V_1 \cup V_2$. Let G_1 be a graph such that $V(G_1) = \{u,v,w\} \cup V_1 \cup V_2$, $E(G_1) = \{ux : x \in V_1\} \cup \{wx : x \in V_2\} \cup \{uw, vw\}$ (see the left of Figure 1). Then G_1 is a connected graph and $uv \notin E(G_1)$. Note that G_1 is a tree such that $\mathrm{te}(G_1;k) = \alpha+1$. On the other hand, G_1+uv has a spanning tree $(G_1+uv)-uw$ such that $\mathrm{te}((G_1+uv)-uw;k) = \alpha$. Let $S = V_2 \cup \{u,v\}$. Then we can see that $|S| = k+1$ and $\Delta_k(S;G_1) = |G_1|-2$. These imply that the degree sum condition of Theorem 2.16 is best possible [1].

Let $\alpha \geq 0$ and $k \geq 2$ be integers, and let G be a connected graph. In [27], Fujisawa et al. showed that if $\alpha(G) \leq k+\alpha$, then G has a spanning tree T with $\mathrm{te}(T;k) \leq \alpha$. Therefore, it is natural to consider the following problem, which corresponds to an improvement of Theorem 2.16.

[1] We can generalize G_1 by replacing each vertex in V_1 with a complete graph.

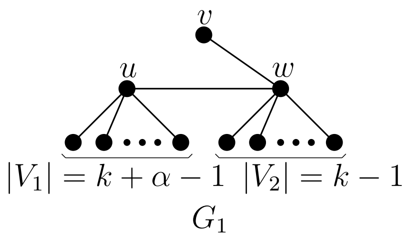


Figure 2.5: A sharpness example G_1 for Theorem 4.

Problem 2.17 *Let $\alpha \geq 0$ and $k \geq 2$ be integers, and let G be a connected graph. Let u and v be two nonadjacent vertices of G. If $\Delta_k(S; G) \geq |G| - 1$ for every independent set S in G of order $k + \alpha + 1$ such that $\{u, v\} \subseteq S$, then G has a spanning tree T such that $te(T; k) \leq \alpha$ if and only if $G + uv$ has a spanning tree T' such that $te(T'; k) \leq \alpha$.*

However, Problem 2.17 is not true for $\alpha > 0$. Let $\alpha > 0$ and $k \geq 2$ be integers, and let S be an independent set of the graph G_1 containing both u and v. Then we can see that $|S| \leq |V_2 \cup \{u, v\}| = k + 1 < k + \alpha + 1$. These imply that G_1 is a counterexample of Problem 2.17[1]. Therefore, we change the condition on S so that S contains at least one of u and v, and prove the following theorem, which is the second main theorem of this paper.

Theorem 2.18 *Let $\alpha \geq 0$ and $k \geq 2$ be integers, and let G be a connected graph. Let u and v be two nonadjacent vertices of G. If $\Delta_k(S; G) \geq |G| - \alpha - 1$ for every independent set S in G of order $k + \alpha + 1$ such that $S \cap \{u, v\} \neq \emptyset$, then G has a spanning tree T such that $te(T; k) \leq \alpha$ if and only if $G + uv$ has a spanning tree T' such that $te(T'; k) \leq \alpha$.*

The degree sum condition of Theorem 2.18 is best possible. Let $k \geq 2$ and $m \geq 1$ be integers, and let G_2 be a complete bipartite graph with bipartite sets A and B such that $|A| = m$ and $|B| = m(k-1)+\alpha+2$. Let u and v be distinct vertices contained in B. Then G_2 is a connected graph and $uv \notin E(G_2)$. Let S be an independent set in G_2 of order $k+\alpha+1$ such that $S \cap \{u, v\} \neq \emptyset$. Then $S \subseteq B$, and hence $\Delta_k(S; G_2) = km = |G_2| - \alpha - 2$. We can easily see that G_2 has a spanning tree T such that $\mathrm{te}(T; k) \leq \alpha$, but $G_2 + uv$ does not have a spanning tree T' such that $\mathrm{te}(T'; k) \leq \alpha$. These imply that the degree sum condition of Theorem 2.18 is best possible.

Moreover, the closure obtained from Theorems 2.16 or 2.18 is well-defined by the similar way to the proof in [42]. We leave checking this to the reader.

Theorem 2.18 is a closure version of the following theorem (in fact, Fujisawa et al. showed a stronger result than Theorem 2.19).

Theorem 2.19 (Fujisawa, Matsumura and Yamashita [27]) *Let $\alpha \geq 0$ and $k \geq 2$ be integers, and let G be a connected graph. If $\Delta_k(S; G) \geq |G| - \alpha - 1$ for every independent set S in G of order $k + \alpha + 1$, then G has a spanning tree T such that $te(T; k) \leq \alpha$.*

Finally, we introduce another result, a corollary of Theorem 2.18. In the workshop on Discrete Mathematics and Its Applications 2018, Hiroshima, Japan, August 20–22, 2018, Matsuda gave a talk on the degree conditions for the existence of spanning k-trees in graphs. In his talk, he mentioned that by using Theorem 2.14, we can easily obtain the following theorem.

Theorem 2.20 (Aung and Kyaw [4]) *Let $k \geq 2$ be an integer, and let G be a connected graph. Let $L = \{v \in V(G) : \deg_G(v) < (|G| - 1)/k\}$. If $L = \emptyset$ or $G[L]$ is complete, then G has a spanning k-tree.*

By the same way as the strategy due to Matsuda, we obtain the following corollary from Theorem 2.18. Note that Corollary 2.21 is a generalization of Theorem 2.20.

Corollary 2.21 *Let $\alpha \geq 0$ and $k \geq 2$ be integers, and let G be a connected graph. Let $L = \{v \in V(G) : \deg_G(v) < (|G| - \alpha - 1)/k\}$. If $L = \emptyset$ or $\alpha(G[L]) \leq \alpha + 1$, then G has a spanning tree T such that $te(T; k) \leq \alpha$.*

Proof. Suppose not. Let G be an edge-maximal counterexample of Corollary 2.21. Let $u, v \in V(G)$ be two non-adjacent vertices of G. Since G is a counterexample of Corollary 2.21, it follows from Theorem 2.18 that there exists an independent set S in G of order $k + \alpha + 1$ such that $S \cap \{u, v\} \neq \emptyset$. Since G is an edge-maximal counterexample of Corollary 2.21, $G + uv$ has a spanning tree T' such that $\mathrm{te}(T'; k) \leq \alpha$. Since $L = \emptyset$ or $\alpha(G[L]) \leq \alpha + 1$, we have $|S \setminus L| \geq k$. This implies that $\Delta_k(S; G) \geq |G| - \alpha - 1$. Since $G + uv$ has a spanning tree T' such that $\mathrm{te}(T'; k) \leq \alpha$, it follows from Theorem 2.18 that G has a spanning tree T such that $\mathrm{te}(T; k) \leq \alpha$. This contradicts that G is a counterexample of Corollary 2.21. ∎

2.4.1 Notation and Lemmas

Let $i \geq 0$, $\alpha \geq 0$ and $k \geq 2$ be integers, and let G be a graph. Let $V_{\geq i}(G) = \{x \in V(G) : \deg_G(x) \geq i\}$, and let $\mathcal{S}(G; k, \alpha)$ be a set of spanning trees T in G such that $\mathrm{te}(T; k) \leq \alpha$. We can easily verify that if $T \in \mathcal{S}(G; k, \alpha)$, then $T \in \mathcal{S}(G + uv; k, \alpha)$ for any two nonadjacent vertices $u, v \in V(G)$. Therefore in our proof of Theorems 2.16 and 2.18, we show only the opposite directions.

In the rest of this section, we prepare lemmas used in the proofs of Theorems 2.16 and 2.18. Let $\alpha \geq 0$ and $k \geq 2$ be integers, and let G be a connected graph. Let u and v be two non-adjacent vertices of G. Suppose that $\mathcal{S}(G + uv; k, \alpha) \neq \emptyset$ but $\mathcal{S}(G; k, \alpha) = \emptyset$. Let

$$\mathcal{T} = \{(T_1, T_2) : \ T_1 \cup T_2 + uv \in \mathcal{S}(G + uv; k, \alpha), \, u \in V(T_1), \, v \in V(T_2)\}.$$

For any $(T_1, T_2) \in \mathcal{T}$, there exist $w_1 \in V(T_1)$ and $w_2 \in V(T_2)$ such that $w_1 w_2 \in E(G)$ because G is connected. Let $T_3 = T_1 \cup T_2 + w_1 w_2$. Choose such $(T_1, T_2) \in \mathcal{T}$, $w_1 \in V(T_1)$ and $w_2 \in V(T_2)$ so that

(T1) $\mathrm{te}(T_3;k)$ is as small as possible.

Note that $\alpha - 1 \leq \mathrm{te}(T_1;k) + \mathrm{te}(T_2;k) \leq \alpha$ because $\mathcal{S}(G;k,\alpha) = \emptyset$ and $T_1 \cup T_2 + uv \in \mathcal{S}(G+uv;k,\alpha)$.

Lemma 2.22 **(i)** *If* $\deg_{T_1}(u) \geq k$, *then* $\deg_{T_2}(v) \leq k-1$ *and* $te(T_1;k)+te(T_2;k) = \alpha-1$.

(ii) *If* $\deg_{T_2}(v) \geq k$, *then* $\deg_{T_1}(u) \leq k-1$ *and* $te(T_1;k) + te(T_2;k) = \alpha - 1$.

Proof. If $\deg_{T_1}(u) \geq k$ and $\deg_{T_2}(v) \geq k$, then $\mathrm{te}(T_1 \cup T_2 + uv;k) = \mathrm{te}(T_1;k) + \mathrm{te}(T_2,k) + 2 \geq \alpha+1$, a contradiction. Hence $\deg_{T_1}(u) \leq k-1$ or $\deg_{T_2}(v) \leq k-1$. This implies that if $\deg_{T_1}(u) \geq k$ or $\deg_{T_2}(v) \geq k$, then $\alpha \leq \mathrm{te}(T_1;k)+\mathrm{te}(T_2;k)+1 = \mathrm{te}(T_1 \cup T_2 + uv;k) \leq \alpha$, that is, $\mathrm{te}(T_1;k) + \mathrm{te}(T_2;k) = \alpha - 1$. ∎

Lemma 2.23 *If* $te(T_1;k) + te(T_2;k) = \alpha - 1$, *then* $d_{T_i}(w_i) \geq k$ *for each* $i \in \{1,2\}$.

Proof. If $\deg_{T_i}(w_i) \leq k-1$ for some $i \in \{1,2\}$, then $\mathrm{te}(T_3;k) \leq \mathrm{te}(T_1;k)+\mathrm{te}(T_2;k)+1 = \alpha$, and hence $\mathcal{S}(G;k,\alpha) \neq \emptyset$, a contradiction. ∎

Lemma 2.24 *For some* $i \in \{1,2\}$, $\deg_{T_i}(w_i) \geq k$.

Proof. If $\deg_{T_i}(w_i) \leq k-1$ for each $i \in \{1,2\}$, then $\mathrm{te}(T_3;k) = \mathrm{te}(T_1;k)+\mathrm{te}(T_2;k) \leq \alpha$, and hence $\mathcal{S}(G;k,\alpha) \neq \emptyset$, a contradiction. ∎

Lemma 2.25 *For some* $i \in \{1,2\}$, *the following statements hold.*

(i) $\deg_{T_i}(w_i) \geq k$ *and* $\deg_{T_3}(w) \leq k-1$, *where* $w \in V(T_i) \cap \{u,v\}$.

(ii) *There exists no tree* S_i *such that* $V(S_i) = V(T_i)$, $te(T_i;k) = te(S_i;k)$ *and* $\deg_{S_i}(w_i) = k-1$.

Proof. By Lemma 2.24 and by the symmetry of T_1 and T_2, we may assume that $d_{T_1}(w_1) \geq k$.

First, suppose that $\deg_{T_1}(u) \geq k$. Then $\deg_{T_2}(v) \leq k-1$ and $\mathrm{te}(T_1;k)+\mathrm{te}(T_2;k) = \alpha-1$ hold by Lemma 2.22 (i). By Lemma 2.23, this implies that $\deg_{T_2}(w_2) \geq k$. Then $v \neq w_2$, and so $\deg_{T_3}(v) \leq k-1$. Suppose that there exists a tree S_2 such that $V(S_2) = V(T_2)$, $\mathrm{te}(T_2;k) = \mathrm{te}(S_2;k)$ and $\deg_{S_2}(w_2) = k-1$. Then $\mathrm{te}(T_1;k) + \mathrm{te}(S_2;k) = \mathrm{te}(T_1;k) + \mathrm{te}(T_2;k) = \alpha - 1$. Since $\deg_{S_2}(w_2) = k-1$, this implies that $T_1 \cup S_2 + w_1w_2 \in \mathcal{S}(G;k,\alpha)$, a contradiction. Hence the statements (i) and (ii) hold for $i = 2$.

Next, suppose that $\deg_{T_1}(u) \leq k-1$. Then $u \neq w_1$, and so $\deg_{T_3}(u) \leq k-1$. Suppose that there exists a tree S_1 such that $V(S_1) = V(T_1)$, $\mathrm{te}(T_1;k) = \mathrm{te}(S_1;k)$ and $\deg_{S_1}(w_1) = k-1$. Then $\deg_{T_2}(w_2) \geq k$ and $\mathrm{te}(T_1;k) + \mathrm{te}(T_2;k) = \alpha$ because $S_1 \cup T_2 + w_1w_2 \notin$

$\mathcal{S}(G;k,\alpha)$. By Lemma 2.22 (i), this implies that $\deg_{T_2}(v) \le k-1$. Then $v \ne w_2$, and so $\deg_{T_3}(v) \le k-1$. If there exists a tree S_2 such that $V(S_2) = V(T_2)$, $\mathrm{te}(T_2;k) = \mathrm{te}(S_2;k)$ and $\deg_{S_2}(w_2) = k-1$, then $S_1 \cup S_2 + w_1w_2 \in \mathcal{S}(G;k,\alpha)$, a contradiction. Hence, the statements (i) and (ii) hold for $i = 2$. ∎

Lemma 2.26 *For each $i \in \{1,2\}$, there exists no tree S_i such that $V(S_i) = V(T_i)$, $te(S_i;k) < te(T_i;k)$ and $te(S_i \cup T_{3-i} + w_1w_2;k) < te(T_3;k)$.*

Proof. By the symmetry of T_1 and T_2, we have only to prove the case $i = 1$. Suppose that there exists a tree S_1 such that $V(S_1) = V(T_1)$, $\mathrm{te}(S_1;k) < \mathrm{te}(T_1;k)$ and $\mathrm{te}(S_1 \cup T_2 + w_1w_2;k) < \mathrm{te}(T_3;k)$. Then $S_1 \cup T_2 + uv \in \mathcal{S}(G+uv;k,\alpha)$ because $\mathrm{te}(S_1;k) < \mathrm{te}(T_1;k)$. By (T1), this implies that $\mathrm{te}(S_1 \cup T_2 + w_1w_2;k) \ge \mathrm{te}(T_3;k)$, a contradiction. ∎

2.4.2 Proof of Theorem 2.16

Let G be a graph which satisfies the assumption of Theorem 2.16. Suppose that $\mathcal{S}(G+uv;k,\alpha) \ne \emptyset$ but $\mathcal{S}(G;k,\alpha) = \emptyset$. We define $\mathcal{T}$ as in Section 2.4.1. Choose $(T_1,T_2) \in \mathcal{T}$, $w_1 \in V(T_1)$ and $w_2 \in V(T_2)$ so that (where we let $T_3 = T_1 \cup T_2 + w_1w_2$)

(T1) $\mathrm{te}(T_3;k)$ is as small as possible.

By the symmetry of T_1 and T_2, we may assume that

$$\text{Lemma 2.25 holds for } i = 1. \tag{2.4}$$

Among all $(T_1,T_2) \in \mathcal{T}, w_1 \in V(T_1)$ and $w_2 \in V(T_2)$ satisfying (T1) and (2.4), we choose w_1 so that

(T2) $\mathrm{dist}_{T_1}(w_1,u)$ is as large as possible.

Claim 2.4.1 *If there exists a tree S_1 which satisfies the following three properties, then $te(S_1;k) = te(T_1;k)$ and $\deg_{T_1}(w_1) \ge k+1$ hold:*

(i) *$V(S_1) = V(T_1)$;*

(ii) *$\deg_{S_1}(w_1) = \deg_{T_1}(w_1) - 1$; and*

(iii) *$te(S_1;k) \le te(T_1;k)$.*

Proof. By (2.4), $\deg_{T_1}(w_1) \ge k$. Let S_1 be a tree which satisfies the properties (i), (ii) and (iii). Suppose that $\mathrm{te}(S_1;k) < \mathrm{te}(T_1;k)$. Then $\mathrm{te}(S_1 \cup T_2 + w_1w_2;k) \ge \mathrm{te}(T_3;k)$ by Lemma 2.26. On the other hand, since $\deg_{T_1}(w_1) \ge k$, it follows from the property (ii) that $\mathrm{te}(S_1 \cup T_2 + w_1w_2;k) < \mathrm{te}(T_3;k)$, a contradiction. Hence by the property (iii),

$\mathrm{te}(S_1;k) = \mathrm{te}(T_1;k)$. If $d_{T_1}(w_1) = k$, then by the property (ii), $\deg_{S_1}(w_1) = k-1$, which contradicts (2.4). ∎

Here we take the outdirected tree with respect to (T_3, w_1). Let $D_1, \ldots, D_l$ be the components of $T_3 - w_1$. Note that $l \geq k+1$ because $\deg_{T_3}(w_1) \geq k+1$. Without loss of generality, we may assume that $u, v \in \bigcup_{1 \leq i \leq k+1} V(D_i)$. For each i $(1 \leq i \leq k+1)$, if $u, v \notin V(D_i)$, then take $x_i \in V(D_i)$ such that $\deg_{T_3}(x_i) \leq k-1$ (since D_i has a leaf of T_3, we can take such a vertex x_i); otherwise, let $\{x_i\} = \{u, v\} \cap V(D_i)$. For each j $(1 \leq j \leq l)$, take $z_j \in V(D_j) \cap N_{T_3}(w_1)$. Let $X = \{x_1, \ldots, x_{k+1}\}$ and $X_k = \{x_1, \ldots, x_k\}$, where $\deg_G(x_{k+1}) = \min\{\deg_G(x_i) : 1 \leq i \leq k+1\}$ (it is possible by changing the indices of D_i, x_i, and z_i). Then $\Delta_k(X;G) = \sum_{x \in X_k} \deg_G(x)$. Suppose that $k = 2$ and $X_2 = \{u, v\}$. Then $\Delta_k(X;G) = \deg_G(u) + \deg_G(v) \geq |G| - 1$. By Theorem 2.15, this implies that $\mathcal{S}(G;k,\alpha) \neq \emptyset$, a contradiction. Hence $X_2 \neq \{u, v\}$ if $k = 2$. This implies that, for $k \geq 2$, we may assume that $x_k \notin \{u, v\}$.

Claim 2.4.2 *For each i, j $(1 \leq i \leq k+1, 1 \leq j \leq l, i \neq j)$, $\deg_{T_3}(x) \geq k$ for all $x \in N_G(x_i) \cap V(D_j)$.*

Proof. Suppose that $\deg_{T_3}(x) \leq k-1$ for some i, j $(1 \leq i \leq k+1, 1 \leq j \leq l, i \neq j)$ and $x \in N_G(x_i) \cap V(D_j)$. If $V(D_i) \subseteq V(T_p)$ and $V(D_j) \subseteq V(T_{3-p})$ hold for some $p \in \{1, 2\}$, then $T_1 \cup T_2 + x_i x \in \mathcal{S}(G;k,\alpha)$, a contradiction (noting that if $x_i = v$ and $\deg_{T_2}(v) \geq k$, then $\mathrm{te}(T_1;k) + \mathrm{te}(T_2;k) = \alpha - 1$ holds by Lemma 2.22 (ii)). Hence $V(D_i) \cup V(D_j) \subseteq V(T_1)$. Then $S_1 = T_1 + x_i x - w_1 z_j$ satisfies the assumption of Claim 2.4.1. Hence $\mathrm{te}(S_1;k) = \mathrm{te}(T_1;k)$ and $\deg_{T_1}(w_1) \geq k+1$. These imply that $\deg_{T_3}(x_i) \geq \deg_{T_1}(x_i) \geq k$ (since otherwise $\mathrm{te}(S_1;k) < \mathrm{te}(T_1;k)$), which contradicts the choice of x_i. ∎

By Claim 2.4.2 and the definition of X, X is an independent set of G. Here we define

$$Y_j = \begin{cases} \bigcup_{1 \leq i \leq k, i \neq j} (N_G(x_i) \cap V(D_j)) & (1 \leq j \leq k) \\ \bigcup_{1 \leq i \leq k-1} (N_G(x_i) \cap V(D_j)) & (k+1 \leq j \leq l), \end{cases}$$

and

$$Y_j^+ = \begin{cases} \bigcup_{y \in Y_j} (N^+_{T_3, x_j}(y) \cap V(D_j)) & (1 \leq j \leq k) \\ \bigcup_{y \in Y_j} (N^+_{T_3, z_j}(y) \cap V(D_j)) & (k+1 \leq j \leq l). \end{cases}$$

Claim 2.4.3 **(i)** *For each j $(1 \leq j \leq k)$, $Y_j^+ \cap N_G(x_j) = \emptyset$.*

(ii) *For each j $(k+1 \leq j \leq l)$, $Y_j^+ \cap N_G(x_k) = \emptyset$.*

Proof. (i) Suppose that there exists $y_j^+ \in Y_j^+ \cap N_G(x_j)$ for some j $(1 \leq j \leq k)$. For convenience, let $y_j = n^-_{T_3, x_j}(y_j^+)$. By the definition of Y_j, there exists i $(1 \leq i \leq k, i \neq j)$

such that $y_j \in N_G(x_i) \cap V(D_j)$. If $V(D_j) \subseteq V(T_p)$ and $V(D_i) \subseteq V(T_{3-p})$ hold for some $p \in \{1, 2\}$, then $T_1 \cup T_2 + x_i y_j + x_j y_j^+ - y_j y_j^+ \in \mathcal{S}(G; k, \alpha)$, a contradiction (noting that if $v \in \{x_j, x_i\}$ and $\deg_{T_2}(v) \geq k$, then $\mathrm{te}(T_1; k) + \mathrm{te}(T_2; k) = \alpha - 1$ holds by Lemma 2.22 (ii)). Hence $V(D_j) \cup V(D_i) \subseteq V(T_1)$. Then $S_1 = T_1 + x_i y_j + x_j y_j^+ - y_j y_j^+ - w_1 z_j$ satisfies the assumption of Claim 2.4.1. Hence $\mathrm{te}(S_1; k) = \mathrm{te}(T_1; k)$ and $\deg_{T_1}(w_1) \geq k + 1$. These imply that $\deg_{T_3}(x_i) \geq k$ or $\deg_{T_3}(x_j) \geq k$, which contradicts the choice of x_i and x_j.

(ii) Suppose that there exists $y_j^+ \in Y_j^+ \cap N_G(x_k)$ for some j $(k + 1 \leq j \leq l)$. Let $y_j = n^-_{T_3, z_j}(y_j^+)$. By the definition of Y_j, there exists i $(1 \leq i \leq k - 1)$ such that $y_j \in N_G(x_i) \cap V(D_j)$. Since $x_k \notin \{u, v\}$, $V(D_k) \subseteq V(T_1)$. If $V(D_i) \subseteq V(T_2)$, then $x_i = v$ and $V(D_j) \subseteq V(T_1)$, and so $T_1 \cup T_2 + x_i y_j + x_k y_j^+ - y_j y_j^+ \in \mathcal{S}(G; k, \alpha)$, a contradiction. Hence $V(D_i) \subseteq V(T_1)$. Suppose that $V(D_j) \subseteq V(T_1)$. Then $S_1 = T_1 + x_i y_j + x_k y_j^+ - y_j y_j^+ - w_1 z_j$ satisfies the assumption of Claim 2.4.1. Hence $\mathrm{te}(S_1; k) = \mathrm{te}(T_1; k)$ and $\deg_{T_1}(w_1) \geq k+1$. These imply that $\deg_{T_3}(x_i) \geq k$ or $\deg_{T_3}(x_k) \geq k$, a contradiction. Hence $V(D_j) \subseteq V(T_2)$. Then $\mathrm{te}(T_1 \cup T_2 + x_i y_j + x_k y_j^+ - y_j y_j^+; k) = \mathrm{te}(T_1; k) + \mathrm{te}(T_2; k) \leq \alpha$. This implies that $\mathcal{S}(G; k, \alpha) \neq \emptyset$, a contradiction. ∎

Claim 2.4.4 *For each i, j $(1 \leq i \leq k, 1 \leq j \leq l, i \neq j)$, $z_j \notin N_G(x_i)$ except for the case $x_i = v$ and $x_j = u$.*

Proof. Suppose that $z_j \in N_G(x_i)$ for some i, j $(1 \leq i \leq k,\ 1 \leq j \leq l,\ i \neq j$ and $(x_i, x_j) \neq (v, u))$. Suppose that $V(D_i) \cup V(D_j) \subseteq V(T_1)$. Then $S_1 = T_1 + x_i z_j - w_1 z_j$ satisfies the assumption of Claim 2.4.1. Hence $\mathrm{te}(S_1; k) = \mathrm{te}(T_1; k)$ and $\deg_{T_1}(w_1) \geq k + 1$. These imply that $\deg_{T_1}(x_i) \geq k$, a contradiction. Thus $V(D_i) \subseteq V(T_p)$ and $V(D_j) \subseteq V(T_{3-p})$ hold for some $p \in \{1, 2\}$. If $p = 1$, then $w_2 = z_j$ and $\mathrm{te}(T_1 \cup T_2 + x_i w_2; k) < \mathrm{te}(T_3; k)$ because $\deg_{T_1}(x_i) \leq k - 1$ and $\deg_{T_1}(w_1) \geq k$, which contradicts (T1). Hence $p = 2$. Note that $x_i = v$ and $x_j \neq u$. Suppose that $\deg_{T_2}(v) \leq k - 1$. Suppose further that either $v \neq w_2$ or $\deg_{T_2}(v) \leq k - 2$. By (T1), $\mathrm{te}(T_3; k) \leq \mathrm{te}(T_1 \cup T_2 + x_i z_j; k)$. This implies that $\deg_{T_2}(w_2) \leq k - 1$ because $\deg_{T_1}(w_1) \geq k$ and $\deg_{T_2}(v) \leq k - 1$. Hence, we have $\mathrm{te}(T_3 + x_i z_j - w_1 z_j; k) \leq \mathrm{te}(T_3; k) - 1 \leq (\alpha + 1) - 1 = \alpha$, a contradiction. Hence $v = w_2$ and $\deg_{T_2}(v) = k - 1$. Let $T_3' = T_1 \cup T_2 + v z_j$ and let $w_1' = z_j$ and $w_2' = w_2$. Then $\mathrm{te}(T_3; k) = \mathrm{te}(T_3'; k)$ and so the choice of T_1, T_2, w_1', and w_2' satisfies the condition (T1). Since $w_2' = v$, Lemma 2.25 holds for $i = 1$ (we regard w_1' and w_2' as w_1 and w_2 in Lemma 2.25 respectively). Then $\mathrm{dist}_{T_1}(w_1, u) < \mathrm{dist}_{T_1}(w_1', u)$, which contradicts (T2). Hence $d_{T_2}(v) \geq k$. Let $w_1'' = z_j$, $w_2'' = v$ and $T_3'' = T_1 \cup T_2 + w_1'' w_2''$. Then $\mathrm{te}(T_3''; k) \leq \mathrm{te}(T_3; k)$. By the condition (T1), this implies that $\mathrm{te}(T_3''; k) = \mathrm{te}(T_3; k)$. Since $\deg_{T_2}(v) \geq k$, Lemma 2.25 holds for $i = 1$ (we regard w_1'' and w_2'' as w_1 and w_2 in Lemma 2.25 respectively). Since $x_j \neq u$, we have $\mathrm{dist}_{T_1}(w_1'', u) > \mathrm{dist}_{T_1}(w_1, u)$, which contradicts (T2). ∎

If $\{u, v\} \subseteq X_k$ and $z_j \in N_G(v)$ hold for some j $(1 \leq j \leq k)$, then we say that $\{u, v\}$ is *bad*. Recall that $X_2 \neq \{u, v\}$. This implies that if $\{u, v\}$ is bad, then $k \geq 3$ holds.

Recall that $Y_j^+ \subseteq V(D_j)$ for $1 \leq j \leq l$. By Claims 2.4.2 and 2.4.4, we obtain the following claim.

Claim 2.4.5 **(i)** *For each j $(1 \leq j \leq k)$,*

$$|Y_j^+| \geq \begin{cases} (k-1)|Y_j| - 1 & (\{u, v\} \text{ is bad, and } x_j = u) \\ (k-1)|Y_j| & (\text{otherwise}). \end{cases}$$

(ii) *For each j $(k+1 \leq j \leq l)$, $|Y_j^+| \geq (k-1)|Y_j|$.*

For each j $(1 \leq j \leq k)$, by Claims 2.4.3 (i) and 2.4.5 (i),

$$\begin{aligned} |N_G(x_j) \cap V(D_j)| &\leq |D_j| - |\{x_j\}| - |Y_j^+| \\ &\leq \begin{cases} |D_j| - (k-1)|Y_j| & (\{u, v\} \text{ is bad, and } x_j = u) \\ |D_j| - 1 - (k-1)|Y_j| & (\text{otherwise}). \end{cases} \end{aligned} \tag{2.5}$$

For each j $(1 \leq j \leq k)$, if $\{u, v\}$ is bad, and $x_j = u$, then by Claim 2.4.4 and $k \geq 3$,

$$\begin{aligned} \sum_{1 \leq i \leq k, i \neq j} |N_G(x_i) \cap V(D_j)| &= \sum_{1 \leq i \leq k, i \neq j} |N_G(x_i) \cap (V(D_j) \setminus \{z_j\})| + \sum_{1 \leq i \leq k, i \neq j} |N_G(x_i) \cap \{z_j\}| \\ &\leq (k-1)|Y_j \setminus \{z_j\}| + 1 \\ &\leq (k-1)|Y_j| - 1. \end{aligned} \tag{2.6}$$

For each j $(1 \leq j \leq k)$, if $\{u, v\}$ is not bad, or $x_j \neq u$, then by Claim 2.4.4,

$$\sum_{1 \leq i \leq k, i \neq j} |N_G(x_i) \cap V(D_j)| \leq (k-1)|Y_j|. \tag{2.7}$$

Hence, it follows from (2.5), (2.6) and (2.7) that for each j $(1 \leq j \leq k)$,

$$\sum_{1 \leq i \leq k} |N_G(x_i) \cap V(D_j)| \leq |D_j| - 1. \tag{2.8}$$

On the other hand, for each j $(k+1 \leq j \leq l)$, by Claims 2.4.3 (ii), 2.4.4 and 2.4.5 (ii),

$$\begin{aligned} |N_G(x_k) \cap V(D_j)| &\leq |D_j| - |\{z_j\}| - |Y_j^+| \\ &\leq |D_j| - 1 - (k-1)|Y_j|, \end{aligned} \tag{2.9}$$

and

$$\sum_{1 \leq i \leq k-1} |N_G(x_i) \cap V(D_j)| \leq (k-1)|Y_j|. \tag{2.10}$$

Hence it follows from (2.9) and (2.10) that for each j $(k+1 \leq j \leq l)$,

$$\sum_{1 \leq i \leq k} |N_G(x_i) \cap V(D_j)| \leq |D_j| - 1. \tag{2.11}$$

Consequently, it follows from (2.8), (2.11) and $l \geq k+1$ that

$$\begin{aligned}\Delta_k(X;G) &= \sum_{x \in X_k} \deg_G(x) \\ &\leq \sum_{1 \leq i \leq k} \Big(|N_G(x_i) \cap \{w_1\}| + \sum_{1 \leq j \leq l} |N_G(x_i) \cap V(D_j)| \Big) \\ &\leq k + \Big(|T_3| - |\{w_1\}| - l \Big) \\ &\leq |G| - 2,\end{aligned}$$

a contradiction. This completes the proof of Theorem 2.16. ∎

2.4.3 Proof of Theorem 2.18

Let G be a graph which satisfies the assumption of Theorem 2.18. Suppose that $\mathcal{S}(G+uv;k,\alpha) \neq \emptyset$ but $\mathcal{S}(G;k,\alpha) = \emptyset$. We define $\mathcal{T}$ as in Section 2. Choose $(T_1, T_2) \in \mathcal{T}$, $w_1 \in V(T_1)$ and $w_2 \in V(T_2)$ so that (where we let $T_3 = T_1 \cup T_2 + w_1w_2$)

(T1) $\mathrm{te}(T_3;k)$ is as small as possible, and

(T2) $\mathrm{te}(T_1;k) + \mathrm{te}(T_2;k)$ is as small as possible, subject to (T1).

Claim 2.4.6 *For each $i \in \{1,2\}$, there exists no tree S_i such that $V(S_i) = V(T_i)$ and $te(S_i;k) < te(T_i;k)$.*

Proof. Suppose that there exists a tree S_i such that $V(S_i) = V(T_i)$ and $\mathrm{te}(S_i;k) < \mathrm{te}(T_i;k)$ for some $i \in \{1,2\}$. Then $\mathrm{te}(S_i \cup T_{3-i} + uv;k) \leq \mathrm{te}(T_1 \cup T_2 + uv;k) \leq \alpha$, and so $(S_i, T_{3-i}) \in \mathcal{T}$. Moreover, $\mathrm{te}(S_i \cup T_{3-i} + w_1w_2;k) \leq \mathrm{te}(T_3;k)$ and $\mathrm{te}(S_i;k) + \mathrm{te}(T_{3-i};k) < \mathrm{te}(T_1;k) + \mathrm{te}(T_2;k)$, which contradicts (T1) or (T2). ∎

Let $V(T_3) = \{y_1, y_2, \ldots, y_{|G|}\}$. Moreover, for each i $(1 \leq i \leq |G|)$, let $d_i = \mathrm{dist}_{T_3}(w_1w_2, y_i)$, which means the distance between the edge w_1w_2 and a vertex y_i in T_3, i.e.,

$$d_i = \min\{\mathrm{dist}_{T_3}(w_1, y_i), \mathrm{dist}_{T_3}(w_2, y_i)\}.$$

Without loss of generality, we may assume that $d_1 \leq \cdots \leq d_{|G|}$. We define a sequence $W(T_3)$ as follows:

$$W(T_3) = (d_{T_3}(y_1), d_{T_3}(y_2), \ldots, d_{T_3}(y_{|G|})).$$

Furthermore, we choose $(T_1, T_2) \in \mathcal{T}$, $w_1 \in V(T_1)$ and $w_2 \in V(T_2)$ so that

(T3) $W(T_3)$ is as large as possible in lexicographic order, subject to (T1) and (T2).

By the symmetry of T_1 and T_2, we may assume that

$$\text{Lemma 2.25 holds for } i = 1. \tag{2.12}$$

Here we take the outdirected tree with respect to (T_3, w_1). Note that $\alpha + 1 \leq \mathrm{te}(T_3; k) \leq \alpha + 2$. For each $y \in V_{\geq k+1}(T_3) \setminus \{w_1\}$, we choose $C(y) \subseteq N^+_{T_3,w_1}(y)$ such that

(I) if $y \neq w_2$ or $\mathrm{te}(T_3; k) = \alpha + 1$, then $|C(y)| = \deg_{T_3}(y) - k$,

(II) if $y = w_2$ and $\mathrm{te}(T_3; k) = \alpha + 2$, then $|C(y)| = \deg_{T_3}(y) - k - 1$, and

(III) there exist two paths from w_1 to u and v in $T_3 - \bigcup_{y \in V_{\geq k+1}(T_3) \setminus \{w_1\}} \{xy : x \in C(y)\}$.

Note that we can choose such $C(y)$ for each $y \in V_{\geq k+1}(T_3) \setminus \{w_1\}$ because $|V(P_{T_3}(w_1, u)) \cap N^+_{T_3,w_1}(y)| \leq 1$, $|V(P_{T_3}(w_1, v)) \cap N^+_{T_3,w_1}(y)| \leq 1$ and $|N^+_{T_3,w_1}(y)| - 1 \geq \deg_{T_3}(y) - 2 \geq \deg_{T_3}(y) - k$ hold. If $\mathrm{te}(T_3; k) = \alpha + 2$, then $w_2 \in V_{\geq k+1}(T_3)$ and $|C(w_2)| = \deg_{T_3}(w_2) - k - 1$. Hence in any case on $\alpha + 1 \leq \mathrm{te}(T_3; k) \leq \alpha + 2$, there exist $k + \alpha + 1$ components in $T_3 - w_1 - \bigcup_{y \in V_{\geq k+1}(T_3) \setminus \{w_1\}} \{xy : x \in C(y)\}$. Let $D_1, \ldots, D_{k+\alpha+1}$ be the components of $T_3 - w_1 - \bigcup_{y \in V_{\geq k+1}(T_3) \setminus \{w_1\}} \{xy : x \in C(y)\}$. Note that $\deg_{D_j}(x) \leq k$ for each j $(1 \leq j \leq k + \alpha + 1)$ and each $x \in V(D_j)$. Without loss of generality, we may assume that $u \in V(D_1)$. Let $x_1 = u$. For each i $(2 \leq i \leq k + \alpha + 1)$, take $x_i \in V(D_i)$ such that $\deg_{T_3}(x_i) \leq k - 1$ (since each D_i has a leaf of T_3, we can take such a vertex x_i) and if $\mathrm{te}(T_3; k) = \alpha + 2$, then we do not choose v as one of $x_2, \ldots, x_{k+\alpha+1}$ (we can choose such $\{x_2, \ldots, x_{k+\alpha+1}\}$ not containing v because if $\mathrm{te}(T_3; k) = \alpha + 2$, then the component of $T_3 - w_1 - \bigcup_{y \in V_{\geq k+1}(T_3) \setminus \{w_1\}} \{xy : x \in C(y)\}$ containing w_2 has at least $k \geq 2$ leaves of T_3). For each j $(1 \leq j \leq k + \alpha + 1)$, take $z_j \in V(D_j)$ so that $|P_{T_3}(w_1, z_j)|$ is as small as possible. Note that $n^-_{T_3,w_1}(z_j) \in V_{\geq k+1}(T_3)$ for each j $(1 \leq j \leq k + \alpha + 1)$. Among all the vertices in $\{x_1, x_2, \ldots, x_{k+\alpha+1}\}$ and for each i $(1 \leq i \leq k + \alpha + 1)$, we change the indices of D_i, x_i, z_i so that $\sum_{1 \leq i \leq k} \deg_G(x_i)$ is as large as possible. Let $X = \{x_1, \ldots, x_{k+\alpha+1}\}$ and $X_k = \{x_1, \ldots, x_k\}$.

Claim 2.4.7 *Suppose that* $\deg_{T_2}(w_2) = k$. *Then the following statements hold:*

(i) *If* $C(w_2) \neq \emptyset$, *then* $te(T_3; k) = \alpha + 1$; *and*

(ii) *There exists no tree* S_2 *such that* $V(S_2) = V(T_2)$, $te(S_2; k) \leq te(T_2; k)$, $\deg_{S_2}(w_2) = k - 1$, *and* $\deg_{S_2}(x) \leq \deg_{T_2}(x)$ *for each* $x \in V(T_2) \setminus X$.

Proof. (i) Note that $\deg_{T_3}(w_2) = k + 1$. Since $C(w_2) \neq \emptyset$, it follows from the definitions (I) and (II) of $C(w_2)$ that $\mathrm{te}(T_3; k) = \alpha + 1$.

(ii) Suppose that there exists a tree S_2 such that $V(S_2) = V(T_2)$, $\mathrm{te}(S_2; k) \leq \mathrm{te}(T_2; k)$, $\deg_{S_2}(w_2) = k - 1$, and $\deg_{S_2}(x) \leq d_{T_2}(x)$ for each $x \in V(T_2) \setminus X$. Since $w_2 \in V_{\geq k+1}(T_3)$ by the assumption of this claim, $\mathrm{te}(T_1 \cup S_2 + w_1w_2; k) \leq \mathrm{te}(T_3; k) - 1$. If $\mathrm{te}(T_3; k) = \alpha + 1$, then $\mathrm{te}(T_1 \cup S_2 + w_1w_2; k) \leq \alpha$, which contradicts $\mathcal{S}(G; k, \alpha) = \emptyset$. Thus $\mathrm{te}(T_3; k) = \alpha + 2$. Then $v \notin X$ by the choice of $x_1, \ldots, x_{k+\alpha+1}$. Since $\deg_{S_2}(x) \leq \deg_{T_2}(x)$ for each $x \in V(T_2) \setminus X$, $\deg_{S_2}(v) \leq \deg_{T_2}(v)$, and so $(T_1, S_2) \in \mathcal{T}$. Therefore $\mathrm{te}(T_1 \cup S_2 + w_1w_2; k) < \mathrm{te}(T_3; k)$, which contradicts (T1). ∎

Claim 2.4.8 *For each i, j $(1 \leq i, j \leq k + \alpha + 1, i \neq j)$, $\deg_{T_3}(x) \geq k$ for all $x \in N_G(x_i) \cap V(D_j)$.*

Proof. Suppose that $\deg_{T_3}(x) \leq k - 1$ for some i, j $(1 \leq i, j \leq k + \alpha + 1, i \neq j)$ and $x \in N_G(x_i) \cap V(D_j)$. If $V(D_i)$ and $V(D_j)$ are contained in different components T_1 and T_2, then $\mathrm{te}(T_1 \cup T_2 + x_ix; k) < \mathrm{te}(T_3; k)$, which contradicts (T1). Thus $V(D_i) \cup V(D_j) \subseteq V(T_s)$ holds for some $s \in \{1, 2\}$. Then the unique cycle of $T_s + x_ix$ contains an edge e that is either $z_in^-_{T_3,w_1}(z_i)$ or $z_jn^-_{T_3,w_1}(z_j)$. Let $T'_s = T_s + x_ix - e$. If either $d_{T_s}(w_s) \geq k + 1$ or w_s is an end-vertex of e, then $V(T'_s) = V(T_s)$ and $\mathrm{te}(T'_s; k) < \mathrm{te}(T_s; k)$, which contradicts Claim 2.4.6. Thus $\deg_{T_s}(w_s) \leq k$ and w_s is an end-vertex of e. Since w_s is an end-vertex of e, we have $C(w_s) \neq \emptyset$, which implies that $\deg_{T_s}(w_s) = k$. Suppose that $s = 2$. Since $\deg_{T_2}(w_2) = k$ and $C(w_2) \neq \emptyset$, it follows from Claim 2.4.7 (i) that $\mathrm{te}(T_3; k) = \alpha + 1$. Note that $\deg_{T'_2}(w_2) = k - 1$ and $\mathrm{te}(T'_2; k) = \mathrm{te}(T_2; k)$. These imply that $\mathrm{te}(T_1 \cup T'_2 + w_1w_2; k) = \alpha$, which contradicts $\mathcal{S}(G; k, \alpha) = \emptyset$. Hence $s = 1$, $\deg_{T_1}(w_1) = k$ and $\deg_{T'_1}(w_1) = k - 1$. Then $V(T'_1) = V(T_1)$, $\mathrm{te}(T'_1; k) = \mathrm{te}(T_1; k)$ and $\deg_{T'_1}(w_1) = k - 1$, which contradicts (2.12). ∎

By Claim 2.4.8 and the definitions of X and X_k, we obtain the following.

Claim 2.4.9 *The set X is an independent set of G, and $\Delta_k(X; G) = \sum_{x \in X_k} \deg_G(x)$.*

Here we define

$$Y_j = \begin{cases} \bigcup_{1 \leq i \leq k, i \neq j} (N_G(x_i) \cap V(D_j)) & (1 \leq j \leq k) \\ \bigcup_{1 \leq i \leq k-1} (N_G(x_i) \cap V(D_j)) & (k + 1 \leq j \leq k + \alpha + 1), \end{cases}$$

and

$$Y_j^+ = \begin{cases} \bigcup_{x \in Y_j} (N^+_{T_3,x_j}(x) \cap V(D_j)) & (1 \leq j \leq k) \\ \bigcup_{x \in Y_j} (N^+_{T_3,x_k}(x) \cap V(D_j)) & (k + 1 \leq j \leq k + \alpha + 1). \end{cases}$$

Claim 2.4.10 **(i)** *For each j $(1 \leq j \leq k)$, $Y_j^+ \cap N_G(x_j) = \emptyset$.*

(ii) *For each j $(k + 1 \leq j \leq k + \alpha + 1)$, $Y_j^+ \cap N_G(x_k) = \emptyset$.*

Proof. (i) Suppose that there exists $y_j^+ \in Y_j^+ \cap N_G(x_j)$ for some j $(1 \le j \le k)$. For convenience, let $y_j = n^-_{T_3,x_j}(y_j^+)$. By the definition of Y_j, y_j is adjacent to some vertex x_m with $1 \le m \le k$ and $m \ne j$ in G. If y_j and x_m are contained in different components T_1 and T_2, then $T_1 \cup T_2 + x_j y_j^+ + x_m y_j - y_j y_j^+ \in \mathcal{S}(G; k, \alpha)$, a contradiction. Thus $\{y_j, x_m\} \subseteq V(T_s)$ holds for some $s \in \{1, 2\}$. Then the unique cycle of $T_s + x_j y_j^+ + x_m y_j - y_j y_j^+$ contains an edge e that is either $z_j n^-_{T_3,w_1}(z_j)$ or $z_m n^-_{T_3,w_1}(z_m)$. Let $T_s' = T_s + x_j y_j^+ + x_m y_j - y_j y_j^+ - e$. If either $\deg_{T_s}(w_s) \ge k+1$ or w_s is not an end-vertex of e in T_s, then $V(T_s') = V(T_s)$ and $\mathrm{te}(T_s'; k) < \mathrm{te}(T_s; k)$, which contradicts Claim 2.4.6. Thus $\deg_{T_s}(w_s) \le k$ and w_s is an end-vertex of e. Then $w_s \in V_{\ge k+1}(T_3)$, which implies that $\deg_{T_s}(w_s) = k$ and $\deg_{T_s'}(w_s) = k-1$. Note that $V(T_s') = V(T_s)$, $\mathrm{te}(T_s'; k) = (T_s; k)$, and $\deg_{T_s'}(x) \le \deg_{T_s}(x)$ for each $x \in V(T_s) \setminus X$. By Claim 2.4.7 (ii), we can see that $s = 1$. But, then we obtain a contradiction to (2.12).

(ii) Suppose that there exists $y_j^+ \in Y_j^+ \cap N_G(x_k)$ for some j $(k+1 \le j \le k+\alpha+1)$. Let $y_j = n^-_{T_3,x_j}(y_j^+)$. By the definition of Y_j, y_j is adjacent to some vertex $x_{m'}$ with $1 \le m' \le k-1$ in G. Suppose that y_j and $x_{m'}$ are contained in different components T_1 and T_2. Then $T_1 \cup T_2 + x_k y_j^+ + x_{m'} y_j - y_j y_j^+ \in \mathcal{S}(G; k, \alpha)$, which contracits $\mathcal{S}(G; k, \alpha) = \emptyset$. Hence $\{y_j, x_{m'}\} \subseteq V(T_s)$ holds for some $s \in \{1, 2\}$. Then the unique cycle of $T_3 + x_k y_j^+ + x_{m'} y_j - y_j y_j^+$ contains an edge e that is $z_\ell n^-_{T_3,w_1}(z_\ell)$ for some ℓ $(1 \le \ell \le k+\alpha+1)$. If $\mathrm{te}(T_3; k) = \alpha + 1$, then $\mathrm{te}(T_3 + x_k y_j^+ + x_{m'} y_j - y_j y_j^+ - e; k) = \alpha$ because $n^-_{T_3,w_1}(z_\ell) \in V_{k+1}(T_3)$ and the degree of $n^-_{T_3,w_1}(z_\ell)$ in $T_3 + x_k y_j^+ + x_{m'} y_j - y_j y_j^+ - e$ is strictly less than $\deg_{T_3}(n^-_{T_3,w_1}(z_\ell))$. This contradicts $\mathcal{S}(G; k, \alpha) = \emptyset$. Hence $\mathrm{te}(T_3; k) = \alpha + 2$. Since $\mathrm{te}(T_1; k) + \mathrm{te}(T_2; k) \le \alpha$, we have $w_1, w_2 \in V_{\ge k+1}(T_3)$. Suppose that $x_k \notin V(T_s)$, and let $T_3' = T_1 \cup T_2 + x_k y_j^+$. Then $\mathrm{te}(T_3'; k) < \mathrm{te}(T_3; k)$ because $\deg_{T_3}(x_k) \le k-1$, which contradicts (T1). Thus $\{y_j, x_{m'}, x_k\} \subseteq V(T_s)$. Let $T_s' = T_s + x_k y_j^+ + x_{m'} y_j - y_j y_j^+ - e$. If either $\deg_{T_s}(w_s) \ge k+1$ or w_s is not an end-vertex of e, then $V(T_s') = V(T_s)$ and $\mathrm{te}(T_s'; k) < \mathrm{te}(T_s; k)$, which contradicts Claim 2.4.6. Thus $\deg_{T_s}(w_s) \le k$ and w_s is an end-vertex of e. Then $w_s \in V_{\ge k+1}(T_3)$, which implies that $\deg_{T_s}(w_s) = k$ and $\deg_{T_s'}(w_s) = k-1$. Note that $V(T_s') = V(T_s)$, $\mathrm{te}(T_s'; k) = (T_s; k)$, and $\deg_{T_s'}(x) \le \deg_{T_s}(x)$ for each $x \in V(T_s) \setminus X$. By Claim 2.4.7 (ii), we can see that $s = 1$. But, then we obtain a contradiction to (2.12). ∎

Claim 2.4.11 *For each i, j $(1 \le i \le k, 1 \le j \le k+\alpha+1, i \ne j)$, $z_j \notin N_G(x_i)$ if $n^-_{T_3,w_1}(z_i) \ne z_j$.*

Proof. Suppose that $z_j \in N_G(x_i)$ and $n^-_{T_3,w_1}(z_i) \ne z_j$ for some i, j $(1 \le i \le k, 1 \le j \le k+\alpha+1, i \ne j)$. Then $d_{T_3}(z_j) \ge k$ by Claim 2.4.8.

First suppose that $\{x_i, z_j\} \subseteq V(T_s)$ holds for some $s \in \{1, 2\}$. Suppose further that $z_j \notin V(P_{T_3}(w_1, x_i))$. Let $T_s' = T_s + x_i z_j - z_j n^-_{T_3,w_1}(z_j)$. If either $\deg_{T_s}(w_s) \ge k+1$ or $n^-_{T_3,w_1}(z_j) \ne w_s$, then $V(T_s') = V(T_s)$ and $\mathrm{te}(T_s'; k) < \mathrm{te}(T_s; k)$, which contradicts Claim 2.4.6. Thus $\deg_{T_s}(w_s) \le k$ and $n^-_{T_3,w_1}(z_j) = w_s$. This implies that $\deg_{T_s}(w_s) = k$ and $\deg_{T_s'}(w_s) = k-1$. Note that $V(T_s') = V(T_s)$, $\mathrm{te}(T_s'; k) \le \mathrm{te}(T_s; k)$, and $\deg_{T_s'}(x) \le$

$\deg_{T_s}(x)$ for each $x \in V(T_s)\backslash X$. If $s = 1$, then we obtain a contradiction to (2.12); if $s = 2$, then we obtain a contradiction to Claim 2.4.7 (ii). Thus $z_j \in V(P_{T_3}(w_1, x_i))$. Then $x_i \neq u$ and $x_i \neq v$ by the definition (III) of $C(*)$. The unique cycle of $T_s + x_i z_j$ contains an edge $e = z_i n^-_{T_3,w_1}(z_i)$. Since $n^-_{T_3,w_1}(z_i) \neq z_j$, z_j is not an end-vertex of e. Let $T''_s = T_s + x_i z_j - e$ (see Fig. 2.6). Then $V(T''_s) = V(T_s)$ and $\mathrm{te}(T''_s; k) \leq \mathrm{te}(T_s; k)$. Note that $z_j \in V_{\geq k}(T_s)$ by Claim 2.4.8. Since $x_i \neq u$ and $x_i \neq v$, this implies that $(T''_s, T_{3-s}) \in \mathcal{T}$ even if $z_j \in \{u, v\}$. Let $T'_3 = T''_s \cup T_{3-s} + w_1 w_2$. Note that $\mathrm{dist}_{T_3}(w_1 w_2, y) = \mathrm{dist}_{T'_3}(w_1 w_2, y)$ holds for any $y \in V(G)$ with $\mathrm{dist}_{T_3}(w_1 w_2, y) \leq \mathrm{dist}_{T_3}(w_1 w_2, z_j)$ or $\mathrm{dist}_{T'_3}(w_1 w_2, y) \leq \mathrm{dist}_{T'_3}(w_1 w_2, z_j)$. Note that $d_{T'_3}(z_j) > d_{T_3}(z_j)$ because z_j is not an end-vertex of e. These contradicts (T3).

Next suppose that x_i and z_j are contained in different components T_1 and T_2. Since $\mathrm{te}(T_3 + x_i z_j - z_j n^-_{T_3,w_1}(z_j); k) < \mathrm{te}(T_3; k)$ and $\mathcal{S}(G; k, \alpha) = \emptyset$, we have $\mathrm{te}(T_3; k) = \alpha + 2$, and hence $w_1, w_2 \in V_{\geq k+1}(T_3)$. Let $S_3 = T_3 + x_i z_j - w_1 w_2$. Let S_1 and S_2 be the components of $S_3 - x_i z_j$. Then $(S_1, S_2) \in \mathcal{T}$ and $\mathrm{te}(S_3; k) < \mathrm{te}(T_3; k)$. This contradicts (T1). ∎

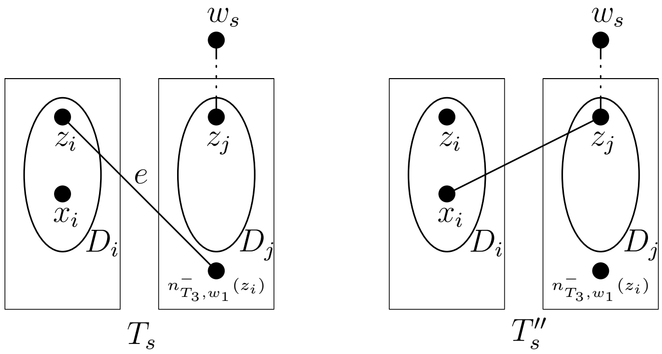


Figure 2.6: Claim 2.4.11 (possibly $n^-_{T_3,w_1}(z_i) \in V(D_j)$)

For each j $(k+1 \leq j \leq k+\alpha+1)$, take $z'_j \in V(D_j)$ so that $|P_{T_3}(x_k, z'_j)|$ is as small as possible.

Claim 2.4.12 *For each j $(k+1 \leq j \leq k+\alpha+1)$, $z'_j \notin N_G(x_k)$ if $n^-_{T_3,w_1}(z_k) \neq z'_j$.*

Proof. Suppose that $n^-_{T_3,w_1}(z_k) \neq z'_j$ and $z'_j \in N_G(x_k)$ hold for some j $(k+1 \leq j \leq k+\alpha+1)$. By Claim 2.4.11, we have only to prove the case $z'_j \neq z_j$. This implies that $z_j, z'_j \in V(P_{T_3}(w_1, x_k))$. Hence $\{x_k, z'_j\} \subseteq V(T_s)$ holds for some $s \in \{1, 2\}$. By the definition (III) of $C(*)$, $x_k \neq u$ and $x_k \neq v$. The unique cycle of $T_s + x_k z'_j$ contains an edge $e = z_k n^-_{T_3,w_1}(z_k)$ and z'_j is an end-vertex of e because $n^-_{T_3,w_1}(z_k) \neq z'_j$. Let $T'_s = T_s + x_k z'_j - e$ (see Fig. 2.7). Then $V(T'_s) = V(T_s)$ and $\mathrm{te}(T'_s; k) \leq \mathrm{te}(T_s; k)$. Note that $z'_j \in V_{\geq k}(T_s)$ by Claim 2.4.8. Since $x_i \neq u$ and $x_i \neq v$, this implies that $(T'_s, T_{3-s}) \in \mathcal{T}$ even if $z'_j \in \{u, v\}$. Let $T'_3 = T'_s \cup T_{3-s} + w_1 w_2$. Note that $\mathrm{dist}_{T_3}(w_1 w_2, y) = \mathrm{dist}_{T'_3}(w_1 w_2, y)$ holds for any $y \in V(G)$ with $\mathrm{dist}_{T_3}(w_1 w_2, y) \leq \mathrm{dist}_{T_3}(w_1 w_2, z'_j)$ or $\mathrm{dist}_{T'_3}(w_1 w_2, y) \leq \mathrm{dist}_{T'_3}(w_1 w_2, z'_j)$. Note that $d_{T'_3}(z_j) > d_{T_3}(z'_j)$ because z'_j is not an end-vertex of e. These contradicts (T3).

∎

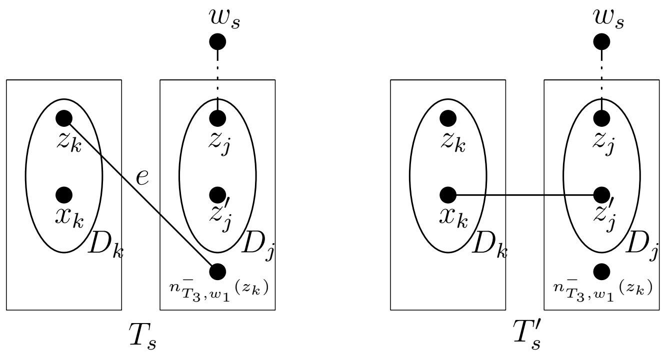


Figure 2.7: Claim 2.4.12 (possibly $n^-_{T_3,w_1}(z_k) \in V(D_j)$)

Claim 2.4.13 *If there exists j $(k+1 \leq j \leq k+\alpha+1)$ such that $z'_j = n^-_{T_3,w_1}(z_k)$, then $w_1 \notin N_G(x_k)$.*

Proof. Suppose that $w_1 \in N_G(x_k)$ and $z'_j = n^-_{T_3,w_1}(z_k)$ hold for some j $(k+1 \leq j \leq k+\alpha+1)$. Then $z'_j \in V_{\geq k+1}(T_3)$ because $z'_j = n^-_{T_3,w_1}(z_k)$. By the definition (III) of $C(*)$, $x_k \neq u$ and $x_k \neq v$. Let $T'_3 = T_3 + x_k w_1 - z'_j z_k$. Let T'_1 and T'_2 be the components of $T'_3 - w_1 w_2$ such that $w_1 \in V(T'_1)$ and $w_2 \in V(T'_2)$ (see Fig. 2.8 and 2.9). By the definition (III) of $C(*)$, the component D of $T_3 - z'_j z_k$ containing z_k does not contain u and v. Hence $u \in V(T'_1)$ and $v \in V(T'_2)$. Since $x_k \neq u$ and $x_k \neq v$, $(T'_1, T'_2) \in \mathcal{T}$. Moreover note that $\mathrm{te}(T'_3;k) \leq \mathrm{te}(T_3;k)$ and $\mathrm{te}(T'_1;k) + \mathrm{te}(T'_2;k) \leq \mathrm{te}(T_1;k) + \mathrm{te}(T_2;k)$. Since $\deg_{T_3}(w_1) < \deg_{T'_3}(w_1)$, we have $W(T_3) < W(T'_3)$. These contradict (T1), (T2), or (T3). ∎

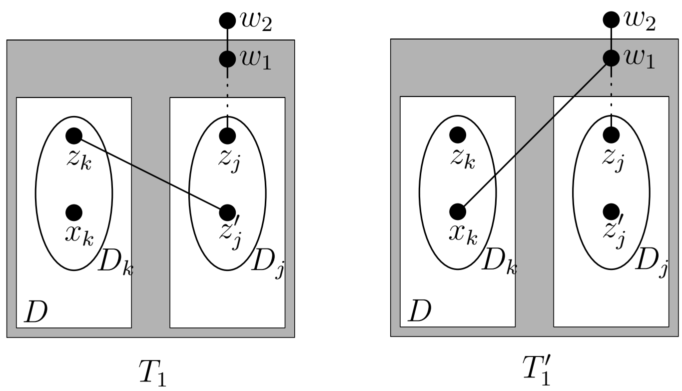


Figure 2.8: Claim 2.4.13 (the case where $w_1 \in P_{T_3}(w_2, z_k)$)

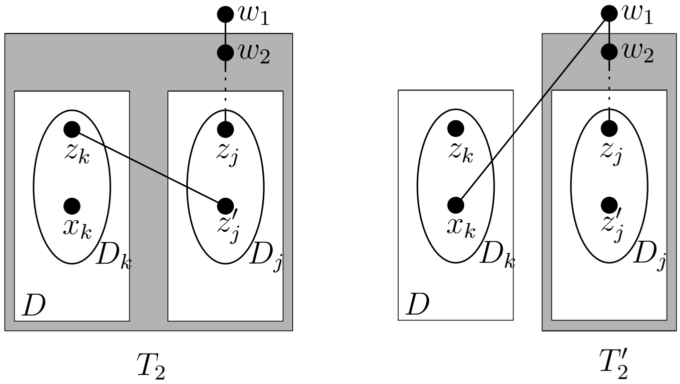


Figure 2.9: Claim 2.4.13 (the case where $w_2 \in P_{T_3}(w_1, z_k)$)

Claim 2.4.14 *For each j $(1 \leq j \leq k+\alpha+1)$, $|Y_j^+| \geq (k-1)|Y_j|$.*

Proof. Let j be an index with $1 \leq j \leq k+\alpha+1$, and let $w = x_j$ if $1 \leq j \leq k$ and $w = x_k$ if $k+1 \leq j \leq k+\alpha+1$. Since D_j is a tree, we have $N^+_{T_3,w}(y_1) \cap N^+_{T_3,w}(y_2) = \emptyset$ for any $y_1, y_2 \in Y_j$ with $y_1 \neq y_2$. Note that if $n^-_{T_3,w_1}(z_i) = z_j$ for some i $(1 \leq i \leq k+\alpha+1, i \neq j)$, then $\deg_{T_3}(z_j) \geq k+1$. Hence by Claims 2.4.8, 2.4.11 and the definitions (I) and (II) of $C(*)$, for each $y \in Y_j$, $|N^+_{T_3,w}(y) \cap V(D_j)| \geq k-1$. Then we obtain $|Y_j^+| = \sum_{y \in Y_j} |N^+_{T_3,w}(y) \cap V(D_j)| \geq (k-1)|Y_j|$. ∎

For each j $(1 \leq j \leq k)$, by Claims 2.4.10 (i) and 2.4.14,

$$\begin{aligned} |N_G(x_j) \cap V(D_j)| &\leq |D_j| - |\{x_j\}| - |Y_j^+| \\ &\leq |D_j| - 1 - (k-1)|Y_j|, \end{aligned} \tag{2.13}$$

and

$$\sum_{1 \leq i \leq k, i \neq j} |N_G(x_i) \cap V(D_j)| \leq (k-1)|Y_j|. \tag{2.14}$$

Hence it follows from (3.1) and (2.14) that for each j $(1 \leq j \leq k)$,

$$\sum_{1 \leq i \leq k} |N_G(x_i) \cap V(D_j)| \leq |D_j| - 1. \tag{2.15}$$

On the other hand, for each j $(k+1 \leq j \leq k+\alpha+1)$, by Claims 2.4.10 (ii), 2.4.12 and 2.4.14,

$$|N_G(x_k) \cap V(D_j)| \leq \begin{cases} |D_j| - |\{z'_j\}| - |Y_j^+| \leq |D_j| - 1 - (k-1)|Y_j| & (n^-_{T_3,w_1}(z_k) \neq z'_j) \\ |D_j| - |Y_j^+| \leq |D_j| - (k-1)|Y_j| & \text{(otherwise)}, \end{cases} \tag{2.16}$$

and

$$\sum_{1\le i\le k-1} |N_G(x_i)\cap V(D_j)| \le (k-1)|Y_j|. \tag{2.17}$$

Hence it follows from (2.16) and (2.17) that for each j $(k+1\le j\le k+\alpha+1)$,

$$\sum_{1\le i\le k} |N_G(x_i)\cap V(D_j)| \le \begin{cases} |D_j|-1 & (n^-_{T_3,w_1}(z_k)\ne z'_j)\\ |D_j| & (\text{otherwise}).\end{cases} \tag{2.18}$$

Note that there exists at most one index j $(k+1\le j\le k+\alpha+1)$ such that $n^-_{T_3,w_1}(z_k)=z'_j$. Consequently, by Claims 2.4.9, 2.4.13, (2.15), and (3.3),

$$\begin{aligned}\Delta_k(X;G) &= \sum_{x\in X_k}\deg_G(x)\\ &\le \sum_{1\le i\le k}\Big(|N_G(x_i)\cap\{w_1\}| + \sum_{1\le j\le k+\alpha+1}|N_G(x_i)\cap V(D_j)|\Big)\\ &\le \begin{cases} k-1+\Big(|T_3|-|\{w_1\}|-(k+\alpha+1-1)\Big) & (n^-_{T_3,w_1}(z_k)=z'_j \text{ for } k+1\le j\le k+\alpha+1)\\ k+\Big(|T_3|-|\{w_1\}|-(k+\alpha+1)\Big) & (\text{otherwise})\end{cases}\\ &\le |G|-\alpha-2,\end{aligned}$$

a contradiction. This completes the proof of Theorem 2.18. ∎

Chapter 3

Spanning trees with some specified properties

In this chapter, we focus on a spanning tree with some specified properties. In Section 3.1, we show some degree conditions for graphs to have a spanning tree with bounded total number of branch vertices and leaves. In Section 3.2, we show a Fan-type condition for graphs to be k-leaf-connected, which is a generalization of Hamilton-connected.

3.1 Degree conditions for graphs to have spanning trees with few branch vertices and leaves

We prove the following theorem, which gives a degree condition for a graph to have a spanning tree with bounded total number of branch vertices and leaves.

Theorem 3.1 *Let $k \geq 2$ be an integer. Suppose that a connected graph G satisfies*

$$\max\{\deg_G(x), \deg_G(y)\} \geq \frac{|G| - k + 1}{2}$$

for every two nonadjacent vertices $x, y \in V(G)$. Then G has a spanning tree T with $|L(T)| + |B(T)| \leq k + 1$.

The lower bound of the degree condition in Theorem 3.1 is sharp as shown in Section 3.1.2. One might conjecture that the sentence "for every two nonadjacent vertices" in Theorem 3.1 can be replaced by "for every two vertices $x, y \in V(G)$ with $\mathrm{dist}_G(x, y) = 2$", which is so-called a Fan-type degree condition.

The following problem assumes a weaker degree condition than Theorem 3.1.

Problem 3.2 *Let $k \geq 2$ be an integer. Let G be a connected graph. Suppose that G satisfies*

$$\max\{\deg_G(x), \deg_G(y)\} \geq \frac{|G|-k+1}{2}$$

for every two vertices $x, y \in V(G)$ with $\mathrm{dist}_G(x,y) = 2$. Does G have a spanning tree T with $|L(T)| + |B(T)| \leq k+1$?

The answer of Problem 3.2 is in the negative and the counterexample for Problem 3.2 is shown in Section 3.1.4. When we restrict ourselves to 2-connected graphs, we also obtain the following result, which contains a Fan-type degree condition.

Theorem 3.3 *Let $k \geq 2$ be an integer. Let G be a 2-connected graph. Suppose that*

$$\max\{\deg_G(x), \deg_G(y)\} \geq \frac{|G|-k+1}{2}$$

for every two vertices $x, y \in V(G)$ with $\mathrm{dist}_G(x,y) = 2$. Then G has a spanning tree T with $|L(T)| + |B(T)| \leq k+1$.

The following two results motivate our results. Theorem 3.4 gives an Ore-type condition for a graph to have a spanning k-ended tree.

Theorem 3.4 (Broersma and Tuinstra [8]) *Let $k \geq 2$ be an integer and let G be a connected graph. If G satisfies $\deg_G(x) + \deg_G(y) \geq |G| - k + 1$ for every two nonadjacent vertices $x, y \in V(G)$, then G has a spanning k-ended tree.*

The following theorem is stronger than Theorem 3.4 although it assumes the same condition as Theorem 3.4.

Theorem 3.5 (Nikoghosyan [46], Saito and Sano [54]) *Let $k \geq 2$ be an integer. If a connected graph G satisfies $\deg_G(x) + \deg_G(y) \geq |G| - k + 1$ for every two nonadjacent vertices $x, y \in V(G)$, then G has a spanning tree T with $|L(T)| + |B(T)| \leq k+1$.*

3.1.1 Preliminary Lemmas

We prove the following lemmas which are used in the proof of Theorems 3.1 and 3.3.

Lemma 3.6 *Let G be a connected graph and let T be a spanning tree of G such that $|L(T)| + |B(T)|$ is minimal. If $B(T) \neq \emptyset$, then $L(T)$ is an independent set of G.*

Proof. Suppose that there exist two vertices $u, v \in L(T)$ with $uv \in E(G)$. Then $T + uv$ contains a unique cycle C. By $B(T) \neq \emptyset$, C has a branch vertex w of T. For $x \in N_T(w) \cap V(C)$, $T' := T + uv - wx$ is a spanning tree of G such that $L(T') \subseteq (L(T) \setminus \{u, v\}) \cup \{x\}$ and $B(T') \subseteq B(T)$. This contradicts the minimality of $|L(T)| + |B(T)|$. ∎

Lemma 3.7 *Let G be a connected graph and let T be a spanning tree of G such that $|L(T)| + |B(T)|$ is minimal. Let x be a leaf of T. Suppose that $B(T) \neq \emptyset$, T is regarded as a rooted spanning tree of G with the root x.*

Then the following two statements hold:

(i) $N_G(x)^- \cap N_G(y) = \emptyset$ *for each* $y \in L(T) \setminus \{x\}$ *and*

(ii) $N_G(x)^- \cap B(T) = \emptyset$.

Proof. (i) Suppose that there exists $y \in L(T) \setminus \{x\}$ such that $N_G(x)^- \cap N_G(y) \neq \emptyset$. Since T is a spanning tree of G such that $\deg_T(x) = \deg_T(y) = 1$ and $B(T) \neq \emptyset$, $P_T(x, y)$ contains a branch vertex v. Let $u \in N_G(x)^- \cap N_G(y)$ and $u^+ \in N_T^+(u) \cap N_G(x)$. Then $T + u^+x + uy - u^+u$ contains a unique cycle C. For $w \in N_T(v) \cap V(C)$, $T' := T + u^+x + uy - u^+u - vw$ is a spanning tree of G with $L(T') \subseteq (L(T) \cup \{w\}) \setminus \{x, y\}$ and $B(T') \subseteq B(T)$. This contradicts the minimality of $|L(T)| + |B(T)|$. Hence $N_G(x)^- \cap N_G(y) = \emptyset$ for each $y \in L(T) \setminus \{x\}$.

(ii) If there exists a vertex $z \in N_G(x)^- \cap B(T)$, then $T' := T + xz^+ - z^+z$ is a spanning tree of G with $L(T') = L(T) \setminus \{x\}$ and $B(T') \subseteq B(T)$. This is a contradiction. Consequently, $N_G(x)^- \cap B(T) = \emptyset$. ∎

Let T be a tree with $B(T) \neq \emptyset$. For all pairs $x \in L(T)$ and $y \in B(T)$ such that $(V(P_T(x, y)) \setminus \{y\}) \cap B(T) = \emptyset$, we delete $V(P_T(x, y)) \setminus \{y\}$ from T. Let T' be the resulting graph. Then T' is a tree and $L(T') \subseteq B(T)$. We say that a leaf of T' is a *peripheral branch vertex* of T. By the definition of T', we obtain the following fact.

Fact 1 *Let T be a tree and let v be a peripheral branch vertex of T. Then the number of leaves x in T satisfying $(V(P_T(x, v)) \setminus \{v\}) \cap B(T) = \emptyset$ equals $\deg_T(v) - 1$.*

Lemma 3.8 *Let G be a connected graph having no Hamiltonian path. Choose a spanning tree T of G such that*

(T1) $|L(T)| + |B(T)|$ *is as small as possible and*

(T2) $\min\{\deg_T(x) : x$ *is a peripheral branch vertex of* $T\}$ *is as small as possible, subject to (T1).*

Let y be a peripheral branch vertex of T such that $\deg_T(y)$ is minimal and let z be a leaf of T such that $(V(P_T(y, z)) \setminus \{y\}) \cap B(T) = \emptyset$. Then $N_G(z) \cap (B(T) \setminus \{y\}) = \emptyset$.

Proof. Suppose that there exists a vertex $w \in N_G(z) \cap (B(T) \setminus \{y\})$. We regard T as a rooted tree with the root z. Then $T' := T + wz - yy^-$ is a spanning tree of G with $L(T') = (L(T) \setminus \{z\}) \cup \{y^-\}$. If $\deg_T(y) = 3$, then $B(T') = B(T) \setminus \{y\}$ and $|L(T')| = |L(T)|$, which is a contradiction to (T1). If $\deg_T(y) \geq 4$, then y is a peripheral branch vertex of T' with $\deg_{T'}(y) < \deg_T(y)$, which is a contradiction to (T2). ∎

3.1.2 Sharpness of Theorem 3.1

In Theorem 3.1, we cannot replace the lower bound $(|G|-k+1)/2$ in the degree condition by $(|G|-k)/2$, which is shown in the following example. Let t be a positive integer and let $k \geq 2$ be an integer. Consider the complete bipartite graph G with partite sets A and B such that $|A| = t$ and $|B| = t + k$. Then $|G| = 2t + k$ and $\max\{\deg_G(x), \deg_G(y)\} \geq t = (|G| - k)/2$ for every two nonadjacent vertices $x, y \in V(G)$. Suppose that G has a spanning tree T with $|L(T)| + |B(T)| \leq k + 1$. If $|L(T)| \leq k$, then $|E(T)| \geq |B \cap L(T)| + 2|B \setminus (B \cap L(T))| = 2|B| - |B \cap L(T)| \geq k + 2t = |G|$. This is a contradiction. If $|L(T)| \geq k + 1$, then $|L(T)| + |B(T)| \geq k + 2$ because T has at least one branch vertex. Hence G has no spanning tree T with $|L(T)| + |B(T)| \leq k + 1$.

3.1.3 Proof of Theorem 3.1

Suppose that a graph G satisfies all the conditions of Theorem 3.1, but has no desired spanning tree. Choose a spanning tree T of G so that

(T1) $|L(T)| + |B(T)|$ is as small as possible and

(T2) $\min\{\deg_T(x) : x$ is a peripheral branch vertex of $T\}$ is as small as possible, subject to (T1).

If $|L(T)| = 2$, then T is a Hamiltonian path of G, which satisfies $|L(T)| + |B(T)| = 2 < k+1$, a contradiction. Hence we may assume that $|L(T)| \geq 3$ and $|B(T)| \geq 1$. By Lemma 3.6 and the assumption of Theorem 3.1, the number of leaves in T having the degree at least $(|G| - k + 1)/2$ in G is at least $|L(T)| - 1$, i.e.,

$$|\{v \in L(T) : \deg_G(v) \geq (|G| - k + 1)/2\}| \geq |L(T)| - 1 \geq 2. \tag{3.1}$$

We divide the proof into two cases according to the value of $|B(T)|$.

Case 3.1.1 $|B(T)| = 1$.

By (3.1), we can choose two distinct vertices $x, y \in L(T)$ which satisfy $\deg_G(x) \geq (|G| - k + 1)/2$ and $\deg_G(y) \geq (|G| - k + 1)/2$. We regard T as a rooted tree with the root x. By Lemma 3.6, $N_G(y) \cap L(T) = \emptyset$. By Lemmas 3.7(i) and (ii), $N_G(x)^- \cap N_G(y) = \emptyset$ and $|N_G(x)^-| = |N_G(x)|$. Hence we obtain

$$\deg_G(x) + \deg_G(y) = |N_G(x)^-| + |N_G(y)| \leq |G| - |L(T)| + |\{x\}|.$$

On the other hand, $\deg_G(x) + \deg_G(y) \geq |G| - k + 1$ by the hypothesis of this theorem. Conbining two inequalities above, we obtain $|L(T)| \leq k$ and hence $|L(T)|+|B(T)| \leq k+1$. This is a contradiction. This completes the proof of Case 3.1.1.

Case 3.1.2 $|B(T)| \geq 2$.

Choose a peripheral branch vertex b_1 of T such that $\deg_T(b_1)$ is as small as possible. By Fact 1, there exist two leaves x_1 and x_2 of T such that $(V(P_T(b_1, x_i))\setminus\{b_1\})\cap B(T) = \emptyset$ for each $i = 1, 2$. By $|B(T)| \geq 2$, there exists a peripheral branch vertex b_2 of T with $b_2 \neq b_1$. Fact 1 implies that there exist two leaves x_3 and x_4 of T such that $(V(P_T(b_2, x_i))\setminus \{b_2\}) \cap B(T) = \emptyset$ for each $i = 3, 4$. By (3.1), without loss of generality, we may assume that $\deg_G(x_i) \geq (|G| - k + 1)/2$ for each $i = 1, 3$. Note that $x_1 \neq x_3$. We regard T as a rooted tree with root x_3. By Lemma 3.6, $N_G(x_1) \cap L(T) = \emptyset$. By Lemmas 3.7(i) and (ii), $N_G(x_1) \cap N_G(x_3)^- = \emptyset$ and $|N_G(x_3)^-| = |N_G(x_3)|$. By Lemma 3.7(ii) and Lemma 3.8, $N_G(x_3)^- \cap B(T) = \emptyset$ and $N_G(x_1) \cap (B(T) \setminus \{b_1\}) = \emptyset$. Hence

$$\begin{aligned}
|N_G(x_1)| + |N_G(x_3)| &= |N_G(x_1)| + |N_G(x_3)^-| \\
&\leq |T| - (|L(T)| - |\{x_3\}| + |B(T)| - |\{b_1\}|) \\
&= |G| - (|L(T)| + |B(T)|) + 2.
\end{aligned}$$

On the other hand, $|N_G(x_1)|+|N_G(x_3)| = \deg_G(x_1)+\deg_G(x_3) \geq |G|-k+1$. Consequently, $|L(T)| + |B(T)| \leq k + 1$. This is a contradiction. This completes the proof of Case 3.1.2. Hence Theorem 3.1 is proved. □

3.1.4 Counterexample of Problem 3.2

For two integers k and t such that $k \geq 2$ and $t \geq k + 1$, denote by K_t a complete graph of order t and denote by $P_i = a_i b_i$ a path of order two for each $i = 1, \ldots, k + 1$.

We define a graph G of order $t + 2k + 2$ as follows:

$$\begin{aligned}
V(G) &= V(K_t) \cup \left(\bigcup_{i=1}^{k+1} V(P_i)\right) \quad \text{and} \\
E(G) &= E(K_t) \cup \left(\bigcup_{i=1}^{k+1} \{xa_i : x \in V(K_t)\}\right) \cup \left(\bigcup_{i=1}^{k+1} E(P_i)\right).
\end{aligned}$$

Then, by $t \geq k+1$, $\max\{\deg_G(x), \deg_G(y)\} \geq t+1 = |G|-2k-1 \geq (|G|-k+1)/2$ for every two vertices $x, y \in V(G)$ with $\mathrm{dist}_G(x, y) = 2$. Since all the vertices in $\{b_1, b_2, \ldots, b_{k+1}\}$ are leaves for each spanning tree T of G, we obtain $|L(T)| \geq k + 1 \geq 3$ and thus $|L(T)| + |B(T)| \geq k + 2$. Therefore the answer for Problem 3.2 is in the negative.

3.1.5 Proof of Theorem 3.3

Suppose that a graph G satisfies all the conditions of Theorem 3.3, but has no desired spanning tree. Let $S = \{v \in V(T) : \deg_G(v) \geq (|G| - k + 1)/2\}$. Choose a spanning tree T of G such that

(T1) $|L(T)| + |B(T)|$ is as small as possible and

(T2) $|S \cap L(T)|$ is as large as possible subject to (T1).

If $|L(T)| = 2$, then T is a Hamiltonian path, which satisfies $|L(T)| + |B(T)| = 2 < k+1$, a contradiction. Hence we consider the case when $|L(T)| \geq 3$ and $|B(T)| \geq 1$.

Claim 3.1.1 *For any leaf x of T, $\deg_G(x) \geq (|G| - k + 1)/2$.*

Proof. Suppose that $\deg_G(x) < (|G| - k + 1)/2$ for some leaf x of T. Choose a vertex $w \in N_G(x)$ such that $|P_T(x, w)|$ is as large as possible. Write $P_T(x, w) = v_1 v_2 \dots v_m$ with $v_1 = x$ and $v_m = w$. Note that $m \geq 3$ because G is 2-connected and $\deg_T(x) = 1$. We regard T as a rooted tree with root v_1.

Subclaim 3.1.1.1 $\{v_2, v_3, \dots, v_m\} \subseteq N_G(v_1)$.

Proof. Suppose that $v_1 v_{i-1} \notin E(G)$ for some i with $v_1 v_i \in E(G)$. Then $\mathrm{dist}_G(v_1, v_{i-1}) = 2$. It follows from the degree condition of this theorem that $\deg_G(v_{i-1}) \geq (|G|-k+1)/2$. Since $v_{i-1} \notin B(T)$ by Lemma 3.7(ii), $T' := T + v_1 v_i - v_i v_{i-1}$ is a spanning tree of G with $L(T') = (L(T) \setminus \{x_1\}) \cup \{v_{i-1}\}$, $B(T') = B(T)$, and $|S \cap L(T')| > |S \cap L(T)|$. This contradicts the choice (T2). Hence $v_1 v_{i-1} \in E(G)$ for all i with $v_1 v_i \in E(G)$. By $v_1 v_m \in E(G)$, this subclaim holds. ∎

By Lemma 3.7(ii) and Subclaim 3.1.1.1, $\{v_1, v_2, \dots, v_{m-1}\} \cap B(T) = \emptyset$.

Subclaim 3.1.1.2 $\deg_G(v_i) < (|G| - k + 1)/2$ *for any v_i with $i = 1, 2, \dots, m-1$.*

Proof. If $\deg_G(v_i) \geq (|G|-k+1)/2$ for some v_i with $i = 2, \dots, m-1$, then $T + v_1 v_{i+1} - v_i v_{i+1}$ contradicts the choice (T2). Hence Subclaim 3.1.1.2 is proved. ∎

We denote by $\mathcal{T}$ the set of spanning trees T_i for $1 \leq i \leq m-1$ such that $L(T_i) = (L(T) \setminus \{x\}) \cup \{v_i\}$, $B(T_i) = B(T)$ and $\max\{|P_{T_i}(v_i, u)| : u \in N_G(v_i)\}$ is as large as possible. Note that each T_i satisfies (T1) and (T2) and $\mathcal{T} \neq \emptyset$. Choose $T_k \in \mathcal{T}$ so that

(T3) $\max\{|P_{T_k}(v_k, u)| : u \in N_G(v_k)\}$ is as large as possible.

Then $v_k \in L(T_k)$ by the choice of T_k and $\deg_G(v_k) < (|G| - k + 1)/2$ by (T2). Hence the role of v_k in T_k is similar to that of v_1 in T. Therefore, without loss of generality, we may assume $k = 1$. Then $|P_{T_1}(v_1, u)|$ is maximal.

Subclaim 3.1.1.3 $N_G(v_i) \subseteq \{v_1, v_2, \dots, v_m\}$ *for each $i = 1, 2, \dots, m-1$.*

Proof. By the definitions of $v_1 = x$ and u, the subclaim holds for $i = 1$. Suppose that v_i is adjacent to $u' \in V(G) \setminus \{v_1, v_2, \dots, v_m\}$ for some $i = 2, \dots, m-1$. By Subclaim 3.1.1.1, $v_1 v_{i+1} \in E(G)$ and let $T' := T_1 + v_1 v_{i+1} - v_i v_{i+1}$. Then $|P_{T'}(v_i, u')| > m = |P_{T_1}(v_1, u)|$, this implies that there exists the tree $T_i \in \mathcal{T}$ such that $\max\{|P_{T_i}(v_i, u)| : u \in N_G(v_i)\} > \max\{|P_{T_1}(v_1, u)| : u \in N_G(v_1)\}$. This contradicts the choice (T3). ∎

By Subclaim 3.1.1.3, v_m is a cut-vertex of G, which contradicts the condition that G is 2-connected. Consequenlty, Claim 3.1.1 is proved. ∎

Take any peripheral branch vertex b of T and put $\deg_T(b) = p$. By Fact 1, T contains $p-1$ leaves $x_1, \ldots, x_{p-1}$ such that $V(P_T(x_i, b)) \cap (B(T) \setminus \{b\}) = \emptyset$ for each $i = 1, \ldots, p-1$. Note that $p-1 = \deg_T(b) - 1 \geq 2$ because b is a branch vertex of T.

Claim 3.1.2 *$N_G(x_i) \cap (B(T) \setminus \{b\}) \neq \emptyset$ for each $i = 1, \ldots, p-1$.*

Proof. Suppose that $N_G(x_i) \cap (B(T) \setminus \{b\}) = \emptyset$ for some $i = 1, \ldots, p-1$. Without loss of generality, we may assume that $i = 1$. We regard T as a rooted tree with root x_2. By Lemma 3.7(ii), we obtain $N_G(x_2)^- \cap B(T) = \emptyset$ and hence $|N_G(x_2)| = |N_G(x_2)^-|$. Moreover, $N_G(x_1) \cap N_G(x_2)^- = \emptyset$ by Lemma 3.7(i) and $N_G(x_1) \cap L(T) = \emptyset$ by Lemma 3.6. Consequently

$$\begin{aligned}
|N_G(x_1)| + |N_G(x_2)| &= |N_G(x_1)| + |N_G(x_2)^-| \\
&\leq |T| - (|L(T)| - |\{x_2\}| + |B(T)| - |\{b\}|) \\
&\leq |G| - k.
\end{aligned}$$

On the other hand, $|N_G(x_1)| + |N_G(x_2)| = \deg_G(x_1) + \deg_G(x_2) \geq |G| - k + 1$ by Claim 3.1.1. This is a contradiction. ∎

For each $i = 1, 2, \ldots, p-2$, let $y_i \in N_T(b) \cap V(P_T(b, x_i))$ and let $b_i \in N_G(x_i) \cap (B(T) \setminus \{b\})$. Then $T' := T + x_1b_1 + \cdots + x_{p-2}b_{p-2} - by_1 - \cdots - by_{p-2}$ is a spanning tree of G with $L(T') \subseteq L(T) \setminus \{x_1, \ldots, x_{p-2}\} \cup \{y_1, \ldots, y_{p-2}\}$ and $B(T') \subseteq B(T) \setminus \{b\}$. This is a contradiction to (T1). Therefore the proof of Theorem 3.3 is completed. ∎

3.2 A Fan-type condition for graphs to be k-leaf-connected

A graph G is said to be *k-leaf-connected* if $|G| > k$ and for each subset S of $V(G)$ with $|S| = k$, G has a spanning tree T precisely S as the set of leaves. By the definition, it is easy to see that "2-leaf-connected" is "Hamilton-connected."

We prove the following theorem, which gives a Fan-type condition for graphs to be k-leaf-connected.

Theorem 3.9 *Let $k \geq 2$ be an integer. Suppose that G is a $(k+1)$-connected graph and that*

$$\max\{\deg_G(u), \deg_G(v)\} \geq \frac{|G|+1}{2}$$

for any vertices u and v in G with $\mathrm{dist}_G(u, v) = 2$. Then G is k-leaf-connected.

3.2.1 Related Results

It is known that many results concerning conditions for a graph to be Hamilton-connected. The property "G is Hamilton-connected" is as same as "G has a spanning tree with two specified endvertices." Moreover, by the definition, it is easy to see that "2-leaf-connected" is "Hamilton-connected." Thus it is natural to look for conditions which ensure the existence of a spanning tree with a specified set of endvertices. This paper is mainly concerned with sufficient conditions for a graph to have a spanning tree with a specified set of endvertices.

The following result motivate our result. Theorem 3.10 is fundemental result, which gives an Ore-type condition for graphs to be k-leaf-connected.

Theorem 3.10 (Egawa, Matsuda, Yamashita, and Yoshimoto [23]) *Let $k \geq 2$ be an integer and let G be a $(k+1)$-connected graph. Suppose that*

$$\deg_G(x) + \deg_G(y) \geq |G| + 1$$

for any two nonajacent vertices x, $y \in V(G)$. Then G is k-leaf-connected.

Theorem 3.9 is a stronger result than Theorem 3.10. In fact, there are infinitely many graphs which satisfy all the conditions of Theorem 3.9, but not satisfy the degree condition of Theorem 3.10.

For example, let $n \geq k+1$ and define K_n as a complete graph of order n with $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and K_{n+1} a complete graph of order $n+1$ with $V(K_{n+1}) = \{v_1, v_2, \ldots, v_{n+1}\}$. Construct a graph G of order $2n+1$ as $V(G) = V(K_n) \cup V(K_{n+1})$ and $E(G) = E(K_n) \cup E(K_{n+1}) \cup \{u_i v_i, u_i v_{i+1} : i = 1, \ldots, n-1\} \cup \{u_n v_n, u_n v_1\}$.

Since K_n and K_{n+1} are complete graphs and $n \geq k+1$, G is $(k+1)$-connected. Moreover, $\max\{\deg_G(u_i), \deg_G(v_j)\} \geq n+1 = (|G|+1)/2$ for any two vertices u_i and v_j with $\mathrm{dist}_G(u_i, v_j) = 2$. In particular, for each $i = 1, 2, \ldots, n$, two vertices u_i and v_{n+1} satisfy $\mathrm{dist}_G(u_i, v_{n+1}) = 2$, $\max\{\deg_G(u_i), \deg_G(v_{n+1})\} = n+1 = (|G|+1)/2$, and $\deg_G(u_i) + \deg_G(v_{n+1}) = 2n+1 \leq |G|$. Thus G satisfies all the conditions of Theorem 3.9, but not satisfy the degree condition of Theorem 3.10. Consequently, Theorem 3.9 can guarantee that G is k-leaf-connected although Theorem 3.10 cannot.

3.2.2 Sharpness of Theorem 3.9

The conditions of Theorem 3.9 are best possible in the following sense:

- We cannot replace the lower bound of the degree condition $(|G|+1)/2$ by $|G|/2$. Consider a complete bipartite graph $|G|$ with partite sets A and B such that $|A| = |B| = n$, where n is an integer with $n \geq k+1$. Then G is $(k+1)$-connected, $|G| = 2n$, and $\max\{\deg_G(x), \deg_G(y)\} \geq |G|/2$ for any vertices x and y of G with $\mathrm{dist}_G(x, y) = 2$. If G is k-leaf-connected, then G has a spanning tree T with $L(T) \subset B$ and $\deg_T(x) \geq 2$ for all $x \in A$. Therefore we have $|E(T)| \geq 2|A| = 2n = |G|$. This contradicts the fact $|E(T)| = |G| - 1$. Hence G is not k-leaf-connected.

- For $k \geq 2$, the condition that G is $(k+1)$-connected is necessary. Consider the graph $G := K_k + (K_1 \cup K_r)$, where $r \geq 2$ is an integer. Then G is k-connected but not $(k+1)$-connected. For two nonadjacent vertices $x \in V(K_r)$ and $y \in V(K_1)$, $\max\{\deg_G(x), \deg_G(y)\} = |G| - 2 \geq (|G|+1)/2$. Note that the last inequality holds by $|G| = k + r + 1 \geq 5$. Since G has no spanning tree T with $L(T) = V(K_k)$, G is not k-leaf-connected.

3.2.3 Proof of Theorem 3.9

We prove Theorem 3.9 by induction on k. Suppose that G satisfies all the conditions of Theorem 3.9. If $k = 2$, then Theorem 3.9 holds by Theorem 1.6 (iii). Thus we consider the case when $k \geq 3$. Suppose that G has no spanning tree T such that $L(T) = S$ and $|S| = k$ for some $S \subset V(G)$. By the induction hypothesis, G has a spanning tree T such that $L(T) \subset S$ and $|L(T)| = |S| - 1$. Denote $\{x_0\} = S - L(T)$ and choose such a spanning tree T so that

(T1) $\deg_T(x_0)$ is as small as possible subject to (T1).

We regard T as a rooted tree with root x_0 in which all the edges are directed away from the root. Write $N_T(x_0) = \{y_1, y_2, \ldots, y_m\}$. By the choice of T, x_0 is not a leaf of T and thus $|N_T(x_0)| = m \geq 2$. Let T_i be the component in $T - \{x_0\}$ containing the vertex y_i for each $i = 1, 2, \ldots, m$ and denote $S_i = S \cap L(T_i)$ for each $i = 1, 2, \ldots, m$.

Claim 3.2.1 $\{y_1, y_2, \ldots, y_m\} \cap S = \emptyset$.

Proof. Suppose that $\{y_1, y_2, \ldots, y_m\} \cap S \neq \emptyset$. Then we may assume that $y_1 \in S$. Since $|S \setminus \{y_1\}| = k - 1$ and G is $(k+1)$-connected, $G - (S \setminus \{y_1\})$ is 2-connected. Hence there exists $z \in V(T_j) \setminus S_j$ with $zy_1 \in E(G)$ for some $j = 2, \ldots, m$. Then $T' = T + y_1z - x_0y_1$ is a spanning tree of G. If $\deg_T(x_0) = 2$, then $L(T') = S$, a contradiction. Hence $\deg_T(x_0) \geq 3$. Then $L(T') = L(T)$ and $\deg_T(x_0) > \deg_{T'}(x_0)$, which contradicts (T1). ∎

Let $\mathcal{T}$ be the set of the all spanning trees T' of G such that $L(T') = L(T)$ and $N_{T'}(x_0) = N_T(x_0)$. Then we can regard T as an arbitrary tree in $\mathcal{T}$.

Claim 3.2.2 *The following four statements hold;*

(i) $\deg_T(y_i) = 2$ *for each* $i = 1, 2, \ldots, m$,

(ii) $B(T)^+ \cap N_G(y_i) = \emptyset$ *for each* $i = 1, 2, \ldots, m$,

(iii) *any vertex* $v \in (N_G(y_i) \cap V(T_i))^-$ *satisfies* $N_G(v) \subseteq S \cup V(T_i)$ *for each* $i = 1, 2, \ldots, m$, *and*

(iv) *no vertex in* $(N_G(y_i) \cap V(T_i))^-$ *is adjacent to a vertex in* $(B(T) \setminus B(T_i))^+$ *for each* $i = 1, 2, \ldots, m$.

Proof. (i) By Claim 3.2.1, $\deg_T(y_i) \geq 2$ for all $i = 1, 2, \ldots, m$. Assume that $\deg_T(y_i) \geq 3$ for some $i = 1, 2, \ldots, m$. Since G is $(k+1)$-connected and $|S| = k$, $G - S$ is connected. Thus, for some i with $1 \leq j \leq m$ and $j \neq i$, there exist two vertices $z_i \in V(T_i) \setminus S_i$ and $z_j \in V(T_j) \setminus S_j$ such that $z_i z_j \in E(G)$. Then $T' := T + z_i z_j - x_0 y_i$ is a spanning tree of G. If $\deg_T(x_0) = 2$, then $L(T') = S$, a contradiction. Hence $\deg_T(x_0) \geq 3$. Then $L(T') = L(T)$ and $\deg_T(x_0) > \deg_{T'}(x_0)$. This contradicts (T1).

(ii) Suppose that there exists a vertex $u \in B(T)^+ \cap N_G(y_i)$. By Claim 3.2.2 (i), $y_i \notin B(T)$ for all i and so $uy_i \notin E(T)$. Then $T' := T + uy_i - uu^-$ is a spanning tree of G with $L(T') = L(T)$, $N_{T'}(x_0) = N_T(x_0)$, and $\deg_{T'}(y_i) \geq 3$ and so $T' \in \mathcal{T}$. This contradicts Claim 3.2.2 (i).

(iii) Suppose that there exists a vertex $v \in N_G(y_i) \cap V(T_i)$ such that v^- is adjacent to a vertex $w \in V(G) \setminus (V(T_i) \cup S)$ for some $i = 1, 2, \ldots, m$. Suppose that $v^- = y_i$. Then $T' := T + v^- w - v^- x_0$ is a spanning tree of G. If $\deg_T(x_0) = 2$, then $L(T') = S$, a contradiction. Hence $\deg_T(x_0) \geq 3$. Then $L(T') = L(T)$ and $\deg_T(x_0) > \deg_{T'}(x_0)$. This contradicts (T1).

Hence we may assume that $v^- \neq y_i$. Then $T' := T + vy_i + v^- w - vv^- - x_0 y_i$ is a spanning tree of G. If $\deg_T(x_0) = 2$, then $L(T') = S$, a contradiction. Hence $\deg_T(x_0) \geq 3$. Then $L(T') = L(T)$ and $\deg_T(x_0) > \deg_{T'}(x_0)$. This contradicts (T1).

(iv) Suppose that for some $i = 1, 2, \ldots, m$, there exists $v \in N_G(y_i) \cap V(T_i)$ such that v^- is adjacent to $w \in (B(T) \setminus B(T_i))^+$. Note that $v^- \neq y_i$ by Claim 3.2.2 (iii). Then $T' := T + vy_i + v^- w - vv^- - ww^-$ is a spanning tree of G with $L(T') = L(T)$, $N_{T'}(x_0) = N_T(x_0)$, and $\deg_{T'}(y_i) \geq 3$ and so $T' \in \mathcal{T}$. This contradicts Claim 3.2.2 (i). ∎

Claim 3.2.3 $\mathrm{dist}_G(y_i, y_j) = 2$ *for each* $1 \leq i < j \leq m$.

Proof. By Claims 3.2.1 and 3.2.2 (iii), y_i and y_j are nonadjacent in G. Since each two vertices y_i and y_j have the common neighbor x_0, $\mathrm{dist}_G(y_i, y_j) = 2$ for each $1 \leq i < j \leq m$. ∎

Claim 3.2.4 $N_T(y_i) \cap S_i = \emptyset$ *for each* $i = 1, 2, \ldots, m$.

Proof. Suppose that $N_T(y_i) \cap S_i \neq \emptyset$ for some $i = 1, 2, \ldots, m$. Then, by Claim 3.2.2 (i), $|S_i| = 1$ and $|T_i| = 2$. Moreover, $N_G(y_i) \subseteq S$ by Claim 3.2.2 (iii). Hence $G - S$ is disconnected because $m \geq 2$. This contradicts the assumption that G is $(k+1)$-connected. ∎

Claim 3.2.5 $|N_G(y_i) \cap V(T_i)| \leq |T_i| - |S_i| - 1$ *for each* $i = 1, 2, \ldots, m$.

Proof. We first assume that T_i which has no branch vertex of T. Then $|S_i| = 1$. Since G is $(k+1)$-connected and $|S| = k$, $G - S$ is connected. Thus, for some $1 \leq j \leq m$ with

$j \neq i$, there exist two vertices $z_i \in V(T_i) \setminus S_i$ and $z_j \in V(T_j) \setminus S_j$ such that $z_i z_j \in E(G)$. By Claim 3.2.2 (iii), we obtain $y_i \neq z_i$ and $N_G(y_i) \cap \{z\}^+ = \emptyset$. Therefore

$$|N_G(y_i) \cap V(T_i)| \leq |T_i| - |\{y_i\} \cup \{z_i\}^+| = |T_i| - 2 = |T_i| - |S_i| - 1.$$

Next, we conside the case when T_i has at least one branch vertex of T. For each vertex $\ell \in S_i$, let $f(\ell)$ denote the unique vertex in $B(T_i)^+$ such that $\mathrm{dist}_T(f(\ell), \ell)$ is as small as possible. Note that $f(\ell) \neq f(\ell')$ for any distinct two vertices ℓ and ℓ' in S_i. By Claim 3.2.2 (ii), $y_i f(\ell) \notin E(G)$ for all $\ell \in S$. Since $y_i \neq f(\ell)$ for all $\ell \in S_i$,

$$|N_G(y_i) \cap V(T_i)| \leq |T_i| - |\{y_i\} \cup S_i| = |T_i| - |S_i| - 1.$$

Hence this claim holds. ∎

By Claims 3.2.2 (iii), 3.2.4, and 3.2.5 for each $i = 1, 2, \ldots, m$,

$$\begin{aligned}\deg_G(y_i) &\leq |N_G(y_i) \cap (S \setminus S_i)| + |N_G(y_i) \cap V(T_i)| \\ &\leq |S| - |S_i| + |T_i| - |S_i| - 1 = |S| + |T_i| - 2|S_i| - 1.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\sum_{i=1}^{m} \deg_G(y_i) &\leq m|S| + \sum_{i=1}^{m} |T_i| - 2\sum_{i=1}^{m} |S_i| - m \\ &= m|S| + |G| - |\{x_0\}| - 2(|S| - |\{x_0\}|) - m \\ &= |G| + (m-2)|S| - m + 1. \qquad (3.1)\end{aligned}$$

On the other hand, by Claim 3.2.3 and the assumption of Theorem 3.9, at least $m - 1$ vertices in $\{y_1, y_2, \ldots, y_m\}$ have degree more than or equal to $(|G| + 1)/2$ in G. Besides, $\delta(G) \geq k + 1 = |S| + 1$ as G is $(k + 1)$-connected. Thus we obtain

$$\sum_{i=1}^{m} \deg_G(y_i) \geq (m-1)\frac{|G|+1}{2} + |S| + 1. \qquad (3.2)$$

By (3.1) and (3.2),

$$(m-3)|S| \geq \frac{m-3}{2}|G| + \frac{3m-1}{2}. \qquad (3.3)$$

We divide the proof into the following two cases according to the value of $m = |N_T(x_0)|$.

Case 3.2.1 $m \geq 3$.

Substituting $m = 3$ into the inequality (3.3), we have $0 \geq 4$, a contradiction. Thus we consider the case $m \geq 4$. By (3.3), we obtain

$$|S| \geq \frac{1}{2}\left(|G| + \frac{3m-1}{m-3}\right) > \frac{1}{2}(|G| + 1).$$

Since G is $(k + 1)$-connected, $\delta(G) \geq k + 1 = |S| + 1 > (|G| + 1)/2 + 1$. Then G satisfies all the conditions of Theorem 3.10 and thus it is k-leaf-connected.

Case 3.2.2 $m = 2$.

By Claim 3.2.3 and the degree condition of Theorem 3.9, at least one of y_1 and y_2 have degree more than or equal to $(|G| + 1)/2$ in G. Without loss of generality, we may assume that

$$\deg_G(y_1) \geq \frac{|G| + 1}{2}.$$

Using the inequality (3.1) with $m = 2$, we have

$$\frac{|G| + 1}{2} + \deg_G(y_2) \leq \deg_G(y_1) + \deg_G(y_2) \leq |G| - 1.$$

Hence $\deg_G(y_2) \leq (|G| - 3)/2$. Since $G - S$ is connected, there exist two vertices $z_1 \in V(T_1) \setminus S_1$ and $z_2 \in V(T_2) \setminus S_2$ with $z_1 z_2 \in E(G)$. Note that $z_i \neq y_i$ for each $i = 1, 2$ by Claims 3.2.1 and 3.2.2 (iii). Since Claim 3.2.2 (iv) asserts that $\{z_2\}^+ \cap N_G(y_2) = \emptyset$, there exists a vertex $z \in V(T_2)$ which is nonadjacent to y_2 in G. Choose such a vertex z so that $|P_T(y_2, z)|$ is as small as possible. By the choice of z, y_2 is adjacent to all the vertices of $V(P_T(y_2, z^-)) \setminus \{y_2\}$ in G. Thus $\mathrm{dist}_G(y_2, z) = 2$. By $\deg_G(y_2) < (|G| + 1)/2$ and the assumption of this theorem, we obtain

$$\deg_G(z) \geq \frac{|G| + 1}{2}.$$

Since y_2 is adjacent to all the vertices of $V(P_T(y_2, z^-)) \setminus \{y_2\}$ in G, it follows from Claim 3.2.2 (ii) that $(V(P_T(y_2, z^-)) \setminus \{z^-\}) \cap B(T_2) = \emptyset$.

Claim 3.2.6 $|N_G(y_1) \cap V(T_1)| + |N_G(z) \cap V(T_1)| \leq |T_1|$.

Proof. To show the claim, suppose first that $(N_G(y_1) \cap V(T_1))^- \cap N_G(z) \neq \emptyset$. Then there exists a vertex $w \in N_G(y_1) \cap V(T_1)$ with $w^- \in N_G(z)$. Then by Claim 3.2.2 (iv), $N_G(w^-) \subseteq S \cup V(T_1)$ and so $z \in S_2$. Since y_2 is adjacent to all the vertices of $V(P_{T_2}(y_2, z^-)) \setminus \{y_2\}$ in G, it follows from Claim 3.2.2 (iii) that $z^- = z_2$. Then $T' := T + y_1 w + w^- z + z_1 z_2 - x_0 y_1 - z z^- - w^- w$ is a spanning tree with $L(T') = S$. This contradicts the assumpution that G has no spanning tree T with $L(T) = S$. Hence $(N_G(y_1) \cap V(T_1))^- \cap N_G(z) = \emptyset$. Since $|N_G(y_1) \cap V(T_1)| = |(N_G(y_1) \cap V(T_1))^-|$ holds by Claim 3.2.2 (ii), we obtain

$$|N_G(y_1) \cap V(T_1)| + |N_G(z) \cap V(T_1)| = |(N_G(y_1) \cap V(T_1))^-| + |N_G(z) \cap V(T_1)| \leq |T_1|.$$

Therefore the claim is proved. ∎

Subcase 3.2.2.1 $B(T_2) = \emptyset$.

By Claim 3.2.2 (iii) and $|S_2| = 1$,

$$|N_G(y_1) \cap V(T_2)| + |N_G(z) \cap V(T_2)| \le |S_2| + |T_2| - |\{z, y_2\}| = |T_2| - 1.$$

The above inequality together with Claim 3.2.6 implies

$$\deg_G(y_1) + \deg_G(z) \le |T_1| + |T_2| - 1 + 2|\{x_0\}| = |G|.$$

This contradicts $\deg_G(y_1) + \deg_G(z) \ge |G| + 1$.

Subcase 3.2.2.2 $B(T_2) \ne \emptyset$.

For any $v \in V(T_2)$, we denote by $S(v)$ the set of vertices ℓ in S_2 such that $P_T(v^-, \ell)$ contains v. In other words, $S(v)$ is defined as the set of leaves in S_2 which exist in the direction away from v in T when v is not a leaf in T_2; otherwise $S(v) = \{v\}$.

Claim 3.2.7 *The following two statements hold for any vertex $v \in B(T_2)^+$,*

(i) $y_2 \notin (N_G(y_1) \cap V(T_2))^- \cup N_G(v)$ *and*

(ii) $(N_G(y_1) \cap V(T_2))^- \cap N_G(v) \subseteq S(v)^-$.

Proof. (i) By Claim 3.2.2 (ii), y_2 is not adjacent to v in G. Assume that there exists $w \in N_G(y_1) \cap V(T_2)$ with $w^- = y_2$. By $B(T_2) \ne \emptyset$ and Claim 3.2.2 (iii), $y_2 \in B(T_2)$. This contradicts Claim 3.2.2 (i).

(ii) Suppose that $((N_G(y_1) \cap V(T_2))^- \cap N_G(v)) \setminus S(v)^- \ne \emptyset$. Let $w \in (N_G(y_1) \cap V(T_2)) \setminus S(v)$ be a vertex such that $vw^- \in E(G)$. Then $w \in S_2$ because Claim 3.2.2 (iii) with y_1 implies $N_G(y_1) \subseteq S \cup V(T_1)$. Note that $P_T(w, v^-)$ does not contain v by $v \notin S(v)$. By $v^- \in B(T_2)$ and Claim 3.2.2 (ii), $w^- \ne v^-$ and thus $vw^- \notin E(T)$. Then $T' := T + wy_1 + vw^- - ww^- - vv^-$ is a spanning tree of G with $L(T') = L(T)$, $N_{T'}(x_0) = N_T(x_0)$, and $\deg_{T'}(y_i) \ge 3$. This yields $T' \in \mathcal{T}$, which contradicts Claim 3.2.2 (i). ∎

Define X as the set of vertices in the path components of $T_2 - z$ containing a vertex in $\{z\}^+$. (In Fig. 3.1, X consists of the black vertices.) Note that $X \cap B(T_2) = \emptyset$ and it might be $X = \emptyset$. Let $x \in (N_G(y_1) \cap V(T_2)) \setminus X$. Since x is adjacent to y_1 in G, we obtain $x \in S_2 \setminus X$ by Claim 3.2.2 (iii). We define a function g from $(N_G(y_1) \cap V(T_2)) \setminus X$ to $V(T_2)$ as follows. If $x \in S(z)$, then by $x \notin X$, $P_T(z, x)$ contains a vertex in $B(T_2)$ and define $g(x) \in B(T_2)^+$ as a vertex such that $|P_T(x, g(x))|$ is as small as possible; otherwise $g(x) := x^-$ (see Fig. 3.1). By Claim 3.2.2 (ii), $g(x) \notin X$ for each $x \in (N_G(y_1) \cap V(T_2)) \setminus X$. Since $x \in S_2 \setminus X$, each pair of two vertices x and $g(x)$ is a one-to-one correspondence. Moreover, $g(x) \ne z$ by the definition of X.

Choose a spanning tree $T \in \mathcal{T}$ so that

(T2) $\sum_{x \in S \setminus \{x_0\}} |P_T(x_0, x)|$ is as small as possible subject to (T1).

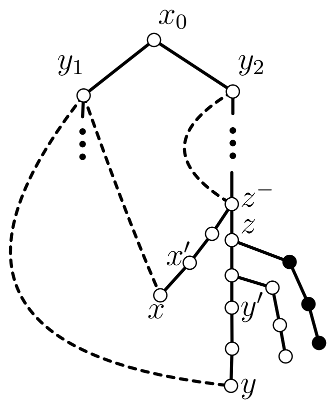


Figure 3.1: A tree T, where dotted lines are the edges not in T, $g(x) = x'$, and $g(y) = y'$.

Claim 3.2.8 *$zg(x) \notin E(G)$ for each $x \in (N_G(y_1) \cap V(T_2)) \setminus X$.*

Proof. If $(N_G(y_1) \cap V(T_2)) \setminus X = \emptyset$, then Claim 3.2.8 holds. Thus we assume that there exists a vertex $x \in (N_G(y_1) \cap V(T_2)) \setminus X$. Suppose first that $x \notin S(z)$. Then $B(T_2) \cap V(P_T(y_2, z^-)) \neq \emptyset$. Since $(V(P_T(y_2, z^-)) \setminus \{z^-\}) \cap B(T_2) = \emptyset$, we obtain $z^- \in B(T_2)$. Since $z \in B(T_2)^+$ and $g(x) = x^-$, it follows from Claim 3.2.7 (ii) that $zg(x) \notin E(G)$.

We next consider the case when $x \in S(z)$. If $zg(x) \in E(G)$, then $T' := T + zg(x) - g(x)g(x)^-$ is a spanning tree of G such that $L(T') = L(T)$, $N_{T'}(x_0) = N_T(x_0)$, and $\sum_{x \in S \setminus \{x_0\}} |P_{T'}(x_0, x)| < \sum_{x \in S \setminus \{x_0\}} |P_T(x_0, x)|$. This contradicts (T2).

Hence Claim 3.2.8 holds. ∎

By Claim 3.2.8, we obtain

$$\begin{aligned} |(N_G(y_1) \cap V(T_2)) \setminus X| + |(N_G(z) \cap V(T_2)) \setminus X| &\leq |T_2| - |X| - |\{y_2, z\}| \\ &= |T_2| - |X| - 2. \end{aligned} \tag{3.4}$$

We shall show that $|N_G(y_1) \cap X| + |N_G(z) \cap X| \leq |X| + 1$. To prove it, we need the following three claims.

Claim 3.2.9 *For any $v \in B(T_2)^+$, $\deg_G(v) < (|G| + 1)/2$ if $|S(v)| = 1$.*

Proof. Suppose that there exists a vertex $v \in B(T_2)^+$ such that $|S(v)| = 1$ and $\deg_G(v) \geq (|G| + 1)/2$. Let ℓ be the unique vertex in $S(v)$. We distinguish two cases.

We first consider the case $v \notin (N_G(y_1) \cap V(T_2))^-$. If $v = \ell$, then by $v \in B(T_2)^+$ and Claim 3.2.2 (ii), we obtain $y_1\ell \notin E(G)$. Hence

$$|N_G(y_1) \cap \{\ell\}| + |N_G(v) \cap \{\ell\}| \leq \begin{cases} 0 & \text{if } v = \ell \\ 2 & \text{otherwise.} \end{cases}$$

Claim 3.2.2 (ii) asserts that $|N_G(y_1) \cap (V(T_2) \setminus \{\ell\})| = |(N_G(y_1) \cap (V(T_2) \setminus \{\ell\}))^-|$ and Claims 3.2.7 (i) and (ii) yield

$$\begin{aligned}
&|N_G(y_1) \cap (V(T_2) \setminus \{\ell\})| + |N_G(v) \cap (V(T_2) \setminus \{\ell\})| \\
=&|(N_G(y_1) \cap (V(T_2) \setminus \{\ell\}))^-| + |N_G(v) \cap (V(T_2) \setminus \{\ell\})| \\
\leq&\begin{cases} |T_2| - |\{y_2, \ell\}| + |S(v)^-| = |T_2| - 1 & \text{if } v = \ell; \\ |T_2| - |\{y_2\}| - |\{\ell, v\}| = |T_2| - 3 & \text{otherwise.} \end{cases}
\end{aligned}$$

Hence we obtain

$$|N_G(y_1) \cap V(T_2)| + |N_G(v) \cap V(T_2)| \leq |T_2| - 1. \tag{3.5}$$

Since $v \in B(T_2)^+$, it follows from Claim 3.2.2 (iv) that $(N_G(y_1) \cap V(T_1))^- \cap N_G(v) = \emptyset$. By Claim 3.2.2 (ii), $|N_G(y_1) \cap V(T_1)| = |(N_G(y_1) \cap V(T_1))^-|$. Hence

$$\begin{aligned}
&|N_G(y_1) \cap V(T_1)| + |N_G(v) \cap V(T_1)| \\
=&|(N_G(y_1) \cap V(T_1))^-| + |N_G(v) \cap V(T_1)| \leq |T_1|.
\end{aligned} \tag{3.6}$$

By (3.5) and (3.6),

$$\deg_G(y_1) + \deg_G(v) \leq |T_1| + |T_2| - 1 + 2|\{x_0\}| = |G|.$$

On the other hand, $\deg_G(y_1) + \deg_G(v) \geq |G| + 1$. This is a contradiction.

Next, we consider the case $v \in (N_G(y_1) \cap V(T_2))^-$. Note that $v = \ell^-$ and $|N_G(y_1) \cap \{\ell\}| + |N_G(v) \cap \{\ell\}| = 2$. By Claim 3.2.2 (ii), $|(N_G(y_1) \cap V(T_2)) \setminus \{\ell\}| = |((N_G(y_1) \cap V(T_2)) \setminus \{\ell\})^-|$ and by Claims 3.2.7 (i) and (ii),

$$\begin{aligned}
&|(N_G(y_1) \cap V(T_2)) \setminus \{\ell\}| + |(N_G(v) \cap V(T_2)) \setminus \{\ell\}| \\
=&|((N_G(y_1) \cap V(T_2)) \setminus \{\ell\})^-| + |(N_G(v) \cap V(T_2)) \setminus \{\ell\}| \\
\leq&|T_2| - |\{y_2, \ell, v\}| + |S(v)^-| = |T_2| - 2.
\end{aligned}$$

Hence we obtain

$$|N_G(y_1) \cap V(T_2)| + |N_G(v) \cap V(T_2)| \leq |T_2|. \tag{3.7}$$

Suppose that v is adjacent to a vertex $z_1' \in V(T_1) \setminus S_1$. Note that $v\ell \in E(T)$. Then $T' := T + y_1\ell + vz_1' - v\ell - x_0y_1$ is a spanning tree of G with $L(T') = S$. Hence T' is a required tree, a contradiction. Hence $(N_G(v) \cap V(T_1)) \subseteq S_1$. By Claim 3.2.5,

$$|N_G(y_1) \cap V(T_1)| + |N_G(v) \cap V(T_1)| = |T_1| - |S_1| - 1 + |S_1| = |T_1| - 1. \tag{3.8}$$

By (3.7) and (3.8),

$$\deg_G(y_1) + \deg_G(v) \leq |T_1| - 1 + |T_2| + 2|\{x_0\}| = |G|.$$

This contradicts $\deg_G(y_1) + \deg_G(v) \geq |G| + 1$. ∎

Claim 3.2.10 $z \notin (N_G(y_1) \cap X)^-$.

Proof. Suppose that $z \in (N_G(y_1) \cap X)^-$. Take $\ell \in N_G(y_1) \cap X$ with $\ell^- = z$. By the assumption of Subcase 3.2.2.2 and Claim 3.2.2 (ii), $z^- \in B(T_2)$. Hence $z \in B(T_2)^+$. By the definition of X, we obtain $z, \ell \notin B(T_2)$. Hence $|S(z)| = 1$. Therefore, by Claim 3.2.9, $\deg_G(z) < (|G| + 1)/2$. This contradicts $\deg_G(z) \geq (|G| + 1)/2$. ∎

Claim 3.2.11 $|(N_G(y_1) \cap X)^- \cap N_G(z)| \leq 1$.

Proof. Suppose that $|(N_G(y_1) \cap X)^- \cap N_G(z)| \geq 2$. Then there exist two distinct vertices $a_1, a_2 \in X$ such that $(N_G(y_1) \cap \{a_i\})^- \cap N_G(z) \neq \emptyset$ for each $i = 1, 2$. Since $a_1, a_2 \in S_2$ by Claim 3.2.2 (iii), we have $z \in B(T_2)$. Furthermore, $a_1, a_2 \notin \{z\}^+$ by Claim 3.2.2 (ii). Let $w_i \in \{z\}^+ \cap V(P_T(z, a_i))$ for each $i = 1, 2$. By Claim 3.2.9, $\deg_G(w_i) < (|G| + 1)/2$ for each $i = 1, 2$. This together with the assumption of the theorem implies $w_1w_2 \in E(G)$. Note that $a_i \neq w_i$ for each $i = 1, 2$. Then $T' := T + a_1y_1 + a_1^- z + w_1w_2 - a_1a_1^- - zw_1 - zw_2$ is a spanning tree of G with $L(T') = L(T)$, $N_{T'}(x_0) = N_T(x_0)$, and $\deg_{T'}(y_1) \geq 3$. This contradicts Claim 3.2.2 (i). ∎

By Claim 3.2.2 (ii), $|N_G(y_1) \cap X| = |(N_G(y_1) \cap X)^-|$. By Claims 3.2.10 and 3.2.11, we obtain

$$|N_G(y_1) \cap X| + |N_G(z) \cap X| = |(N_G(y_1) \cap X)^-| + |N_G(z) \cap X| \leq |X| + 1.$$

By (3.4), Claim 3.2.6, and the above inequality, we obtain

$$\deg_G(y_1) + \deg_G(z) \leq |T_1| + |T_2| - 1 + 2|\{x_0\}| = |G|.$$

This contradicts $\deg_G(y_1) + \deg_G(z) \geq |G| + 1$. The proof of Subcase 3.2.2.2 is shown.

This completes the proof of Theorem 3.9. ∎

Chapter 4

Long paths in bipartite graphs

4.1 A Hamilton path in bipartite graphs

In 1963, Moon and Moser obtained a degree condition for bipartite graphs to have a Hamiton cycle (resp. path). For a bipartite graph G with bipartition (A, B), we define

$$\sigma_{1,1}(G) = \min\Big\{\deg_G(x) + \deg_G(y) : x \in A\ y \in B, xy \notin E(G)\Big\}$$

if G is a complete bipartite, then $\sigma_{1,1}(G) = \infty$.

Theorem 4.1 (Moon and Moser [44]) *Let G be a connected bipartite graph with bipartition (A, B).*

(i) *If $|A| \leq |B| \leq |A| + 1$ and $\sigma_2(G) \geq |B|$, then G has a Hamilton path.*

(ii) *If $|A| = |B| = n \geq 2$ and $\sigma_{1,1}(G) \geq n + 1$, then G has a Hamilton cycle.*

Note that the conditions $|A| \leq |B| \leq |A| + 1$ and $|A| = |B|$ are necessary conditions for bipartite graphs to have a Hamilton path and a Hamilton cycle, respectively.

4.2 Long paths in bipartite graphs and path-bistar bipartite Ramsey numbers

To find a long path in graphs is one of generalizations of finding a Hamilton path. Inspired by Theorems 4.1, we study a Fan-type condition for long paths in bipartite graphs in this section.

In Graph Theory, many types of degree conditions were studied for some important properties. We explain it with the Hamiltonicity of graphs as an example. Dirac [20] proved that if a graph G of order $n \geq 3$ satisfies $\deg_G(x) \geq \frac{n}{2}$ for all $x \in V(G)$, then G has a Hamilton cycle. This result influenced sufficient conditions for the existence of a Hamilton cycle with many extensions, for example, degree-sum condition, neighborhood-union condition, and so on (see a survey [37]). One of important extensions is a Fan-type degree condition that we introduce in Chapter 2. In Graph Theory, similar situations occur, i.e., a minimum degree condition is frequently replaced by a Fan-type condition, that is a condition concerning $\max\{\deg_G(x), \deg_G(y)\}$ for non-adjacent vertices x and y (see, for example, [40, 43, 57]). We carry the concept to bipartite graphs. The following is one of our main results.

Theorem 4.2 *Let m and n be positive integers with $n \geq m$. Let G be a bipartite graph having partite sets X_1 and X_2 with $|X_1| = |X_2| = n$. If*

(D1) $\max\{\deg_G(x_1), \deg_G(x_2)\} \geq m$ *or*

(D2) $\min\{\deg_G(x_1), \deg_G(x_2)\} \geq \frac{n+1}{2}$

for all vertices $x_1 \in X_1$ and $x_2 \in X_2$ with $x_1x_2 \notin E(G)$, then G contains a path P with $|V(P)| \geq 2m$.

The condition (D1) in Theorem 4.2 is best possible because $G = K_{n,n} - E(K_{m-1,m-1} \cup K_{n-m+1,n-m+1})$ satisfies $\max\{\deg_G(x_1), \deg_G(x_2)\} \geq m - 1$ for all vertices $x_1 \in X_1$ and $x_2 \in X_2$ with $x_1x_2 \notin E(G)$, and any paths of G have at most $2m - 1$ vertices.

One of our main targets in this section is the bipartite Ramsey number. Let H^r and H^b be bipartite graphs. The following fact is obtained by similar argument in the original Ramsey's theorem: there exists a positive integer N such that for any edge-disjoint spanning subgraphs G^r and G^b of $K_{N,N}$ with $E(G^r) \cup E(G^b) = E(K_{N,N})$, $H^r \subset G^r$ or $H^b \subset G^b$. The smallest value of N satisfying the above property is called the *bipartite Ramsey number* with respect to H^r and H^b and denoted by $b(H^r, H^b)$. Note that $b(H^r, H^b) = b(H^b, H^r)$. If H^b is a star, then the determination problem of $b(H^r, H^b)$ is reduced to a problem of finding H^r under a high minimum degree condition. Thus the bipartite Ramsey numbers involving stars tend to be simply determined. For example, Harary et al. [29] proved that $b(K_{1,s}, K_{1,t}) = s + t - 1$ and Hattingh and Henning [30] completely determined the value $b(P_s, K_{1,t})$ for $s \geq 2$ and $t \geq 2$. Further results for the bipartite Ramsey number related to stars were given in [17, 53]. As we mentioned above, some bipartite Ramsey numbers involving stars are determined using a high minimum degree condition problem. We will later show that a Fan-type condition gives manageable objects which can be replaced by stars.

Let n_1 and n_2 be non-negative integers, and let S_1 and S_2 be two vertex-disjoint stars having $n_1 + 1$ vertices and $n_2 + 1$ vertices, respectively. The (n_1, n_2)-*bistar*, denoted by B_{n_1,n_2}, is the graph obtained from S_1 and S_2 by joining their centers. Note that the $(n_1, 0)$-bistar is the star having $n_1 + 2$ vertices and the $(0, 0)$-bistar is the connected graph of order two. Recently, Hattingh and Joubert [31] proved that $b(B_{s,s}, B_{t,t}) = s + t + 1$, and Alm et al. [2] extended the result as $b(B_{s_1,s_2}, B_{t_1,t_2}) = s_1 + t_1 + 1$ for $s_1 \geq s_2$ and $t_1 \geq t_2$. In particular, we obtain $b(K_{1,s}, K_{1,t}) = b(B_{s-1,s-1}, B_{t-1,t-1})$. Hence the bipartite Ramsey number involving bistars seems to be related to one involving stars.

Recall that $b(P_s, K_{1,t+1})$ $(= b(P_s, B_{t,0}))$ was determined by Hattingh and Henning [30]. In this paper, using Theorem 4.2, we extend their result and determine the value $b(P_s, B_{t_1,t_2})$ as following.

Theorem 4.3 *Let s, t_1 and t_2 be integers with $s \geq 2$ and $t_1 \geq t_2 \geq 0$. Then the following hold.*

(i) *If $t_1 = t_2$, then $b(P_s, B_{t_1,t_2}) = \lfloor \frac{s-1}{2} \rfloor + t_1 + 1$.*

(ii) *Assume that $t_1 > t_2$.*

(ii-a) *If $t_1 \geq \lfloor \frac{s-1}{2} \rfloor$, then*

$$b(P_s, B_{t_1,t_2}) = \begin{cases} \lfloor \frac{s-1}{2} \rfloor + t_1 + 1 & (s \text{ is even, or } s \text{ is odd and } t_1 \equiv 0 \ (mod \ \frac{s-1}{2})) \\ \lfloor \frac{s-1}{2} \rfloor + t_1 & (otherwise). \end{cases}$$

(ii-b) *If $t_1 < \lfloor \frac{s-1}{2} \rfloor$, then*

$$b(P_s, B_{t_1,t_2}) = \begin{cases} 2t_1 + 1 & (2t_1 - t_2 \geq \lfloor \frac{s-1}{2} \rfloor) \\ \lfloor \frac{s-1}{2} \rfloor + t_2 + 1 & (otherwise). \end{cases}$$

4.2.1 Proof of Theorem 4.2

We start with two lemmas. The following lemma is well-known (see, for example, [30]).

Lemma 4.4 *Let m be a positive integer, and let G be a bipartite graph. If $\deg_G(x) \geq m$ for all $x \in V(G)$, then G contains a path P such that $|V(P)| \geq 2m$.*

Lemma 4.5 *Let m be a positive integer. Let G be a connected bipartite graph having partite sets X_1 and X_2 with $|X_1| \geq |X_2|$, and let $x_1 \in X_1$. If $\deg_G(x) \geq m$ for all $x \in X_1$, then G contains a path P such that x_1 is an end-vertex of P and $|V(P)| \geq 2m$.*

Proof. We proceed by induction on m. It is clear that the theorem holds for $m = 1$. Thus we may assume that $m \geq 2$.

Let $H_0 = G - \{x_1, y : y \in N_G(x_1),\ \deg_G(y) = 1\}$. Since $|V(H_0)| \geq |X_1 - \{x_1\}| \geq |X_2| - 1 \geq \deg_G(x_1) - 1 \geq m - 1 \geq 1$, H_0 is non-empty. Since $|V(H_0) \cap X_1| = |X_1| - 1 \geq |X_2| - 1 \geq |V(H_0) \cap X_2| - 1$, there exists a component H_1 of H_0 such that $|V(H_1) \cap X_1| \geq |V(H_1) \cap X_2| - 1$. Since G is connected, it follows from the definition of H_0 that there exists a vertex $x_2 \in N_G(x_1) \cap V(H_1)$ and $|V(H_1)| \geq 2$.

Since $|V(H_1 - x_2) \cap X_1| = |V(H_1) \cap X_1| \geq |V(H_1) \cap X_2| - 1 = (|V(H_1 - x_2) \cap X_2| + 1) - 1$, there exists a component H_2 of $H_1 - x_2$ such that $|V(H_2) \cap X_1| \geq |V(H_2) \cap X_2|$. Since $\deg_G(x_2) \geq 2$, there exists a vertex $x_3 \in N_G(x_2) \cap V(H_2)$. Note that $x_3 \in X_1$ and $\deg_{H_2}(x) = \deg_G(x) - |N_G(x) - V(H_2)| \geq m - |N_G(x) \cap \{x_2\}| \geq m - 1$ for all $x \in V(H_2) \cap X_1$. By the induction hypothesis, H_2 contains a path Q such that x_3 is an end-vertex of Q and $|V(Q)| \geq 2(m-1)$. Then the path $P = x_1x_2x_3Q$ is a desired path. ∎

Proof of Theorem 4.2. Let m, n, G, X_1 and X_2 be as in Theorem 4.2. By way of contradiction, suppose that every path of G has at most $2m-1$ vertices. Let $P = y_1y_2\cdots y_l$ be a longest path of G. Then $l \leq 2m-1$. Note that $V(G)-V(P) \neq \emptyset$ because $|V(G)| = 2n$. Without loss of generality, we may assume that $y_1 \in X_1$.

Since P is a longest path, all neighbors of y_1 are contained in $V(P) \cap X_2$. So, if $\deg_G(y_1) \geq m$, then $|V(P)| = |V(P) \cap X_1| + |V(P) \cap X_2| \geq 2|V(P) \cap X_2| \geq 2\deg_G(y_1) \geq 2m$, a contradiction. Thus, we have $\deg_G(y_1) \leq m-1$.

Suppose that there exists a vertex $u \in X_2 - V(P)$ such that (D2) $\min\{\deg_G(y_1), \deg_G(u)\} \geq \frac{n+1}{2}$ holds. Let $I_1 = \{1 \leq i \leq \frac{l}{2} : y_1y_{2i} \in E(G)\}$ and $I_2 = \{1 \leq i \leq \frac{l}{2} : uy_{2i-1} \in E(G)\}$. Note that $|I_1| = \deg_G(y_1) \geq \frac{n+1}{2}$ and since y_l is not a neighbor of u, $|I_2| = \deg_G(u) - \deg_{G-V(P)}(u) \geq \frac{n+1}{2} - |X_1 - V(P)|$. Thus,

$$\begin{aligned} n - |X_1 - V(P)| &= |X_1 \cap V(P)| \geq \frac{l}{2} \geq |I_1 \cup I_2| \\ &= |I_1| + |I_2| - |I_1 \cap I_2| \geq n + 1 - |X_1 - V(P)| - |I_1 \cap I_2|. \end{aligned}$$

This implies $I_1 \cap I_2 \neq \emptyset$, say $i \in I_1 \cap I_2$. Then $y_ly_{l-1}\cdots y_{2i}y_1y_2\cdots y_{2i-1}u$ is a path longer than P, a contradiction.

Therefore, for $u \in X_2 - V(P)$, (D1) $\max\{\deg_G(y_1), \deg_G(u)\} \geq m$ holds. Since $\deg_G(y_1) \leq m-1$, we have $\deg_G(u) \geq m$ for $u \in X_2 - V(P)$. Since $|X_1| = |X_2|$ and $|V(P) \cap X_1| \geq |V(P) \cap X_2|$, there exists a component H_0 of $G - V(P)$ such that $|V(H_0) \cap X_2| \geq |V(H_0) \cap X_1|$. Let $h = \max\{|N_G(u) \cap V(P)| : u \in V(H_0) \cap X_2\}$. Take a vertex $u^* \in V(H_0) \cap X_2$ so that $|N_G(u^*) \cap V(P)| = h$. Since $|V(P) \cap X_1| \leq \frac{l+1}{2} \leq \frac{2m}{2}$ and $u^*y_1 \notin E(G)$, we have $0 \leq h \leq m-1$. For $u \in V(H_0) \cap X_2$, since $\deg_G(u) \geq m$,

$$\deg_{H_0}(u) = \deg_G(u) - |N_G(u) \cap V(P)| \geq m - h \ (\geq 1).$$

Then by Lemma 4.5, there exists a path P' of H_0 such that u^* is an end-vertex of P' and $|V(P')| \geq 2(m-h)$. If $h = 0$, then $|V(P')| \geq 2m$, which is a contradiction. Thus $h \geq 1$.

Note that $N_G(u^*) \cap V(P) \subseteq V(P) \cap (X_1 - \{y_1\})$ $(= \{y_{2j-1} : j \geq 2\})$. Let j be the maximum integer satisfying $u^*y_{2j-1} \in E(G)$. Since $|N_G(u^*) \cap V(P)| = h$, we have $j \geq h+1$. Let P'' be the path as $P'' = y_1Py_{2j-1}u^*P'$. Then $|V(P'')| \geq (2j-1)+2(m-h) \geq (2(h+1)-1)+2(m-h) > 2m$, which is a contradiction. This completes the proof of Theorem 4.2. ∎

4.2.2 Proof of Theorem 4.3

In this section, we prove Theorem 4.3.

Lemma 4.6 *Let N be a positive integer, and let t_1 and t_2 be non-negative integers with $N \geq t_1 \geq t_2$. Let X_1 and X_2 be the partite sets of $K_{N,N}$. Let G^r and G^b be edge-disjoint spanning subgraphs of $K_{N,N}$ with $E(G^r) \cup E(G^b) = E(K_{N,N})$. If $B_{t_1,t_2} \not\subseteq G^b$, then*

(N1) $\max\{\deg_{G^r}(x_1), \deg_{G^r}(x_2)\} \geq N - t_2$ *or*

(N2) $\min\{\deg_{G^r}(x_1), \deg_{G^r}(x_2)\} \geq N - t_1$

for all vertices $x_1 \in X_1$ and $x_2 \in X_2$ such that $x_1x_2 \notin E(G^r)$.

Proof. Let $x_1 \in X_1$ and $x_2 \in X_2$ be vertices such that $x_1x_2 \notin E(G^r)$. Since $B_{t_1,t_2} \not\subset G^b$, $\deg_{G^b}(x_1) \leq t_j$ or $\deg_{G^b}(x_2) \leq t_{3-j}$ for each $j \in \{1, 2\}$. Since $\deg_{G^r}(x_i) + \deg_{G^b}(x_i) = N$, this implies that

$$\deg_{G^r}(x_1) \geq N - t_j \text{ or } \deg_{G^r}(x_2) \geq N - t_{3-j} \quad \text{for each } j \in \{1, 2\}. \tag{4.1}$$

If $\deg_{G^r}(x_1) \geq N - t_2$ or $\deg_{G^r}(x_2) \geq N - t_2$, then (N1) holds. Thus we may assume that $\deg_{G^r}(x_1) < N - t_2$ and $\deg_{G^r}(x_2) < N - t_2$. Then by (4.1), we have $\deg_{G^r}(x_1) \geq N - t_1$ and $\deg_{G^r}(x_2) \geq N - t_1$, which implies (N2). ∎

Lemma 4.7 *Let s be an integer with $s \geq 2$, and let t_1 and t_2 be non-negative integers with $t_1 \geq t_2$. Then $b(P_s, B_{t_1,t_2}) \leq \lfloor \frac{s-1}{2} \rfloor + t_1 + 1$.*

Proof. Let $N = \lfloor \frac{s-1}{2} \rfloor + t_1 + 1$. Let X_1 and X_2 be the partite sets of $K_{N,N}$. Let G^r and G^b be edge-disjoint spanning subgraphs of $K_{N,N}$ with $E(G^r) \cup E(G^b) = E(K_{N,N})$. Suppose that $B_{t_1,t_2} \not\subset G^b$. It suffices to show that $P_s \subset G^r$. Since $t_1 \geq t_2$, it follows from Lemma 4.6 that $\max\{\deg_{G^r}(x_1), \deg_{G^r}(x_2)\} \geq N - t_1 = \lfloor \frac{s-1}{2} \rfloor + 1$ for all vertices $x_1 \in X_1$ and $x_2 \in X_2$ with $x_1x_2 \notin E(G^r)$. Since $N \geq \lfloor \frac{s-1}{2} \rfloor + 1$, applying Theorem 4.2 with $n = N$ and $m = \lfloor \frac{s-1}{2} \rfloor + 1$, we obtain a path P in G^r with

$$|V(P)| \geq 2\left(\left\lfloor \frac{s-1}{2} \right\rfloor + 1\right) \geq 2\left(\frac{s-2}{2} + 1\right) = s,$$

as desired. ∎

Lemma 4.8 *Let s be an odd integer with $s \geq 3$, and let t_1 and t_2 be non-negative integers such that $t_1 > t_2$ and $t_1 \not\equiv 0 \pmod{\frac{s-1}{2}}$. Then $b(P_s, B_{t_1,t_2}) \leq \frac{s-1}{2} + t_1$.*

Proof. Let $N = \frac{s-1}{2} + t_1$. Let X_1 and X_2 be the partite sets of $K_{N,N}$. Let G^r and G^b be edge-disjoint spanning subgraphs of $K_{N,N}$ with $E(G^r) \cup E(G^b) = E(K_{N,N})$. By way of contradiction, suppose that $P_s \not\subset G^r$ and $B_{t_1,t_2} \not\subset G^b$. Since $t_1 > t_2$, it follows from Lemma 4.6 that $\max\{\deg_{G^r}(x_1), \deg_{G^r}(x_2)\} \geq N - t_1 = \frac{s-1}{2}$ for all vertices $x_1 \in X_1$ and $x_2 \in X_2$ with $x_1x_2 \notin E(G^r)$.

Claim 4.2.1 *If a component H of G^r contains a path of order $s-1$, then $|V(H)| = s-1$.*

Proof. Suppose that H contains a path $P = y_1y_2\cdots y_{s-1}$. Without loss of generality, we may assume that $y_1 \in X_1$. Note that $y_{s-1} \in X_2$. Since H contains no path of order s, $N_H(y_1) \subseteq V(P) \cap X_2$ and $N_H(y_{s-1}) \subseteq V(P) \cap X_1$. If $y_1y_{s-1} \notin E(H)$, then $\deg_H(y_1) \le |V(P) \cap (X_2 - \{y_{s-1}\})| = \frac{s-3}{2}$, $\deg_H(y_{s-1}) \le |V(P) \cap (X_1 - \{y_1\})| = \frac{s-3}{2}$, which contradicts the fact that $\max\{\deg_H(y_1), \deg_H(y_{s-1})\} \ge \frac{s-1}{2}$. Thus $y_1y_{s-1} \in E(H)$. In particular, $y_1y_2\cdots y_{s-1}y_1$ is a cycle of H. Since G^r contains no path of order s, it follows that $N_H(y_i) \subseteq V(P)$ for all i $(1 \le i \le s-1)$. In particular, $H[V(P)] = H$. ∎

Since $N = \frac{s-1}{2} + t_1 \ge \frac{s-1}{2}$, applying Theorem 4.2 with $n = N$ and $m = \frac{s-1}{2}$, we obtain a path P in G^r with $|V(P)| \ge 2 \cdot \frac{s-1}{2} = s-1$. It follows from Claim 4.2.1 that $G^r[V(P)]$ is a component of G^r. In particular, $\deg_{G^r[V(P)]}(x) \le \frac{s-1}{2} = N - t_1 < N - t_2$ for all $x \in V(P)$. This together with Lemma 4.6 implies that $\deg_{G^r}(u) \ge N - t_1$ for all $u \in V(G^r) - V(P)$.

Since $N - \frac{s-1}{2} = t_1 \ge 1$, $V(G^r) - V(P) \ne \emptyset$. Let H be a component of G^r other than $G^r[V(P)]$. Since $\deg_{G^r}(u) \ge N - t_1 = \frac{s-1}{2}$ for every $u \in V(H)$, it follows from Lemma 4.4 that H contains a path of order $s-1$. Then by Claim 4.2.1, $|V(H)| = s-1$ (i.e., $|V(H) \cap X_1| = \frac{s-1}{2}$). Since H is arbitrary, N $(= |X_1|)$ is a multiple of $\frac{s-1}{2}$, which contradicts the assumption that $t_1 \not\equiv 0 \pmod{\frac{s-1}{2}}$. ∎

Lemma 4.9 *Let s be an integer with $s \ge 2$, and let t_1 and t_2 be non-negative integers with $\lfloor \frac{s-1}{2} \rfloor > t_1 > t_2$. Then*

$$b(P_s, B_{t_1,t_2}) \le \begin{cases} 2t_1 + 1 & (2t_1 - t_2 \ge \lfloor \frac{s-1}{2} \rfloor) \\ \lfloor \frac{s-1}{2} \rfloor + t_2 + 1 & (\textit{otherwise}). \end{cases}$$

Proof. Let $N = \lfloor \frac{s-1}{2} \rfloor + t_2 + 1 + \max\{2t_1 - t_2 - \lfloor \frac{s-1}{2} \rfloor, 0\}$. Let X_1 and X_2 be the partite sets of $K_{N,N}$. Let G^r and G^b be edge-disjoint spanning subgraphs of $K_{N,N}$ with $E(G^r) \cup E(G^b) = E(K_{N,N})$. Suppose that $B_{t_1,t_2} \not\subset G^b$ as a subgraph. It suffices to show that $P_s \subset G^r$. Note that $N - t_2 \ge (\lfloor \frac{s-1}{2} \rfloor + t_2 + 1) - t_2 = \lfloor \frac{s-1}{2} \rfloor + 1$ and $N - t_1 \ge \frac{N+1}{2}$ because

$$\begin{aligned} 2(N - t_1) - (N+1) &= N - 2t_1 - 1 \\ &= \left\lfloor \frac{s-1}{2} \right\rfloor + t_2 + \max\left\{2t_1 - t_2 - \left\lfloor \frac{s-1}{2} \right\rfloor, 0\right\} - 2t_1 \\ &= \begin{cases} 0 & (2t_1 - t_2 \ge \lfloor \frac{s-1}{2} \rfloor) \\ \lfloor \frac{s-1}{2} \rfloor - (2t_1 - t_2) > 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

This together with Lemma 4.6 implies that, for all vertices $x_1 \in X_1$ and $x_2 \in X_2$ with $x_1x_2 \notin E(G^r)$,

- $\max\{\deg_{G^r}(x_1), \deg_{G^r}(x_2)\} \ge N - t_2 \ge \lfloor \frac{s-1}{2} \rfloor + 1$ or

- $\min\{\deg_{G^r}(x_1), \deg_{G^r}(x_2)\} \geq N - t_1 \geq \frac{N+1}{2}$.

Since $N = \lfloor \frac{s-1}{2} \rfloor + t_2 + 1 + \max\{2t_1 - t_2 - \lfloor \frac{s-1}{2} \rfloor, 0\} \geq \lfloor \frac{s-1}{2} \rfloor + 1$, applying Theorem 4.2 with $n = N$ and $m = \lfloor \frac{s-1}{2} \rfloor + 1$, we obtain a path P in G^r with

$$|V(P)| \geq 2\left(\left\lfloor \frac{s-1}{2} \right\rfloor + 1\right) \geq 2\left(\frac{s-2}{2} + 1\right) = s,$$

as desired. ∎

Proof of Theorem 4.3. Let s, t_1 and t_2 be as in Theorem 4.3. We first prove the theorem for the case where $s = 2$, i.e., $b(P_2, B_{t_1,t_2}) = t_1 + 1$. By Lemma 4.7, we have $b(P_2, B_{t_1,t_2}) \leq t_1 + 1$. Now we prove that $b(P_2, B_{t_1,t_2}) \geq t_1 + 1$. Let X_1 and X_2 be the partite sets of K_{t_1,t_1}. Let G^r be the graph obtained from K_{t_1,t_1} by deleting all edges, and let $G^b = K_{t_1,t_1}$. Then it is clear that $P_2 \not\subseteq G^r$ and $B_{t_1,t_2} \not\subseteq G^b$, and so $b(P_2, B_{t_1,t_2}) \geq t_1 + 1$. Thus we may assume that $s \geq 3$. Let $q \in \mathbb{N} \cup \{0\}$ and r $(0 \leq r \leq \lfloor \frac{s-1}{2} \rfloor - 1)$ be the integers satisfying $t_1 = \lfloor \frac{s-1}{2} \rfloor q + r$.

(i) Suppose that $t_1 = t_2$. Let $N = \lfloor \frac{s-1}{2} \rfloor + t_1 + 1$. By Lemma 4.7, we have $b(P_s, B_{t_1,t_2}) \leq N$. Now we prove that $b(P_s, B_{t_1,t_2}) \geq N$. Let X_1 and X_2 be the partite sets of $K_{N-1,N-1}$. We partition X_i into $q+2$ sets $X_i^0, X_i^1, \dots, X_i^{q+1}$ with $|X_i^0| = |X_i^1| = \cdots = |X_i^q| = \lfloor \frac{s-1}{2} \rfloor$ and $|X_i^{q+1}| = r$. Note that $X_i^{q+1} = \emptyset$ if and only if $t_1 \equiv 0 \pmod{\lfloor \frac{s-1}{2} \rfloor}$. Let G^r be the spanning subgraph of $K_{N-1,N-1}$ such that

$$E(G^r) = \bigcup_{0 \leq j \leq q+1} \{x_1x_2 : x_1 \in X_1^j,\ x_2 \in X_2^j\},$$

and let $G^b = K_{N-1,N-1} - E(G^r)$. Then the order of longest paths of G^r is at most $2\lfloor \frac{s-1}{2} \rfloor$ $(\leq s-1)$. Furthermore, since $\min\{\deg_{G^b}(x_1), \deg_{G^b}(x_2)\} \leq (N-1) - \lfloor \frac{s-1}{2} \rfloor = t_1$ $(= t_2)$ for every edge $x_1x_2 \in E(G^b)$, we see that $B_{t_1,t_2} \not\subseteq G^b$. Therefore $b(P_s, B_{t_1,t_2}) \geq N$.

(ii-a) Suppose that $t_1 > t_2$ and $t_1 \geq \lfloor \frac{s-1}{2} \rfloor$. Note that $q \geq 1$. Let

$$N = \begin{cases} \lfloor \frac{s-1}{2} \rfloor + t_1 + 1 & (s \text{ is even, or } s \text{ is odd and } t_1 \equiv 0 \pmod{\frac{s-1}{2}}) \\ \lfloor \frac{s-1}{2} \rfloor + t_1 & (\text{otherwise}). \end{cases}$$

By Lemmas 4.7 and 4.8, we have $b(P_s, B_{t_1,t_2}) \leq N$. Now we prove that $b(P_s, B_{t_1,t_2}) \geq N$. Let X_1 and X_2 be the partite sets of $K_{N-1,N-1}$.

If s is even, or s is odd and $t_1 \equiv 0 \pmod{\frac{s-1}{2}}$, we partition X_i into $q+2$ sets $X_i^0, X_i^1, \dots, X_i^{q+1}$ with $|X_i^0| = |X_i^1| = \cdots = |X_i^q| = \lfloor \frac{s-1}{2} \rfloor$ and $|X_i^{q+1}| = r$; otherwise, we partition X_i into $q+2$ sets $X_i^0, X_i^1, \dots, X_i^{q+1}$ with

- $|X_i^j| = \frac{s-1}{2}$ for $i \in \{1,2\}$ and j $(0 \leq j \leq q)$ with $(i,j) \notin \{(1,0),(2,1)\}$,

- $|X_1^0| = |X_2^1| = \frac{s-3}{2}$ and
- $|X_1^{q+1}| = |X_2^{q+1}| = r$.

Note that $X_i^{q+1} = \emptyset$ if and only if $t_1 \equiv 0 \pmod{\lfloor \frac{s-1}{2} \rfloor}$. Let G^r be the spanning subgraph of $K_{N-1,N-1}$ obtained by

- joining all vertices in X_1^0 to all vertices in $X_2^0 \cup X_2^{q+1}$,
- joining all vertices in X_2^1 to all vertices in $X_1^1 \cup X_1^{q+1}$ and
- for each j $(2 \leq j \leq q)$, joining all vertices in X_1^j to all vertices in X_2^j,

and let $G^b = K_{N-1,N-1} - E(G^r)$.

If s is even, then the order of longest paths of G^r is at most $2\lfloor \frac{s-1}{2} \rfloor + 1 = 2 \cdot \frac{s-2}{2} + 1 = s-1$; if s is odd and $t_1 \equiv 0 \pmod{\frac{s-1}{2}}$, then the order of longest paths of G^r is $2\lfloor \frac{s-1}{2} \rfloor = 2 \cdot \frac{s-1}{2} = s-1$; if s is odd and $t_1 \not\equiv 0 \pmod{\frac{s-1}{2}}$, then the order of longest paths of G^r is at most

$$\max\left\{2 \cdot \frac{s-1}{2}, 2 \cdot \frac{s-3}{2} + 1\right\} = s-1.$$

Furthermore, since we easily check that $\deg_{G^b}(x) \leq t_1$ for all $x \in V(G^b)$, $B_{t_1,t_2} \not\subset G^b$. Therefore $b(P_s, B_{t_1,t_2}) \geq N$.

(ii-b) Suppose that $t_1 > t_2$ and $t_1 < \lfloor \frac{s-1}{2} \rfloor$. Let $N = \lfloor \frac{s-1}{2} \rfloor + t_2 + 1 + \max\{2t_1 - t_2 - \lfloor \frac{s-1}{2} \rfloor, 0\}$. By Lemma 4.9, we have $b(P_s, B_{t_1,t_2}) \leq N$. Now we prove that $b(P_s, B_{t_1,t_2}) \geq N$. Let X_1 and X_2 be the partite sets of $K_{N-1,N-1}$.

If $2t_1 - t_2 \geq \lfloor \frac{s-1}{2} \rfloor$ (i.e., $N - 1 = 2t_1$), we partition X_i into two sets X_i^1 and X_i^2 with $|X_i^1| = |X_i^2| = t_1$; otherwise (i.e., $N = \lfloor \frac{s-1}{2} \rfloor + t_2$), we partition X_i into two sets X_i^1 and X_i^2 with $|X_i^1| = \lfloor \frac{s-1}{2} \rfloor$ and $|X_i^2| = t_2$. Let G^r be the spanning subgraph of $K_{N-1,N-1}$ such that

$$E(G^r) = \bigcup_{j \in \{1,2\}} \{x_1x_2 : x_1 \in X_1^j,\ x_2 \in X_2^j\},$$

and let $G^b = K_{N-1,N-1} - E(G^r)$.

Since $t_2 < t_1 < \lfloor \frac{s-1}{2} \rfloor$, the order of longest paths of G^r is at most $2\lfloor \frac{s-1}{2} \rfloor$ $(\leq 2 \cdot \frac{s-1}{2} = s-1)$. Furthermore, if $2t_1 - t_2 \geq \lfloor \frac{s-1}{2} \rfloor$, then $\deg_{G^b}(x) = (N-1) - t_1 = t_1$ for all $x \in V(G^b)$; if $2t_1 - t_2 < \lfloor \frac{s-1}{2} \rfloor$, then $\min\{\deg_{G^b}(x_1), \deg_{G^b}(x_2)\} = t_2$ for every edge $x_1x_2 \in E(G^b)$. In either case, $B_{t_1,t_2} \not\subset G^b$. Therefore $b(P_s, B_{t_1,t_2}) \geq N$.

This completes the proof of Theorem 4.3. ∎

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