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THE FOURIER AND GROVER WALKS ON THE TWO-DIMENSIONAL LATTICE AND TORUS

By

Masahiro Asano, Takashi Komatsu, Norio Konno,
and Akihiro Narimatsu

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Abstract. In this paper, we consider discrete-time quantum walks with moving shift (MS) and flip-flop shift (FF) on two-dimensional lattice $\mathbb{Z}^2$ and torus $\pi_N^2 = (\mathbb{Z}/N)^2$. Weak limit theorems for the Grover walks on $\mathbb{Z}^2$ with MS and FF were given by Watabe et al. and Higuchi et al., respectively. The existence of localization of the Grover walks on $\mathbb{Z}^2$ with MS and FF was shown by Inui et al. and Higuchi et al., respectively. Non-existence of localization of the Fourier walk with MS on $\mathbb{Z}^2$ was proved by Komatsu and Tate. Here our simple argument gave non-existence of localization of the Fourier walk with both MS and FF. Moreover we calculate eigenvalues and the corresponding eigenvectors of the (k_1, k_2)-space of the Fourier walks on π_N^2 with MS and FF for some special initial conditions. The probability distributions are also obtained. Finally, we compute amplitudes of the Grover and Fourier walks on π_2^2.

1. Introduction

The notion of quantum walks (QWs) was introduced by Aharonov et al. [1] as a quantum version of the usual random walks. QWs have been intensively studied from various fields. For example, quantum algorithm [11], the topological insulator [6], and radioactive waste reduction [10]. As for books and review papers on QWs, see [2, 8, 12, 13].

In the present paper, we consider QWs on the two-dimensional lattice, $\mathbb{Z}^2$, and $N \times N$ torus, π_N^2 with moving shift (MS) and flip-flop shift (FF).

The properties of QWs in the one dimension, e.g., ballistic spreading and localization, are well studied, see Konno [7]. On the other hand, the corresponding properties of QWs in the two-dimensions have not been clarified. However, here are some results for the Grover walk and Fourier walk on the two-dimensional lattice. Weak limit theorems for the Grover walks on $\mathbb{Z}^2$ with MS and FF were obtained by Watabe et al. [14] and Higuchi et al. [3], respectively. Inui et al. [4] and Higuchi et al. [3] showed that localization occurs for the Grover walks on $\mathbb{Z}^2$

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with MS and FF, respectively. Komatsu and Tate [5] proved that localization does not occur for the Fourier walk on $\mathbb{Z}^2$ with MS. It is not known for any result on the Fourier walk on $\mathbb{Z}^2$ with FF. Hence we proved that localization does not occur for the Fourier walk on $\mathbb{Z}^2$ with both MS and FF by using a simple contradiction argument which is different from the method based on the Fourier analysis given by [5]. Time-averaged limit measure of the Grover walk with FF on π_N^2 was obtained analytically by Marquezino et al. [9]. The expression of the time averaged limit measure of the Grover walk on π_N^2 with both MS and FF is not known.

In this paper, we compute eigenvalues and the corresponding eigenvectors of the (k_1, k_2)-space, which is equivalent to $\{0,\ 1,\ \dots,\ N-1\}^2$, of the Fourier walks on π_N^2 with both MS and FF for the special initial conditions, for examples, $k_1 = k_2$ or $k_1 + k_2 \equiv 0 \ (\mathrm{mod} N)$. By using these results, we obtain the measures at time n for the walks. Moreover, we calculate amplitudes of the Fourier walks on π_2^2 (i.e., $N = 2$) with MS and FF. We also compute amplitudes of the Grover walks on π_2^2 with MS and FF, and discuss a difference between the Fourier and Grover walks.

The rest of the paper is organized as follows. Section 2 is devoted to the definition of QWs on π_N^2. In Section 3, we consider the Fourier walks on π_N^2 and $\mathbb{Z}^2$ with both MS and FF. Section 4 deals with the Fourier and Grover walks on π_2^2. In Section 5, we summarize our results.

2. QWs on π_N^2

This section presents the definition of QWs on π_N^2. Let U be a 4×4 unitary matrix which is the coin operator of QW. For a coin operator U, we introduce $U_j = P_j U \ (j = 1, 2, 3, 4)$, where

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In this paper, we consider two types of shift operator, i.e., moving shift (MS) and flip-flop shift (FF). First we introduce the time evolution of QW with MS as follows: for $x_1,\ x_2 \in \pi_N^2$,

$$\Psi_{n+1}^{(m)}(x_1,x_2) = U_1^{(m)}\Psi_n^{(m)}(x_1+1,x_2) + U_2^{(m)}\Psi_n^{(m)}(x_1-1,x_2) + U_3^{(m)}\Psi_n^{(m)}(x_1,x_2+1) + U_4^{(m)}\Psi_n^{(m)}(x_1,x_2-1),$$

where $U^{(m)} = U$. Next we introduce the time evolution of QW with FF as follows:

$$\Psi_{n+1}^{(f)}(x_1,x_2) = U_1^{(f)}\Psi_n^{(f)}(x_1+1,x_2) + U_2^{(f)}\Psi_n^{(f)}(x_1-1,x_2) + U_3^{(f)}\Psi_n^{(f)}(x_1,x_2+1) + U_4^{(f)}\Psi_n^{(f)}(x_1,x_2-1).$$

Here $U^{(f)}$ is given by

$$U^{(f)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} U^{(m)}.$$

Put $\mathbb{K}_N = \{0,1,\ldots,N-1\}$, we define the Fourier transform by

$$\hat{\Psi}_n^{(j)}(k_1,k_2) = \frac{1}{N}\sum_{(x_1,x_2)\in\pi_N^2} \omega^{-(k_1x_1+k_2x_2)}\Psi_n^{(j)}(x_1,x_2) \quad (j=m,f),$$

where $(k_1,k_2) \in \mathbb{K}_N^2$ and $\omega = \exp(2\pi i/N)$. The time evolution of QW on (k_1,k_2)-space is written as

$$\hat{\Psi}_{n+1}^{(j)}(k_1,k_2) = U^{(j)}(k_1,k_2)\hat{\Psi}_n^{(j)}(k_1,k_2) \quad (j=m,f),$$

where

$$U^{(j)}(k_1,k_2) = \begin{bmatrix} \omega^{k_1} & 0 & 0 & 0 \\ 0 & \omega^{-k_1} & 0 & 0 \\ 0 & 0 & \omega^{k_2} & 0 \\ 0 & 0 & 0 & \omega^{-k_2} \end{bmatrix} U^{(j)} \quad (j=m,f).$$

Remark that

$$\hat{\Psi}_n^{(j)}(k_1,k_2) = \left(U^{(j)}(k_1,k_2)\right)^n \hat{\Psi}_0^{(j)}(k_1,k_2) \quad (j=m,f). \tag{1}$$

Since $U^{(j)}(k_1,k_2)$ is unitary, we have the following spectral decomposition as follows:

$$U^{(j)}(k_1,k_2) = \sum_{i=0}^{3} \lambda_i^{(j)}(k_1,k_2)|v_i^{(j)}(k_1,k_2)\rangle\langle v_i^{(j)}(k_1,k_2)| \quad (j=m,f). \tag{2}$$

where $\lambda_i^{(j)}(k_1,k_2)$ is eigenvalue of $U^{(j)}(k_1,k_2)$ and $|v_i^{(j)}(k_1,k_2)\rangle$ is the corresponding eigenvector for $i=0,1,2,3$. Therefore, combining Eq.(1) with Eq.(2), we obtain

$$\hat{\Psi}_n^{(j)}(k_1,k_2)=\sum_{i=0}^{3}\lambda_i^{(j)}(k_1,k_2)^n|v_i^{(j)}(k_1,k_2)\rangle\langle v_i^{(j)}(k_1,k_2)|\hat{\Psi}_0^{(j)}(k_1,k_2)\ \ (j=m,f).$$

3. The Fourier walk on π_N^2

In this section, we present the definition of the Fourier walks with moving and flip-flop shifts on π_N^2.

3.1 The Fourier walk with MS

This subsection deals with the Fourier walk with MS whose coin operator is defined by

$$U^{(m)}=\frac{1}{2}\begin{bmatrix}1&1&1&1\\1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\end{bmatrix}.$$

Then $U^{(m)}(k_1,k_2)$ for $k_1,k_2\in\mathbb{K}_N=\{0,1,\dots,N-1\}$ is given by

$$U^{(m)}(k_1,k_2)=\frac{1}{2}\begin{bmatrix}\omega^{k_1}&\omega^{k_1}&\omega^{k_1}&\omega^{k_1}\\\omega^{-k_1}&i\omega^{-k_1}&-\omega^{-k_1}&-i\omega^{-k_1}\\\omega^{k_2}&-\omega^{k_2}&\omega^{k_2}&-\omega^{k_2}\\\omega^{-k_2}&-i\omega^{-k_2}&-\omega^{-k_2}&i\omega^{-k_2}\end{bmatrix}.$$

Moreover we can compute the characteristic polynomial as follows.

$$\begin{aligned}&\det(\lambda I_4-U^{(m)}(k_1,k_2))\\&=\lambda^4-\frac{1+i}{2}\Big(\cos\tilde{k}_1+\sin\tilde{k}_1+\cos\tilde{k}_2+\sin\tilde{k}_2\Big)\lambda^3-\frac{1-i}{2}\Big(1+\cos\Big(\tilde{k}_1-\tilde{k}_2\Big)\Big)\lambda^2\\&\quad+\frac{1+i}{2}\Big(\cos\tilde{k}_1+\sin\tilde{k}_1+\cos\tilde{k}_2+\sin\tilde{k}_2\Big)\lambda-i,\end{aligned}\tag{3}$$

with $\tilde{k}_j=2\pi k_j/N$. Since $\lambda\neq 0$, dividing both sides of Eq.(3) by λ^2, we have

$$\begin{aligned}&\frac{\det(\lambda I_4-U^{(m)}(k_1,k_2))}{\lambda^2}\\&=\lambda^2-\frac{1+i}{2}\Big(\cos\tilde{k}_1+\sin\tilde{k}_1+\cos\tilde{k}_2+\sin\tilde{k}_2\Big)\lambda-\frac{1-i}{2}\Big(1+\cos\Big(\tilde{k}_1-\tilde{k}_2\Big)\Big)\\&\quad+\frac{1+i}{2}\Big(\cos\tilde{k}_1+\sin\tilde{k}_1+\cos\tilde{k}_2+\sin\tilde{k}_2\Big)\lambda^{-1}-i\lambda^{-2}.\end{aligned}\tag{4}$$

Let $x = \Re(\lambda)$ and $y = \Im(\lambda)$, where λ is an eigenvalue of $U^{(m)}(k_1, k_2)$. Here $\Re(z)$ is the real part of z and $\Im(z)$ is the imaginary part of $z \in \mathbb{C}$. We should remark that Eq.(4) implies that x and y satisfy the following equation.

$$x^2 - y^2 - 2xy + Ay - B = 0, \tag{5}$$

where $A = \cos\tilde{k}_1 + \sin\tilde{k}_1 + \cos\tilde{k}_2 + \sin\tilde{k}_2$ and $B = \{1 + \cos(\tilde{k}_1 - \tilde{k}_2)\}/2$. It would be difficult to get an explicit form of $\lambda = x + iy$ for any $(k_1, k_2) \in \mathbb{K}_N^2$ by using Eq.(5). Therefore we consider the proper subsets $\mathcal{A}$ of $\mathbb{K}_N^2$. In this model, we deal with the following two cases; (**a**) $\mathcal{A} = \{(k_1, k_2) = (0, 0)\}$ and (**b**) $\mathcal{A} = \{(k_1, k_2) \in \mathbb{K}_N^2 : k_1 = k_2\}$. Let $\lambda_j^{(m)}(k_1, k_2)$ denote the eigenvalues of $U^{(m)}(k_1, k_2)$ and $v_j^{(m)}(k_1, k_2)$ be the corresponding eigenvectors for $j = 0, 1, 2, 3$. We should note that case (**a**) related to a QW starting from uniform initial state given by Eq.(10), and case (**b**) is related to a QW starting from restricted uniform initial state $(x_1 + x_2 = N)$ given by Eq.(11).

(**a**) $(k_1, k_2) = (0, 0)$ case
The eigenvalues of $U^{(m)}(0, 0)$ are

$$\lambda_0^{(m)}(0,0) = \lambda_1^{(m)}(0,0) = 1,\ \lambda_2^{(m)}(0,0) = -1,\ \lambda_3^{(m)}(0,0) = i, \tag{6}$$

and the corresponding eigenvectors are

$$\begin{aligned} v_0^{(m)}(0,0) &= \frac{1}{2}^T \begin{bmatrix} 1 & 1 & -1 & 1 \end{bmatrix},\ v_1^{(m)}(0,0) = \frac{1}{\sqrt{2}}^T \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \\ v_2^{(m)}(0,0) &= \frac{1}{2}^T \begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix},\ v_3^{(m)}(0,0) = \frac{1}{\sqrt{2}}^T \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}. \end{aligned} \tag{7}$$

(**b**) $(k_1, k_2) = (k, k)$ case
The eigenvalues are

$$\lambda_0^{(m)}(k,k) = 1,\ \lambda_1^{(m)}(k,k) = \omega^k,\ \lambda_2^{(m)}(k,k) = -1,\ \lambda_3^{(m)}(k,k) = i\omega^{-k}, \tag{8}$$

and the corresponding eigenvectors are

$$\begin{aligned} v_0^{(m)}(k,k) &= \frac{1}{2}^T \begin{bmatrix} \omega^k & 1 & -\omega^k & 1 \end{bmatrix},\ v_1^{(m)}(k,k) = \frac{1}{\sqrt{2}}^T \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \\ v_2^{(m)}(k,k) &= \frac{1}{2}^T \begin{bmatrix} \omega^k & -1 & -\omega^k & -1 \end{bmatrix},\ v_3^{(m)}(k,k) = \frac{1}{\sqrt{2}}^T \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}. \end{aligned} \tag{9}$$

From now on, we calculate $\Psi_n^{(m)}(x_1, x_2)$ for two cases. Specific initial states by Eqs.(6), (7), (8) and (9).
(i) Here we consider with uniform initial state $\Psi_0^{(m)}(x_1, x_2)$, i.e.,

$$\Psi_0^{(m)}(x_1, x_2) = \frac{1}{N}^T \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix}\ ((x_1, x_2) \in \pi_N^2), \tag{10}$$

where $|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2=1$ with $\alpha_j\in\mathbb{C}$ $(j=1,2,3,4)$. By the Fourier transform, we see that the initial state of (k_1,k_2)-space becomes

$$\hat{\Psi}_0^{(m)}(k_1,k_2)=\begin{cases}\dfrac{1}{N}\,{}^T\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{bmatrix} & (k_1,k_2)=(0,0)\\[2ex] {}^T\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix} & (k_1,k_2)\neq(0,0)\end{cases}.$$

Thus we have

$$\begin{aligned}\Psi_n^{(m)}(x_1,x_2)&=\frac{1}{N}\sum_{k_1,k_2\in\mathbb{K}_N}\omega^{k_1x_1+k_2x_2}\sum_{i=0}^{3}\lambda_i(k_1,k_2)^n|v_i(k_1,k_2)\rangle\langle v_i(k_1,k_2)|\phi\rangle\\&=\frac{1}{N}\sum_{i=0}^{3}\lambda_i(0,0)^n|v_i(0,0)\rangle\langle v_i(0,0)|\phi\rangle,\end{aligned}$$

where $\phi={}^T[\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4]$. From Eqs.(6) and (7), we obtain the desired result as follows.

$$\begin{aligned}&\Psi_n^{(m)}(x_1,x_2)\\&=\frac{1}{4N}\begin{bmatrix}\big(3+(-1)^n\big)\alpha_1+\big(1-(-1)^n\big)\alpha_2+(1-(-1)^n)\alpha_3+\big(1-(-1)^n\big)\alpha_4\\ \big(1-(-1)^n\big)\alpha_1+\big(1+(-1)^n+2i^n\big)\alpha_2+(-1+(-1)^n)\alpha_3+\big(1+(-1)^n-2i^n\big)\alpha_4\\ \big(1-(-1)^n\big)\alpha_1+\big(-1+(-1)^n\big)\alpha_2+\big(3+(-1)^n\big)\alpha_3+\big(-1+(-1)^n\big)\alpha_4\\ \big(1-(-1)^n\big)\alpha_1+\big(1+(-1)^n-2i^n\big)\alpha_2+\big(-1+(-1)^n\big)\alpha_3+\big(1+(-1)^n+2i^n\big)\alpha_4\end{bmatrix},\end{aligned}$$

for $(x_1,x_2)\in\pi_N^2$. Hence we find that the amplitude $\Psi_{n+4}^{(m)}(x_1,x_2)=\Psi_n^{(m)}(x_1,x_2)$ for $(x_1,x_2)\in\pi_N^2$ and $n\in\mathbb{Z}_{\geq}$. That is to say, the Fourier walk with MS starting from uniform initial state has the period 4.

(ii) We consider a QW with restricted uniform initial state $\Psi_0^{(m)}(x_1,x_2)$ given by

$$\Psi_0^{(m)}(x_1,x_2)=\begin{cases}\dfrac{1}{\sqrt{N}}\,{}^T\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{bmatrix} & (x_1+x_2=N)\\[2ex] {}^T\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix} & (x_1+x_2\neq N)\end{cases}, \qquad (11)$$

where $|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2=1$. The Fourier transform implies that the initial state of (k_1,k_2)-space becomes

$$\hat{\Psi}_0^{(m)}(k_1,k_2)=\begin{cases}\dfrac{1}{\sqrt{N}}\,{}^T\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{bmatrix} & (k_1=k_2)\\[2ex] {}^T\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix} & (k_1\neq k_2)\end{cases}.$$

Therefore we have

$$\Psi_n^{(m)}(x_1,x_2)=\frac{1}{N^{3/2}}\sum_{k\in\mathbb{K}_N}\omega^{(x_1+x_2)k}\sum_{i=0}^{3}\lambda_i(k,k)^n|v_i(k,k)\rangle\langle v_i(k,k)|\phi\rangle, \tag{12}$$

where $\phi = {}^T\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{bmatrix}$. Combining Eq.(8) with Eq.(12), we get

$$\begin{aligned}&\Psi_n^{(m)}(x_1,x_2)\\&=\frac{1}{N^{3/2}}\sum_{k=0}^{N-1}\omega^{(x_1+x_2)k}\Big\{1^n\langle v_0^{(m)}(k,k)|\phi\rangle|v_0^{(m)}(k,k)\rangle+\omega^{nk}\langle v_1^{(m)}(k,k)|\phi\rangle|v_1^{(m)}(k,k)\rangle\\&\quad+(-1)^n\langle v_2^{(m)}(k,k)|\phi\rangle|v_2^{(m)}(k,k)\rangle+i^n\omega^{-nk}\langle v_3^{(m)}(k,k)|\phi\rangle|v_3^{(m)}(k,k)\rangle\Big\}.\end{aligned} \tag{13}$$

From Eq.(9), we see that

$$\begin{aligned}\langle v_0^{(m)}(k,k)|\phi\rangle&=\frac{1}{2}\big\{(\alpha_1-\alpha_3)\omega^{-k}+\alpha_2+\alpha_4\big\},\quad \langle v_1^{(m)}(k,k)|\phi\rangle=\frac{1}{\sqrt{2}}(\alpha_1+\alpha_3),\\ \langle v_2^{(m)}(k,k)|\phi\rangle&=\frac{1}{2}\big\{(\alpha_1-\alpha_3)\omega^{-k}-\alpha_2-\alpha_4\big\},\quad \langle v_3^{(m)}(k,k)|\phi\rangle=\frac{1}{\sqrt{2}}(\alpha_2-\alpha_4).\end{aligned} \tag{14}$$

Inserting Eq.(14) to Eq.(13) gives

$$\Psi_n^{(m)}(x_1,x_2)=\frac{1}{4N^{3/2}}\sum_{k=0}^{N-1}\omega^{(x_1+x_2)k}$$

$$\begin{bmatrix}\alpha_1-\alpha_3+(\alpha_2+\alpha_4)\omega^k+2(\alpha_1+\alpha_3)\omega^{nk}+(-1)^n(\alpha_1-\alpha_3)-(-1)^n(\alpha_2+\alpha_4)\omega^k\\(\alpha_1-\alpha_3)\omega^{-k}+\alpha_2+\alpha_4-(-1)^n\{(\alpha_1-\alpha_3)\omega^{-k}-\alpha_2-\alpha_4\}+2i^n(\alpha_2-\alpha_4)\omega^{-nk}\\-\alpha_1+\alpha_3-(\alpha_2+\alpha_4)\omega^k+2(\alpha_1+\alpha_3)\omega^{nk}-(-1)^n\{\alpha_1-\alpha_3-(\alpha_2+\alpha_4)\omega^k\}\\(\alpha_1-\alpha_3)\omega^{-k}+\alpha_2+\alpha_4-(-1)^n\{(\alpha_1-\alpha_3)\omega^{-k}-\alpha_2-\alpha_4\}-2i^n(\alpha_2-\alpha_4)\omega^{-nk}\end{bmatrix} \tag{15}$$

Then Eq.(15) can be rewritten as

$$\begin{aligned}\Psi_n^{(m)}(x_1,x_2)=\frac{1}{4\sqrt{N}}\Bigg\{&\begin{bmatrix}\alpha_1-\alpha_3\\\alpha_2+\alpha_4\\-(\alpha_1-\alpha_3)\\\alpha_2+\alpha_4\end{bmatrix}\{1+(-1)^n\}\delta_{j,-j}(x_1,x_2)\\&+\begin{bmatrix}\alpha_2+\alpha_4\\0\\-(\alpha_2+\alpha_4)\\0\end{bmatrix}\{1+(-1)^n\}\delta_{j,-j-1}(x_1,x_2)+\begin{bmatrix}0\\\alpha_1-\alpha_3\\0\\\alpha_1-\alpha_3\end{bmatrix}\{1-(-1)^n\}\delta_{j,-j+1}(x_1,x_2)\\&+\begin{bmatrix}\alpha_1+\alpha_3\\0\\\alpha_1+\alpha_3\\0\end{bmatrix}2\delta_{j,-j-n}(x_1,x_2)+\begin{bmatrix}0\\\alpha_2-\alpha_4\\0\\-(\alpha_2-\alpha_4)\end{bmatrix}2i^n\delta_{j,-j+n}(x_1,x_2)\Bigg\}\quad(j\in\mathbb{K}_N),\end{aligned}$$

where

$$\delta_{a,b}(x_1,x_2) = \begin{cases} 1 \ \big((x_1,x_2)=(a,b)\big) \\ 0 \ \big((x_1,x_2)\neq(a,b)\big) \end{cases}.$$

The first, second and third terms of the equation mean that the walker is trapped around $x_1 + x_2 \equiv 0 \pmod{N}$. The forth and fifth terms of the equation mean that the walker keeps on moving straightly.

3.2 The Fourier walk with FF

In this subsection, we consider the Fourier walk with FF whose coin operator is defined by

$$U_F^{(f)} = \frac{1}{2}\begin{bmatrix} 1 & i & -1 & -i \\ 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Then $U^{(f)}(k_1,k_2)$ for $k_1,k_2 \in \mathbb{K}_N = \{0,1,\ldots,N-1\}$ is given by

$$U^{(f)}(k_1,k_2) = \frac{1}{2}\begin{bmatrix} \omega^{k_1} & i\omega^{k_1} & -\omega^{k_1} & -i\omega^{k_1} \\ \omega^{-k_1} & \omega^{-k_1} & \omega^{-k_1} & \omega^{-k_1} \\ \omega^{k_2} & -i\omega^{k_2} & -\omega^{k_2} & i\omega^{k_2} \\ \omega^{-k_2} & -\omega^{-k_2} & \omega^{-k_2} & -\omega^{-k_2} \end{bmatrix}.$$

The eigenvalues of $U^{(f)}(k_1,k_2)$ are the roots of the following polynomial:

$$\begin{aligned}\det(\lambda I_4 - U^{(f)}(k_1,k_2)) = \lambda^4 - \Big(\cos\tilde{k}_1 - \cos\tilde{k}_2\Big)\lambda^3 + \frac{1-i}{2}\Big(1-\cos\Big(\tilde{k}_1 - \tilde{k}_2\Big)\Big)\lambda^2 \\ + i\Big(\cos\tilde{k}_1 - \cos\tilde{k}_2\Big)\lambda - i. \qquad (16)\end{aligned}$$

We put $x = \Re(\lambda)$ and $y = \Im(\lambda)$, where λ is an eigenvalue of $U^{(f)}(k_1,k_2)$. We should remark that Eq.(16) implies that x and y satisfy the following equation:

$$x^2 - y^2 - 2xy - C\big(x-y\big) + D = 0,$$

where $C = \cos\tilde{k}_1 - \cos\tilde{k}_2$ and $D = \{1-\cos(\tilde{k}_1 - \tilde{k}_2)\}/2$. It would be hard to get solution for any $(k_1,k_2) \in \mathbb{K}_N^2$, so we consider suitable proper subsets $\mathcal{B} \subset \mathbb{K}_N^2$, as in the case of MS model. In this model, we deal with the following two cases: (**a**) $\mathcal{B} = \{(k_1,k_2) \in \mathbb{K}_N^2 : k_1 = k_2\}$ and (**b**) $\mathcal{B} = \{(k_1,k_2) \in \mathbb{K}_N^2 : k_1 + k_2 \equiv 0 \pmod{N}\}$. Let $\lambda_j^{(f)}(k_1,k_2)$ be the eigenvalues of $U^{(f)}(k_1,k_2)$ and $v_j^{(f)}(k_1,k_2)$ be the corresponding eigenvectors for $j = 0,1,2,3$.

(**a**) $(k_1, k_2) = (k, k)$ case
The eigenvalues are

$$\lambda_0^{(f)}(k,k) = e^{\pi i/8},\ \lambda_1^{(f)}(k,k) = e^{5\pi i/8},\ \lambda_2^{(f)}(k,k) = e^{9\pi i/8},\ \lambda_3^{(f)}(k,k) = e^{13\pi i/8},$$

and the corresponding eigenvectors are

$$v_j^{(f)}(k,k) = \frac{1}{Z_j(k,k)} \begin{bmatrix} \omega^k(\lambda_j^{(f)}(k,k)^2 + i\omega^{-2k})(\lambda_j(k,k) + \omega^k) \\ \omega^{-k}(\lambda_j^{(f)}(k,k)^2 + \omega^{2k})(\lambda_j(k,k) + \omega^{-k}) \\ -\omega^k(\lambda_j^{(f)}(k,k)^2 + i\omega^{-2k})(\lambda_j(k,k) - \omega^k) \\ \omega^{-k}(\lambda_j^{(f)}(k,k)^2 + \omega^{2k})(\lambda_j(k,k) - \omega^{-k}) \end{bmatrix},$$

where $Z_j(k,k)$ is a normalized constant.

(**b**) $(k_1, k_2) = (k, N-k)$ case
The eigenvalues λ satisfy the following equation with fourth order;

$$\lambda^4 + (1-i)\sin^2\tilde{k}\lambda^2 - i = 0.$$

Thus we get

$$\lambda_j^{(f)}(k, N-k) = \pm\frac{\sqrt{2 - \sin^2\tilde{k} + \sqrt{2 - \sin^4\tilde{k}}} + i\sqrt{2 + \sin^2\tilde{k} - \sqrt{2 - \sin^4\tilde{k}}}}{2},$$

$$\pm\frac{\sqrt{2 - \sin^2\tilde{k} - \sqrt{2 - \sin^4\tilde{k}}} - i\sqrt{2 + \sin^2\tilde{k} + \sqrt{2 - \sin^4\tilde{k}}}}{2},$$

and corresponding eigenvectors are

$$v_j^{(f)}(k, N-k) \\ = \frac{1}{Z_j(k, N-k)} \begin{bmatrix} (\lambda_j^{(f)}(k, N-k) + i\sin\tilde{k})(1 + \lambda_j^{(f)}(k, N-k)\omega^k) \\ (\lambda_j^{(f)}(k, N-k) + i\sin\tilde{k})(1 + \lambda_j^{(f)}(k, N-k)\omega^{-k}) \\ (\lambda_j^{(f)}(k, N-k) + i\sin\tilde{k})(1 - \lambda_j^{(f)}(k, N-k)\omega^{-k}) \\ -(\lambda_j^{(f)}(k, N-k) + i\sin\tilde{k})(1 - \lambda_j^{(f)}(k, N-k)\omega^k) \end{bmatrix},$$

where $Z_j(k, N-k)$ is a normalized constant.

3.3 Non-existence of localization

In this subsection, we prove that localization does not occur for the Fourier walks on $\mathbb{Z}^2$ with both MS and FF. According to Komatsu and Tate [5], if a

QW has localization, then the characteristic polynomial of the quantum coin in (k_1, k_2)-space has greater than one constant roots. Assume that the Fourier walk with MS has a constant root λ with $|\lambda| = 1$. In a similar way, by using characteristic polynomial (3), we have

$$\lambda^4 - \frac{1+i}{2}\Big(\cos k_1 + \sin k_1 + \cos k_2 + \sin k_2\Big)\lambda^3 - \frac{1-i}{2}\Big(1 + \cos\Big(k_1 - k_2\Big)\Big)\lambda^2 + \frac{1+i}{2}\Big(\cos k_1 + \sin k_1 + \cos k_2 + \sin k_2\Big)\lambda - i = 0, \tag{17}$$

where $(k_1, k_2) \in (-\pi, \pi]^2$. Since the constant root λ does not depend on k_1, we obtain the following equation by differentiating Eq.(17) with respect to k_1.

$$-\frac{1+i}{2}\Big(-\sin k_1 + \cos k_1\Big)\lambda^3 + \frac{1-i}{2}\Big(\sin\Big(k_1 - k_2\Big)\Big)\lambda^2 + \frac{1+i}{2}\Big(-\sin k_1 + \cos k_1\Big)\lambda = 0. \tag{18}$$

Hence Eq.(18) can be rewritten as

$$\Big(-\sin k_1 + \cos k_1\Big)\Big(\lambda^2 - 1\Big) - \lambda \sin\Big(k_1 - k_2\Big) + i\Big\{\Big(-\sin k_1 + \cos k_1\Big)\Big(\lambda^2 - 1\Big) + \lambda \sin\Big(k_1 - k_2\Big)\Big\} = 0.$$

Therefore we have $\lambda \sin\big(k_1 - k_2\big) = 0$ for any $(k_1, k_2) \in (-\pi, \pi]^2$. This contradicts $|\lambda| = 1$. Thus we conclude that Eq.(17) does not have any constant root, so non-existence of localization for the Fourier walk with MS is shown.

In a similar fashion, we can prove that localization does not occur for the Fourier walk with FF. Eq.(16) gives the characteristic polynomial for this model is as follows.

$$\lambda^4 - \Big(\cos k_1 - \cos k_2\Big)\lambda^3 + \frac{1-i}{2}\Big(1 - \cos\Big(k_1 - k_2\Big)\Big)\lambda^2 + i\Big(\cos k_1 - \cos k_2\Big)\lambda - i = 0, \tag{19}$$

where $(k_1, k_2) \in (-\pi, \pi]^2$. By differentiating Eq.(19) with respect to k_1, we have

$$\Big(\sin k_1\Big)\lambda^3 + \frac{1-i}{2}\Big(\sin\Big(k_1 - k_2\Big)\Big)\lambda^2 - i\Big(\sin \tilde{k}_1\Big)\lambda = 0. \tag{20}$$

Therefore Eq.(20) becomes

$$2\lambda^2 \sin k_1 + \lambda \sin\Big(k_1 - k_2\Big) - i\Big\{2 \sin k_1 + \lambda \sin\Big(k_1 - k_2\Big)\Big\} = 0.$$

Then we obtain $2\sin k_1 \pm \sin\Big(k_1 - k_2\Big) = 0$ for any $(k_1, k_2) \in (-\pi, \pi]^2$, this is contradiction. Hence we show non-existence of localization for the Fourier walk with FF.

4. π_2^2 case

In this section, we compute $\Psi_n(x_1, x_2)$ of the Fourier and Grover walks with both MS and FF when $N = 2$, for all $n \in \mathbb{Z}_{\geq}$ and $(x_1, x_2) \in \pi_2^2$. As the initial state, we take

$$\Psi_0^{(j)}(x_1, x_2) = \begin{cases} {}^T\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{bmatrix} & \big((x_1, x_2) = (0,0)\big) \\ \\ {}^T\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix} & \big((x_1, x_2) \neq (0,0)\big) \end{cases} \qquad (j = m, f)$$

for $|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 = 1$ with $\alpha_\ell \in \mathbb{C}$ $(\ell = 1, 2, 3, 4)$.

4.1 The Fourier walk on π_2^2

This subsection deals with $\Psi_n^{(j)}(x_1, x_2)$ $(j = m, f)$ of the Fourier walk.
(**a**) MS case
By Eq.(17), we get the eigenvalues of $U^{(m)}(k_1, k_2)$ as follows.

$$\begin{aligned}
&\lambda_0^{(m)}(0,0) = 1,\ \lambda_1^{(m)}(0,0) = -1,\ \lambda_2^{(m)}(0,0) = 1,\ \lambda_3^{(m)}(0,0) = i,\\
&\lambda_0^{(m)}(1,1) = 1,\ \lambda_1^{(m)}(1,1) = -1,\ \lambda_2^{(m)}(1,1) = -1,\ \lambda_3^{(m)}(1,1) = -i,\\
&\lambda_0^{(m)}(0,1) = e^{\pi i/8},\ \lambda_1^{(m)}(0,1) = e^{5\pi i/8},\ \lambda_2^{(m)}(0,1) = e^{9\pi i/8},\ \lambda_3^{(m)}(0,1) = e^{13\pi i/8},\\
&\lambda_0^{(m)}(1,0) = e^{\pi i/8},\ \lambda_1^{(m)}(1,0) = e^{5\pi i/8},\ \lambda_2^{(m)}(1,0) = e^{9\pi i/8},\ \lambda_3^{(m)}(1,0) = e^{13\pi i/8},
\end{aligned}$$

and corresponding eigenvectors are

$$\begin{aligned}
&v_0^{(m)}(0,0) = \frac{1}{2}\,{}^T\begin{bmatrix}1 & 1 & -1 & 1\end{bmatrix},\ v_1^{(m)}(0,0) = \frac{1}{2}\,{}^T\begin{bmatrix}1 & -1 & -1 & -1\end{bmatrix},\\
&v_2^{(m)}(0,0) = \frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix},\ v_3^{(m)}(0,0) = \frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}0 & 1 & 0 & -1\end{bmatrix},\\
&v_0^{(m)}(1,1) = -\frac{1}{2}\,{}^T\begin{bmatrix}1 & -1 & -1 & -1\end{bmatrix},\ v_1^{(m)}(1,1) = -\frac{1}{2}\,{}^T\begin{bmatrix}1 & 1 & -1 & 1\end{bmatrix},\\
&v_2^{(m)}(1,1) = \frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix},\ v_3^{(m)}(1,1) = \frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}0 & 1 & 0 & -1\end{bmatrix},
\end{aligned}$$

$$v_j^{(m)}(0,1) = \frac{\sqrt{2(2+\lambda_j^{(m)}+\bar{\lambda}_j^{(m)})}}{4\lambda_j^{(m)}(\lambda_j^{(m)}+1)} \begin{bmatrix} \lambda_j(\lambda_j^{(m)}+1) \\ \lambda_j^{(m)^3}+1 \\ \lambda_j^{(m)}(\lambda_j^{(m)}-1) \\ \lambda_j^{(m)^3}-1 \end{bmatrix},$$

$$v_j^{(m)}(1,0) = \frac{\sqrt{2(2-\lambda_j^{(m)}-\bar{\lambda}_j^{(m)})}}{4\lambda_j^{(m)}(\lambda_j^{(m)}+1)} \begin{bmatrix} \lambda_j^{(m)}(\lambda_j^{(m)}-1) \\ -(\lambda_j^{(m)^3}+1) \\ \lambda_j^{(m)}(\lambda_j^{(m)}+1) \\ -(\lambda_j^{(m)^3}-1) \end{bmatrix}.$$

Then we obtain $\Psi_n^{(m)}(x_1,x_2)$ for all $n \in \mathbb{Z}_{\geq}$ and $(x_1,x_2) \in \pi_2^2$.

$$\Psi_{4k}^{(m)}(0,0) = \frac{1+i^k}{2} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}, \ \Psi_{4k}^{(m)}(1,1) = \frac{1-i^k}{2} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix},$$

$$\Psi_{4k}^{(m)}(0,1) = \Psi_{4k}^{(m)}(1,0) = {}^T\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Psi_{4k+1}^{(m)}(0,1) = \frac{1}{4} \begin{bmatrix} 1-i^k & 1-i^k & 1-i^k & 1-i^k \\ 1-i^k & i-i^{k+1} & -1+i^k & -i+i^{k+1} \\ 1+i^k & -1-i^k & 1+i^k & -1-i^k \\ 1+i^k & -i-i^{k+1} & -1-i^k & i+i^{k+1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix},$$

$$\Psi_{4k+1}^{(m)}(1,0) = \frac{1}{4} \begin{bmatrix} 1+i^k & 1+i^k & 1+i^k & 1+i^k \\ 1+i^k & i+i^{k+1} & -1-i^k & -i-i^{k+1} \\ 1-i^k & -1+i^k & 1-i^k & -1+i^k \\ 1-i^k & -i+i^{k+1} & -1+i^k & i-i^{k+1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix},$$

$$\Psi_{4k+1}^{(m)}(0,0) = \Psi_{4k+1}^{(m)}(1,1) = {}^T\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Psi_{4k+2}^{(m)}(0,0) = \frac{1}{4} \begin{bmatrix} 2 & i^k(1+i) & 0 & -i^k(-1+i) \\ i^k(-1+i) & 0 & -i^k(-1+i) & 2 \\ 0 & i^k(-1+i) & 2 & -i^k(1+i) \\ i^k(-1+i) & 2 & -i^k(1+i) & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix},$$

$$\Psi_{4k+2}^{(m)}(1,1) = \frac{1}{4} \begin{bmatrix} 2 & -i^k(1+i) & 0 & i^k(-1+i) \\ -i^k(-1+i) & 0 & i^k(-1+i) & 2 \\ 0 & -i^k(-1+i) & 2 & i^k(1+i) \\ -i^k(-1+i) & 2 & i^k(1+i) & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix},$$

$$\Psi^{(m)}_{4k+2}(0,1)=\Psi^{(m)}_{4k+2}(1,0)={}^T\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix},$$

$$\Psi^{(m)}_{4k+3}(0,1)=\frac{1}{4}\begin{bmatrix}1-i^{k+1} & 1-i^{k+1} & 1+i^{k+1} & 1+i^{k+1}\\ 1-i^{k+1} & -i-i^{k} & -1-i^{k+1} & i-i^{k}\\ 1-i^{k+1} & -1+i^{k+1} & 1+i^{k+1} & -1-i^{k+1}\\ 1-i^{k+1} & i+i^{k} & -1-i^{k+1} & -i+i^{k}\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi^{(m)}_{4k+3}(1,0)=\frac{1}{4}\begin{bmatrix}1+i^{k+1} & 1+i^{k+1} & 1-i^{k+1} & 1-i^{k+1}\\ 1+i^{k+1} & -i+i^{k} & -1+i^{k+1} & i+i^{k}\\ 1+i^{k+1} & -1-i^{k+1} & 1-i^{k+1} & -1+i^{k+1}\\ 1+i^{k+1} & i-i^{k} & -1+i^{k+1} & -i-i^{k}\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi^{(m)}_{4k+3}(0,0)=\Psi^{(m)}_{4k+3}(1,1)={}^T\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix},$$

where $k\in\mathbb{Z}_{\geq}$.

(**b**) FF case
The eigenvalues are

$$\begin{aligned}
&\lambda^{(f)}_0(0,0)=e^{\pi i/8},\ \lambda^{(f)}_1(0,0)=e^{5\pi i/8},\ \lambda^{(f)}_2(0,0)=e^{9\pi i/8},\ \lambda^{(f)}_3(0,0)=e^{13\pi i/8},\\
&\lambda^{(f)}_0(1,1)=e^{\pi i/8},\ \lambda^{(f)}_1(1,1)=e^{5\pi i/8},\ \lambda^{(f)}_2(1,1)=e^{9\pi i/8},\ \lambda^{(f)}_3(1,1)=e^{13\pi i/8},\\
&\lambda^{(f)}_0(0,1)=1,\ \lambda^{(f)}_1(0,1)=1,\ \lambda^{(f)}_2(0,1)=e^{\pi i/4},\ \lambda^{(f)}_3(0,1)=e^{5\pi i/4},\\
&\lambda^{(f)}_0(1,0)=-1,\ \lambda^{(f)}_1(1,0)=-1,\ \lambda^{(f)}_2(1,0)=-e^{\pi i/4},\ \lambda^{(f)}_3(1,0)=-e^{5\pi i/4},
\end{aligned}$$

and the corresponding eigenvectors are

$$v^{(f)}_j(0,0)=\frac{\sqrt{2(2+\lambda^{(f)}_j+\bar{\lambda}^{(f)}_j)}}{4(1+\lambda^{(f)}_j)}\begin{bmatrix}1+\lambda^{(f)}_j\\ -i{\lambda^{(f)}_j}^2(1+\lambda^{(f)}_j)\\ 1-\lambda^{(f)}_j\\ i{\lambda^{(f)}_j}^2(1-\lambda^{(f)}_j)\end{bmatrix},$$

$$v^{(f)}_j(1,1)=\frac{\sqrt{2(2-\lambda^{(f)}_j-\bar{\lambda}^{(f)}_j)}}{4(1-\lambda^{(f)}_j)}\begin{bmatrix}1-\lambda^{(f)}_j\\ -i{\lambda^{(f)}_j}^2(1-\lambda^{(f)}_j)\\ 1+\lambda^{(f)}_j\\ i{\lambda^{(f)}_j}^2(1+\lambda^{(f)}_j)\end{bmatrix},$$

$$v^{(f)}_0(0,1)=\frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}1 & 0 & -1 & 0\end{bmatrix},\ v^{(f)}_1(0,1)=\frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}0 & 1 & 0 & 1\end{bmatrix},$$

$$v_2^{(f)}(0,1) = \frac{1}{2}\,{}^T\big[1 \quad e^{7\pi i/4} \quad 1 \quad -e^{7\pi i/4}\big],\ v_3^{(f)}(0,1) = \frac{1}{\sqrt{2}}\,{}^T\big[1 \quad e^{3\pi i/4} \quad 1 \quad -e^{3\pi i/4}\big].$$

Thus we have $\Psi_n(x_1, x_2)$ as below.

$$\Psi_{4k}^{(f)}(0,0) = \frac{1}{4}\begin{bmatrix} 2i^k+1+(-1)^k & 0 & -1+(-1)^k & 0 \\ 0 & 2i^k+1+(-1)^k & 0 & 1-(-1)^k \\ -1+(-1)^k & 0 & 2i^k+1+(-1)^k & 0 \\ 0 & 1-(-1)^k & 0 & 2i^k+1+(-1)^k \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi_{4k}^{(f)}(1,1) = \frac{1}{4}\begin{bmatrix} 2i^k-(1+(-1)^k) & 0 & -(-1+(-1)^k) & 0 \\ 0 & 2i^k-(1+(-1)^k) & 0 & -(1-(-1)^k) \\ -(-1+(-1)^k) & 0 & 2i^k-(1+(-1)^k) & 0 \\ 0 & -(1-(-1)^k) & 0 & 2i^k-(1+(-1)^k) \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi_{4k}^{(f)}(0,1) = \Psi_{4k}^{(f)}(1,0) = {}^T\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix},$$

$$\Psi_{4k+1}^{(f)}(0,1) = \frac{1}{4}\begin{bmatrix} -1+i^k & (-1)^{k+1}i+i^{k+1} & 1-i^k & (-1)^k i-i^{k+1} \\ (-1)^{k+1}+i^k & -1+i^k & (-1)^{k+1}+i^k & -1+i^k \\ 1+i^k & (-1)^{k+1}i-i^{k+1} & -1-i^k & (-1)^k i+i^{k+1} \\ (-1)^k+i^k & -1-i^k & (-1)^k+i^k & -1-i^k \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi_{4k+1}^{(f)}(1,0) = \frac{1}{4}\begin{bmatrix} 1+i^k & (-1)^k i+i^{k+1} & -1-i^k & (-1)^{k+1}i-i^{k+1} \\ (-1)^k+i^k & 1+i^k & (-1)^k+i^k & 1+i^k \\ -1+i^k & (-1)^k i-i^{k+1} & 1-i^k & (-1)^{k+1}i+i^{k+1} \\ (-1)^{k+1}+i^k & 1-i^k & (-1)^{k+1}+i^k & 1-i^k \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi_{4k+1}^{(f)}(0,0) = \Psi_{4k+1}^{(f)}(1,1) = {}^T\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix},$$

$$\Psi_{4k+2}^{(f)}(0,0) = \frac{1}{4}\begin{bmatrix} 1+(-1)^k i & 2i^{k+1} & -1+(-1)^k i & 0 \\ 2i^k & 1+(-1)^k i & 0 & 1-(-1)^k i \\ -1+(-1)^k i & 0 & 1+(-1)^k i & -2i^{k+1} \\ 0 & 1-(-1)^k i & -2i^k & 1+(-1)^k i \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi_{4k+2}^{(f)}(1,1) = \frac{1}{4}\begin{bmatrix} -1-(-1)^k i & 2i^{k+1} & 1-(-1)^k i & 0 \\ 2i^k & -1-(-1)^k i & 0 & -1+(-1)^k i \\ 1-(-1)^k i & 0 & -1-(-1)^k i & -2i^{k+1} \\ 0 & -1+(-1)^k i & -2i^k & -1-(-1)^k i \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi_{4k+2}^{(f)}(0,1) = \Psi_{4k+2}^{(f)}(1,0) = {}^T\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix},$$

$$\Psi_{4k+3}^{(f)}(0,1) = \frac{1}{4}\begin{bmatrix} -1+i^{k+1} & (-1)^k+i^{k+1} & 1+i^{k+1} & (-1)^{k+1}+i^{k+1} \\ (-1)^{k+1}i+i^k & -1+i^{k+1} & (-1)^{k+1}i-i^k & -1-i^{k+1} \\ 1-i^{k+1} & (-1)^k+i^{k+1} & -1-i^{k+1} & (-1)^{k+1}+i^{k+1} \\ (-1)^k i-i^k & -1+i^{k+1} & (-1)^k i+i^k & -1-i^{k+1} \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi_{4k+3}^{(f)}(1,0) = \frac{1}{4}\begin{bmatrix} 1+i^{k+1} & (-1)^{k+1}+i^{k+1} & -1+i^{k+1} & (-1)^k+i^{k+1} \\ (-1)^k i+i^k & 1+i^{k+1} & (-1)^k i-i^k & 1-i^{k+1} \\ -1-i^{k+1} & (-1)^{k+1}+i^{k+1} & 1-i^{k+1} & (-1)^k+i^{k+1} \\ (-1)^{k+1}i-i^k & 1+i^{k+1} & (-1)^{k+1}i+i^k & 1-i^{k+1} \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi_{4k+3}^{(f)}(0,0) = \Psi_{4k+3}^{(f)}(1,1) = {}^T\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}.$$

Then we see that the Fourier walks with MS and FF on π_2^2 have period 16 i.e., for all $\Psi_{n+16}^{(j)} = \Psi_n^{(j)}$ $(j = m, f)$ for $n \in \mathbb{Z}_{\geq}$. Where $\Psi_n^{(j)}$ is the state of the walk at time n.

4.2 The Grover walk on π_2^2

In this subsection, we will check the probability amplitudes of the Grover walk to compare with that of the Fourier walk for the following same initial state:

$$\Psi_0^{(j)}(x_1,x_2) = \begin{cases} {}^T\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{bmatrix} & \big((x_1,x_2)=(0,0)\big) \\ \\ {}^T\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix} & \big((x_1,x_2)\neq(0,0)\big) \end{cases} \qquad (j=m,f)$$

for $|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2+|\alpha_4|^2=1$ with $\alpha_\ell \in \mathbb{C}$ $(\ell=1,2,3,4)$.
Here, the coin operators of the Grover walk with MS and FF are given by

$$U^{(m)}=\frac{1}{2}\begin{bmatrix}-1 & 1 & 1 & 1\\ 1 & -1 & 1 & 1\\ 1 & 1 & -1 & 1\\ 1 & 1 & 1 & -1\end{bmatrix},$$

and

$$U^{(f)}=\begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}U^{(m)}.$$

(**a**) MS case

$$\begin{aligned}
&\lambda_0^{(m)}(0,0)=1,\ \lambda_1^{(m)}(0,0)=\lambda_2^{(m)}(0,0)=\lambda_3^{(m)}(0,0)=-1,\\
&\lambda_0^{(m)}(1,1)=\lambda_1^{(m)}(1,1)=\lambda_2^{(m)}(1,1)=1,\ \lambda_3^{(m)}(1,1)=-1,\\
&\lambda_0^{(m)}(0,1)=1,\ \lambda_1^{(m)}(0,1)=-1,\ \lambda_2^{(m)}(0,1)=i,\ \lambda_3^{(m)}(0,1)=-i,\\
&\lambda_0^{(m)}(1,0)=1,\ \lambda_1^{(m)}(1,0)=-1,\ \lambda_2^{(m)}(1,0)=i,\ \lambda_3^{(m)}(1,0)=-i,
\end{aligned}$$

the eigenvectors are

$$\begin{aligned}
&v_0^{(m)}(0,0)=\frac{1}{2}\,{}^T\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix},\ v_1^{(m)}(0,0)=\frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix},\\
&v_2^{(m)}(0,0)=\frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}0 & 0 & 1 & -1\end{bmatrix},\ v_3^{(m)}(0,0)=\frac{1}{2}\,{}^T\begin{bmatrix}1 & 1 & -1 & -1\end{bmatrix},\\
&v_0^{(m)}(1,1)=\frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix},\ v_1^{(m)}(1,1)=\frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}0 & 0 & 1 & -1\end{bmatrix},\\
&v_2^{(m)}(1,1)=\frac{1}{2}\,{}^T\begin{bmatrix}1 & 1 & -1 & -1\end{bmatrix},\ v_3^{(m)}(1,1)=\frac{1}{2}\,{}^T\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix},\\
&v_0^{(m)}(0,1)=\frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}0 & 0 & 1 & -1\end{bmatrix},\ v_1^{(m)}(0,1)=\frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix},\\
&v_2^{(m)}(0,1)=\frac{1}{2}\,{}^T\begin{bmatrix}1 & 1 & i & i\end{bmatrix},\ v_3^{(m)}(0,1)=\frac{1}{2}\,{}^T\begin{bmatrix}1 & 1-i & -i\end{bmatrix},\\
&v_0^{(m)}(1,0)=\frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix},\ v_1^{(m)}(1,0)=\frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}0 & 0 & 1 & -1\end{bmatrix},\\
&v_2^{(m)}(1,0)=\frac{1}{2}\,{}^T\begin{bmatrix}1 & 1 & -i & -i\end{bmatrix},\ v_3^{(m)}(1,0)=\frac{1}{2}\,{}^T\begin{bmatrix}1 & 1 & i & i\end{bmatrix}.
\end{aligned}$$

Then we get

$$\Psi_n^{(m)}(0,0)=\frac{1+(-1)^n}{8}\begin{bmatrix} i^n+3 & i^n-1 & 0 & 0 \\ i^n-1 & i^n+3 & 0 & 0 \\ 0 & 0 & i^n+3 & i^n-1 \\ 0 & 0 & i^n-1 & i^n+3 \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi_n^{(m)}(1,1)=\frac{(1-i^n)(1+(-1)^n)}{8}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi_n^{(m)}(0,1)=\frac{1+(-1)^{n+1}}{8}\begin{bmatrix} 0 & 0 & 1+i^{n+1} & 1+i^{n+1} \\ 0 & 0 & 1+i^{n+1} & 1+i^{n+1} \\ 1-i^{n+1} & 1-i^{n+1} & -2 & 2 \\ 1-i^{n+1} & 1-i^{n+1} & 2 & -2 \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi_n^{(m)}(1,0)=\frac{1+(-1)^{n+1}}{8}\begin{bmatrix} -2 & 2 & 1-i^{n+1} & 1-i^{n+1} \\ 2 & -2 & 1-i^{n+1} & 1-i^{n+1} \\ 1+i^{n+1} & 1+i^{n+1} & 0 & 0 \\ 1+i^{n+1} & 1+i^{n+1} & 0 & 0 \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

(**b**) FF case

$$\begin{aligned}
&\lambda_0^{(f)}(0,0)=\lambda_1^{(f)}(0,0)=\lambda_2^{(f)}(0,0)=1,\ \lambda_3^{(f)}(0,0)=-1,\\
&\lambda_0^{(f)}(1,1)=1,\ \lambda_1^{(f)}(1,1)=\lambda_2^{(f)}(1,1)=\lambda_3^{(f)}(1,1)=-1,\\
&\lambda_0^{(f)}(0,1)=1,\ \lambda_1^{(f)}(0,1)=-1,\ \lambda_2^{(f)}(0,1)=i,\ \lambda_3^{(f)}(0,1)=-i,\\
&\lambda_0^{(f)}(1,0)=1,\ \lambda_1^{(f)}(1,0)=-1,\ \lambda_2^{(f)}(1,0)=i,\lambda_3^{(f)}(1,0)=-i,
\end{aligned}$$

and the corresponding eigenvectors are

$$\begin{aligned}
&v_0^{(f)}(0,0)=\frac{1}{\sqrt{2}}{}^T\begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix},\ v_1^{(f)}(0,0)=\frac{1}{\sqrt{2}}{}^T\begin{bmatrix}0 & 0 & 1 & -1\end{bmatrix},\\
&v_2^{(f)}(0,0)=\frac{1}{2}{}^T\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix},\ v_3^{(f)}(0,0)=\frac{1}{2}{}^T\begin{bmatrix}1 & 1 & -1 & -1\end{bmatrix},\\
&v_0^{(f)}(1,1)=\frac{1}{2}{}^T\begin{bmatrix}1 & 1 & -1 & -1\end{bmatrix},\ v_1^{(f)}(1,1)=\frac{1}{\sqrt{2}}{}^T\begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix},\\
&v_2^{(f)}(1,1)=\frac{1}{\sqrt{2}}{}^T\begin{bmatrix}0 & 0 & 1 & -1\end{bmatrix},\ v_3^{(f)}(1,1)=\frac{1}{2}{}^T\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix},
\end{aligned}$$

$$v_0^{(f)}(0,1) = \frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix},\ v_1^{(f)}(0,1) = \frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}0 & 0 & 1 & -1\end{bmatrix},$$
$$v_2^{(f)}(0,1) = \frac{1}{2}\,{}^T\begin{bmatrix}1 & 1 & i & i\end{bmatrix},\ v_3^{(f)}(0,1) = \frac{1}{2}\,{}^T\begin{bmatrix}1 & 1-i & -i\end{bmatrix},$$
$$v_0^{(f)}(1,0) = \frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}0 & 0 & 1 & -1\end{bmatrix},\ v_1^{(f)}(1,0) = \frac{1}{\sqrt{2}}\,{}^T\begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix},$$
$$v_2^{(f)}(1,0) = \frac{1}{2}\,{}^T\begin{bmatrix}1 & 1 & -i & -i\end{bmatrix},\ v_3^{(f)}(1,0) = \frac{1}{2}\,{}^T\begin{bmatrix}1 & 1 & i & i\end{bmatrix}.$$

Thus we have

$$\Psi_n^{(f)}(0,0) = \frac{1+(-1)^n}{8}\begin{bmatrix} i^n+3 & i^n-1 & 0 & 0 \\ i^n-1 & i^n+3 & 0 & 0 \\ 0 & 0 & i^n+3 & i^n-1 \\ 0 & 0 & i^n-1 & i^n+3 \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi_n^{(f)}(1,1) = \frac{(1-i^n)(1+(-1)^n)}{8}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi_n^{(f)}(0,1) = \frac{1+(-1)^{n+1}}{8}\begin{bmatrix} 0 & 0 & 1+i^{n+1} & 1+i^{n+1} \\ 0 & 0 & 1+i^{n+1} & 1+i^{n+1} \\ 1-i^{n+1} & 1-i^{n+1} & 2 & -2 \\ 1-i^{n+1} & 1-i^{n+1} & -2 & 2 \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix},$$

$$\Psi_n^{(f)}(1,0) = \frac{1+(-1)^{n+1}}{8}\begin{bmatrix} 2 & -2 & 1-i^{n+1} & 1-i^{n+1} \\ -2 & 2 & 1-i^{n+1} & 1-i^{n+1} \\ 1+i^{n+1} & 1+i^{n+1} & 0 & 0 \\ 1+i^{n+1} & 1+i^{n+1} & 0 & 0 \end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix}.$$

Compared with the Fourier walks, we see that the Grover walks with both MS and FF on π_2^2 have period 4, i.e., $\Psi_{n+4} = \Psi_n$ for $n \in \mathbb{Z}_{\geq}$. Where Ψ_n is the state of the walk at time n.

5. Summary

In this paper we considered discrete-time QWs with MS and FF on $\mathbb{Z}^2$ and π_N^2. We showed that localization does not occur for the Fourier walk on $\mathbb{Z}^2$ with MS and FF by using our contradiction argument which is different from the method based on the Fourier analysis by Komatsu and Tate [5]. Moreover we computed eigenvalues and the corresponding eigenvectors of the (k_1, k_2)-space of the Fourier walks on π_N^2 with MS and FF for some special initial conditions, for instance, $k_1 = k_2$ or $k_1 + k_2 \equiv 0 \ (\mathrm{mod} N)$. We derived the measure at time n from these results. In addition, we calculated amplitudes of the Grover and Fourier walks on π_2^2 and clarified the difference between both walks, for example, the Grover walks with MS and FF have period 4, and the Fourier walks with MS and FF have period 16. One of the interesting future problems would be to obtain the measure at time n of the Fourier walks on $\mathbb{Z}^2$ and π_N^2 for any initial state.

References

[1] D. Aharonov, A. Ambainis, J. Kempe, and U. V. Vazirani, Quantum walks on graphs, Proceedings of ACM Symposium on Theory of Computation (STOC'01), July 2001, 50–59.

[2] M. J. Cantero, F. A. Grünbaum, L. Moral, and L. Velázquez, The CGMV method for quantum walks. *Quantum Inf. Process.*, **11** (2012), 1149–1192.

[3] Y. Higuchi, N. Konno, I. Sato, and E. Segawa, Spectral and asymptotic properties of Grover walks on crystal lattice. *J. Funct. Anal.*, **267** (2014), 4197–4235.

[4] N. Inui, Y. Konishi, and N. Konno, Localization of two-dimensional quantum walks. *Phys. Rev. A*, **69** (2004), 052323.

[5] T. Komatsu, and T. Tate, Eigenvalues of quantum walks of Grover and Fourier types. *J. Fourier Anal. Appl.*, (in press).

[6] T. Kitagawa, Topological phenomena in quantum walks: Elementary introduction to the physics of topological phases. *Quantum Inf. Process.*, **11** (2012), 1107–1148.

[7] N. Konno, Quantum Walks. *Lecture Notes in Mathematics, Vol.1954*, 309–452, Springer, 2008.

[8] K. Manouchehri, and J. Wang, *Physical Implementation of Quantum Walks.* Springer, 2014.

[9] F. L. Marquezino, R. Portugal, and G. Abal, Mixing times in quantum walks on two-dimensional grids. *Phys. Rev. A*, **82** (2010), 042341.

[10] L. Matsuoka, T. Kasajima, M. Hashimoto, and K. Yokoyama, Numerical study on quantum walks implemented on the cascade rotational transitions in a diatomic molecule. *J. Korean Phys. Soc.*, **59** (2011), 2897–2900.

[11] R. Portugal, *Quantum Walks and Search Algorithms.* second edition, Springer, 2018.

[12] S. E. Venegas-Andraca, *Quantum Walks for Computer Scientists.* Morgan and Claypool, 2008.

[13] S. E. Venegas-Andraca, Quantum walks: a comprehensive review. *Quantum Inf. Process.*, **11** (2012), 1015–1106.

[14] K. Watabe, N. Kobayashi, M. Katori, and N. Konno, Limit distributions of two-dimensional quantum walks. *Phys. Rev. A*, **77** (2008), 062331.

Masahiro Asano
E-mail: `asano-masahiro-kc@ynu.jp`,

Takashi Komatsu
E-mail: `komatsu-takashi-fn@ynu.ac.jp`,

Norio Konno
E-mail: `konno-norio-bt@ynu.ac.jp`,

Akihiro Narimatsu
E-mail: `narimatsu-akihiro-pd@ynu.jp`
(e-mail of the corresponding author)

Department of Applied Mathematics,
Faculty of Engineering,
Yokohama National University
79-5 Tokiwadai, Hodogaya, Yokohama, 240-8501,
Japan