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ON THE CRITICAL PROBABILITY AND THE UNIQUENESS THRESHOLD FOR THE PRODUCT GRAPH OF A REGULAR TREE AND A LINE

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Abstract. We consider Bernoulli bond percolation on the product graph of a regular tree and a line. We show the critical probability p_c and the uniqueness threshold p_u can be expressed by using a certain function $\alpha(p)$, which is defined by an exponential decay rate of probability that two vertices of the same layer are connected, and was originally introduced by Schonmann to show that there are a.s. infinitely many infinite clusters at $p = p_u$.

1. Introduction

Let $G = (V, E)$ be a connected, locally finite and infinite graph, where V is the set of vertices, E is the set of edges. In Bernoulli bond percolation, each edge will be open with probability p, and closed with probability $1-p$ independently, where $p \in [0,1]$ is a fixed parameter. Let $\Omega = \{0,1\}^E$ be the set of samples, where $\omega(e) = 1$ means e is open. Each $\omega \in \Omega$ is regarded as a subgraph of G consisting of all open edges. The connected components of ω are referred to as clusters. Let $p_c = p_c(G)$ be the critical probability for Bernoulli bond percolation on G, that is,

$$p_c = \inf \{p \in [0,1] \mid \text{there exists an infinite cluster } \mathbb{P}_p\text{-almost surely}\},$$

and let $p_u = p_u(G)$ be the uniqueness threshold for Bernoulli bond percolation on G, that is,

$$p_u = \inf \{p \in [0,1] \mid \text{there exists a unique infinite cluster } \mathbb{P}_p\text{-almost surely}\}.$$

One of the most popular graphs in the theory of percolation is the Euclidean lattice $\mathbb{Z}^d$. In 1980 Kesten [11] proved that $p_c = 1/2$ in the case of two dimensions. But in the case of three dimensions or more, as a numerical value, the

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critical probability is not quite clear. Regarding the uniqueness threshold of the Euclidean lattice, in 1987 Aizenman, Kesten, and Newman [2] proved that there exists at most one infinite cluster almost surely for all $d \geq 1$, that is, they showed that $p_c = p_u$ for all $d \geq 1$. The product graph of a d-regular tree and a line $T_d \square \mathbb{Z}$ was presented as a first example of a graph with $p_c < p_u < 1$ by Grimmett and Newman [7] in 1990, where a product graph means a Cartesian product graph. They showed that $p_c < p_u$ holds when d is sufficiently large. After this article had appeared, percolation on $T_d \square \mathbb{Z}$ has become a popular topic. However, the critical probability of $T_d \square \mathbb{Z}$ is, as a value, also not quite clear. In this paper we study Bernoulli bond percolation on $T_d \square \mathbb{Z}$. Our goal is to write the critical probability and the uniqueness threshold by using a certain function $\alpha(p)$.

We denote the probability measure associated with Bernoulli percolation process by $\mathbb{P}_p$ or $\mathbb{P}_p^G$. Let $(x \leftrightarrow y)$ be an event that there exists an open path between x and y for two vertices $x, y \in V$. Similarly, Let $(K \leftrightarrow L)$ be an event that there exists two vertices $x \in K, y \in L$ and an open path between x and y for two sets of vertices $K, L \subset V$. If either K or L is finite in addition, then an event $(K \leftrightarrow L)$ is called a connection event. The function $\alpha(p)$, which was appeared in [13], is defined by

$$\alpha(p) = \alpha_d(p) = \lim_{n \to \infty} \mathbb{P}_p(o \leftrightarrow (v_n, 0))^{\frac{1}{n}},$$

where v_n is a vertex on T_d with distance n from the origin o. From a homogeneity of T_d, $\alpha(p)$ does not depend on a choice of v_n. We abbreviate v_n as n. We check the existence of the above limit. From the FKG inequality, we have

$$\mathbb{P}_p(o \leftrightarrow (n+l, 0)) \geq \mathbb{P}_p(o \leftrightarrow (n, 0)) \mathbb{P}_p(o \leftrightarrow (l, 0))$$

for all $n, l \geq 0$. By using Fekete's subadditive lemma, the existence of the limit is ensured, and we have

$$\alpha(p) = \lim_{n \to \infty} \mathbb{P}_p(o \leftrightarrow (n, 0))^{\frac{1}{n}} = \sup_{n \geq 1} \mathbb{P}_p(o \leftrightarrow (n, 0))^{\frac{1}{n}}.$$

Letting $B(k)$ be a k-ball of $T_d \square \mathbb{Z}$ whose center is o, we have

$$\alpha(p) = \sup_{n \geq 1} \sup_{k \geq 1} \mathbb{P}_p^{B(k)}(o \leftrightarrow (n, 0))^{\frac{1}{n}},$$

and observe that we are taking the supremum of a continuous function of p. Therefore $\alpha(p)$ is lower semi-continuous and, since it is clearly non-decreasing, it is also left-continuous. By using the function $\alpha(p)$, we have new characterizations of p_c and p_u as in the following theorem.

Theorem 1.1. *For all $d \geq 3$, we have*

$$p_c(T_d \square \mathbb{Z}) = \alpha^{-1}\left(\frac{1}{d-1}\right),$$
$$p_u(T_d \square \mathbb{Z}) = \alpha^{-1}\left(\frac{1}{\sqrt{d-1}}\right).$$

To prove this theorem, we require the following lemmas.

Lemma 1.2. *For all $d \geq 3$, we have*

$$\alpha(p_c(T_d \square \mathbb{Z})) = \frac{1}{d-1},$$
$$\alpha(p_u(T_d \square \mathbb{Z})) = \frac{1}{\sqrt{d-1}}.$$

Lemma 1.3. *The function $\alpha(p)$ is strictly increasing in $[0, p_u]$.*

From Lemma 1.3, the inverse function of α can be defined in $\{\alpha(p) \mid p \in [0, p_u]\}$.

As for the critical probability p_c, Hutchcroft showed the following theorem.

Theorem 1.4. ([8]) *Let G be a quasi-transitive graph with exponential growth. Then*

$$\kappa_{p_c}(n) = \inf\{\tau_{p_c}(x, y) \mid x, y \in V, d(x, y) \leq n\} \leq \mathrm{gr}(G)^{-n}$$

for all $n \geq 1$, where $\tau_p(x, y) = \mathbb{P}_p(x \leftrightarrow y)$ and $\mathrm{gr}(G) = \liminf_{r \to \infty} |B(x, r)|^{1/r}$.

The following lemma can be shown by using a similar argument of this theorem.

Lemma 1.5. *Let $G = T_d \square \mathbb{Z}$. Then we have*

$$\alpha(p_c) \leq \mathrm{gr}(G)^{-1} = \frac{1}{d-1}.$$

On the other hand, when $p > p_c$, we have the following lemma.

Lemma 1.6. *For all $p > p_c$, we have*

$$\alpha(p) \geq \frac{1}{d-1}.$$

To obtain $\alpha(p_c) = 1/(d-1)$, we require the following lemma.

Lemma 1.7. *The function $\alpha(p)$ is continuous at $p \in [0, p_c]$.*

Hence by using Lemmas 1.5 and 1.6, we have $\alpha(p_c) = 1/(d-1)$.

We turn to the uniqueness threshold p_u. A part of Lemma 1.2 was already obtained by Schonman[13], who showed there are a.s. infinitely many infinite clusters at $p = p_u$.

THEOREM 1.8. ([13]) *Let p_u be the uniqueness threshold. Then we have*

$$\alpha(p_u) \leq \frac{1}{\sqrt{d-1}}.$$

To obtain $\alpha(p_u) = 1/\sqrt{d-1}$, we require the following lemma.

LEMMA 1.9. *For all $p \geq p_u$, we have*

$$\alpha(p) \geq \frac{1}{\sqrt{d-1}}.$$

We will show Lemma 1.5 and Lemma 1.6 in Section 2, and show Lemma 1.3 in Section 3. Section 4, Section 5 and Section 6 will be devoted to prove Lemma 1.7. In Section 7, we will show Lemma 1.9, and we will complete the proof of Lemma 1.2.

2. Proof of Lemma 1.5 and Lemma 1.6

We require the following well-known theorem.

THEOREM 2.1. ([1], [3]) *Let G be a quasi-transitive graph, and o be a fixed vertex of G. Then we have*

$$\sum_{x \in V} \tau_p(o, x) < \infty$$

for all $p < p_c$.

This theorem was proven in the transitive case by Aizenman and Barsky [1], and in the quasi-transitive case by Antunović and Veselić [3].

Proof of Lemma 1.5. Let $S(n)$ be a set of vertices of $T_d \times \{0\}$ with distance n from the origin. For all $p \in [0,1]$ and all $n \geq 1$, we have

$$\tau_p(o, (n, 0)) \cdot |S(n)| = \sum_{x \in T_d, |x| = n} \tau_p(o, (x, 0)) \leq \sum_{x \in T_d \square \mathbb{Z}} \tau_p(o, x).$$

By using Theorem 2.1, the right-hand side is finite when $p < p_c$. We know $|S(n)| = d(d-1)^{n-1}$. Then we have

$$\lim_{n \to \infty} \tau_p(o, (n, 0))^{\frac{1}{n}} \leq \lim_{n \to \infty} \left(\frac{\sum_{x \in T_d \square \mathbb{Z}} \tau_p(o, x)}{|S(n)|} \right)^{\frac{1}{n}} = \frac{1}{d-1}.$$

This means that $\alpha(p) \leq 1/(d-1)$ for all $p < p_c$. Since $\alpha(p)$ is left-continuous, we have $\alpha(p_c) \leq 1/(d-1)$, which completes the proof of Lemma 1.5. $\square$

We prepare some tools to show Lemma 1.6. Let $G_{\bullet,m} = T_d \square [-m,m]$ for each $m \geq 0$. Similarly to $\alpha(p)$, we define $\alpha_m(p)$ by

$$\alpha_m(p) = \lim_{n\to\infty} \mathbb{P}_p^{G_{\bullet,m}}(o \leftrightarrow (n,0))^{\frac{1}{n}} = \sup_{n\geq 1} \mathbb{P}_p^{G_{\bullet,m}}(o \leftrightarrow (n,0))^{\frac{1}{n}}.$$

Schonmann proved the following lemma.

Lemma 2.2. ([13]) *The function $\alpha(p)$ is given by taking a limit of $\alpha_m(p)$, that is,*

$$\lim_{m\to\infty} \alpha_m(p) = \alpha(p)$$

for all $p \in [0,1]$.

Proof. It is clear that for all $m \geq 0, \alpha_m(p) \leq \alpha_{m+1}(p) \leq \alpha(p)$. Thus we have $\{\alpha_m(p)\}_{m\geq 0}$ converges and $\lim_{m\to\infty} \alpha_m(p) \leq \alpha(p)$. On the other hand, by definition of $\alpha(p)$, for any small $\epsilon > 0$, there is an n such that

$$\alpha(p) - \epsilon \leq \mathbb{P}_p(o \leftrightarrow (n,0))^{\frac{1}{n}}.$$

By the definition of $\alpha_m(p)$, for any $n \geq 1$, we have

$$\mathbb{P}_p^{G_{\bullet,m}}(o \leftrightarrow (n,0)) \leq \alpha_m(p)^n.$$

From these two inequalities, we have

$$(\alpha(p) - \epsilon)^n \leq \mathbb{P}_p(o \leftrightarrow (n,0)) = \lim_{m\to\infty} \mathbb{P}_p^{G_{\bullet,m}}(o \leftrightarrow (n,0)) \leq \lim_{m\to\infty} \alpha_m(p)^n.$$

Hence, we have $\alpha(p) - \epsilon \leq \lim_{m\to\infty} \alpha_m(p)$, which completes the proof. $\square$

Let π be a natural projection from $T_d \square \mathbb{Z}$ to T_d, and $\tau_p(o, \pi^{-1}(x)) = \mathbb{P}_p(o \leftrightarrow \pi^{-1}(x))$ for $x \in V(T_d)$. We define functions $\alpha'(p), \alpha'_m(p)$, similarly to $\alpha(p), \alpha_m(p)$.

$$\alpha'(p) = \sup_{n\geq 1} \mathbb{P}_p(o \leftrightarrow \pi^{-1}(n))^{\frac{1}{n}},$$
$$\alpha'_m(p) = \lim_{n\to\infty} \mathbb{P}_p^{G_{\bullet,m}}(o \leftrightarrow \pi^{-1}(n))^{\frac{1}{n}} = \sup_{n\geq 1} \mathbb{P}_p^{G_{\bullet,m}}(o \leftrightarrow \pi^{-1}(n))^{\frac{1}{n}}.$$

We check the existence of a limit defining the function $\alpha_m(p)$. Let E_n be an event that all edges of $\pi^{-1}(n) \cap G_{\bullet,m}$ are open. By using the FKG inequality, we have

$$\begin{aligned} \mathbb{P}_p^{G_{\bullet,m}}(o \leftrightarrow \pi^{-1}(n+l)) &\geq \mathbb{P}_p^{G_{\bullet,m}}(o \leftrightarrow \pi^{-1}(n+l) \cap E_n) \\ &= \mathbb{P}_p^{G_{\bullet,m}}((o \leftrightarrow \pi^{-1}(n)) \cap E_n \cap ((n,0) \leftrightarrow \pi^{-1}(n+l))) \\ &\geq p^{2m} \mathbb{P}_p^{G_{\bullet,m}}(o \leftrightarrow \pi^{-1}(n)) \mathbb{P}_p^{G_{\bullet,m}}(o \leftrightarrow \pi^{-1}(l)) \end{aligned}$$

for all $n, l \geq 0$. By using Fekete's subadditive lemma, the existence of the limit is ensured, and we have

$$\alpha'_m(p) = \sup_{n \geq 1} p^{\frac{2m}{n}} \mathbb{P}_p^{G_\bullet,m}(o \leftrightarrow \pi^{-1}(n))^{\frac{1}{n}} = \sup_{n \geq 1} \mathbb{P}_p^{G_\bullet,m}(o \leftrightarrow \pi^{-1}(n))^{\frac{1}{n}}.$$

Similarly to Lemma 2.2, we can show $\lim_{m\to\infty} \alpha'_m(p) = \alpha'(p)$.

Lemma 2.3. *For all $p \in [0,1]$, we have $\alpha(p) = \alpha'(p)$.*

Proof. It is clear that $\alpha_m(p) \leq \alpha'_m(p)$. On the other hand, we have

$$\mathbb{P}_p^{G_\bullet,m}(o \leftrightarrow \pi^{-1}(n)) \leq \sum_{|k| \leq m} \mathbb{P}_p^{G_\bullet,m}(o \leftrightarrow (n,k)) \leq (2m+1)\alpha_m(p)^n,$$

$$\alpha'_m(p) = \lim_{n\to\infty} \mathbb{P}_p^{G_\bullet,m}(o \leftrightarrow \pi^{-1}(n))^{\frac{1}{n}} \leq \lim_{n\to\infty} (2m+1)^{\frac{1}{n}} \alpha_m(p) = \alpha_m(p).$$

Then $\alpha_m(p) = \alpha'_m(p)$ holds for all $m \geq 0$. By taking the limit, we have $\alpha(p) = \alpha'(p)$. □

Proof of Lemma 1.6. We use another definition of the critical probability. Let $(o \leftrightarrow \infty)$ be the event that there exists an infinite open path from the origin. Then we have

$$p_c = \sup \{ p \in [0,1] \mid \mathbb{P}_p(o \leftrightarrow \infty) = 0 \}.$$

Let $B(n) \subset T_d$ be a n-ball whose center is the origin, and we set $G_{n,\bullet} = B(n)\square\mathbb{Z}$. If $(o \leftrightarrow \infty)$ occurs on $T_d\square\mathbb{Z}$, then $(o \leftrightarrow \partial B(n)\square\mathbb{Z})$ or $(o \leftrightarrow \infty)$ occur on $G_{n,\bullet}$. It is clear that $p_c(G_{n,\bullet}) = 1$. Hence, by using Lemma 2.3, we have

$$\begin{aligned} \mathbb{P}_p(o \leftrightarrow \infty) &\leq \mathbb{P}_p(o \leftrightarrow \partial B(n)\square\mathbb{Z}) + \mathbb{P}_p^{G_{n,\bullet}}(o \leftrightarrow \infty) \\ &\leq \sum_{x \in \partial B(n)} \mathbb{P}_p(o \leftrightarrow \pi^{-1}(x)) \leq d(d-1)^{n-1}\alpha(p)^n \end{aligned}$$

for all $p < 1$. The right-hand side goes to 0 if $\alpha(p) < 1/(d-1)$. Since $\mathbb{P}_p(o \leftrightarrow \infty) > 0$ holds when $p > p_c$, Then we have $\alpha(p) \geq 1/(d-1)$ for all $p > p_c$. □

3. Extension of some theorems

In Bernoulli percolation, some theorems can only be applied to events which depend on finitely many edges. For an edge subset F, let $[\omega]_F$ be a subset of Ω whose elements have the same configuration as ω on F. An event A is said to depend on (only) finitely many edges if there exists a finite edge set F such that

$[\omega]_F \subset A$ or $[\omega]_F \cap A = \emptyset$ holds for all $\omega \in \Omega$. For $\omega, \tau \in \Omega$, we write $\omega \leq \tau$ if $\omega(e) \leq \tau(e)$ holds for all $e \in E$. An event A is called increasing if $\tau \in A$ whenever $\omega \in A$ and $\omega \leq \tau$.

Theorem 3.1. ([6] (2.39)) *Let A be an increasing event which depends on finitely many edges. Then we have*

$$\mathbb{P}_{p^\gamma}(A) \leq \mathbb{P}_p(A)^\gamma$$

for all $0 < p < 1$ and $\gamma \geq 1$.

For two events A and B, $A \circ B$ is defined as the event that A and B occur on disjoint edge sets, formulated by

$$A \circ B = \{\omega \in \Omega \mid \exists \text{finite disjoint } K, L \subset E \text{ s.t. } [\omega]_K \subset A, [\omega]_L \subset B\}.$$

Theorem 3.2. (the BK inequality [5]) *Let A, B be increasing events which depends on finitely many edges. Then we have*

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

We will extend these two theorems so that it can be applied to certain events which depends on infinitely many edges. In section 1, we defined connection events. It is clear that a connection event is an increasing event, and depends on infinitely many edges in general. For example $(o \leftrightarrow x)$ and $(o \leftrightarrow \pi^{-1}(x))$ are connection events which are depends on infinitely many edges.

Lemma 3.3. *Let A be a connection event. Then we have*

$$\mathbb{P}_{p^\gamma}(A) \leq \mathbb{P}_p(A)^\gamma$$

for all $0 < p < 1$ and $\gamma \geq 1$.

Lemma 3.4. *Let A, B be connection events. Then we have*

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

Proof of Lemma 3.3 and Lemma 3.4. Let $A = (K \leftrightarrow L)$, and Γ be a set of all paths between K and L. Then we have

$$A = \bigcup_{q \in \Gamma} (q : open).$$

Each event $(q : open)$ is increasing and it depends on finitely many edges. For any $\epsilon > 0$, there exists a finite subset $\Gamma' \subset \Gamma$ such that

$$\mathbb{P}_{p^\gamma}(A) - \epsilon \leq \mathbb{P}_{p^\gamma}\left(\bigcup_{q \in \Gamma'} (q : open)\right).$$

The event in the right-hand side is increasing and it depends on finitely many edges. Then by using Theorem 3.1, we have

$$\mathbb{P}_{p^\gamma}(A) - \epsilon \le \mathbb{P}_p\left(\bigcup_{q\in\Gamma'}(q : open)\right)^\gamma \le \mathbb{P}_p(A)^\gamma.$$

It completes the proof of Lemma 3.3. Next we will show Lemma 3.4. Let $A_i = (K_i \leftrightarrow L_i)$, $\Gamma_i^{(n)}$ be a set of all paths between K_i and L_i with length n or less, $C_i^{(n)} = \bigcup_{q\in\Gamma_i^{(n)}} (q : open)$ for $i = 1, 2$. Then we have

$$A_i = \bigcup_{n\ge1} C_i^{(n)}.$$

If $\omega \in A_1 \circ A_2$, then there exists finite disjoint subsets $F_1, F_2 \subset E$ such that $[\omega]_{F_i} \subset A_i$. We take $n = \max\{|F_i|\}$, then we have $[\omega]_{F_i} \subset C_i^{(n)}$, that is $\omega \in C_1^{(n)} \circ C_2^{(n)}$. Hence, we have

$$A_i \circ A_2 \subset \bigcup_{n\ge1}\left(C_1^{(n)} \circ C_2^{(n)}\right).$$

Since K_1, K_2 are finite, each of the events $C_1^{(n)}$ and $C_2^{(n)}$ is increasing and depends on finite edges. Then by using Theorem 3.2, we have

$$\begin{aligned}&\mathbb{P}_p(A_1 \circ A_2)\\ &\le \lim_{n\to\infty}\mathbb{P}_p\left(C_1^{(n)} \circ C_2^{(n)}\right) \le \lim_{n\to\infty}\left(\mathbb{P}_p\left(C_1^{(n)}\right)\mathbb{P}_p\left(C_2^{(n)}\right)\right) = \mathbb{P}_p(A_1)\mathbb{P}_p(A_2).\end{aligned}$$

□

At the end of this section, we show Lemma 1.3. By using Lemma 3.3, we have

$$\alpha(p^\gamma) = \lim_{n\to\infty}\mathbb{P}_{p^\gamma}(o \leftrightarrow (n,0))^{\frac{1}{n}} \le \lim_{n\to\infty}\mathbb{P}_p(o \leftrightarrow (n,0))^{\frac{\gamma}{n}} = \alpha(p)^\gamma.$$

We know $\alpha(p) < 1$ for all $p \le p_u$ from Theorem 1.8. Then we have

$$\alpha(p^\gamma) \le \alpha(p)^\gamma < \alpha(p)$$

for all $p \le p_u$ and $\gamma > 1$. Therefore $\alpha(p)$ is a strictly increasing function on $[0, p_u]$.

4. Connection event

In Section 3, we prepared some lemmas concerning connection events. In this section, we prepare one more lemma concerning connection events. A graph G

is called nonamenable if the Cheeger constant of G, defined by

$$h(G) = \inf \left\{ \frac{|\partial S|}{|S|} \mid S \subset V, |S| < \infty \right\},$$

is positive.

THEOREM 4.1. ([4]) *Let G be a nonamenable Cayley graph. Then we have*

$$\mathbb{P}_{p_c}(o \leftrightarrow \infty) = 0.$$

It is well-known that $h(T_d \square \mathbb{Z}) = d - 2$, that is, $T_d \square \mathbb{Z}$ is a nonamenable graph for all $d \geq 3$. Also, let $S = \{a_1, \ldots, a_d, b\}$ be a generating set, and $\Gamma = \langle a_i, b | a_i^{-1} = a_i, a_i b = b a_i \rangle$ be a group generated by S, then $T_d \square \mathbb{Z}$ is a Cayley graph of (Γ, S). Therefore, we can use this theorem for $T_d \square \mathbb{Z}$.

LEMMA 4.2. *Let $G = T_d \square \mathbb{Z}$, and A be a connection event. Then $\mathbb{P}_p(A)$ is continuous at $p \in [0, p_c]$.*

Proof. It is clear that $\mathbb{P}_p(A)$ is left-continuous similar to $\alpha(p)$, since

$$\mathbb{P}_p(A) = \sup_{k \geq 1} \mathbb{P}_p^{B(k)}(A)$$

where $B(k)$ is a k-ball. We will prove that $\mathbb{P}_p(A)$ is right-continuous at $p \in [0, p_c]$ in this section. We prepare another definition of $\mathbb{P}_p$ which is found in [6]. Let $\Omega' = [0, 1]^E$, μ_e be a uniform distribution on $[0, 1]$ for each $e \in E$, and $\mu = \prod_{e \in E} \mu_e$ be a probability measure on Ω'. For any $p \in [0, 1]$ and $\{X_e\}_{e \in E} \in \Omega'$, let ω_p be a configuration defined by

$$\omega_p(e) = \mathbb{1}_{\{X_e < p\}}$$

for any $e \in E$. We define a map f_p from Ω' to $\Omega = \{0, 1\}^E$, by $f_p(\{X_e\}) = \omega_p$. Then the pushforward measure of μ is the same as $\mathbb{P}_p$, that is,

$$f_{p*}(\mu) = \mathbb{P}_p.$$

By using this equation, we have

$$\mathbb{P}(A) = \mu(\omega_p \in A).$$

Fix $p_0 \in [0, p_c]$ arbitrarily. For any $p > p_0$, we have

$$\mathbb{P}_p(A) - \mathbb{P}_{p_0}(A) = \mu(\omega_p \in A, \omega_{p_0} \notin A).$$

Hence, by taking the limit, we have

$$\lim_{p\downarrow p_0}(\mathbb{P}_p(A)-\mathbb{P}_{p_0}(A))=\mu(\forall p>p_0,\omega_p\in A,\omega_{p_0}\notin A).$$

Let $A=(K\leftrightarrow L)$, and suppose that $(\omega_{p_0}\notin A)$ occurs. By Theorem 4.1, there exists no infinite path from x on ω_{p_0} almost surely for any $x\in K$ and any $p_0\in[0,p_c]$. Hence, connected components containing elements in K are finite. Let H be a finite subgraph which contains all of these connected components. If $\omega_p\in A$ holds, then there exists at least one edge e on H such that $\omega_{p_0}(e)=0$ and $\omega_p(e)=1$. If $\omega_p\in A$ holds for all $p>p_0$, then there exists at least one edge e on H such that $X_e=p_0$. Hence, we have

$$\lim_{p\downarrow p_0}(\mathbb{P}_p(A)-\mathbb{P}_{p_0}(A))\le\sum_{e\in E(H)}\mu(X_e=p_0)\le 0,$$

which completes the proof. □

5. Another function $\beta(p)$

In this section, we prove Lemma 1.7. We prepare another function $\beta(p)$ similar to $\alpha(p)$, defined by

$$\beta(p)=\lim_{m\to\infty}\mathbb{P}_p(o\leftrightarrow(0,m))^{\frac{1}{m}}=\sup_{m\ge1}\mathbb{P}_p(o\leftrightarrow(0,m))^{\frac{1}{m}}.$$

By using the FKG inequality and the homogeneity of $T_d\square\mathbb{Z}$, we have

$$\mathbb{P}_p(o\leftrightarrow(2n,0))\ge\mathbb{P}_p\left((o\leftrightarrow(n,m))\cap((n,m)\leftrightarrow(2n,0))\right)\ge\mathbb{P}_p(o\leftrightarrow(n,m))^2.$$

Hence, we have

$$\mathbb{P}_p(o\leftrightarrow(n,m))\le\mathbb{P}_p(o\leftrightarrow(2n,0))^{\frac{1}{2}}\le\alpha(p)^n$$

for each (n,m). Similarly, we have

$$\mathbb{P}_p(o\leftrightarrow(n,m))\le\beta(p)^m.$$

For each $n\ge1$, we define $I_n(p)$ by

$$I_n(p)=\sum_{k\in\mathbb{Z}}\mathbb{P}_p(o\leftrightarrow(n,k)).$$

Since $\mathbb{P}_p(o\leftrightarrow(n,m))\le\beta(p)^m$, it is well-defined when $\beta(p)<1$.

Lemma 5.1. *For any $p < p_u$, we have*

$$\beta(p) < 1.$$

This lemma will be shown in the next section. We assume Lemma 5.1 holds, and only consider when $p < p_u$.

Lemma 5.2. *For any $n, l \geq 1$, we have*

$$I_{n+l}(p) \leq I_n(p) I_l(p).$$

Proof. By using Lemma 3.4, we have

$$\begin{aligned}
&I_{n+l}(p) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{P}_p(o \leftrightarrow (n+l, k)) = \sum_{k \in \mathbb{Z}} \mathbb{P}_p\left(\bigcup_{t \in \mathbb{Z}} (o \leftrightarrow (n,t)) \circ ((n,t) \leftrightarrow (n+l,k)) \right) \\
&\leq \sum_{k \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \mathbb{P}_p(o \leftrightarrow (n,t)) \mathbb{P}_p(o \leftrightarrow (l, k-t)) \leq I_n(p) I_l(p).
\end{aligned}$$

□

Therefore, we define the function $\eta(p)$ by

$$\eta(p) = \lim_{n \to \infty} I_n(p)^{\frac{1}{n}} = \inf_{n \geq 1} I_n(p)^{\frac{1}{n}}.$$

Lemma 5.3. *The function $\eta(p)$ is right-continuous at $p \in [0, p_c]$.*

Proof. We define $I_n^{(m)}(p)$ by

$$I_n^{(m)}(p) = \sum_{|k| \leq m} \mathbb{P}_p(o \leftrightarrow (n,k))$$

for all $n \geq 1, m \geq 0$. By Lemma 4.2, $I_n^{(m)}(p)$ is continuous at $p \in [0, p_c]$. For any $p_0 \in (p_c, p_u)$, we have

$$\begin{aligned}
I_n(p) - I_n^{(m)}(p) &= \sum_{|k| > m} \mathbb{P}_p(o \leftrightarrow (n,k)) \leq 2 \sum_{k > m} \beta(p)^k \\
&= 2 \cdot \frac{\beta(p)^{m+1}}{1 - \beta(p)} \leq 2 \cdot \frac{\beta(p_0)^{m+1}}{1 - \beta(p_0)}.
\end{aligned}$$

Since $\beta(p_0) < 1$, the right-hand side goes to 0 as $m \to \infty$. Then $\{I_n^{(m)}(p)\}_{m \geq 0}$ uniformly converges to $I_n(p)$ on $[0, p_0]$. Hence, $I_n(p)$ is continuous at $p \in [0, p_c]$. The function $\eta(p)$ is defined as the pointwise infimum of continuous functions. Therefore, $\eta(p)$ is right-continuous at $p \in [0, p_c]$. □

Now we know that $\alpha(p)$ is left-continuous on $[0,1]$ and $\eta(p)$ is right-continuous at $p \in [0, p_c]$.

LEMMA 5.4. *For any $p \in [0, p_u)$, we have $\alpha(p) = \eta(p)$. In particular, $\alpha(p)$ is continuous at $p \in [0, p_c]$.*

Proof. We define $\eta_m(p)$ by

$$\eta_m(p) = \liminf_{n\to\infty} I_n^{(m)}(p)^{\frac{1}{n}}$$

for all $m \geq 0, n \geq 1, p \in [0, p_u)$. It is clear that $I_n^{(m)}(p)^{1/n} \leq I_n^{(m+1)}(p)^{1/n} \leq I_n(p)^{1/n}$ holds. Then $\{\eta_m(p)\}_{m\geq 0}$ converges and we have $\lim_{m\to\infty} \eta_m(p) \leq \eta(p)$. First, we show that $\lim_{m\to\infty} \eta_m(p) = \eta(p)$. By the definition of $\eta_m(p)$, for any $\epsilon > 0, m \geq 0$, there exists an $n \geq 1$ such that

$$I_n^{(m)}(p)^{\frac{1}{n}} - \epsilon \leq \eta_m(p).$$

By the definition of $\eta(p)$, for all $n \geq 1$, we have

$$\eta(p)^n \leq I_n(p) = \lim_{m\to\infty} \sum_{|k|\leq m} \mathbb{P}_p(o \leftrightarrow (n,k)) = \lim_{m\to\infty} \left(I_n^{(m)}(p)^{\frac{1}{n}}\right)^n,$$

$$\eta(p) \leq \lim_{m\to\infty} I_n^{(m)}(p)^{\frac{1}{n}}.$$

Therefore, we have

$$\eta(p) - \epsilon \leq \lim_{m\to\infty} \eta_m(p)$$

for any $\epsilon > 0$. It completes the proof of $\lim_{m\to\infty} \eta_m(p) = \eta(p)$. Next, for all $n \geq 1$, it is clear that $\mathbb{P}_p(o \leftrightarrow (n,0))^{1/n} \leq I_n(p)^{1/n}$. Then we have $\alpha(p) \leq \eta(p)$. For any $\epsilon > 0$, there exists m such that

$$\eta(p) - \frac{\epsilon}{2} \leq \eta_m(p).$$

By the definition of $\eta_m(p)$, there exists $N \geq 1$ such that

$$\eta_m(p) - \frac{\epsilon}{2} \leq I_n^{(m)}(p)^{\frac{1}{n}}$$

for all $n \geq N$. By the inequality $\mathbb{P}_p(o \leftrightarrow (n,k)) \leq \alpha(p)^n$, we have

$$I_n^{(m)}(p) \leq (2m+1)\alpha(p)^n.$$

Therefore, from above three inequalities, we have

$$\eta(p) - \epsilon \leq (2m+1)^{\frac{1}{n}}\alpha(p)$$

for any $\epsilon > 0$. The right-hand side goes to $\alpha(p)$ as $n \to \infty$, which completes the proof. □

6. Proof of Lemma 5.1

It is left to prove Lemma 5.1 to show Lemma 1.7. Our proof is an adaption of the method in [12], which is about contact process, to percolation process.

LEMMA 6.1. *For any $p \in [0, p_u)$, we have*

$$\inf_{m \geq 0} \mathbb{P}_p(o \leftrightarrow (0,m)) = 0.$$

Proof. We recall $G_{n,\bullet}$ is a subgraph defined by $G_{n,\bullet} = B(n)\square\mathbb{Z}$, where $B(n)$ is an n-ball whose center is the origin. Since $p_c(G_{n,\bullet}) = 1$, we have

$$\inf_{m \geq 0} \mathbb{P}_p^{G_{n,\bullet}}(o \leftrightarrow (0,m)) = 0$$

for any $p < 1$. If $(o \leftrightarrow (0,m))$ occurs on G, then $(o \leftrightarrow (0,m))$ occurs on $G_{n,\bullet}$ or there exists $x \in \partial B(n)$ such that $(o \leftrightarrow \pi^{-1}(x))$ and $((0,m) \leftrightarrow \pi^{-1}(x))$ occur on disjoint edge subsets. The latter occurs when there exists an open path between o and $(0,m)$ which is not contained $G_{n,\bullet}$. Then we have

$$\begin{aligned}
&\mathbb{P}_p(o \leftrightarrow (0,m)) \\
&\quad \leq \mathbb{P}_p^{G_{n,\bullet}}(o \leftrightarrow (0,m)) + \mathbb{P}_p\left(\bigcup_{x \in \partial B(n)} (o \leftrightarrow \pi^{-1}(x)) \circ ((0,m) \leftrightarrow \pi^{-1}(x))\right)
\end{aligned}$$

for all $n \geq 1$. By Lemma 2.3 and Lemma 3.4, we have

$$\begin{aligned}
&\mathbb{P}_p\left(\bigcup_{x \in \partial B(n)} (o \leftrightarrow \pi^{-1}(x)) \circ ((0,m) \leftrightarrow \pi^{-1}(x))\right) \\
&\quad \leq \sum_{x \in \partial B(n)} \mathbb{P}_p(o \leftrightarrow \pi^{-1}(x))\mathbb{P}_p((0,m) \leftrightarrow \pi^{-1}(x)) \\
&\quad \leq d(d-1)^{n-1}\alpha(p)^{2n}.
\end{aligned}$$

By Theorem 1.8 and Lemma 1.3, we have

$$\alpha(p) < \frac{1}{\sqrt{d-1}}$$

for all $p \in [0, p_u)$. Therefore, we have

$$\inf_{m \geq 0} \mathbb{P}_p(o \leftrightarrow (0,m)) \leq d(d-1)^{n-1}\alpha(p)^{2n} \to 0$$

as $n \to \infty$, which completes the proof. □

Liggett used the level function. We define the level difference function $L(x,y)$ from $T_d \times T_d$ to $\mathbb{Z}$, which is used by Hutchcroft [10]. Let ξ be a fixed end of T_d. The parent of a vertex $x \in T_d$ is the unique neighbor of x that is closer to ξ than x is. We call the other vertices of x its children. If y is a parent of x, then we define $L(x,y) = 1$. If y is a child of x, then we define $L(x,y) = -1$. In general cases, for any x, y, there exists a unique geodesic $\{x_i\}_{i=0}^n$ such that $x_0 = x$ and $x_n = y$, then we define

$$L(x,y) = \sum_{i=1}^{n} L(x_{i-1}, x_i).$$

Note that $L(x,z) = L(x,y) + L(y,z)$ and $L(y,x) = -L(x,y)$ for any $x, y, z \in V(T_d)$. For $n \geq 0, z \in \mathbb{R}_{>0}$, we define $a_n(z)$ by

$$a_n(z) = \sum_{\substack{x \in V(T_d) \\ |x|=n}} z^{L(o,x)}.$$

Stacey [14] has computed the number of vertices $x \in T_d$ satisfying $|x| = n$, $L(o,x) = 2t - n$ as

$$\begin{cases} b^n & (t = 0), \\ (b-1)b^{n-t-1} & (1 \leq t \leq n-1), \\ 1 & (t = n), \end{cases}$$

where $b = d - 1$. By using this formula, Liggett [12] showed the following equations.

$$\begin{aligned} a_n(z) &= (bz^{-1})^n + \sum_{t=1}^{n-1} (b-1) b^{n-t-1} z^{2t-n} + z^n \\ &= \begin{cases} \frac{b^{n-1} z^{-n} (b^2 - z^2) + z^n (1 - z^2)}{b - z^2} & (z^2 \neq b), \\ \sqrt{b}^n ((n+1) - b^{-1}(n-1)) & (z^2 = b), \end{cases} \\ a_n(1/bz) &= a_n(z). \end{aligned}$$

For $r > 0$ and $z > 0$ we define $J(r,z)$ by

$$J(r,z) = \sum_{x \in V(T_d)} r^{|x|} z^{L(o,x)} = \sum_{n \geq 0} r^n a_n(z).$$

Lemma 6.2. *For any $r < 1/\sqrt{b}$ and $z \in (br, 1/r)$, we have $J(r,z) < \infty$.*

Proof. If $z = \sqrt{b}$, then we have

$$\begin{aligned} J(r,z) &= \sum_{n\geq 0} r^n a_n(z) \\ &= \frac{b-1}{b} \sum_{n\geq 0} n r^n \sqrt{b}^n + \frac{b+1}{b} \sum_{n\geq 0} r^n \sqrt{b}^n \\ &= \frac{b-1}{b} \frac{r\sqrt{b}}{(1-r\sqrt{b})^2} + \frac{b+1}{b} \frac{1}{r\sqrt{b}} < \infty. \end{aligned}$$

If $z \in (br, 1/r)$ and $z \neq \sqrt{b}$, then we have

$$\begin{aligned} J(r,z) &\leq \sum_{n\geq 0} r^n a_n(z) \\ &= \frac{b^2 - z^2}{b(b-z^2)} \sum_{n\geq 0} r^n (bz^{-1})^n + \frac{1-z^2}{b-z^2} \sum_{n\geq 0} r^n z^n \\ &= \frac{b^2 - z^2}{b(b-z^2)} \cdot \frac{1}{1-rbz^{-1}} + \frac{1-z^2}{b-z^2} \cdot \frac{1}{1-rz} < \infty, \end{aligned}$$

which completes the proof. □

For $m \geq 0$, we define $J_m(p,z)$ by

$$J_m(p,z) = \sum_{x\in V(T_d)} \mathbb{P}_p(o \leftrightarrow (x,m)) z^{L(o,x)} = \sum_{n\geq 0} \mathbb{P}_p(o \leftrightarrow (x_n, m)) a_n(z),$$

where $x_n \in V(T_d)$ such that $|x_n| = n$. From Lemma 6.2, if $\alpha(p) < 1/\sqrt{b}$ and $z \in (b\alpha(p), 1/\alpha(p))$, we have

$$J_m(p,z) \leq \sum_{n\geq 0} \alpha(p)^n a_n(z) = J(\alpha(p), z) < \infty.$$

Therefore, $J_m(p,z)$ is well-defined for $\alpha(p) < 1/\sqrt{b}$ and $z \in (b\alpha(p), 1/\alpha(p))$.

Lemma 6.3. *For any $m, l \geq 0$, we have $J_{m+l}(p,z) \leq J_m(p,z) J_l(p,z)$.*

Proof. By a homogeneity of $T_d \square \mathbb{Z}$, we can write

$$J_l(p,z) = \sum_{x\in T_d} \mathbb{P}_p((y,m) \leftrightarrow (x, m+l)) z^{L(y,x)}.$$

By Lemma 3.4, we have

$$\begin{aligned}
J_{m+l}(p,z) &= \sum_{x\in T_d} \mathbb{P}_p\left(\bigcup_{y\in T_d} (o \leftrightarrow (y,m)) \circ ((y,m) \leftrightarrow (x,m+l))\right) z^{L(o,x)} \\
&\le \sum_{y\in T_d} \mathbb{P}_p(o \leftrightarrow (y,m)) z^{L(o,y)} \sum_{x\in T_d} \mathbb{P}_p((y,m) \leftrightarrow (x,m+l)) z^{L(y,x)} \\
&= J_m(p,z) J_l(p,z),
\end{aligned}$$

which completes the proof. □

From this lemma, we can define $\phi(p,z)$ by

$$\phi(p,z) = \lim_{m\to\infty} J_m(p,z)^{\frac{1}{m}} = \inf_{m\ge 0} J_m(p,z)^{\frac{1}{m}}.$$

By definition of $\phi(p,z)$, we have

$$\phi(p,z)^m \le J_m(p,z).$$

Since $L(o,o) = 0$, we have $\mathbb{P}_p(o \leftrightarrow (0,m)) \le J_m(p,z)$. Then $\beta(p) \le \phi(p,z)$. Therefore, if there exists z such that $\inf_{m\ge 0} J_m(p,z) < 1$, then $\phi(p,z) < 1$. Hence, we obtain $\beta(p) < 1$. The next lemma completes the proof of Lemma 5.1.

Lemma 6.4. *For any $z \in (b\alpha(p), 1/\alpha(p))$, we have*

$$\inf_{m\ge 0} J_m(p,z) = 0.$$

Proof. Since $a_n(1/bz) = a_n(z)$, we have $J_m(p,1/bz) = J_m(p,z)$. Then we only consider $z \in [\sqrt{b}, 1/\alpha(p))$. For $z \ne \sqrt{b}$ and any $z_0 \in (z, 1/\alpha(p))$, we have

$$\begin{aligned}
\frac{a_n(z)}{a_n(z_0)} &= \frac{b-z_0^2}{b-z^2} \cdot \frac{b^{n-1}z^{-n}(b^2-z^2) + z^n(1-z^2)}{b^{n-1}z_0^{-n}(b^2-z_0^2) + z_0^n(1-z_0^2)} \\
&= \frac{b-z_0^2}{b-z^2} \cdot \frac{b^{-1}(b/z^2)^n(b^2-z^2) + (1-z^2)}{b^{-1}(b/zz_0)^n(b^2-z_0^2) + (z_0/z)^n(1-z_0^2)} \\
&\to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

Similarly, $a_n\left(\sqrt{b}\right)/a_n(z_0)$ goes to 0 as $n \to \infty$ for $z_0 \ne \sqrt{b}$. Therefore, for any $z \in (b\alpha(p), 1/\alpha(p))$ and $\epsilon > 0$, there exist z_0 and $N \in \mathbb{N}$ such that

$$\frac{a_n(z)}{a_n(z_0)} \le \frac{\epsilon}{J(\alpha(p), z_0)}$$

for all $n \geq N$, where $J(\alpha(p), z)$ is a constant which does not depend on m such that $J_m(p, z_0) \leq J(\alpha(p), z_0)$. Then we have

$$\begin{aligned} J_m(p,z) &= \sum_{n\geq 0} a_n(z)\mathbb{P}_p(o \leftrightarrow (x_n, m)) \\ &= \sum_{n\geq N} \frac{a_n(z)}{a_n(z_0)} \cdot a_n(z_0)\mathbb{P}_p(o \leftrightarrow (x_n, m)) + \sum_{n<N} a_n(z)\mathbb{P}_p(o \leftrightarrow (x_n, m)) \\ &\leq \frac{\epsilon}{J(\alpha(p), z_0)} J_m(p, z_0) + \sum_{n<N} a_n(z)\mathbb{P}_p(o \leftrightarrow (x_n, m)), \end{aligned}$$

where $x_n \in V(T_d)$ such that $|x_n| = n$. From Lemma 6.1 we have

$$\inf_{m\geq 0} J_m(p,z) \leq \epsilon + \sum_{n<N} a_n(z) \inf_{m\geq 0} \mathbb{P}_p(o \leftrightarrow (0, 2m))^{\frac{1}{2}} = \epsilon,$$

which completes the proof. $\square$

7. Proof of Lemma 1.9

It is left to prove Lemma 1.9. The level difference function L can be extended to $T_d\square\mathbb{Z}$ naturally. Let π be a natural projection from $T_d\square\mathbb{Z}$ to T_d, and L_T be the traditional level difference function on T_d. Then we extend the level difference function as $L(x,y) = L_T(\pi(x), \pi(y))$. Similarly, we have $L(x,z) = L(x,y) + L(y,z)$ and $L(y,x) = -L(x,y)$ for any $x, y, z \in V(T_d\square\mathbb{Z})$. We define $\Delta(x,y)$ by

$$\Delta(x,y) = (d-1)^{L(x,y)}$$

for all $x, y \in V(T_d\square\mathbb{Z})$. Note that $\Delta(x,z) = \Delta(x,y)\Delta(y,z)$ and $\Delta(y,x) = \Delta(x,y)^{-1}$ for any $x, y, z \in V(T_d)$ because $L(x,z) = L(x,y)+L(y,z)$ and $L(y,x) = -L(x,y)$. Our method is based on [9]. The tilted susceptibility is defined by

$$\chi_{p,1/2}(o) = \sum_{x\in V(T_d\square\mathbb{Z})} \mathbb{P}_p(o \leftrightarrow x)\Delta(o,x)^{1/2}.$$

For detail on this quantity, see [9]. Hutchcroft showed the following theorem.

Theorem 7.1. ([9]) *The set* $\left\{p \in [0,1] \mid \chi_{p,1/2}(o) < \infty\right\}$ *is open in* $[0,1]$.

By using this theorem, we show the following lemma.

Lemma 7.2. *We have* $\chi_{p_u,1/2}(o) = \infty$.

Proof. When $p > p_u$, there exists a.s. only one infinite cluster. Hence, if both $(x \leftrightarrow \infty)$ and $(y \leftrightarrow \infty)$ occur, then $(x \leftrightarrow y)$ must occur. Thus, we have

$$\mathbb{P}_p(x \leftrightarrow y) \geq \mathbb{P}_p\left((x \leftrightarrow \infty) \cap (y \leftrightarrow \infty)\right) \geq \mathbb{P}_p(x \leftrightarrow \infty)^2 > 0$$

for all $x, y \in V$. Therefore, $\mathbb{P}_p(x \leftrightarrow y)$ has a uniform bound. Then we have

$$\chi_{p,1/2}(o) \geq \mathbb{P}_p(o \leftrightarrow \infty)^2 \sum_x \Delta(o,x)^{1/2} = \mathbb{P}_p(o \leftrightarrow \infty)^2 \sum_{n \geq 0} a_n \left(\sqrt{d-1}\right) = \infty.$$

On the other hand, when $p < p_u$, we have $\alpha(p) < 1/\sqrt{d-1}$. Then we obtain

$$\begin{aligned}
\chi_{p,1/2}(o) &= \sum_{x \in V(T_d)} \Delta(o,x)^{1/2} \sum_{m \in \mathbb{Z}} \tau_p(o,(x,m)) \\
&= \sum_{x \in V(T_d)} \sqrt{d-1}^{L(o,x)} I_x(p) \\
&= \sum_{n \geq 0} I_{x_n}(p) a_n \left(\sqrt{d-1}\right)
\end{aligned}$$

where $x_n \in V(T_d)$ such that $|x_n| = n$. By using the Cauchy root test, $\chi_{p,1/2}(o) < \infty$ because

$$\lim_{n \to \infty} \left(I_{x_n}(p) a_n \left(\sqrt{d-1}\right)\right)^{\frac{1}{n}} = \alpha(p)\sqrt{d-1} < 1.$$

Therefore, from Theorem 7.1, we have $\chi_{p_u,1/2}(o) = \infty$. □

Proof of Lemma 1.9. Similarly to the proof of Lemma 7.2, if $\alpha(p_u) < 1/\sqrt{d-1}$, then we have $\chi_{p_u,1/2}(o) < \infty$. It is contrary to Lemma 7.2. Thus we have $\alpha(p_u) \geq 1/\sqrt{d-1}$. □

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