Facially-constrained colorings of triangulations on closed surfaces

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Preface

This thesis is written on the subject "Facially-constrained colorings of triangulations on closed surfaces" and it is to be submitted to get the degree of Doctor of Philosophy at Yokohama National University.

I like to listen to music and play the guitar. When I was a first year student at Ochanomizu University, I knew that Pythagorean tuning had been constructed by using mathematics. I was interested in the relationship between music and mathematics. However, there is little information about it in the Internet. Moreover, no professors in Ochanomizu University were familiar with such a topic.

When I was a second year student at Ochanomizu University, I found a symposium held at Yokohama National University by chance. Though that symposium is for mathematics, Ms. Sachiko Nakajima who is a gold medalist of International Mathematical Olympiad and a jazz pianist would appear. I expected that if I attend the symposium, then I could obtain some information about music and mathematics, and hence I decided to attend it.

In that symposium, my heart was moved by Professor Seiya Negami's talk. He talked about his policy "mathematics without calculating" and my prejudice of mathematics was broken by him. After the symposium, I talked to Professor Negami and talked that I wanted to study the relationship between music and mathematics. He told me that he had discovered the relationship between musical chords and the Möbius band with his student and that if I would come to his laboratory, then I might be able to study what I wanted. Thus, I decided to enter a master course in Yokohama National University and choice Professor Negami as my supervisor.

In the first year of the graduate school, Professor Negami gave me a new idea of colorings of triangulations which is motivated by musical chords and called it a "triad coloring". When I heard it, I was very interested in it and I wanted to develop it. Since Professor Negami is familiar with topology, I was taught some topological and algebraic topological method by him, and I constructed the theory of the new coloring, using them.

On the other hand, when I was the first year student in a doctor course, Naoki Matsumoto, an assistant professor in Keio University, gave me some problems about colorings of triangulations called a "facial complete coloring". He invited me for studying together and we obtained some results about it. I will present my work on these topics in this thesis, entitled "*facially-constrained colorings of triangulations on closed surfaces*" which belongs to topological graph theory.

Finally, I am grateful to Professor Negami not only for his kindness and advice but also for his teaching me the possibility of mathematics. Thanks to him, I noticed that I can enjoy talking with foreign people through mathematics and it made me want to go abroad actively. Moreover, I thank to Professor Naoki Matsumoto. Though I had mistakes in studying and in writing the papers, he taught me kindly and politely. Then, I am thankful to Professor Atsuhiro Nakamoto and Professor Kenta Ozeki. They gave me many useful advices for my study and my presentations.

The author
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March 2020

Papers underlying the thesis

- S. Negami and Y. Ohno, Triad colorings of triangulations on closed surfaces, *J. Nonlinear Convex Anal.* **19** (2018), 1775–1780.
- N. Matsumoto and Y. Ohno, Facial achromatic number of triangulations on the sphere, *Discrete Math.* **343** (2020), 111651.

Contents

Introduction

A *graph* consists of finitely many points and lines, each of which joins a pair of vertices. They are called *vertices* and *edges*. The sets of vertices and edges of a graph G are usually denoted by $V(G)$ and $E(G)$, respectively. A graph is often represented by a figure drawn on paper and such a figure may have edge crossings. When we consider only the combinatorial structure of graphs, we do not care about whether graphs have edge crossings or not. However, in topological graph theory, which the author is majoring in, we deal with graphs drawn on a closed surface with no edge crossings and we study properties or structures of such graphs by using topology. If a graph G is drawn on a closed surface F^2 with no edge crossings, G is said to be *embedded* on F^2 and we simply say "a graph G on a closed surface". For a graph G on a closed surface F^2, each component of $F^2 - G$ is called a *face* of G. Faces of graphs on closed surfaces play an important role in topological graph theory.

In this thesis, we deal with "colorings of graphs". Let G be a graph. A *coloring* of G is defined as an assignment of colors to each of vertices in G. In particular, a coloring is said to be *proper* if every pair of vertices joined by an edge receive different colors. In what follows, "a coloring" means a proper coloring implicitly unless otherwise stated. Colors are often regarded as numbers and the set of n colors is defined as the set $\{1, \ldots, n\}$ in studies on colorings of graphs. A coloring of G using n colors is called an *n-coloring*, which is usually defined as a function $c : V(G) \to \{1, \ldots, n\}$. If we assign different colors to all vertices in G, then such an assignment of colors is a coloring clearly. Since we allow the same color to be used for vertices which are not joined by edges, we may be able to decrease the number of colors to make a coloring of G. Thus, we should discuss the minimum number n such that G has an n-coloring. Such an invariant of graphs is called the *chromatic number*, denoted by $\chi(G)$.

Studies on colorings of graphs started with the famous problem called "Four Color Problem"; can we color a map by four colors so that countries sharing a common boundary do not receive the same color? This problem was given by Francis Guthrie in 1852. Representing each country by a vertex and joining two vertices if countries corresponding to them share a common boundary, we obtain a graph embedded on the plane, which is called a *plane graph*. Thus, we can rephrase Four Color Problem with a problem on a coloring of a plane graph; does any plane graph G have a 4-coloring? Though Four Color Problem fascinated many researchers, it had not been solved for more than 100 years. In

1976, Appel and Haken solved affirmatively Four Color Problem by using computers and their result has been called "Four Color Theorem":

Theorem 0.1 (Appel and Haken [5, 6, 7])**.** *Any plane graph has a* 4*-coloring.*

Four Color Theorem implies that the chromatic number of any plane graph does not exceed 4. Moreover, the upper bound of the chromatic number for graphs on closed surfaces other than the sphere was evaluated as "Map Color Theorem":

Theorem 0.2 (Ringel [72])**.** *Let G be a graph on a closed surface F^2 other than the sphere and the Klein bottle. Then the maximum of $\chi(G)$ taken over all graphs G on F^2 is equal to $\left\lfloor \frac{7+\sqrt{49-24\varepsilon(F^2)}}{2} \right\rfloor$, where $\varepsilon(F^2)$ stands for the Euler characteristic of F^2. For the Klein bottle, the maximum is* 6*.*

Moreover, studies on colorings of graphs with some additional conditions have been considered. For example, a *distinguishing coloring* is a coloring of a graph breaking its symmetry [2, 10, 21, 30, 65, 73, 82, 83] and an *equitable coloring* is a coloring of a graph such that the numbers of vertices in any two color classes differ by at most one [17, 18, 29, 36, 52, 53, 54, 57, 59, 62]. In particular, there have been studied in topological graph theory those colorings of graphs on closed surfaces that satisfy suitable conditions on colors appearing around each face. Such a coloring is called a *facially-constrained coloring* of a graph on a closed surface generally and at least 100 papers about facially-constrained colorings have already been published. In this thesis, we consider two facially-constrained colorings of triangulations on closed surfaces, where a *triangulation* on a closed surface F^2 is a graph on F^2 such that each face is triangular.

The first facially-constrained coloring of triangulations on closed surface is called a *triad coloring*. Let G be a triangulation on a closed surface. We use the cyclic group $\mathbb{Z}_n$ ($n \geq 3$) as the color set $\{1, \ldots, n\}$ with $n \equiv 0 \pmod{n}$ to define an algebraic property of a coloring. Put $\mathcal{T}_n = \{\{i, i+1, i+2\} \mid i \in \mathbb{Z}_n\}$ and call it the set of *triads*. A function $c : V(G) \to \mathbb{Z}_n$ is called an *n-triad coloring* if $\{c(u), c(v), c(w)\}$ belongs to $\mathcal{T}_n$ for each face uvw of G. Roughly speaking, a triad coloring is a coloring of G such that vertices on the boundary of any face of G have three consecutive colors. Note that triads $\{n-2, n-1, 0\}, \{n-1, 0, 1\}$ and $\{0, 1, 2\}$ are the elements of $\mathcal{T}_n$ since we define the colors modulo n.

It is clear that an n-triad coloring of G is also an n-coloring since $c(u) - c(v) \not\equiv 0 \pmod{n}$ for any edge uv. If $n = 3$ or 4, then the set of triads $\mathcal{T}_n$ contains all 3-element subsets in $\mathbb{Z}_n$ and hence any 3- or 4-coloring of G becomes a 3- or 4-triad coloring, respectively. However, if $n \geq 5$, then there are many 3-element subsets in $\mathbb{Z}_n$ not belonging to $\mathcal{T}_n$. Thus, an n-coloring of G cannot be always regarded as an n-triad coloring of G for $n \geq 5$. Therefore, we would like to know the set of numbers n such that G has an n-triad coloring and we define $\mathrm{TCS}(G)$ as such a set meaning "Triad Coloring Set of G".

If the chromatic number of a triangulation G on a closed surface is 3, then a 3-coloring of G becomes an n-triad coloring for any $n \geq 3$ since we may assume that three colors used in the 3-coloring are $0, 1$ and 2. That is, $\mathrm{TCS}(G)$ contains all positive integers more than 2 in such a case. In addition, for a triangulation G with $\chi(G) = 4$, $\mathrm{TCS}(G)$ contains 4 as its element since a 4-triad coloring of G is a 4-coloring. However, in this case, we do not know which n belongs to $\mathrm{TCS}(G)$ for $n \geq 5$ immediately. Similarly, the elements of $\mathrm{TCS}(G)$ are not obvious if the chromatic number of G is at least 5. To investigate the elements of the Triad Coloring Set, we first prove the following theorem for a triangulation on the sphere or the projective plane.

Theorem 0.3. *Let n be any natural number ≥ 5. A triangulation G on the sphere or the projective plane has an n-triad coloring if and only if G has a 3-coloring.*

By Theorem 0.3, the elements of $\mathrm{TCS}(G)$ are determined completely for a triangulation G on the sphere or the projective plane, that is, $\mathrm{TCS}(G)$ contains only 4 as its element if $\chi(G) = 4$ and $\mathrm{TCS}(G)$ has no element if $\chi(G) \geq 5$. To prove this theorem, we use some notions in algebraic topology. The set of triads $\mathcal{T}_n$ induces naturally a combinatorial simplicial 2-complex, called a *triad complex*, and its underlying space X is homeomorphic to a Möbius band if $n \geq 5$ is odd and to an annulus if $n \geq 6$ is even. By regarding faces of a triangulation G on a closed surface F^2 as a 2-dimensional simplex, G naturally induces a simplicial 2-complex and there exists a continuous simplicial map between it and the triad complex when G has a triad coloring. Moreover, such a map induces a continuous map from F^2 to X naturally, too. Under this situation, we show Theorem 0.3 discussing the fundamental groups and covering spaces. In particular, it is important that the fundamental groups π_1 of the Möbius band and the annulus are isomorphic to the cyclic group $\mathbb{Z}$, while that the order of $\pi_1(F^2)$ is finite when F^2 is the sphere or the projective plane. It seems to be difficult to prove Theorem 0.3 by only combinatorial methods.

Since the order of the fundamental group of a closed surface other than the sphere or the projective plane is infinite, Theorem 0.3 does not hold for a triangulation on the closed surface. In fact, there exists a triangulation on the torus for which the same argument as in our proof of the theorem does not work. Though it is difficult to determine completely the elements of $\mathrm{TCS}(G)$ for a triangulation G on a closed surface F^2 other than the sphere and the projective plane, we investigate them partially as follows by using the continuous map between F^2 and X described above. Our proofs of the theorems and the details of methods in algebraic topology used for them are introduced in Chapter 3.

Theorem 0.4. *Let G be a triangulation on a closed surface.*

(i) *If $\chi(G) = 3$, then $\mathrm{TCS}(G)$ consists of all natural numbers $n \geq 3$.*

(ii) *If $\chi(G) = 4$, then there exists the maximum number $n \geq 4$ with $n \not\equiv 0 \pmod 3$ such that G has an n-triad coloring. For this number n, $\mathrm{TCS}(G)$ includes all natural numbers k such that $4 \leq k \leq n$ and $k \equiv n \pmod 3$. Furthermore, if $n \geq 8$, then there is a natural number $m \geq \lfloor n/2 \rfloor - 1$ with $m \equiv -n \pmod 3$ such that $\mathrm{TCS}(G)$*

includes all natural numbers k with $4 \leq k \leq m$ and $k \equiv m \pmod 3$, and $\mathrm{TCS}(G)$ *includes no other numbers.*

(iii) *If $\chi(G) = 5$, then either $\mathrm{TCS}(G) = \{5\}$ or $\mathrm{TCS}(G) = \emptyset$.*

(iv) *If $\chi(G) \geq 6$, then $\mathrm{TCS}(G) = \emptyset$.*

The second facially-constrained coloring of triangulations on closed surfaces is a *facial complete coloring*. This coloring is an extension of a coloring called a *complete coloring* defined as follows:

Let G be a graph. A *complete n-coloring* of G is an n-coloring such that each pair of colors appears on at least one edge. A $\chi(G)$-coloring of G is necessarily a complete $\chi(G)$-coloring, for if a pair (i, j) of colors did not appear on any edge, we could obtain a proper $(\chi(G) - 1)$-coloring of G by recoloring all vertices with color j by color i, which is contrary to $\chi(G)$ being the minimum number of colors in colorings. We define the *achromatic number* of G, denoted by $\psi(G)$, to be the maximum number n for which G has a complete n-coloring.

Complete colorings and the achromatic number were introduced by Harary and Hedetniemi [39]. They gave a general upper bound for the achromatic number of graphs by using the maximum number of vertices which are not joined by edges mutually, called the *independence number* of G and denoted by $\alpha(G)$:

Theorem 0.5 (Harary and Hedetniemi [39])**.** *For any graph G, the following equality holds:*

$$\psi(G) \leq |V(G)| - \alpha(G) + 1.$$

The achromatic number of graphs on closed surfaces also has been studied (see [37] and [38], for example). Hara [37] completely characterized triangulations on a closed surface having the achromatic number 3, as follows. Here, $K_{n,n,n}$ denotes the *complete tripartite graphs*, described in Section 1.1 of Chapter 1. See the survey [49] for other studies on complete colorings and the achromatic number.

Theorem 0.6 (Hara [37])**.** *Let G be a triangulation on a closed surface. Then $\psi(G) = 3$ if and only if G is isomorphic to $K_{n,n,n}$ for some $n \geq 1$.*

We introduce a "facial complete coloring" by a slightly general form. Let G be a graph on a closed surface and t be a positive integer. An n-coloring $c : V(G) \to \{1, 2, \ldots, n\}$, which is not necessarily proper, is a *facial t-complete n-coloring* if for any t-element subset X of $\{1, \cdots, n\}$, there exists at least one face of G such that the set of colors assigned to the vertices lying along its boundary includes X. The *facial t-achromatic number* of G, denoted by $\psi_t(G)$, is defined as the maximum number n such that G has a facial t-complete n-coloring. Similarly, if we deal only with proper colorings as c, then a *proper facial t-complete n-coloring* and the *proper facial t-achromatic number* $\psi_t^p(G)$ are defined as well as in the previous. A (proper) facial 1-complete coloring is just a (proper) coloring using each color at least once, and a proper facial 2-complete coloring is a complete coloring. In this thesis, we concentrate on a (proper) facial 3-complete coloring of a triangulation on the sphere.

Recall that every graph G has a complete n-coloring for some $n \geq \chi(G)$. However, there exists a triangulation G on the sphere which has no (proper) facial 3-complete n-coloring for any $n \geq \chi(G)$. On the other hand, every even triangulation G on the sphere has at least one proper facial 3-complete 3-coloring, since it has a proper 3-coloring [81], where a triangulation G on a closed surface is *even* if every vertex of G is incident to even number of edges. Thus, in this thesis, we principally focus on the (proper) facial 3-achromatic number of even triangulations on the sphere.

By the definition of a facial complete coloring, we intuitively see that the greater the number of mutually vertex disjoint faces of an even triangulation on the sphere becomes, the larger its facial 3-achromatic number is, where faces f_1 and f_2 of a graph on a closed surface are said to be *vertex disjoint* if the boundaries of f_1 and f_2 contain no common vertices. In fact, we can easily see that if G has at least $\binom{n}{3}$ such faces, then the facial 3-achromatic number of G is at least n; assign all triples of colors to such faces of G one by one and then assign the same color to other uncolored vertices. Moreover, we can also have a similar result with the restriction to proper colorings as follows.

Theorem 0.7. *Let G be an even triangulation on the sphere and k be the maximum number of faces which are vertex disjoint in G. If $k \geq 4\binom{n}{3}$, then $\psi_3^p(G) \geq n$.*

In addition, we characterize even triangulations G on the sphere with $\chi(G) = \psi_3^p(G) = 3$, which is an analogue of Theorem 0.6. Here, the *double wheel DW_n* for $n \geq 3$ is a triangulation on the sphere which is obtained from the cycle C_n by adding two extra vertices x and y and joining them to all vertices of C_n (see the left of Figure 1). When n is even, DW_n is an even triangulation on the sphere.

Theorem 0.8. *Let G be an even triangulation on the sphere. The proper facial 3-achromatic number of G is exactly 3 if and only if G is isomorphic to the double wheel DW_{2n} for $n \geq 2$ or one of the two graphs OC_1 and Q_3 given in Figure 1.*

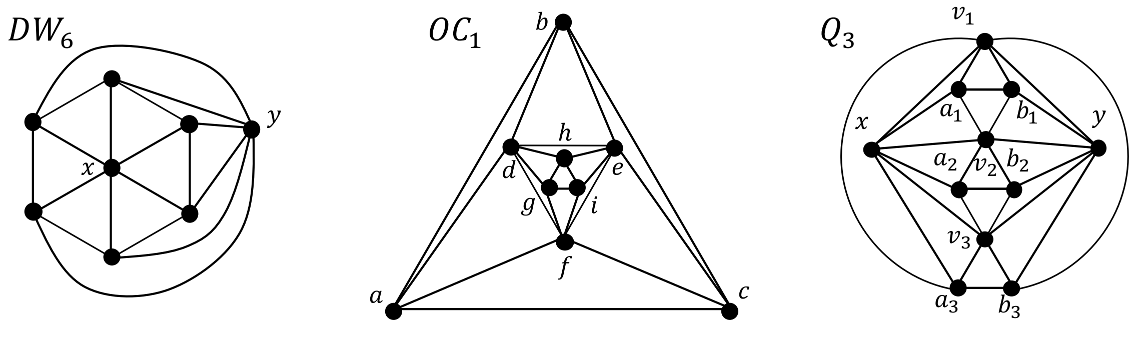

Figure 1: The double wheel DW_6 and graphs G with $\psi_3^p(G) = 3$

Though many facially-constrained colorings have been defined for general graphs on closed surfaces, we especially focus on those of triangulations on closed surfaces in this thesis. By that restriction, we can apply effectively some notions of algebraic topology to consider triad colorings. Moreover, we see the fascinating fact that there are triangulations G on closed surfaces which have no (proper) facial 3-complete n-colorings for any $n \geq \chi(G)$, in contrast to complete colorings.

This thesis consists of some chapters as follows: In Chapter 1, we prepare some terminologies of graph theory and topological graph theory. Moreover, we prepare some notions of algebraic topology for our results. In Chapter 2, we see some facts on colorings of graphs and a short survey of facially-constrained colorings. In Chapter 3, we prove Theorems 0.3 and 0.4, and consider some examples of a triad coloring for a triangulation on the torus. In Chapter 4, we introduce some results of facial complete colorings and show Theorems 0.7 and 0.8.

Chapter 1

Foundations

In this chapter, we shall give the foundations of this thesis. We confirm basic terminologies of graph theory, topology and algebraic topology. We refer to [25, 33, 34, 35, 76].

1.1 Graphs

A *graph* G is a pair of two sets $V(G)$ and $E(G)$. The elements of $V(G)$ are called *vertices* and those of $E(G)$ are called *edges*, where $E(G)$ is a set of 2-element subsets of $V(G)$. Note that an element of $E(G)$ admits multisets or 1-element subsets of $V(G)$. That is, a graph is a figure which consists of vertices and edges as shown in Figure 1.1. We use a notation $|X|$ for a set X to represent the cardinality of X.

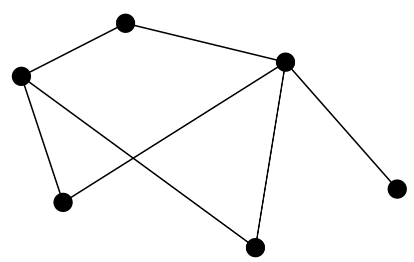

Figure 1.1: A graph

If two vertices u and v are joined by an edge, then we say that u is *adjacent* to v or that $uv \in E(G)$ is *incident* to u and v, where uv often denotes the edge which joines *endvertices* u and v. Two adjacent vertices are referred to as a *neighbor* of each other. The set of neighbors of a vertex v is called the *open neighborhood* of v (or simply the *neighborhood* of v) and is denoted by $N_G(v)$ or simply $N(v)$. If two vertices u and v are joined by two or more edges, then these edges are called *multiple edges* and a *loop* is an edge which joins one vertex to itself as shown in Figures 1.2 and 1.3, respectively. If a graph G has neither multiple edges nor loops, then G is called *simple* and we deal with a simple graph in this thesis unless otherwise stated.

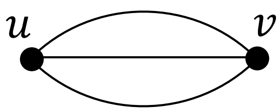

Figure 1.2: Multiple edges

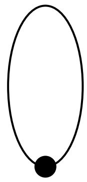

Figure 1.3: A loop

Let G_1 and G_2 be graphs. An *isomorphism* of G_1 and G_2 is a bijection $f : V(G_1) \to V(G_2)$ such that any two vertices u and v of G_1 are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent in G_2. If there exists an isomorphism of G_1 and G_2, then we say that G_1 and G_2 are *isomorphic* and it is denoted by $G_1 \cong G_2$. Roughly speaking, if each pair of adjacent vertices of G_1 is adjacent in G_2, then G_1 and G_2 are called isomorphic. Generally, an isomorphism $f : G \to G$ which carries a graph G to G itself is called an *automorphism*.

Let G and G' be two graphs consists of $V(G)$, $E(G)$ and $V(G')$, $E(G')$, respectively. If $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$, then G' is called a *subgraph* of G. For a subgraph G' of G, if $V(G') = V(G)$, then G' is called a *spanning subgraph* of G. Let G' be a subgraph of G and S be a subset of $V(G)$. If $V(G') = S$ and every edge $uv \in E(G)$ for $u, v \in S$ is in $E(G')$, then G' is called an *induced subgraph* of G or G' is *induced* by S, denoted by $\langle S \rangle$. A *subdivision* of a graph G is obtained from G by replacing edges of G with paths of length at least 1. Note that G is also a subdivision of G.

The number of edges incident to v of a graph G is called the *degree* of v and denoted by $\deg_G(v)$. In particular, if degrees of all the vertices of a graph G are r for $r \geq 1$, then G is called *r-regular*. A 3-regular graph is often called *cubic*. The *maximum degree* and the *minimum degree* of G are the maximum and minimum degree of vertices in G and denoted by $\Delta(G)$ and $\delta(G)$, respectively. The following theorem, which is well known as *Handshaking lemma*, is the fundamental and important one in Graph Theory.

Theorem 1.1. *Let G be a graph. Then* $\sum_{v \in V(G)} \deg_G(v) = 2|E(G)|$.

Moreover, we obtain the following corollary by Theorem 1.1 immediately.

Corollary 1.2. *Every graph has an even number of vertices of odd degree.*

A *walk* is a sequence of vertices $W = v_0 v_1 \dots v_n$ such that the vertices v_{j-1} and v_j are adjacent for $j = 1, \dots, n$. In particular, if a walk W has no overlap vertices, then W is called a *path* as shown in Figure 1.4. If there exists a path P between two vertices u and v, then u and v are called *connected* by P. A walk $W = v_0 \cdots v_n$ is *closed* if $v_0 = v_n$ and a closed walk W is called a *cycle* if W has no overlap vertices as shown in Figure 1.5. The *length* of a walk, a path or a cycle is the number of edges in the walk, the path or the cycle, respectively. If there are k edges in a path (resp., cycle), then we say that it is a *k-path* (resp., *k-cycle*). Moreover, a k-path (resp., k-cycle) is often denoted by P_k (resp., C_k). A cycle C is said to be a *Hamilton cycle* if C passes through each vertex of G exactly once.

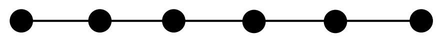

Figure 1.4: A path P_5

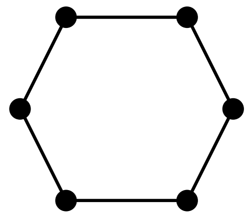

Figure 1.5: A cycle C_6

Let G be a graph and let S be a subset of $V(G)$. We write $G - S = \langle V(G) \setminus S \rangle$. In particular, if S contains only one vertex v, then we write $G - v$ simply. If each two vertices of a graph G are connected by a path, then G is called *connected*. Otherwise, G can be resolved into some connected parts and each of them are called *connected component*. A *cut set* S of a connected graph G is a set of vertices such that $G - S$ is disconnected. In particular, if a cut set of G contains only one vertex v, then v is called a *cut vertex*. If $|V(G)| > k$ and $G - S$ is connected for $S \subseteq V(G)$ with $|S| < k$, then G is called *k-connected*.

For any two vertices u and v in a graph G, the *distance* of u and v, denoted by $\mathrm{dist}_G(u, v)$, is the length of a shortest path which connects u and v. If there is no path between u and v, then we define $\mathrm{dist}_G(u, v) = \infty$. Note that $\mathrm{dist}_G(u, u) = 0$.

A set of vertices $S \subseteq V(G)$ is an *independent set* if for any vertices x and y in S, x and y are not adjacent in G. The maximum number of vertices in an independent set of G is called the *independence number*, denoted by $\alpha(G)$.

As typical graphs, we introduce some kinds of graphs. A *tree* is a connected graph which has no cycle and a *forest* is a graph whose components are trees as shown in Figure 1.6. A *complete graph* is a graph in which every distinct two vertices are adjacent as shown in Figure 1.7. A complete graph with k vertices is denoted by K_k. For a graph G, if $V(G)$ is divided into k subsets $X_1, \ldots, X_k$ for $k \geq 2$ such that any adjacent vertices of G belong to different subsets, then G is called a *k-partite graph*. In particular, if any pair of vertices belonging to different subsets of a k-partite graph G are adjacent, then G is called a *complete k-partite graph* as shown in Figure 1.8 and denoted by $K_{n_1,\ldots,n_k}$, where $n_i = |X_i|$ for $i = 1, \cdots, k$.

A 2-partite graph is often called a *bipartite graph*. For a bipartite graph, the following theorem is well known.

Theorem 1.3. *A graph G is a bipartite graph if and only if G contains no odd cycle.*

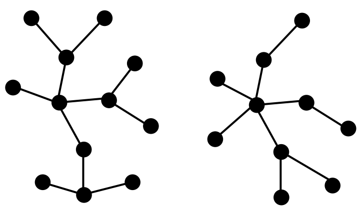

Figure 1.6: A forest

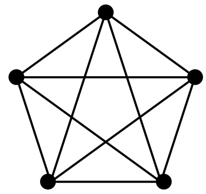

Figure 1.7: A complete graph K_5

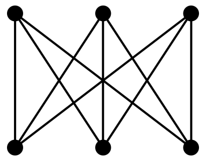

Figure 1.8: A complete bipartite graph $K_{3,3}$

Proof. If a bipartite graph has an odd cycle, then the vertices of such a cycle are not divided into two partite sets. Thus, the necessity is clear. Therefore, it suffices to show that if a graph G has no odd cycle, then G is a bipartite graph. Let G be a graph with no odd cycle. We may assume that G is connected. First, fix a vertex u of G and suppose that a vertex v is in a set X if $\mathrm{dist}_G(u,v)$ is odd and that v is in a set Y if $\mathrm{dist}_G(u,v)$ is even. Note that u is in Y since $\mathrm{dist}_G(u,u) = 0$. Since G is connected, every vertex of G is either in X or in Y. Suppose that there is a pair of two adjacent vertices x and y which are in the same set, that is, at least one of x and y is in both X and Y. By symmetry, we may assume that $\mathrm{dist}_G(u,x) \geq \mathrm{dist}_G(u,y)$. Let $P = u, u_1, u_2, \cdots, x$ be a shortest path between u and x and let $P' = u, v_1, v_2, \cdots, y$ be one between u and y. Let $W = P \cup P' \cup xy$ be a closed walk. Since the length of P and that of P' have same parity, the length of W is odd. It is easy to see that W contains an odd cycle, a contradiction. $\square$

1.2 Embeddings

Throughout this thesis, we call a connected compact 2-dimensional manifold without boundaries a *closed surface*. There are two classes of closed surfaces, *orientable* ones and *non-orientable* ones. On orientable closed surfaces, we can compatibly prescribe clockwise and counter clockwise orientations around all the points on it. On the other hand, we cannot do on non-orientable closed surfaces. For example, on a Möbius band, we cannot give compatible clockwise orientations to points on the center line of the Möbius band as shown in Figure 1.9. In fact, a closed surface is orientable if and only if it does not include a Möbius band.

Let F_1^2 and F_2^2 be two closed surfaces. The closed surface obtained from F_1^2 with a disk removed and F_2^2 with a disk removed by pasting them along their boundaries is called a *connected sum* of F_1^2 and F_2^2, denoted by $F_1^2 \# F_2^2$. We can characterize orientable and non-orientable closed surfaces, as follows. A closed surface is an *orientable closed surface of genus* $g \geq 1$, denoted by $\mathbb{S}_g$, if F^2 is homeomorphic to $\underbrace{T^2 \# \cdots \# T^2}_{g}$, where T^2 stands for the torus. Note that the sphere is regarded as a connected sum of no torus and denoted by $\mathbb{S}_0$. On the other hand, a closed surface is a *non-orientable closed surface of genus* (or

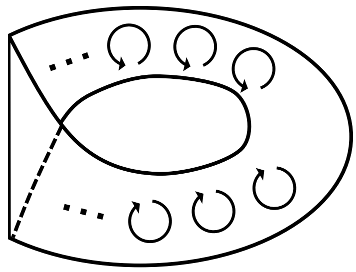

Figure 1.9: A Möbius band

cross-cap number) $k \geq 1$, denoted by $\mathbb{N}_k$, if F^2 is homeomorphic to $\underbrace{P^2\#\cdots\#P^2}_{k}$, where P^2 is the projective plane. Equivalently, $\mathbb{N}_k$ is obtained from the sphere with k pairwise disjoint disk removed by attaching k Möbius bands to each boundary of the punctured sphere. For example, $\mathbb{S}_0, \mathbb{S}_1, \mathbb{N}_1$ and $\mathbb{N}_2$ denote the sphere, the torus, the projective plane and the Klein bottle, respectively.

By the classification theorem of closed surfaces, it is known that every closed surface is homeomorphic to either an orientable closed surface or a non-orientable closed surface with some genus. For non-orientable closed surfaces, it is also known that $\mathbb{N}_3$ and $\mathbb{N}_4$ are homeomorphic to $T^2\#P^2$ and $T^2\#K^2$, respectively, where K^2 stands for the Klein bottle. In general, for any positive integer k and any even integer $0 \leq k' < k$, $\mathbb{N}_k$ is homeomorphic to $\mathbb{N}_{k-k'}\#\mathbb{S}_{\frac{k'}{2}}$.

A *closed curve* on a closed surface F^2 is a continuous function $l : S^1 \to F^2$ or its image, where S^1 is the 1-dimensional sphere, that is, $\{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$. A closed curve l is called *simple* if the function l is an injection. A simple closed curve l on a closed surface F^2 is called *separating* (resp., *non-separating*) if $F^2 - l$ is disconnected (resp., connected). A simple closed curve l on a closed surface F^2 is said to be *trivial* (or *contractible*) if l bounds a 2-cell on F^2. Otherwise, l is said to be *essential* (or *non-contractible*). Among essential simple closed curves, one with an annular neighborhood is called 2-*sided* while one whose tubular neighborhood forms a Möbius band is called 1-*sided*. Two closed curves l_1 and l_2 on a closed surface F^2 are said to be *homotopic* to each other on F^2 if there exists a continuous function $\phi : [0, 1] \times S^1 \to F^2$ such that $\phi(0, x) = l_1(x)$ and $\phi(1, x) = l_2(x)$ for each $x \in S^1$.

When we discuss embeddings of graphs into closed surfaces, we regard graphs as 1-dimensional topological spaces, not only as combinatorial objects. Roughly speaking, to *embed* a graph into a closed surface F^2 is to draw the graph on F^2 without crossing edges. It is sometimes effective to regard an embedding as an injective continuous map $f : G \to F^2$. We deal with G and $f(G)$ as the same object intuitively. However, to distinguish G from the embedded one $f(G)$, we sometimes call G an *abstract graph* while we call $f(G)$ an *embedding*. In this thesis, we often denote an embedded graph by G.

When G is embedded on a closed surface F^2, G can be regarded as a subset of F^2. Each component of $F^2 - G$ is called a *face* of G on F^2. A closed walk W (resp., cycle C) of G which bounds a face F of G is called the *boundary walk* (resp., *boundary cycle*) of F. An embedded graph G is said to be a 2-*cell embedding*, or G is said to be 2-*cell embedded* in F^2 if each face of G is homeomorphic to an *open* 2-*cell*, that is, $\{(x, y) \in R^2 | x^2 + y^2 < 1\}$. After this, we simply call 2-cell embeddings *embeddings.* An *even-(resp., odd-)embedding* on a closed surface is a graph such that each face is bounded by a cycle of length even (resp., odd). For a graph G on a closed surface F^2, we denote the face set of G by $F(G)$, and denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. Moreover, for any face (or a 2-cell region) f in a graph G on a closed surface, ∂f denotes the boundary walk of f.

Let G_1 and G_2 be two graphs on closed surfaces F_1^2 and F_2^2, respectively. Two graphs G_1 and G_2 are said to be *homeomorphic* to each other if there exists a homeomorphism $h : F_1^2 \to F_2^2$ with $h(G_1) = G_2$ which induces an isomorphism from G_1 to G_2. In this case, we also say that $G_1 \subset F_1^2$ and $G_2 \subset F_2^2$ are the same ones *up to homeomorphism.*

For a given graph G on a closed surface, the *dual graph* of G is defined as follows: A vertex is placed in each face of G and two distinct vertices are joined by an edge for each common edge on the boundaries of the two corresponding faces of G. Lastly, by deleting G, we obtain a dual graph of G.

So far, we have not referred to the orientability of closed surfaces or used *Euler's formula.* To make it explicit, the *Euler characteristic* $\varepsilon(F^2)$ of a closed surface F^2 is defined as

$$\varepsilon(F^2) = \begin{cases} 2 - 2g & (\text{if} F^2 = \mathbb{S}_g) \\ 2 - k & (\text{if} F^2 = \mathbb{N}_k) \end{cases}$$

We introduce the following theorem that is well known as "Euler's formula". (Throughout this thesis, Euler's formula means the following equation.)

Theorem 1.4. *Let G be a connected graph (might not be simple) on a closed surface F^2. Then, the following holds: $|V(G)| + |E(G)| - |F(G)| = \varepsilon(F^2)$.*

A *triangulation* G on a closed surface F^2 is a simple graph on F^2 such that every face is bounded by a cycle of length 3 and any two faces of G share at most one edge. A triangulation is *even* (or *Eulerian*) if every vertex has even degree. For an even triangulation on the sphere, the following proposition holds by using Euler's formula.

Proposition 1.5. *Let G be an even triangulation on the sphere. Then G has at least six vertices of degree* 4.

Proof. Let V, E and F be the number of the vertices, edges and faces of G. Since G is an even triangulation, it holds that $3F = 2E$. Thus, we obtain that $3V - E = 6$ by Euler's formula. Moreover, by Handshaking lemma, the equation is replaced by $6V - \sum_{v \in V(G)} \deg_G(v) = 12$. Let V_i be the number of vertices of degree i. Using such a notation, the above equation can be changed to $\sum_{i=3}^{\Delta(G)} 6V_i - \sum_{i=3}^{\Delta(G)} iV_i = 12$ since G is a simple graph. Moreover, since G is an even triangulation, the degrees of the vertices in G are even. Thus, the equation represents that $2V_4 - (2V_8 + 4V_{10} + \dots) = 12$. Clearly, V_i for any $i = 1, \dots, \Delta(G)$ is at least 0 and hence, we obtain that $2V_4 \geq 12$. Therefore, there exist at least six vertices of degree 4 in G. $\square$

In the end of this section, we introduce the following two theorems which state fundamental properties of topology of the sphere.

Theorem 1.6 (Veblen [84]). *Any simple closed curve C on the plane divides the plane into exactly two connected components, the interior and the exterior. Both of these regions have C as the boundary.*

Theorem 1.7 (Thomassen [79]). *The interior of any simple closed curve on the plane is homeomorphic to an open 2-cell.*

1.3 Algebraic topology

Let N be a set of points, possible infinitely many. An (abstract) *simplex* is a nonempty finite subset of N, in particular, if a simplex consists of $s+1$ points, then it is called an *s-dimensional simplex.* For example, 0-dimensional simplexes are regarded as vertices, 1-dimensional simplexes are edges, 2-dimensional simplexes are triangles and 3-dimensional simplexes are tetrahedrons. The *dimension* of an s-dimensional simplex is defined as s. Here, let σ be an s-dimensional simplex and let σ' be any subset of σ. In this case, σ' becomes also a simplex and σ' is called a *face* of σ denoted by $\sigma' \prec \sigma$.

An (abstract) *simplicial complex* K over N is a collection of simplexes which are formed by subsets of N such that any face of every simplex in K is also in K. If the maximum dimension of simplexes in K is equal to s, then K is called a *simplicial s-complex.* The collection of simplexes whose dimensions are at most s in K is called the *s-skeleton* of K. In particular, the 1-skeleton of any simplicial complex can be regarded as a simple graph.

Now, let K and K' be two simplicial complexes over N and N'. A map $c : N \to N'$ is said to be *simplicial* if c sends each simplex in K to a simplex in K'. In particular, if c sends any simplex in K to one in K' which has the same dimension, then c is said to be *non-degenerate*.

Suppose that the topological spaces X and X' which are exhibited by K and K'. A simplicial map $c : N \to N'$ naturally induces a continuous map $f_c : X \to X'$. If a simplicial map $c : N \to N'$ is non-degenerate, then the regions in X and X' corresponding to each simplex σ and $c(\sigma)$ are homeomorphic since their dimensions are the same. In this case, we say that a continuous map f_c is *simplicial* and *non-degenerate*, too.

Let X be a topological space. A *curve* in X is a continuous map $\gamma : I \to X$, where $I = [0, 1]$. We say that γ is a curve from the point $x = \gamma(0)$ to $x' = \gamma(1)$ and x is called an *initial point* and x' is called a *terminal point.* A curve whose initial point and terminal point are the same is called a *closed curve* and its initial point is called a *base point.*

If γ is a curve from x to x' and γ' is a curve from x' to x'', then there is a *product* curve $\gamma \cdot \gamma'$ from x to x''. A product curve is defined as follows: if $0 \le t \le \frac{1}{2}$, then $\gamma \cdot \gamma'(t) = \gamma(2t)$ and if $\frac{1}{2} \le t \le 1$, then $\gamma \cdot \gamma'(t) = \gamma'(2t-1)$. For any point $x \in X$, a *constant* curve ε_x is defined as $\varepsilon_x(t) = x$ $(0 \le t \le 1)$. If γ is a curve from x to x', then there is an *inverse* curve γ^{-1} from x' to x such that $\gamma^{-1}(t) = \gamma(1-t)$ $(0 \le t \le 1)$.

Let $\gamma, \gamma' : I \to X$ be two closed curves with a base point x. If there is a continuous map $H : I \times I \to X$ such that $H(0,t) = \gamma(t)$ $(0 \le t \le 1)$, $H(1,t) = \gamma'(t)$ $(0 \le t \le 1)$ and $H(s,0) = H(s,1) = x$ $(0 \le s \le 1)$, then γ and γ' are called *homotopic* denoted by $\gamma \simeq \gamma'$ and $H : I \times I \to X$ is called a *homotopy* between γ and γ'. Roughly speaking, if one of two closed curves can be transformed continuously into the other fixing their base point, then they are homotopic. Since the relation of homotopic is an equivalence relation, then we can consider the equivalence class of closed curves. A set of closed curves which is homotopic with γ is called a *homotopy class* of γ and denoted by $[\gamma]$.

Let X be a topological space and let x be a point in X. The *fundamental group* of X with a base point x is defined as the set of homotopy class of closed curves with a base point x and it is usually denoted by $\pi_1(X)$. Here, we define an operation of homotopy class as $[\gamma] \cdot [\gamma'] = [\gamma \cdot \gamma']$ and define identity element as $e = [\varepsilon_x]$. Then, the above operation $\cdot$ makes $\pi_1(X)$ into a group.

If there is only one homotopy class in a topological space X, that is, any closed curve can be transformed into other one, then $\pi_1(X)$ is called a *trivial group* and it is denoted by $\pi_1(X) = \{1\}$. In this case, X is said to be *simply connected.* If there is a continuous map $f : X \to X'$ between two topological spaces X and X', then it naturally induces a group homomorphism $f_\# : \pi_1(X) \to \pi_1(X')$. In the following, $f_\#$ denotes an induced group homomorphism by f.

For example, the fundamental group of the sphere $\pi_1(\mathbb{S}_0)$ is isomorphic to the trivial group $\{1\}$ since any closed curve with a base point x in $\mathbb{S}_0$ can be continuously transformed into ε_x. Here, we shall confirm the fundamental group of the projective plane, an *annulus* and a *Möbius band* for later arguments. An *annulus* is obtained by pasting side edges of a rectangle band and a *Möbius band* is obtained by twisting one of the side edges of a rectangle band by 180° and pasting them.

Figures 1.10 and 1.11 show closed curves on the projective plane, whish is obtained by identifying the antipodal points of the circumference. A closed curve in Figure 1.10 is called e and one in Figure 1.11 is called a here. Now, we shall consider $a \cdot a$. Two closed curves like a can be transformed as in Figure 1.12. Therefore, we have $a \cdot a = e$. Thus, the fundamental group of the projective plane $\pi_1(P^2)$ is generated by loops $[e]$ and $[a]$ and it is isomorphic to $\mathbb{Z}_2$. Note that e and a are not isomorphic each other.

Now, to confirm the fundamental group of an annulus and a Möbius band, we shall define a *deformation retract.* Let X and Y be two topological spaces and let $f, f' : X \to Y$ be two continuous maps. If a continuous map $F : I \times X \to Y$ satisfies the following two

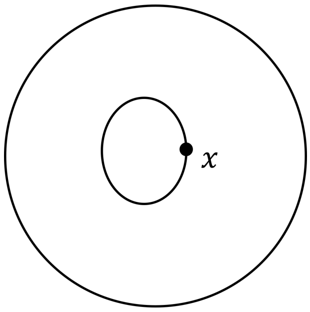

Figure 1.10: A loop e

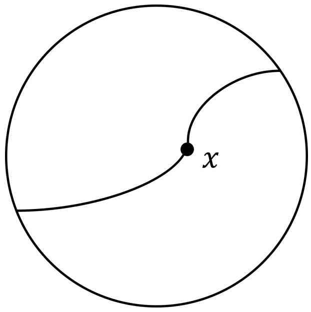

Figure 1.11: A loop a

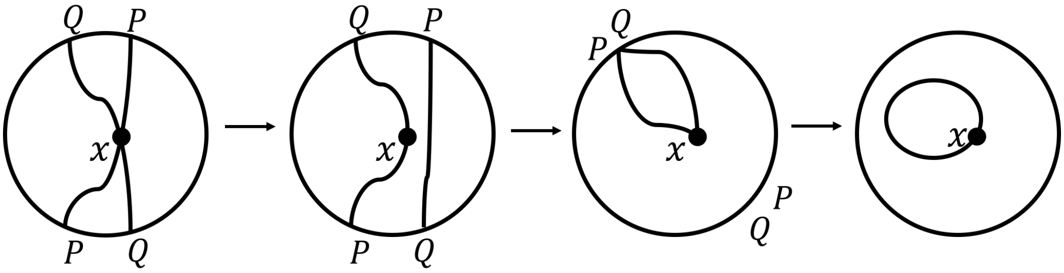

Figure 1.12: A transformation of $a \cdot a$

conditions for any $x \in X$, then F is called a *homotopy* between f and f' and we say that f and f' are *homotopic*:

(i) $F(0, x) = f(x)$.

(ii) $F(1, x) = f'(x)$.

Let A be a subset of X. If a continuous map $r : X \to A$ satisfies $r(a) = a$ for all $a \in A$ and if r is homotopic to the identity map of X, then A is called a *deformation retract*. It is known that if A is a deformation retract of X, then the fundamental group of A and that of X are isomorphic. A circle S^1 is a deformation retract of an annulus and a Möbius band, and the fundamental group of a circle is isomorphic to $\mathbb{Z}$. Therefore, the fundamental group of an annulus and a Möbius band is isomorphic to $\mathbb{Z}$, too.

Let X and $\tilde{X}$ be two topological spaces. A continuous map $p : \tilde{X} \to X$ is called a *covering projection* if there is an open neighborhood U of any point $x \in X$ such that $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$, each of which is mapped homeomorphically onto U by p. If there is a covering projection $p : \tilde{X} \to X$, then $\tilde{X}$ or $(\tilde{X}, p)$ is called a *covering space* of X. In particular, if $\tilde{X}$ is simply connected, then it is called a *universal covering*.

Let $(\tilde{X}, p)$ be a covering space of X. A continuous map $f : Y \to X$ is said to be *lifted* to $\tilde{X}$ if there is a continuous map $\tilde{f} : Y \to \tilde{X}$ such that $p\tilde{f} = f$. We call $\tilde{f}$ a *lift* of f. Here, we shall give the fact in algebraic topology called "Map Lifting Property".

Theorem 1.8 (Map Lifting Property). *Let X and Y be two topological spaces and let $(\tilde{X}, p)$ be a covering space of X. Suppose that f be a continuous map. Points $x \in X, \tilde{x} \in \tilde{X}, y \in Y$ hold $p(\tilde{x}) = f(y) = x$. A continuous map $f : Y \to X$ can be lifted to a covering space $\tilde{X}$ if and only if $f_{\#}(\pi_1(Y)) < p_{\#}(\pi_1(\tilde{X}))$, up to conjugate.*

Chapter 2

Facially-constrained colorings

A *facially-constrained coloring* of a graph G on a closed surface is a coloring with some additional restriction concerning faces of G. In this section, we first confirm some notations for colorings of graphs and introduce a short survey of facially-constrained colorings of graphs on closed surfaces. For more details of facially-constrained colorings, we refer to a survey [22].

2.1 Colorings

Let G be a graph. A map $c : V(G) \to \{1, 2, \cdots, k\}$ is called a *vertex k-coloring*. In particular, a vertex coloring c is *proper* if $c(u) \neq c(v)$ for any vertices $u, v \in V(G)$ such that $uv \in E(G)$. In what follows, "a coloring" is implied a proper vertex coloring unless otherwise stated. If G has a k-coloring, then we say that G is *k-colorable*. The *chromatic number* of G denoted by $\chi(G)$ is the minimum number k such that G is k-colorable. We often call a graph G with $\chi(G) = k$ a *k-chromatic graph*.

It is clear that $\chi(G) \leq n$, where n is the number of vertices of G. Moreover, by using the induction on the number of vertices in G, we can easily prove that $\chi(G) \leq \Delta(G) + 1$. Brooks [16] showed that almost all graphs G are $\Delta(G)$-colorable as follows.

Theorem 2.1 (Brooks [16])**.** *Let G be a connected graph other than a cycle with odd length or a complete graph. Then, $\chi(G) \leq \Delta(G)$ holds.*

On the other hands, an edge coloring has been considered as the same as vertex one. A map $c' : E(G) \to \{1, 2, \cdots, k\}$ is called an *k-edge-coloring*. In particular, an edge coloring c' is called a *proper* if $c'(e_1) \neq c'(e_2)$ for any $e_1, e_2 \in E(G)$ such that e_1 and e_2 are incident to the same vertex. The minimum number k such that G has a proper k-edge-coloring is called the *chromatic index* denoted by $\chi'(G)$.

By the definition of a proper edge coloring, it is clear that $\Delta(G) \leq \chi'(G)$. Moreover, Vizing [85] showed the upper bound of $\chi'(G)$ as follows.

Theorem 2.2 (Vizing [85])**.** *Let G be a graph. Then, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ holds.*

If $\chi'(G) = \Delta(G)$, then we say that G is in *class* 1 and if $\chi'(G) = \Delta(G) + 1$, then we say that G is in *class* 2. By Theorem 2.2, every graph is classified into either class 1 or class 2. In particular, a 3-regular graph G which is in class 2 is often called a *snark*.

2.2 Rainbow coloring

A face of a graph on a closed surface is *rainbow* if all vertices of its boundary walk have different colors. A *rainbow coloring* (originally called a *cyclic coloring*) of a graph on a closed surface is a coloring such that all faces are rainbow as shown in Figure 2.1, where Figure 2.1 shows a rainbow coloring with 6 colors. The minimum number of colors which are used in a rainbow coloring of a graph G called the *rainbowness* of G denoted by $\mathrm{rb}(G)$.

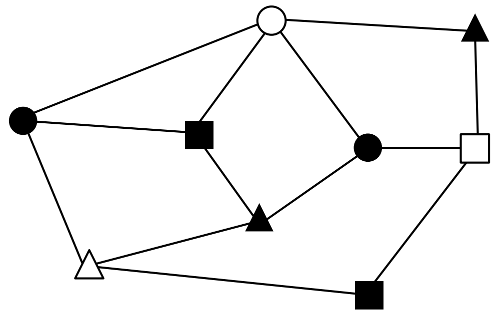

Figure 2.1: A rainbow coloring of a plane graph

A rainbow coloring was introduced by Ore and Plummer [68]. If a graph G on a closed surface is 2-connected, then it is clear that $\mathrm{rb}(G) \geq \Delta^*(G)$, where $\Delta^*(G)$ is the longest length of a boundary cycle in G. Borodin [11] conjectured that for every 2-connected plane graph G, $\mathrm{rb}(G) \leq \lfloor \frac{3\Delta^*(G)}{2} \rfloor$. Sanders and Zhao [74] showed $\mathrm{rb}(G) \leq \lceil \frac{5\Delta^*(G)}{3} \rceil$ for every 2-connected plane graph G and this value is the currently best known result. For small value of $\Delta^*(G)$, there are results for the conjecture as follows. If $\Delta^*(G) = 3$, then the conjecture is for plane triangulations and hence, the conjecture holds by Four Color Theorem. Borodin [11, 12] showed that $\mathrm{rb}(G) \leq 6$ if $\Delta^*(G) = 4$ and Hebdige and Král' [42] proved that $\mathrm{rb}(G) \leq 9$ if $\Delta^*(G) = 6$. Moreover, when $\Delta^*(G) = 5$ and 7, $\mathrm{rb}(G)$ is at most 8 [13] and 11 [41] are shown, respectively.

On the other hand, it is conjectured in [69] that every 3-connected plane graph has a rainbow coloring with at most $\Delta^*(G)+2$ colors. This conjecture is true when $\Delta^*(G) = 3, 4$ or $\Delta^*(G) \geq 18$ [46, 47, 48]. Moreover, if $\Delta^*(G) \geq 60$ or vertices of all faces whose length of the boundary cycle is four or more are mutually disjoint, then $\mathrm{rb}(G) \leq \Delta^*(G) + 1$ [28, 8], that is, the conjecture is strengthen in such a supposition. However, for every 3-connected plane graph, the best known upper bound is $\mathrm{rb}(G) \leq \Delta^*(G) + 5$ in general [27].

There are some studies for graphs on closed surface other than plane ones. Schumacher [75] showed that for a graph on the projective plane with $\Delta^*(G) = 4$, $\mathrm{rb}(G) \leq 7$. Borodin et al. [13] proved that $\mathrm{rb}(G)$ is at most 8 for a graph on the projective plane with $\Delta^*(G) = 5$. Moreover, Nakamoto et al. [61] showed the necessary and sufficient condition for a graph G on closed surface with $\Delta^*(G) \leq 4$ to have a cyclic 4-coloring.

2.3 Antirainbow coloring

An *antirainbow coloring* is a coloring of a graph on a closed surface, which is not necessarily proper, such that no face in the graph is rainbow as shown in Figure 2.2, which shows an antirainbow coloring with 5 colors. The maximum number of colors which are used in an antirainbow coloring of a graph G on a closed surface is called the *antirainbowness* denoted by $\mathrm{arb}(G)$. The length of the shortest boundary walk of a face in G on a closed surface is called the *girth* denoted by $g(G)$.

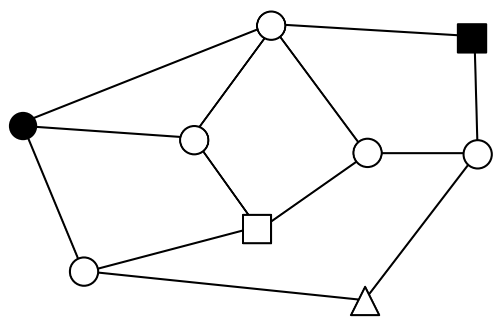

Figure 2.2: An antirainbow coloring of a plane graph

Ramamurthi and West [70, 71] showed that $\mathrm{arb}(G) \geq \alpha(G) + 1$ for every plane graph G with $\alpha(G) \leq |V(G)| - 1$, where $\alpha(G)$ is the independence number of G. Since the set of vertices colored by the same color for a coloring of a graph is an independent set, we obtain that $\mathrm{arb}(G) \geq \lceil \frac{|V(G)|}{4} \rceil + 1$ for every plane graph G by Four Color Theorem and $\mathrm{arb}(G) \geq \lceil \frac{|V(G)|}{3} \rceil + 1$ for every plane graph G with $g(G) \geq 4$ by Grötzsch's Theorem. Jungič, Král' and Škrekovski proved that for every plane graph G with $g(G) \geq 5$, if $g(G)$ is odd, then $\mathrm{arb}(G) \geq \lceil \frac{g(G)-3}{g(G)-2} n - \frac{g(G)-7}{2(g(G)-2)} \rceil$ and if $g(G)$ is even, then $\mathrm{arb}(G) \geq \lceil \frac{g(G)-3}{g(G)-2} n - \frac{g(G)-6}{2(g(G)-2)} \rceil$. There are some results for the upper bound of $\mathrm{arb}(G)$, see [26].

Let G be a triangulation on a closed surface and $c : V(G) \to \{1, \ldots, k+3\}$ be a coloring of G, which is not necessarily proper, such that c is a surjection. For any assignment of c, if there exists a rainbow face in G, then G is called *k-loosely tight* and the minimum number k such that G is k-loosely tight is called the *looseness* of G denoted by $\xi(G)$. It holds that $\mathrm{arb}(G) = \xi(G) - 1$ and the looseness is introduced in [66]. The authors of the paper showed that $\xi(G) \geq \alpha(G) - 1$. Moreover, Nakamoto et al. [60] proved the nontrivial upper bound of the looseness of a triangulation G on the sphere, that is, $\xi(G) \leq \frac{11\alpha(G)-10}{6}$. They also showed the upper bound of the looseness of a triangulation on a closed surface generally in the same paper. For other studies of the looseness, see [23, 64, 78].

2.4 Polychromatic coloring

A *polychromatic n-coloring* of G on a closed surface is a coloring of G, which is not necessarily proper, such that all n colors appear on the vertices of the boundary walk of each face of G. (Figure 2.3 represents a polychromatic 4-coloring of a plane graph.) The *polychromatic number* of G is the maximum number n such that G has a polychromatic n-coloring denoted by $p(G)$.

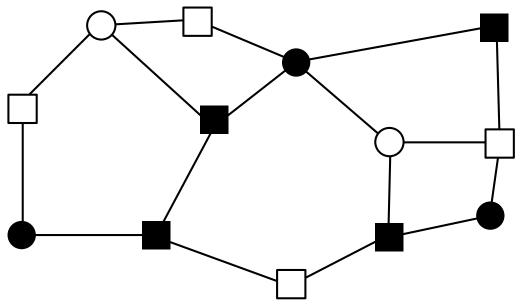

Figure 2.3: A polychromatic 4-coloring of a plane graph

A polychromatic coloring was introduced by Alon et al. [3] and it is related to *guarding problems* as follows. Guarding problems are problems which ask for a small set of vertices that "see" a given domain, for example, a polygon, a plane graph and so on. Guarding problems for a polygon are known as *art gallery problems* [19, 32].

In a guarding problem for a plane graph G, we want to know the smallest set $S \subseteq V(G)$ such that every face is incident to at least one of the vertices in S. Such a set is called a *guarding set* of G and the minimum number of vertices which are in a guarding set of G is called the *guarding number* denoted by $\mathrm{guard}(G)$ in this thesis. Observe that each *color class*, which is a set of vertices colored by the same color, of a polychromatic coloring becomes a guarding set. That is, $\mathrm{guard}(G) \leq \frac{n}{p(G)}$ holds, where n is the number of vertices in G.

It is NP-hard to determine whether $p(G) \geq 3$ for a graph G in general [3] and there are many studies about polychromatic colorings of plane graphs. In 1969, Lovász [58] showed that $p(G) \geq 2$ for any plane graph G. Clearly, $p(G) \leq g(G)$ holds. Alon et al. [3] proved that $p(G) \geq \lfloor \frac{3g(G)-5}{4} \rfloor$. Bose et al. [15] proved that every plane graph G with $g(G) \geq 3$ has a polychromatic 2-coloring. Though they had proved it by using Four Color Theorem, Bose et al. [14] showed it without the theorem later.

Polychromatic colorings of plane graphs with some degree conditions have been studied. Horev and Krakovski [45] proved that for a plane graph G with $\Delta(G) \geq 3$ other than K_4 and a subdivision of K_4 on five vertices, $p(G) \geq 3$. Horev et al. [44] also showed that every cubic bipartite plane graph has a polychromatic 4-coloring.

For triangulations G on the sphere, $p(G) = 3$ if and only if G is even since even triangulations on the sphere are 3-colorable [81]. Hoffman and Kriegel [43] proved that every 2-connected bipartite plane graph can be transformed into an even triangulation by adding edges only. It follows that for a 2-connected bipartite plane graph G, $p(G) \geq 3$ by 3-colorability of even triangulations on the sphere. Polychromatic colorings have been studied not only for graphs on the sphere as the above but also for those on the projective plane [55].

Chapter 3

Triad colorings

In this chapter, we consider a facially-constrained coloring of a triangulation on a closed surface called a *triad coloring*.

3.1 Foundations

3.1.1 Motivation and Definition

First, we introduce a motivation of a triad coloring, which is affected by some musical phenomenon. The interesting fact connecting music and mathematics is told in the textbook [67] as follows.

There are seven musical notes which are made by playing white keys of piano and called do, re, mi, fa, so, la and ci. We shall consider consonances constructed of three musical notes from above skipping one musical note, that is, made by piling three musical notes on a score as shown in Figure 3.1. For example, the leftmost consonance is well-known one called "do mi so".

For each consonance, we consider a triangle which has three musical notes consisting of the consonance on its corners as shown in Figure 3.2 and then we can make seven triangles. Considering to paste edges of them whose endpoints are two common musical notes, we can obtain a figure as shown in Figure 3.3. Side edges of Figure 3.3 have common musical notes and they can be pasted by twisting, that is, Figure 3.3 becomes a Möbius band.

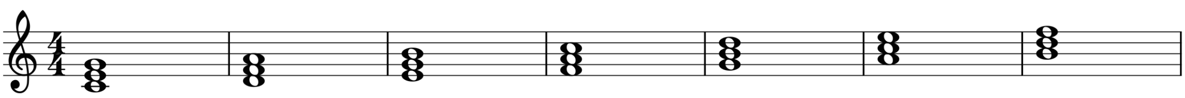

Figure 3.1: Seven consonances

The author was impressed by the above fact and considered to improve it in mathematics. If we replace musical notes do, re, mi, $\cdots$ with numbers $0, 1, 2, \cdots$ in

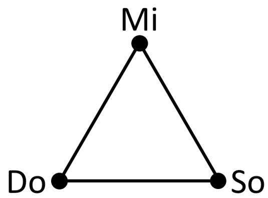

Figure 3.2: A triangle with three musical notes

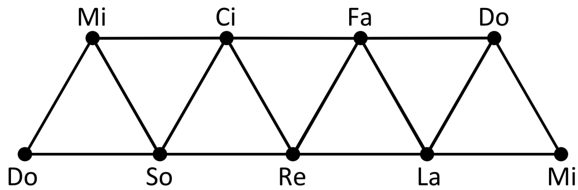

Figure 3.3: Pasted seven triangles

order, then the above musical triangles consist of three numbers $i, i+2$ and $i+4$ with modulo 7 for $i \in \{0, \cdots, 6\}$. On the other hand, if we replace musical notes with numbers $0, 4, 1, 5, 2, 6, 3$ in order, then the musical triangles consist $i \equiv i, i+1 \equiv i+1$ and $i+2 \equiv i+2 \pmod 7$ for $i \in \{0, \cdots, 6\}$. For the latter sequence of numbers, if we twice each number, then the resulting sequence is the former ones in modulo 7. Thus, each number in the latter sequence corresponds to the former one by one. By this idea, we define a facially-constrained coloring of triangulations on closed surfaces called a *triad coloring* as follows.

Let G be a triangulation on a closed surface F^2. Here we use $\mathbb{Z}_n$ $(n \geq 3)$ as the color set $\{1, \ldots, n\}$ with $n \equiv 0 \pmod n$ to define an algebraic property. Put $\mathcal{T}_n = \{\{i, i+1, i+2\} \mid i \in \mathbb{Z}_n\}$ and call it the set of *triads*. A function $c : V(G) \to \mathbb{Z}_n$ is called an *n-triad coloring* if $\{c(u), c(v), c(w)\}$ belongs to $\mathcal{T}_n$ for each face uvw of G. If G has an n-triad coloring, then G is said to be *n-triad colorable*.

Let G be a triangulation on a closed surface and define $\mathrm{TCS}(G)$ as the set of numbers n such that G has an n-triad coloring. We call it the *triad coloring set*. If a graph G is m-colorable in the ordinary sense, then G is n-colorable for any natural number $n \geq m$. However, triad colorings defined as above do not have such a property. So a natural question arises; for what number n, a triangulation is n-triad colorable if it is m-triad colorable? Thus, we would like to investigate the set of such m's, that is, the elements of $\mathrm{TCS}(G)$.

3.1.2 Observations

Here, we confirm some observations of triad colorings of triangulations on closed surfaces. For an n-triad coloring c of a triangulation G on a closed surface, since $c(u) - c(v) \not\equiv 0 \pmod n$ for any edge uv of G, the following observation is obtained clearly.

Observation 3.1. *Let G be a triangulation on a closed surface. An n-triad coloring of G is also an n-coloring for $n \geq 3$.*

Moreover, if $n = 3$ or 4, then the set of triads $\mathcal{T}_n$ contains all 3-element subsets in $\mathbb{Z}_n$ and hence, the following observation holds.

Observation 3.2. *Let G be a triangulation on a closed surface. If $n = 3$ or 4, then an n-coloring of G is equivalent to an n-triad coloring.*

If $n \geq 5$, then there are many 3-element subsets in $\mathbb{Z}_n$ not belonging to $\mathcal{T}_n$. Thus, an n-colorable triangulation is not always n-triad colorable for $n \geq 5$.

In a sense of triad coloring sets, we obtain the following observations clearly.

Observation 3.3. *Let G be a triangulation on a closed surface. If $\chi(G) = 3$, then* $\mathrm{TCS}(G) = \{3, 4, 5, \cdots\}$.

Observation 3.4. *Let G be a triangulation on a closed surface. If $\chi(G) = 4$, then* $\mathrm{TCS}(G)$ *contains* 4 *as its element.*

3.2 Triad complexes and coverings

To investigate elements of $\mathrm{TCS}(G)$ of a triangulation G on a closed surface, we prepare the special structure called a *triad complex* using some notions of algebraic topology.

We regard $\mathbb{Z}_n$ as a set of points $1, \ldots, n$ to define a simplicial 2-complex over the color set. Let $K(\mathcal{T}_n)$ denote the simplicial 2-complex induced from $\mathbb{Z}_n$ by adding all triads $\{i, i+1, i+2\} \in \mathcal{T}_n$ and their subsets, that is, their 0-dimensional simplexes and 1-dimensional simplexes. We call it the *triad complex* of $\mathcal{T}_n$. Then $K(\mathcal{T}_n)$ can be regarded as a triangulation on the closed surface obtained from all triangles having three vertices $\{i, i+1, i+2\}$ by pasting them along edges $\{i, i+1\}$ for $i \in \mathbb{Z}_n$. We call the surface the *triad space* of $\mathcal{T}_n$ and denote it by $X(\mathcal{T}_n)$. It is clear that $X(\mathcal{T}_3)$ is a triangle and that $X(\mathcal{T}_4)$ is the tetrahedron homeomorphic to the sphere. When $n \geq 5$, $X(\mathcal{T}_n)$ is homeomorphic to an annulus if n is even and to a Möbius band if n is odd.

Let G be a triangulation on a closed surface F^2. Since G is a simple graph, G itself has the structure of a simplicial 1-complex over $V(G)$. Let $K(G)$ denote the simplicial 2-complex obtained from G by adding all faces as 2-dimensional simplexes.

Suppose that G has an n-triad coloring $c : V(G) \to \mathbb{Z}_n$ for some $n \geq 3$. Then it is clear that c extends naturally to a simplicial map $c : K(G) \to K(\mathcal{T}_n)$, which is non-degenerate, that is, it sends each simplex of $K(G)$ to a simplex of the same dimension in $K(\mathcal{T}_n)$. Furthermore, c induces a continuous map $f_c : F^2 \to X(\mathcal{T}_n)$. Conversely, if we have a continuous map $f : F^2 \to X(\mathcal{T}_n)$ which induces a simplicial map between $K(G)$ and $K(\mathcal{T}_n)$, then it induces an n-triad coloring $c : V(G) \to \mathbb{Z}_n$ of G with $f = f_c$.

Now we consider a covering space of the triad space $X(\mathcal{T}_n)$, which is a pair of a topological space $\tilde{X}$ and a locally homeomorphic surjection $p : \tilde{X} \to X(\mathcal{T}_n)$. Recall that $X(\mathcal{T}_n)$ is homeomorphic to either an annulus or a Möbius band if $n \geq 5$. Then its covering space also is either an annulus or a Möbius band which winds around it several times via the projection if the space is compact. There is a unique non-compact covering space of $X(\mathcal{T}_n)$, which is a strip of infinite length homeomorphic to $\mathbb{R} \times [0, 1]$ and which winds around $X(\mathcal{T}_n)$ infinite times. We call it the *universal covering space* of $X(\mathcal{T}_n)$ and denote it by $\tilde{X}_{\{1\}}(\mathcal{T}_n)$. Note that $\pi_1(X(\mathcal{T}_n)) \cong \mathbb{Z}$ and $\pi_1(\tilde{X}_{\{1\}}(\mathcal{T}_n)) \cong \{1\}$.

Let $p : \tilde{X}_{\{1\}}(\mathcal{T}_n) \to X(\mathcal{T}_n)$ be the covering projection of the universal covering space $\tilde{X}_{\{1\}}(\mathcal{T}_n)$ on $X(\mathcal{T}_n)$. Pulling back each simplex in $K(\mathcal{T}_n)$ by p, we obtain a simplicial complex of $\tilde{X}_{\{1\}}(\mathcal{T}_n)$ which contains an infinite number of simplexes. We denote this simplicial complex by $K(\tilde{X}_{\{1\}}(\mathcal{T}_n))$. Then this can be regarded as a simplicial 2-complex over $\mathbb{Z}$ with 2-dimensional simplexes $\{i, i+1, i+2\}$ $(i \in \mathbb{Z})$. Then p works as a simplicial map from $K(\tilde{X}_{\{1\}}(\mathcal{T}_n))$ to $K(\mathcal{T}_n)$, which sends each 2-dimensional simplex $\{i, i+1, i+2\}$ $(i \in \mathbb{Z})$ to a 2-dimensional simplex $\{i, i+1, i+2\}$ $(i \in \mathbb{Z}_n)$.

3.3 Triad coloring sets of triangulations on closed surfaces

In this section, we prove Theorem 0.4 and Theorem 0.3 described in Introduction.

3.3.1 Lemmas

First, we show some lemmas for triad colorings by using triad complexes.

Lemma 3.5. *Let $n \geq 6$ be a natural number. If a triangulation G on a closed surface is n-triad colorable, then G is 4-colorable. In particular, if $n \equiv 0 \pmod 3$, then G is 3-colorable.*

Proof. Suppose that G has an n-triad coloring $c : V(G) \to \mathbb{Z}_n$ for a natural number $n \geq 6$. Consider the square of the cycle over $\mathbb{Z}_n$, say C_n^2, which is the 1-skeleton of $K(\mathcal{T}_n)$. If $n \equiv 0 \pmod 3$, then there is a 3-coloring $c_3 : \mathbb{Z}_n \to \{1, 2, 3\}$ of C_n^2. Otherwise, it is easy to find a 4-coloring $c_4 : \mathbb{Z}_n \to \{1, 2, 3, 4\}$ of C_n^2. Then the composition $c_k\, c : V(G) \to \{1, 2, \ldots, k\}$ of two colorings c and c_k ($k = 3$ or 4) becomes a k-coloring of G since f_c sends each edge in G to an edge in C_n^2. Thus, the lemma follows. □

Lemma 3.6. *Let $n \geq 6$ be a natural number. If a triangulation G on a closed surface is n-triad colorable, then G is $(n-3)$-triad colorable.*

Proof. Suppose that G has an n-triad coloring $c : V(G) \to \mathbb{Z}_n$ for a natural number $n \geq 6$. Define a map $g_n : \mathbb{Z}_n \to \mathbb{Z}_{n-3}$ by $g_n(x) = x$ for $x = 0, 1, \ldots, n-4$, $g_n(n-3) = 0$, $g_n(n-2) = 1$ and $g_n(n-1) = 2$. Then it is easy to see that g_n induces a non-degenerate simplicial map between $K(\mathcal{T}_n)$ and $K(\mathcal{T}_{n-3})$. The composition $g_n c : V(G) \to \mathbb{Z}_{n-3}$ becomes an $(n-3)$-triad coloring of G. □

Lemma 3.7. *Let $n \geq 3$ and $m \geq 3$ be natural numbers such that m divides n. If a triangulation G on a closed surface is n-triad colorable, then G is m-triad colorable.*

Proof. Since m divides n, we can define a map $h_{n,m} : \mathbb{Z}_n \to \mathbb{Z}_m$ by $h_{n,m}(x) \equiv x \pmod m$ and $h_{n,m}$ induces a non-degenerate simplicial map from $K(\mathcal{T}_n)$ to $K(\mathcal{T}_m)$. Composing an n-triad coloring $c : V(G) \to \mathbb{Z}_n$ with this $h_{n,m}$, we obtain an m-triad coloring $h_{n,m}c : V(G) \to \mathbb{Z}_m$. □

Lemma 3.8. *If a triangulation G on a closed surface has an n-triad coloring for a natural number $n \geq 5$ and if there is a triad in $\mathcal{T}_n$ which does not appear at any face of G in the coloring, then G is 3-colorable.*

Proof. Let $c : V(G) \to \mathbb{Z}_n$ be an n-triad coloring of a triangulation G on a closed surface F^2 and suppose that the triad $\{1, 2, 3\} \in \mathcal{T}_n$ does not appear at any face of G in the triad coloring c. That is, the simplicial map $f_c : K(G) \to K(\mathcal{T}_n)$ induced by c carries any face to a triad different from $\{1, 2, 3\}$.

Suppose that there is a vertex v in G which f_c sends to the vertex 2 in $K(\mathcal{T}_n)$, that is, $f_c(v) = 2$. Let C_k be the link of v in $K(G)$, which is a cycle consisting of all neighbors of v and surrounds v on the surface. Since C_k is connected, the whole of $f_c(C_k)$ must be contained in exactly one of edges 01 and 34 in $K(\mathcal{T}_n)$. Split the vertex 2 into two distinct vertices $2'$ and $2''$ to obtain another 2-dimensional simplicial complex K', where each of $\{0, 1, 2'\}$ and $\{2'', 3, 4\}$ becomes a 2-dimensional simplex in K'. Then we can modify the simplicial map f_c to be a simplicial map $f'_c : K(G) \to K'$.

The underlying space of K' is homeomorphic to a disk and the 1-skeleton of K' has a 3-coloring $c' : V(K') \to \{1, 2, 3\}$. Pulling back this 3-coloring c' via f'_c, we obtain a 3-coloring $c' f'_c : V(G) \to \{1, 2, 3\}$ of G. □

Lemma 3.9. *Let G be a triangulation on a closed surface F^2 and suppose that G has an n-triad coloring $c : V(G) \to \mathbb{Z}_n$ for some natural number $n \geq 5$. If $f_{c\#}(\pi_1(F^2))$ is a trivial subgroup in $\pi_1(X(\mathcal{T}_n))$, then G is 3-colorable.*

Proof. Suppose that $f_{c\#}(\pi_1(F^2)) = \{1\} < \pi_1(X(\mathcal{T}_n))$ and consider the universal covering space $\tilde{X}_{\{1\}}(\mathcal{T}_n)$ of $X(\mathcal{T}_n)$. Since $\pi_1(\tilde{X}_{\{1\}}(\mathcal{T}_n))$ is trivial, we have $p_\#(\pi_1(\tilde{X}_{\{1\}}(\mathcal{T}_n)))$ must be the trivial subgroup in $\pi_1(X(\mathcal{T}_n))$, which contains $f_{c\#}(\pi_1(F^2))$. Then f_c can be lifted to $\tilde{X}_{\{1\}}(\mathcal{T}_n)$ by Map Lifting Property.

Let $\tilde{f} : F^2 \to \tilde{X}_{\{1\}}(\mathcal{T}_n)$ be the lift of f_c and let $c_3 : \mathbb{Z} \to \mathbb{Z}_3$. Since $\mathbb{Z}$ forms the vertex set of $K(\tilde{X}_{\{1\}}(\mathcal{T}_n))$ as in the previous, c_3 can be regarded as a 3-coloring of its 1-skeleton; i and $i + 1$ get two different colors and so do i and $i + 2$. Then $c_3 \tilde{f}|_{V(G)} : V(G) \to \mathbb{Z}_3$ becomes a 3-coloring of G. □

3.3.2 Proof of Theorem 0.3

We first show Theorem 0.3 to investigate the elements of $\mathrm{TCS}(G)$ for a triangulation G on the sphere and the projective plane. We recall the theorem for the readability as follows.

Theorem 0.3. *Let n be any natural number ≥ 5. A triangulation G on the sphere or the projective plane has an n-triad coloring if and only if G has a 3-coloring.*

Proof of Theorem 0.3. Let G be a triangulation on a closed surface F^2. If G has a 3-coloring $c : V(G) \to \{1, 2, 3\}$, then we can define an n-triad coloring $c_n : V(G) \to \mathbb{Z}_n$ by $c_n(v) \equiv c(v) \pmod{n}$. Thus, the sufficiency holds.

Now suppose that G has an n-triad coloring $c : V(G) \to \mathbb{Z}_n$. Then this extends to a simplicial non-degenerate map $f_c : F^2 \to X(\mathcal{T}_n)$, which induces a group homomorphism $f_{c\#} : \pi_1(F^2) \to \pi_1(X(\mathcal{T}_n))$. If F^2 is homeomorphic to the sphere, then $\pi_1(F^2)$ is trivial, and hence $f_{c\#}(\pi_1(F^2))$ must be trivial. If F^2 is homeomorphic to the projective plane, then $\pi_1(F^2) \cong \mathbb{Z}_2$ and $f_{c\#}(\pi_1(F^2))$ must be trivial since $\pi_1(X(\mathcal{T}_n)) \cong \mathbb{Z}$ has no torsion for $n \geq 5$. Therefore, G is 3-colorable by Lemma 3.9 and the necessity holds. □

By Four Color Theorem, the chromatic number of any triangulation on the sphere is equal to 3 or 4. Therefore, we can conclude the following theorem:

Theorem 3.10. *Let G be a triangulation on the sphere.*

- *If $\chi(G) = 3$, then G has an n-triad coloring for any natural number $n \geq 3$.*
- *If $\chi(G) = 4$, then G has a 4-triad coloring and no n-triad coloring for $n \neq 4$.*

There are three options for the projective plane since the chromatic numbers of its triangulations can be more than 4:

Theorem 3.11. *Let G be a triangulation on the projective plane.*

- *If $\chi(G) = 3$, then G has an n-triad coloring for any natural number $n \geq 3$.*
- *If $\chi(G) = 4$, then G has a 4-triad coloring and no n-triad coloring for $n \neq 4$.*
- *If $\chi(G) \geq 5$, then G has no n-triad coloring for any natural number $n \geq 3$.*

Proof of Theorems 3.10 *and* 3.11. Let G be a triangulation on the sphere or the projective plane. Then we have $\chi(G) = 3$ or 4 for the sphere while $3 \leq \chi(G) \leq 6$ for the projective plane [72]. The same argument as in the first paragraph of the previous proof implies that if G is 3-colorable, that is, if $\chi(G) = 3$, then G has an n-triad coloring for any natural number $n \geq 3$. On the other hand, if $\chi(G) \geq 4$, then G has no n-triad coloring for any natural number $n \geq 5$ by Theorem 0.3. If $\chi(G) = 4$, then G has a 4-coloring, which can be regarded as a 4-triad coloring. If $\chi(G) \geq 5$, then G has neither 3- nor 4-coloring and hence it has no n-triad coloring for any $n \geq 3$. The last case happens only for the projective plane. □

Theorem 0.3 implies that a 3-colorable triangulation has an n-triad coloring for some $n \geq 5$, but the n-triad coloring constructed in the proof contains only 3 colors 1, 2 and 3. Is there an n-triad coloring which contains all n colors? In fact, we can make such an n-triad coloring by "attaching an octahedron" shown as in Figure 3.4. Here let G be a triangulation on a closed surface which has an n-triad coloring. Suppose that the three vertices of a face uvw of G are colored by $i, i+1$ and $i+2$, respectively. Attach an octahedron inside the face and color the added vertices by $i+1, i+2$ and $i+3$ as in Figure 3.4. If the new color $i+3$ does not exceed n, then the resulting coloring becomes an n-triad coloring of the new triangulation. Repeating this operation, we can obtain a triangulation on the same closed surface and its n-triad coloring which contains all n colors.

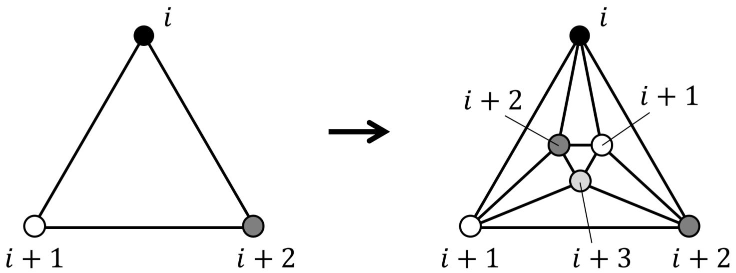

Figure 3.4: Attaching an octahedron

3.3.3 Proof of Theorem 0.4

Next, we prove Theorem 0.4. To show it, we see the following lemma.

Lemma 3.12. *Let $n \geq 4$ be a natural number. If $n \not\equiv 0 \pmod 3$, then there is a natural number $m \geq \lfloor n/2 \rfloor - 1$ such that m divides either n or $n-3$ and that $m \equiv -n \pmod 3$.*

Proof of Theorems 3.10 *and* 3.11. If n is an even number $2m$, then we have $n = 2m \equiv -m \pmod 3$ and hence $m \equiv -n \pmod 3$ and $m = n/2 > \lfloor n/2 \rfloor - 1$. On the other hand, if n is an odd number, then $n-3$ is an even number $2m$. In this case, we have $n \equiv n-3 = 2m \equiv -m \pmod 3$. Thus, $m \equiv -n \pmod 3$ and $m = (n-3)/2 = \lfloor n/2 \rfloor - 1$. □

Now we recall Theorem 0.4 and prove it.

Theorem 0.4. *Let G be a triangulation on a closed surface.*

(i) *If $\chi(G) = 3$, then $\mathrm{TCS}(G)$ consists of all natural numbers $n \geq 3$.*

(ii) *If $\chi(G) = 4$, then there exists the maximum number $n \geq 4$ with $n \not\equiv 0 \pmod 3$ such that G has an n-triad coloring. For this number n, $\mathrm{TCS}(G)$ includes all natural numbers k such that $4 \leq k \leq n$ and $k \equiv n \pmod 3$. Furthermore, if $n \geq 8$, then there is a natural number $m \geq \lfloor n/2 \rfloor - 1$ with $m \equiv -n \pmod 3$ such that $\mathrm{TCS}(G)$ includes all natural numbers k with $4 \leq k \leq m$ and $k \equiv m \pmod 3$, and $\mathrm{TCS}(G)$ includes no other numbers.*

(iii) *If $\chi(G) = 5$, then either $\mathrm{TCS}(G) = \{5\}$ or $\mathrm{TCS}(G) = \emptyset$.*

(iv) *If $\chi(G) \geq 6$, then $\mathrm{TCS}(G) = \emptyset$.*

Proof of Theorem 0.4. (i) First suppose that G has a 3-coloring $c : V(G) \to \{0, 1, 2\}$. Since $\mathbb{Z}_n$ contains 0, 1 and 2 as three distinct elements for any $n \geq 3$, the 3-coloring c can be regarded as an n-triad coloring of G. Thus, if $\chi(G) = 3$, then we have $\mathrm{TCS}(G) = \{n \in \mathbb{N} : n \geq 3\}$.

(ii) Suppose that $\chi(G) = 4$ and that G has an n-triad coloring for some $n \geq 4$. Since $K(\mathcal{T}_n)$ has exactly n 2-dimensional simplexes, if n is more than the number of faces of G, then there is a triad in $K(\mathcal{T}_n)$ which is not assigned to any face of G. By Lemma 3.8, G would be 3-colorable in this case, which is contrary to our assumption. Thus, the number of faces will be an upper bound for n.

By Lemma 3.6, if $n \geq 6$ belongs to $\mathrm{TCS}(G)$, then so does any integer $k \geq 3$ with $k \equiv n \pmod 3$. However, if $n \equiv 0 \pmod 3$, then 3 would belong to $\mathrm{TCS}(G)$ and G would be 3-colorable. Therefore, the maximum element in $\mathrm{TCS}(G)$, say n, is congruent to 1 or 2 modulo 3 and $\mathrm{TCS}(G)$ includes a series of decreasing numbers $n = n_0, n_1, \ldots \geq 4$ with $n_i \equiv n \pmod 3$.

By Lemma 3.12, there is an integer $m \geq \lfloor n/2 \rfloor - 1$ such that m divides either n or $n-3$ and that $m \equiv -n \pmod 3$. If $n \geq 8$, then we have $m \geq 4$. Since n and $n-3$ belong to $\mathrm{TCS}(G)$, so does m by Lemma 3.7. Consider the maximum of such m's. Then $\mathrm{TCS}(G)$ includes another series of decreasing numbers $m = m_0, m_1, \ldots \geq 4$ with $m_i \equiv m \pmod 3$. One of the two series of decreasing numbers ends at 4.

(iii) and (iv) By Lemma 3.5, if $\mathrm{TCS}(G)$ contains $n \geq 6$, then $\chi(G) \leq 4$. Thus, if $\chi(G) \geq 5$, then $\mathrm{TCS}(G)$ does not contain any integer $n \geq 6$. Therefore, we have $\mathrm{TCS}(G) = \{5\}$ or $= \emptyset$ in this case. □

3.3.4 Triad colorings of triangulations on the torus

Triangulations on the sphere or the projective plane whose chromatic numbers are more than 3 have no n-triad colorings for $n \neq 4$ by Theorem 0.3. However, it does not hold for other closed surfaces in general. Moreover, in the proof of Theorem 0.4 (ii), there are two series of decreasing integers $n = n_0, n_1, \ldots$ and $m = m_0, m_1, \ldots$. However, it is difficult to estimate the gap between n and m more precisely. For example, we do not know whether G has $(n-1)$- or $(n-2)$-triad colorings for the maximum element $n \in \mathrm{TCS}(G)$. In this subsection, we shall show some examples of triangulations on the torus corresponding to the above problems and each case of Theorem 0.4.

6-regular triangulations on the torus have been classified in [4] and [63], and can be described using three parameters p, q and r. We use the notation given in the latter as follows. Prepare a rectangle subdivided by $(p+1)\times(r+1)$ grid having the vertices $v_{(x,y)}$ for $x = 0, \ldots, r$ and $y = 0, \ldots p$, and add the diagonal $v_{(x,y)}v_{(x+1,y+1)}$ of slope 1 in each small square. First identify the pair of horizontal sides of length $r+1$ to obtain a cylinder having two cycles of length p at its ends. The left cycle consists of $v_{(0,0)} = v_{(0,p)}, v_{(0,1)}, \ldots, v_{(0,p-1)}$ and the right cycle has $v_{(r,0)} = v_{(r,p)}, v_{(r,1)}, \ldots, v_{(r,p-1)}$. Identify these cycles at both ends of the cylinder so that $v_{(0,y)} = v_{(r,y-q)}$ afterward, where $y - q$ is considered in modulo p. Then we obtain a 6-regular triangulation on the torus and denote it by $T(p, q, r)$. If one starts at $v_{(0,0)}$ on the left cycle and go along the path corresponding to the horizontal side toward the right cycle, then he will reach $v_{(r,0)} = v_{(0,q)}$. (See Figure 3.5 for $T(5, 3, 4)$.)

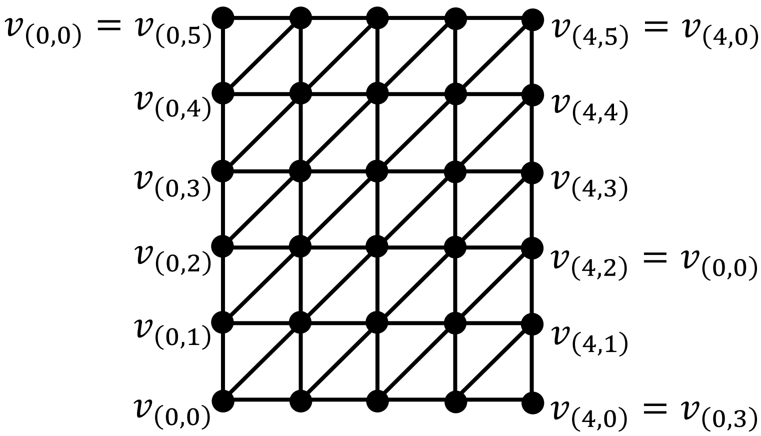

Figure 3.5: The 6-regular triangulation $T(5, 3, 4)$ on the torus

First, we show a 6-regular triangulation on the torus which is a counterexample of Theorem 0.4 for a graph on the torus. Let G be a 6-regular triangulation $T(5, 0, 5)$ on the torus as shown in Figure 3.6. The chromatic number of $\chi(T(p, q, r))$ have been already determined completely by the results in [20, 80, 86] and G is known as 4-colorable. It is easy to see that if $T(p, q, r)$ is 3-colorable, then 3 divides p. Since 3 does not divide 5, we obtain that $\chi(G) = 4$. Moreover, Figure 3.6 shows a 5-triad coloring of G. This fact means that Theorem 0.3 does not always hold on triangulations on closed surfaces other than the sphere and the projective plane.

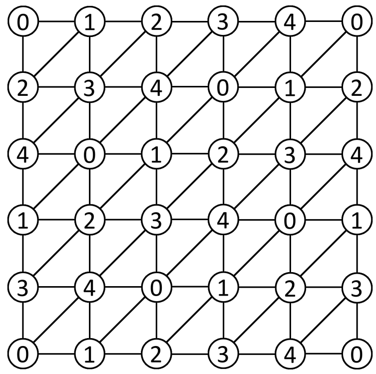

Figure 3.6: A 5-triad colorable triangulation G on the torus with $\chi(G) = 4$

To analyze triad colorings of $T(p, q, r)$, we shall prepare several lemmas. The next one will be used to give an upper bound for elements $n \in \mathrm{TCS}(T(p, q, r))$. Here, a *zigzag path*

$Z = v_0v_1 \dots v_k$ for $k \geq 1$ of a 6-regular triangulation G on the torus is a path in G such that every angle between v_i and v_{i+1} for $i = 0, \dots, k-1$ is $60°$.

Lemma 3.13. *Let G be a 6-regular triangulation on the torus and suppose that G has an n-triad coloring $c : V(G) \to \mathbb{Z}_n$ for some $n \geq 5$. If the two ends of a zigzag path $Z = w_0w_1 \cdots w_k$ of length $k < n$ get the same color, then $k \equiv 0 \pmod 3$.*

Proof. Consider the non-degenerate simplicial map $f_c : K(G) \to K(\mathcal{T}_n)$ induced by an n-triad coloring $c : V(G) \to \mathbb{Z}_n$ and suppose that $c(w_0) = c(w_k) = 0$. Since $k < n$, the path $f_c(Z)$ in $K(\mathcal{T}_n)$ cannot reach the vertex $0 \in \mathbb{Z}_n$, going around the annulus or the Möbius band. It follows that we can trace the image of $f_c(Z)$ as a walk Z' in the zigzag ladder complex K' over $[-k, k] = \{-k, -k+1, \dots, -1, 0, 1, \dots, k\}$ with triangles $\Delta_i = \{i, i+1, i+2\}$. That is, the natural projection $p : [-k, k] \to \mathbb{Z}_n$ maps Z' onto $f_c(Z)$ in $K(\mathcal{T}_n)$. The walk Z' in K' starts at the vertex 0 and comes back to 0. This may not be a zigzag path, but turns $60°$ at each corner if we draw each face Δ_i as an equilateral triangle.

Fold the end of the zigzag ladder complex K' with a crease $\{k-1, k-2\}$ to obtain a shorter zigzag ladder complex K'' with the most right triangle Δ_{k-2} missing. Then Z' will be deformed into a similar walk in K''. Repeat such folding of the zigzag ladder complex at both ends until only one triangle $\Delta_{-1} = \{-1, 0, 1\}$ remains. Then Δ_{-1} includes a walk of length k starting and ending at 0 as a trace of $f_c(Z)$ and the walk must turn $60°$ at each corner of Δ_{-1}. This implies that it goes around the triangle $k/3$ times in one direction. Therefore we have $k \equiv 0 \pmod 3$. □

Lemma 3.14. *Let G be a 6-regular triangulation on the torus and suppose that G has an n-triad coloring for a natural number $n \geq 5$. For each triad which appears on vertices of the boundary walk of faces in G, it appears at least four times in G.*

Proof. Let $c : V(G) \to \mathbb{Z}_n$ be an n-triad coloring and take a face $u_0v_0w_0$ of G which gets a triad $\{1, 2, 3\} \in \mathcal{T}_n$. We may assume that $c(u_0) = 1$, $c(v_0) = 2$ and $c(w_0) = 3$. The face $u_0v_0w_0$ is surrounded by twelve faces and by a cycle of length 9. Let $v_1u_2w_3u_1w_2v_3w_1v_2u_3$ be the cycle and let $u_0v_0w_2$, $v_0w_0u_2$ and $w_0u_0v_2$ be the three faces adjacent to $u_0v_0w_0$. Then the remaining nine faces will be automatically labeled as $u_iv_jw_k$.

First look at $u_0w_0v_2$. Since $\{1, 2, 3\}$ is the unique triad in $\mathcal{T}_n$ containing $\{1, 3\}$ if $n \geq 5$, the face $u_0w_0v_2$ necessarily gets the triad $\{1, 2, 3\}$. Thus, it suffices to find two more faces which have $\{1, 2, 3\}$ in the triad coloring c.

There are only two triads in $\mathcal{T}_n$ containing $\{1, 2\}$, namely $\{0, 1, 2\}$ and $\{1, 2, 3\}$. This implies that either $c(w_2) = 0$ or 3. If $c(w_2) = 3$, then $u_0v_0w_2$ gets the triad $\{1, 2, 3\}$ and so does $u_0w_2v_3$; we found two in this case. Thus we may assume that $c(w_2) = 0$ and conclude that $w_2u_1v_0$ gets the triad $\{0, 1, 2\}$ and $c(u_1) = 1$.

Similarly, if $u_2v_0w_0$ gets the triad $\{1, 2, 3\}$, then so does $u_2v_1w_0$. Otherwise, $v_0w_0u_2$ gets the triad $\{2, 3, 4\}$ and we have $c(w_3) = 3$. Therefore, we found the third face $u_1v_0w_3$ having $\{1, 2, 3\}$ with $c(u_1) = 1$ and $c(w_3) = 3$.

Carrying out the same argument for the face $u_1v_0w_3$ as for $u_0v_0w_0$, we conclude that if we never found two more $\{1,2,3\}$, then the face meeting u_1w_3 different from $u_1v_0w_3$ gets the triad $\{1,2,3\}$. This is the fourth face having the triad $\{1,2,3\}$. The simpleness of G guarantees that the four faces we found are all different. $\square$

Theorem 3.15. *If $T(p,q,r)$ is not 3-colorable, then the maximum element in* $\mathrm{TCS}(T(p,q,r))$ *does not exceed $pr/2$.*

Proof. The 6-regular triangulation $T(p,q,r)$ on the torus has exactly $2pr$ faces. If it has an n-triad coloring and is not 3-colorable, then any triad in $\mathcal{T}_n$ appears at four or more faces by Lemmas 3.8 and 3.14. This implies that $4n \leq 2pr$ and hence $n \leq pr/2$. $\square$

In what follows, we show some examples of 6-regular triangulations $T(p,q,r)$ on the torus to see various patterns of $\mathrm{TCS}(T(p,q,r))$.

Example 1. If $p \equiv 0$ and $r+q \equiv 0 \pmod 3$, then $\chi(T(p,q,r)) = 3$ and $\mathrm{TCS}(T(p,q,r)) = \{n \in \mathbb{N} : n \geq 3\}$.

It is not so easy to determine $\mathrm{TCS}(T(p,q,r))$ if it is not 3-colorable in general. So we shall discuss $T(p,q,r)$ here only for a few concrete parameters (p,q,r).

Example 2. $\chi(T(5,3,4)) = 4$ and $\mathrm{TCS}(T(5,3,4)) = \{4,5,7,10\}$. The maximum element 10 in this is congruent to 1 modulo 3 and it includes two series of decreasing numbers $10,7,4$ and 5.

Proof. Assign $0,2,4,6$ and 8 to $v_{(0,0)},\ldots,v_{(0,4)}$, $1,3,5,7$ and 9 to $v_{(1,1)},\ldots,v_{(1,5)}$, $0,2,4,6$ and 8 to $v_{(2,1)},\ldots,v_{(2,5)}$ and $1,3,5,7$ and 9 to $v_{(3,2)},\ldots,v_{(3,1)}$ in order. This assignment extends naturally to a 10-triad coloring of $T(5,3,4)$ and hence $\mathrm{TCS}(T(5,3,4))$ contains the decreasing numbers 10, 7 and 4 by Lemma 3.6 and 5 by Lemma 3.7. Since $T(5,3,4)$ is not 3-colorable and $5 \times 4/2 = 10$, it does not have any n-triad coloring for $n > 10$ by Theorem 3.15. Thus, it suffices to show that 8 does not belong to $\mathrm{TCS}(T(5,3,4))$.

Suppose that $T(5,3,4)$ has an 8-triad coloring. Since it has exactly 20 vertices, at least one of eight colors must be used for more than two vertices. Assume that color 0 is such a color. Since it is vertex-transitive, we may assume that $v_{(2,2)}$ gets color 0. Then the six neighbors of $v_{(2,2)}$ cannot get color 0. Furthermore, Lemma 3.13 forbids all vertices, except $v_{(2,2)}$ and $v_{(0,1)}$, to have color 0. Thus, there are at most two vertices having color 0. However, this is contrary to our assumption on color 0. Therefore, $T(5,3,4)$ does not have an 8-triad coloring. $\square$

Example 3. $\chi(T(22,16,1)) = 4$ and $\mathrm{TCS}(T(22,16,1)) = \{4,5,8,11\}$. The maximum element 11 in this is congruent to 2 modulo 3 and it includes two series of decreasing numbers $\{11,8,5\}$ and $\{4\}$.

Proof. The 6-regular triangulation $T(22, 16, 1)$ on the torus has a hamilton cycle, namely $v_{(0,0)}v_{(0,1)} \cdots v_{(0,21)}$ and we can assign $0, 2, \dots, 10, 1, 3, \dots, 9$ twice to the vertices along this hamilton cycle to obtain an 11-triad coloring. Since $T(22, 16, 1)$ is not 3-colorable and $22 \times 1/2 = 11$, the maximum element in $\mathrm{TCS}(T(22, 16, 1))$ is 11 by Theorem 3.15. It follows that $\mathrm{TCS}(T(22, 16, 1))$ contains 11, 8 and 5. Furthermore, $T(22, 16, 1)$ has a 4-coloring. Since any 4-coloring can be regarded as a 4-triad coloring, $\mathrm{TCS}(22, 16, 1) \supset \{4, 5, 8, 11\}$. Thus, it suffices to show that it does not contain 7, which implies that it does not contain 10 by Lemma 3.6.

Suppose that there is a 7-triad coloring $c : V(T(22, 16, 1)) \to \mathbb{Z}_7$. Since $22/7 > 3$, at least one color, say color 0, must appear at four or more vertices. Choose two vertices having color 0 to minimize the distance between them along the hamilton cycle. Since $22/4 < 6$, we may assume that $c(v_{(0,0)}) = c(v_{(0,t)}) = 0$ and $t \le 5$. However, we must have $t = 3$ by Lemma 3.13 while there is a zigzag path of length 5 between $v_{(0,3)}$ and $v_{(1,6)}$. The latter is contrary to Lemma 3.13 since $v_{(1,6)} = v_{(0,0)}$ must have color 0. Therefore, $T(22, 16, 1)$ does not have any 7-triad coloring. □

Example 4. $\chi(T(14, 10, 1)) = 4$ and $\mathrm{TCS}(T(14, 10, 1)) = \{4, 7\}$. This contains only two congruent integers modulo 3 and hence the condition of $n \ge 8$ in Theorem 0.4 (ii) cannot be omitted.

Proof. The 6-regular triangulation $T(14, 10, 1)$ on the torus has the hamilton cycle $v_{(0,0)}v_{(0,1)} \cdots v_{(0,13)}$. By Theorem 3.15, the maximum element in $\mathrm{TCS}(T(14, 10, 1))$ does not exceed $14 \times 1/2 = 7$. Assign $2i \pmod 7$ to $v_{(0,i)}$. Such an assignment becomes a 7-triad coloring and hence it suffices to show that $5 \notin \mathrm{TCS}(T(14, 10, 1))$.

Suppose that $T(14, 10, 1)$ has a 5-triad coloring $c : V(T(14, 10, 1)) \to \mathbb{Z}_5$. We may assume that $c(v_{(0,0)}) = 0$. Investigating the ends of zigzag path starting at $v_{(0,0)}$ and at $v_{(1,4)} = v_{(0,0)}$, we conclude that the four vertices $v_{(0,1)}$ to $v_{(0,4)}$ cannot get color 0 by Lemma 3.13. This implies that each pair of vertices having color 0 have distance at least 5 along the hamilton cycle $v_{(0,0)}v_{(0,1)} \cdots v_{(0,13)}$ of length 14 and there are at most two vertices having color 0. This is the same for other colors and hence $T(14, 10, 1)$ would have at most 10 vertices, a contradiction. Therefore, there does not exist its 5-triad coloring. □

By the same logic using Lemma 3.13 and Theorem 3.15 as in the previous examples, we can conclude the following. Since the chromatic number of any graph on the torus does not exceed 7 by Map Color Theorem [72], there are only four cases for the chromatic numbers of triangulations on the torus, namely $\chi(G) = 3, 4$, 5, 6 and 7. By the result in [80], $T(11, 7, 1)$ is the unique 6-chromatic 6-regular graph on the torus and it has been denoted by J in [1]. Also $T(7, 2, 1)$ is the unique 7-chromatic one and is isomorphic to K_7. Both $T(10, 7, 1)$ and $T(3, 1, 3)$ contain K_5 as their subgraphs and hence they are not 4-colorable.

- $\chi(G) = 3$: $\quad \mathrm{TCS}(T(p, q, r)) = \{n \in \mathbb{N} : n \ge 3\}$.

- $\chi(G) = 4$: $\quad \mathrm{TCS}(T(4,1,3)) = \{4\}$, $\mathrm{TCS}(T(5,2,3)) = \{4,5\}$
- $\chi(G) = 5$: $\quad \mathrm{TCS}(T(10,7,1)) = \{5\}$, $\mathrm{TCS}(T(3,1,3)) = \emptyset$
- $\chi(G) = 6$: $\quad \mathrm{TCS}(T(11,7,1)) = \emptyset$
- $\chi(G) = 7$: $\quad \mathrm{TCS}(T(7,2,1)) = \emptyset$

Chapter 4

Facial complete colorings

In this chapter, we consider a facially-constrained coloring of a triangulation on a closed surface called a *facial complete coloring.*

4.1 Definition and observations

Let G be a graph on a closed surface. In this chapter, a coloring is not necessarily proper and we say a proper coloring if we assume that a coloring is proper. For a positive integer t, an n-coloring $c : V(G) \to \{1, 2, \ldots, n\}$ is a *facial t-complete n-coloring* if for any t-element subset X of n colors, there exists at least one face such that X is a subset of colors assigned to the vertices lying along its boundary walk. The *facial t-achromatic number* of G, denoted by $\psi_t(G)$, is the maximum number n such that G has a facial t-complete n-coloring. Similarly, for a proper n-coloring, a *proper facial t-complete n-coloring* and the *proper facial t-achromatic number* $\psi_t^p(G)$ are defined as well as non-proper ones.

We first give several observations for a (proper) facial complete coloring and the (proper) facial achromatic number. The following observations are trivial from the definitions.

Observation 4.1. *For any graph G on a closed surface, we have*

$$\psi_t(G) \geq \psi_t^p(G)$$

if G has a proper facial t-complete coloring.

Observation 4.2. *Let G be a graph on a closed surface and let $h(G)$ be the length of the longest boundary walk of G. If G has a facial t-complete coloring, then $t \leq h(G)$.*

Observation 4.3. *Let G be a triangulation on a closed surface. If $\chi(G) = 3$, then we have*

$$\psi_3(G) \geq \psi_3^p(G) \geq 3.$$

Note that it is well known that an even triangulation on the sphere is 3-colorable [81] and hence, $\psi_3(G) \geq \psi_3^p(G) \geq 3$ holds for such a graph.

Observation 4.4. *Let G be a triangulation on a closed surface. If $\psi_3^p(G) \geq k$, then the number of faces of G is at least $\binom{k}{3}$.*

Next, we introduce two graph families which are related to graphs shown in Figure 1. Let $v_1v_2v_3$ be a triangular face in a triangulation G with $\deg(v_1) = \deg(v_2) = \deg(v_3) = 4$. Suppose that $v_1v_2v_3$ is surrounded by a 3-cycle $v_1'v_2'v_3'$ and that $v_iv_jv_k'$ and $v_iv_j'v_k'$ are faces for $\{i, j, k\} = \{1, 2, 3\}$. The *octahedron removal* is removing v_1, v_2 and v_3 from G as shown in Figure 4.1. The inverse operation of the octahedron removal is called the *octahedron addition*. (This operation was introduced in [9].)

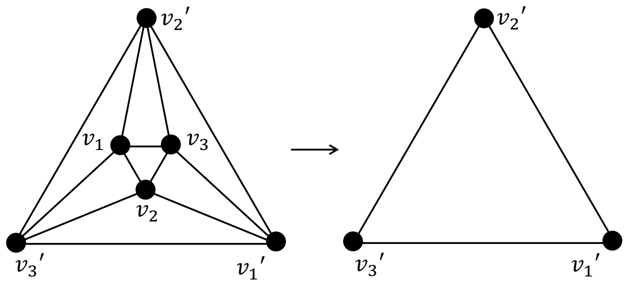

Figure 4.1: The octahedron removal

The *octahedron cylinder* OC_n is an even triangulation on the sphere which is obtained from the octahedron by repeatedly applying an octahedron addition to a face xyz with $\deg(x) = \deg(y) = \deg(z) = 4$ for $n \geq 0$ times. Note that $DW_4 = OC_0$ is the octahedron and the graph shown in the center of Figure 1 is OC_1.

Proposition 4.5. *If $n \in \{0, 1\}$, then $\psi_3^p(OC_n) = 3$.*

Proof. Since OC_0 is isomorphic to DW_4, it is easy to check that it has no proper facial 3-complete n-coloring for $n \geq 4$; note that a face which consists of three distinct colors assigned to vertices on the rim cannot appear in DW_4. (The details of the check are described in Proposition 4.8.)

Suppose that $G = OC_1$ in which vertices in G are labelled as in the center of Figure 1. Since the number of faces of G is $14 < \binom{6}{3}$, G may have a proper facial 3-complete 4- or 5-coloring. We first show that G has no proper facial 3-complete 4-coloring. Without loss of generality, we color the vertices a, b, c, d, e and f by colors $1, 2, 3, 4, 1$ and 2, respectively. (If we color a, b, c, d, e and f by colors only $1, 2$ and 3, then two of triples of 4 colors containing color 4 cannot appear.) In this case, we have two triples $\{1, 2, 3\}$ and $\{1, 2, 4\}$, but in any proper coloring of g, h and i, one of $\{1, 3, 4\}$ and $\{2, 3, 4\}$ cannot appear.

Next, we show that G has no proper facial 3-complete 5-coloring. If we color the octahedron by using five colors $1, 2, 3, 4$ and 5, then at most four kinds of triples of colors appear. (For example, color a, b, c, d, e and f by color $1, 2, 3, 4, 1$ and 5, respectively.) Since OC_1 is obtained from two octahedrons by identifying one face of each octahedron, at most eight kinds of triples of colors can appear, and hence, we cannot obtain a proper facial 3-complete 5-coloring. $\square$

The *split double wheel* Q_n for $n \geq 2$ is an even triangulation on the sphere which is obtained from a *quadrangulation* on the sphere (i.e., a graph on a closed surface such that every face is quadrilateral) whose faces are $v_1xv_2y, v_2xv_3y, \ldots, v_{n-1}xv_ny, v_nxv_1y$ by adding two adjacent vertices a_i and b_i to inside a face $v_ixv_{i+1}y$ and adding six edges $a_iv_i, a_iv_{i+1}, b_iv_i, b_iv_{i+1}, a_ix$ and b_iy; see the right of Figure 1. (The split double wheel was introduced in [51] in detail.) Note that $DW_6 = Q_2$ and the graph shown in the right of Figure 1 is Q_3. In what follows, the inside of the disk surrounded by a contractible cycle $v_1 \cdots v_k$ for $k \geq 3$ is called a *region*.

Proposition 4.6. *If* $n \in \{2, 3\}$, *then* $\psi_3^p(Q_n) = 3$.

Proof. Since Q_2 is isomorphic to DW_6, it is easy to check that it has no proper facial 3-complete n-coloring for $n \geq 4$ similar to DW_4 in Proposition 4.5. (The details of the check are described in Proposition 4.8 in Section 4.2.)

Suppose that $G = Q_3$ in which vertices in G are labelled as in the right of Figure 1. Since the number of faces of G is $18 < \binom{6}{3}$, G may have a proper facial 3-complete m-coloring for $m \in \{4, 5\}$. For the case when $m = 4$, if we color x and y by the same color 1 and v_1 by color 2, then v_2 and v_3 must be colored by color 2. Thus, each face has a vertex colored by 2, and so, we cannot obtain a proper facial 3-complete 4-coloring. On the other hand, if we color x and y by color 1 and 2, respectively, and v_1 by color 3, then v_2 and v_3 are colored by color 3 or 4. If both v_2 and v_3 are colored by 3, then we are done as the previous case. Thus, by symmetry, we assume that exactly one of v_2 and v_3, say v_2, is colored by 4. The inside of two quadrilateral regions v_1xv_2y and v_2xv_3y are colored uniquely, and hence, two triples $\{1, 2, 3\}$ and $\{1, 2, 4\}$ appear. Though the inside of v_3xv_1y is not colored uniquely, at most two of triples of colors can appear. Moreover, $\{1, 2, 3\}$ certainly appears inside of it. Since at most three triples of colors can appear in total, G has no proper facial 3-complete 4-coloring.

For $m = 5$, we consider the colors assigned to v_1, v_2 and v_3. If we color v_1, v_2 and v_3 by different three colors j, k and l for $\{j, k, l\} \subseteq \{1, 2, 3, 4, 5\}$, then a triples of colors $\{j, k, l\}$ cannot appear on any face of G. If two of v_1, v_2 and v_3 have the same color j and the other is colored by k (possibly $j = k$), then a triple which consists of three colors in $\{1, 2, 3, 4, 5\} \setminus \{j, k\}$ does not appear. Hence, G has no proper facial 3-complete 5-coloring. $\square$

On the other hand, there are triangulations which have no facial 3-complete coloring, as follows.

Proposition 4.7. *There exist infinitely many triangulations on the sphere which have no (resp., proper) facial* 3*-complete* n*-coloring for any* $n \geq 5$ *(resp.,* $n \geq 3$*).*

Proof. Consider the double wheel DW_{2n+1} for any integer $n \geq 1$. Assume that the two vertices not on the rim are colored by color i and j for $i, j \in \{1, \ldots, n\}$. (Note that i and j may be the same.) In this case, there exists no face whose vertices are all colored by colors except i and j. Therefore, for any n-coloring of DW_{2n+1} with $n \geq 5$, three colors used only on the rim cannot appear on any face. Moreover, $\chi(DW_{2n+1}) = 4$ and at least three colors need to properly color the rim, and hence, we have the same conclusion for proper colorings as above. □

Proposition 4.8. *There exist infinitely many even triangulations on the sphere which have no (resp., proper) facial* 3*-complete* n*-coloring for any* $n \geq 5$ *(resp.,* $n \geq 4$*).*

Proof. We can show the proposition similarly to Proposition 4.7 by considering the double wheel DW_{2n} for any integer $n \geq 2$. □

4.2 Proof of Theorem 0.7

It is known that every even triangulation on the sphere can be obtained from the octahedron OC_0 by repeatedly applying octahedron addition and 4-*splitting* [9]. (The 4-*contraction* is removing a vertex v with degree 4, identifying the vertices b and d and replacing the two pairs of multiple edges with two single edges as shown in Figure 4.2. The 4-*splitting* is the inverse operation of the 4-contraction.) Note that an octahedron addition and a 4-splitting do not decrease the maximum number of faces whose boundary cycles are vertex disjoint. In what follows, faces f_1 and f_2 of a graph on a closed surface are called *vertex disjoint* if vertices of the boundary walk of f_1 and those of f_2 are distinct. If an octahedron addition is applied to OC_0, then we have OC_1 and it has three faces which are vertex disjoint. On the other hand, if we apply the 4-splitting to OC_0, then the double wheel DW_6 is obtained. It is easy to check that an even triangulation on the sphere obtained from DW_{2m} for $m \geq 2$ by applying the 4-splitting has exactly three faces which are vertex disjoint or is $DW_{2(m+1)}$ by symmetry of the graph. Therefore, an even triangulation on the sphere with exactly two faces which are vertex disjoint is isomorphic to the double wheel, and hence, we obtain the following corollary.

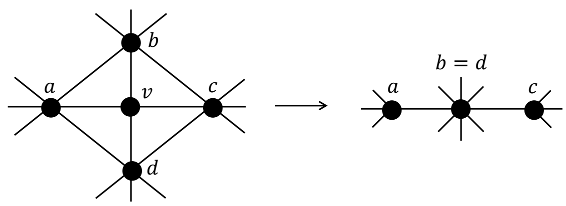

Figure 4.2: The 4-contraction

Corollary 4.9. *Let G be an even triangulation on the sphere. If G has at most two faces which are vertex disjoint, then G has no (resp., proper) facial 3-complete n-coloring for any $n \geq 5$ (resp., $n \geq 4$).*

Now we shall show Theorem 0.7.

Theorem 0.8. *Let G be an even triangulation on the sphere and k be the maximum number of faces which are vertex disjoint in G. If $k \geq 4\binom{n}{3}$, then $\psi_3^p(G) \geq n$.*

Proof of Theorem 0.7. Since G is 3-colorable, we properly assign colors $1, 2$ and 3 to the vertices of G. Let T be the set of faces which are vertex disjoint of G with $|T| = k \geq 4\binom{n}{3}$. Let G' be the graph obtained from G by contracting each face in T to a single vertex and removing the vertices of G which are not on the boundary walk of the faces in T. (Note that G' may have multiple edges and no loops.) Since G' is also planar, G' is 4-colorable by the Four Color Theorem [5], and hence, $\alpha(G') \geq \frac{k}{4} \geq \binom{n}{3}$ by assumption.

Let $S \subseteq T$ be the subset corresponding to the maximum independent set of G' and let N be $\binom{\{1,\ldots,n\}}{3}$. Since $|S| = \alpha(G') \geq \binom{n}{3}$, there exists an surjection $f : N \to S$. Thus, according to f, we assign each 3-element subset of $\{1, \ldots, n\}$ to one of the faces in S, keeping the original color of any vertex colored $1, 2$ or 3. More formally, for any element $x \in N$ which contains at least one of $1, 2$ and 3, the recoloring of the face $t = f(x)$ preserves $1, 2$ or 3 appearing on t which belongs to x. Since the vertices of faces in S are not adjacent in G, the obtained coloring is a proper facial 3-complete n-coloring. □

The order of k in Theorem 0.7 is best possible in general (the coloring is not necessarily proper), as follows: Let G be a triangulation on the sphere obtained from a double wheel DW_{2m} for $m \geq 2$ by adding an octahedron piece into each face of DW_{2m} incident to x as shown in Figure 4.3. (Figure 4.3 represents a triangulation which is obtained from DW_6.) Assume that the number of these added octahedron pieces in G is $\mathcal{O}(n^l)$ for $l < 3$ and that we color x and y by color 1 and 2, respectively. The number of faces of G which contains neither x nor y is four times the number of octahedron pieces, that is, it is $\mathcal{O}(n^l)$ for $l < 3$. Since the number of triples of colors which contain neither 1 nor 2 as its element is $\binom{n-2}{3}$ and such triples must appear on faces which include neither x nor y, at least one of such triples cannot appear on faces of G when n is sufficiently large. Thus, G has no facial 3-complete n-coloring, and the order of k in Theorem 0.7 is best possible.

By Theorem 0.7 and Corollary 4.9, we see that faces which are vertex disjoint in an even triangulation G on the sphere play an important role to construct a proper facial 3-complete coloring. However, they are not available for a proper facial 3-complete 4-coloring of 4-chromatic triangulations on the sphere.

Theorem 4.10. *For any integer $k \geq 3$, there exists a triangulation on the sphere with k faces which are vertex disjoint, which has no proper facial 3-complete 4-coloring.*

Proof. Let G be the graph shown in the left of Figure 4.4. Without loss of generality, we color the vertices a, b and c by color $1, 2$ and 3, respectively. Thus, d and e must be

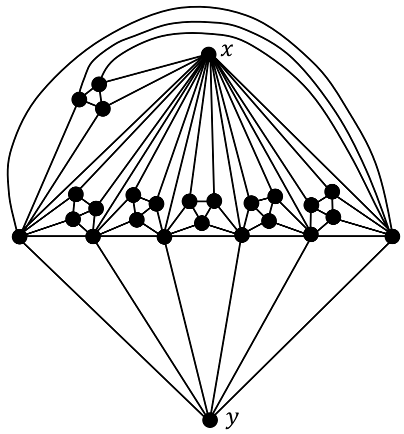

Figure 4.3: A triangulation obtained from DW_6 by adding octahedron pieces

colored by colors 4 and 2, respectively. If we color f by color 4, then we cannot obtain a proper 4-coloring of G. Thus, f must be colored by color 2, and hence, we cannot obtain a proper facial 3-complete 4-coloring since a triple of colors $\{1, 3, 4\}$ cannot appear.

Let G' be the graph shown in the right of Figure 4.4. We can obtain G' from G by repeatedly adding a copy of the rectangle region $bafc$ of G with all inner vertices and edges to the triangle region acf identifying ba with fa and bc with fc. Though G' has $\frac{n-3}{4} + 1$ faces which are vertex disjoint, where $n = |V(G')|$, we see that G' has no proper facial 3-complete 4-coloring similarly to G. □

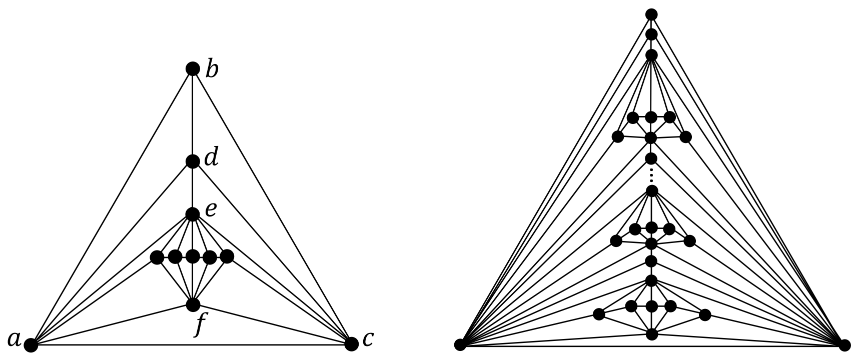

Figure 4.4: Triangulations which have no proper facial 3-complete 4-coloring

4.3 Proof of Theorem 0.8

To prove Theorem 0.8, we first prepare the following lemmas.

Lemma 4.11 (Komuro et al. [56]). *Let G be a triangulation on a closed surface with minimum degree at least 4 and H be a component of the subgraph induced by the vertices of degree 4 in G. Then one of the following holds.*

(i) H is a path $v_1, \dots, v_s$ with $s \geq 1$ and there are four other vertices forming a cycle $abcd$ of length 4 such that a and c are adjacent to all of $v_1, \dots, v_s$ and $bv_1 \cdots v_s d$ forms a path.

(ii) H is a triangle $v_1v_2v_3$ and there are three other vertices forming a cycle $a_1a_2a_3$ of length 3 such that a_i is adjacent to v_j and v_k for $\{i, j, k\} = \{1, 2, 3\}$.

(iii) H is a cycle $v_1 \cdots v_s$ with $s \geq 5$ and G is a double wheel with rim H.

(iv) $H = G$ is the octahedron.

The *diamond graph* is a complete graph K_4 minus one edge as shown in the left of Figure 4.5, denoted by K_4^-. The right of Figure 4.5 is the double wheel DW_{s+2} minus one edge from its rim, denoted by DW_{s+2}^-.

Lemma 4.12. *Let R be a quadrilateral region $abcd$ in an even triangulation G on the sphere. If R is isomorphic to neither K_4^- nor DW_n^- for any $n \geq 3$, then there exists at least one face xyz inside of $abcd$ with x, y and z being different from a, b, c and d.*

Proof. We prove the lemma by induction on the number of vertices inside of R. Suppose that R is isomorphic to neither K_4^- nor DW_n^- for any $n \geq 3$. If R contains no vertex, i.e., it has exactly two faces, then R is isomorphic to K_4^-, a contradiction. Thus, we may assume that R contains at least one vertex inside of R.

By Euler's formula, R contains a vertex of degree 4 [77, Lemma 5]. So let u be a vertex of degree 4 and let $u_1u_2u_3u_4$ anticlockwise be the link of u. If the link of u coincides with the boundary of R, then R is clearly DW_3^-, a contradiction. On the other hand, if at most one vertex of neighbors of u lies on the boundary of R, then we can find a desired face.

So we first suppose that exactly two vertices of neighbors of u lie on the boundary of R. By symmetry, if $\{u_1, u_2\} = \{a, b\}$ or $\{u_1, u_2\} = \{a, c\}$, then the face uu_3u_4 is a desired one. Thus, we may assume that $u_1 = a$ and $u_3 = c$. Let $R_1 = abcu_2$ and $R_2 = au_4cd$ be quadrilateral regions inside of R. By inductive hypothesis, each of R_1 and R_2 is isomorphic to K_4^- or DW_n^- for some $n \geq 3$. (Otherwise, we can find a desired face.) By symmetry, if $R_1 = DW_n^-$ for some $n \geq 4$, then vertices of degree 4 inside of R_1 are adjacent to both a and c, since otherwise, we can find a desired face inside of R_1. Moreover, if $R_1 = K_4^-$, then R_1 must have u_2b since G is an even triangulation. Therefore, in this case, R is isomorphic to DW_n^- for some $n \geq 5$, a contradiction.

Next suppose that exactly three vertices of neighbors of u lie on the boundary of R. By symmetry, we assume that $u_1 = a, u_3 = c$ and $u_4 = d$. Let $R' = abcu_2$ be a quadrilateral region inside of R. Similarly to the previous case, we can find a desired face or have a contradiction, by applying induction to R'.

Otherwise, i.e., we assume that $u_1 = a, u_2 = b$ and $u_4 = c$ by symmetry and we prove that the degree of u_3 cannot be even in this case. Let F be a region $u_2u_3u_4$ and suppose to the contrary that the degree of u_3 in R is even. If there is no vertex inside of F, then the degree of u_3 in R is odd, a contradiction. Thus, there exists at least one vertex inside of F and hence F forms a triangulation on the sphere. Since the degree of u_3 in R is even, that in F is odd. Since the degree of all vertices inside of F are even, one of the degrees of u_2 and that of u_4 in F is odd by the handshaking lemma. However, this is a contradiction by the fact that if a triangulation on the sphere has exactly two vertices of odd degree, then they are not adjacent [31]. Therefore, the lemma holds. $\square$

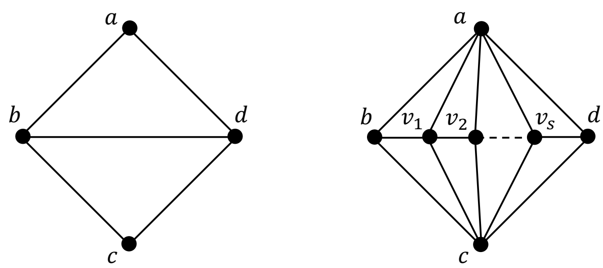

Figure 4.5: The diamond graph K_4^- and DW_{s+2}^-

Lemma 4.13. *Let G be an even triangulation on the sphere with exactly six vertices of degree 4 and such that the subgraph induced by the vertices of degree 4 in G is the union of two triangles. Then G is isomorphic to OC_n for some $n \geq 1$.*

Proof. Let $v_1v_2v_3$ and $v_4v_5v_6$ be triangular faces in G with $\deg(v_i) = 4$ for any $i \in \{1, 2, \dots, 6\}$. By Lemma 4.11, there exists a 3-cycle $v_1'v_2'v_3'$ (resp., $v_4'v_5'v_6'$) which surrounds $v_1v_2v_3$ (resp., $v_4v_5v_6$) such that $v_iv_jv_k'$ and $v_iv_j'v_k'$ are faces for $\{i, j, k\} = \{1, 2, 3\}$ (resp., $\{i, j, k\} = \{4, 5, 6\}$), where the degrees of v_i''s for each $i \in \{1, \dots 6\}$ are exactly 6, by Euler's formula and the assumption. We apply octahedron removal to $v_1v_2v_3$. After that, the degrees of v_1', v_2' and v_3' are reduced to 4 and those of other vertices do not change. That is, the number of vertices of degree 4 is still exactly six and the induced subgraph of them is the union of two triangles or is isomorphic to the octahedron. Thus, by repeating the application of octahedron removal, G can be reduced to the octahedron, that is, G is isomorphic to OC_n for some $n \geq 1$. $\square$

Now we shall prove Theorem 0.8.

Theorem 0.9. *Let G be an even triangulation on the sphere. The proper facial 3-achromatic number of G is exactly 3 if and only if G is isomorphic to the double wheel DW_{2n} for $n \geq 2$ or one of the two graphs shown in the center and the right in Figure 4.6.*

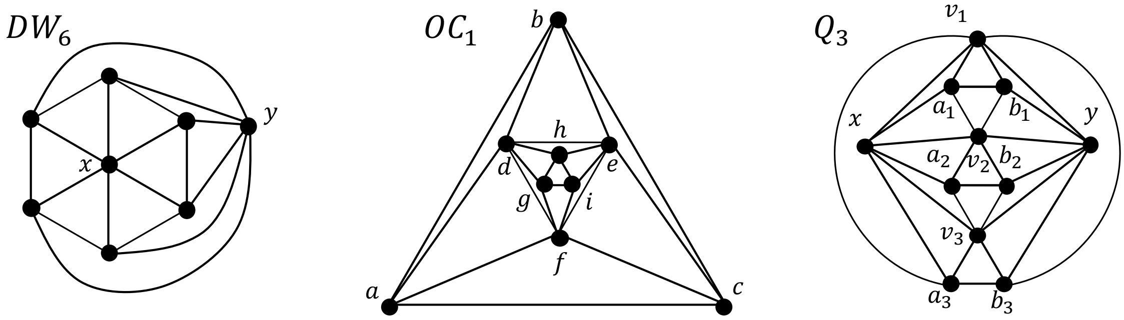

Figure 4.6: The double wheel DW_6 and graphs G with $\psi_3^p(G) = 3$

Proof of Theorem 0.8. If G is isomorphic to one of the double wheel DW_{2n} for $n \geq 2$, the octahedron cylinder OC_1 and the split double wheel Q_3, then G has no proper facial 3-complete n-coloring for $n \geq 4$ by Propositions 4.8 and 4.9, 4.5 and4.6, respectively. Hence the "if" part holds.

We shall prove the "only-if" part, that is, if G is isomorphic to none of the three exceptions, then $\psi_3^p(G) \geq 4$. Since G is an even triangulation on the sphere, G has a proper 3-coloring $f : V(G) \to \{1, 2, 3\}$. In what follows, by recoloring some vertices using four colors $\{1, 2, 3, 4\}$, we construct a proper facial 3-complete 4-coloring $f' : V(G) \to \{1, 2, 3, 4\}$.

Let $H_1, H_2, \ldots, H_k$ be components of the subgraph induced by the vertices of degree 4 in G and let $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$. Note that $k \geq 1$ since G has at least six vertices of degree 4 by Proposition 1.5. Since G is isomorphic to a double wheel, it suffices to consider that H_i is either a path or a triangle for each $i \in \{1, \ldots, k\}$, by Lemma 4.11.

Case 1. $\mathcal{H}$ contains at least three triangles.

Without loss of generality, we may suppose that $H_1 = u_1u_2u_3, H_2 = v_1v_2v_3$ and $H_3 = w_1w_2w_3$ are triangles and that $f(v_i) = f(u_i) = f(w_i) = i$ for each $i \in \{1, 2, 3\}$. Recoloring u_1, v_2 and w_3 by color 4 as shown in Figure 4.7, we obtain a desired 4-coloring f'.

Case 2. $\mathcal{H}$ has at least one path.

Suppose that H_1 is a path $v_1v_2 \ldots v_{l+1}$ of length l and $abcd$ be a cycle of length 4 of G such that a and c are adjacent to all of $v_1, \ldots, v_{l+1}$ and $bv_1 \ldots v_{l+1}d$ forms a path. Without loss of generality, we may suppose that $f(a) = f(c) = 1$ and $f(b) = 2$ ($f(d) = 2$ or 3 depending on the parity of l).

Subcase 1. $l \geq 2$.

Since G is an even triangulation on the sphere and b is not included in H_1, the degree of b is at least 6. Thus, there exists at least one vertex x with $f(x) = 1$ other than a and c, which is adjacent to b. Since such a vertex is adjacent to neither v_1 nor v_3, by

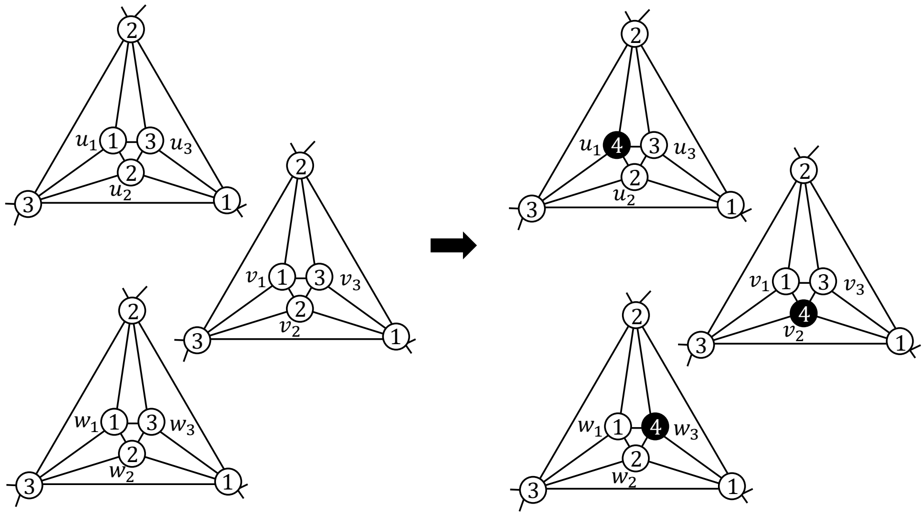

Figure 4.7: Recoloring of three triangles

recoloring v_1, v_3 and x by color 4 and v_2 by color 3 as shown in Figure 4.8, we have a desired 4-coloring f'.

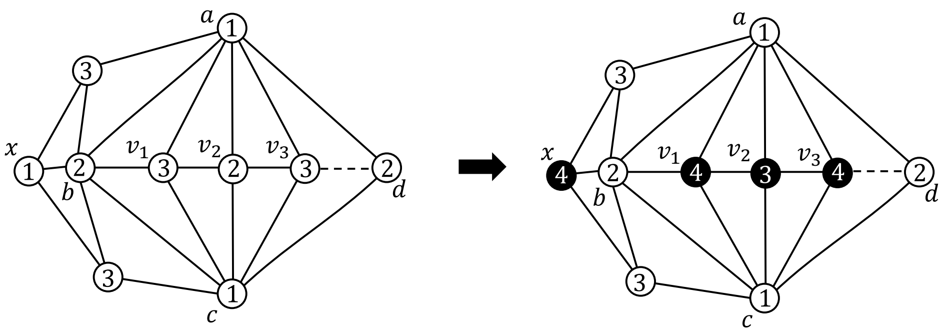

Figure 4.8: Recoloring of G when $l \geq 2$

Subcase 2. $l = 1$.

If there exists at least one vertex x with $f(x) = 1$ and $x \notin \{a, c\}$, which is adjacent to exactly one of b or d, say b, then we recolor v_1, d and x by color 4 and v_2 by color 3 similarly to Subcase 1. The resulting 4-coloring is a desired one. Otherwise, we have $\deg(b) = \deg(d)$ since all neighborhoods of b with color 1 are adjacent to d. In this case, we can represent the structure around H_1 as shown in Figure 4.9. Inside of shaded regions in Figure 4.9 are triangulated suitably.

If at least one of the shaded regions is isomorphic to DW_m^- for some $m \geq 3$, then G has a path $H_i \in \mathcal{H}$ of length at least 2, which is a degenerate case (or G has a vertex of odd degree, a contradiction). Thus, we assume that each shaded region is not isomorphic to DW_m^-.

Suppose that there exists at least one shaded region, say R, which are not isomorphic to K_4^-. In this case, we recolor v_2 by color 3, vertices which are not inside of R with color 1 by color 4, and v_1 and d by color 1. Since there exists at least one face colored by $1, 2$ and 3 in R by Lemma 4.12, we have a desired 4-coloring f'.

Now we may suppose that each shaded region is isomorphic to K_4^-. If the degrees of b and d are at least 8, then we can obtain a desired 4-coloring f' by recoloring vertices as shown in Figure 4.10. When the degrees of b and d are exactly 6, G is isomorphic to Q_3, a contradiction.

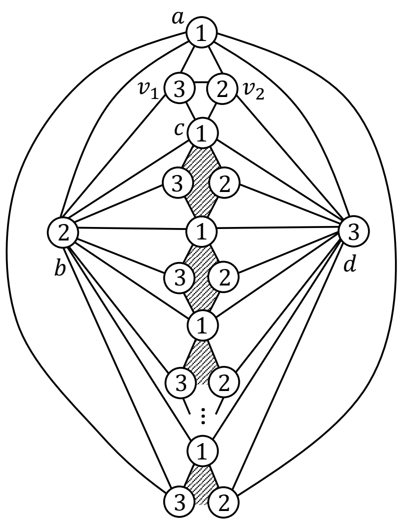

Figure 4.9: A structure around H_1

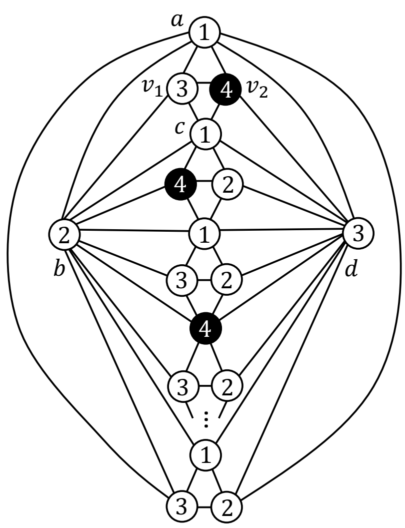

Figure 4.10: Recoloring of G when $l = 1$ and $\deg(b) = \deg(d) \geq 8$

Subcase 3. $l = 0$.

Let $N_i(v)$ be the set of the neighborhoods of a vertex v which are colored by color i. In this subcase, we consider three cases based on the relation between $N_1(b)$ and $N_1(d)$: $N_1(b) = N_1(d), N_1(b) \subsetneq N_1(d)$ (or $N_1(d) \subsetneq N_1(b)$) and otherwise.

Case (i). $N_1(b) = N_1(d)$.

Let $b_1 b_2 \cdots b_m$ be the link of b in anticlockwise order and $d_1 d_2 \cdots d_m$ be the link of d in clockwise order for $m \geq 6$, where $a = b_1 = d_1$ and $v_1 = b_m = d_m$. In this case, there are two quadrilateral regions $ab_2b_3d_2$ and $cb_{m-2}b_{m-3}d_{m-2}$. (Figure 4.11 shows the structure of around H_1 when $\deg(b) = \deg(d) = 6$.) In this case, if at least one of such quadrilateral regions is isomorphic to DW_n^- for some n, then $\mathcal{H}$ has a path of length at least 1. (Such a region cannot be K_4^- since otherwise two vertices with the same color must be adjacent.) Hence, there exists a face inside of each quadrilateral region each of whose vertices does not coincide with any of vertices on the boundary cycle by Lemma 4.12. Thus, by recoloring such a face by color $\{2, 3, 4\}$ in one quadrilateral region, and recolor around H_1 as shown in Figure 4.11, and then we obtain a desired 4-coloring f'.

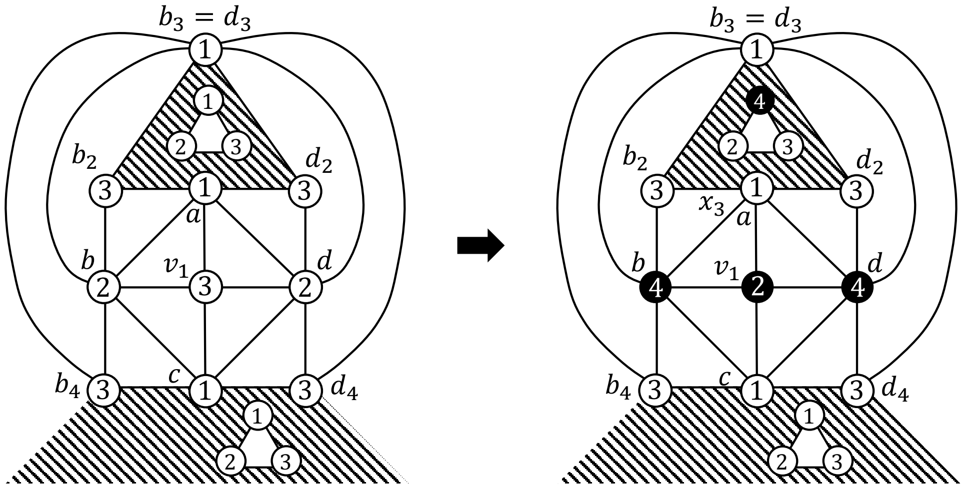

Figure 4.11: Recoloring of G when $N_1(b) = N_1(d)$

Case (ii). $N_1(b) \subsetneq N_1(d)$ **or** $N_1(d) \subsetneq N_1(b)$.

In this case, we may suppose by symmetry that $N_1(b) \subsetneq N_1(d)$. Let $b_1b_2 \cdots b_m$ for $m \geq 6$ and $d_1d_2 \cdots d_l$ for $l \geq 6$ be the links of b and d in anti-clockwise and clockwise, respectively. Suppose that $b_1 = d_1 = v_1$, $b_2 = d_2 = a$ and $b_m = d_l = c$ as shown in Figure 4.12. Since all of the degrees of a, b, c and d are at least 6 and G is on the sphere, all of the b_3, b_{m-1}, d_3 and d_{l-1} are mutually distinct. There exist vertices d_i and d_j for $4 \leq i \leq j \leq l-2$ such that $b_4 = d_i$ and $b_{m-2} = d_j$ since $N_1(b) \subsetneq N_1(d)$, $\deg(a) \geq 6$ and $\deg(c) \geq 6$. Moreover, there exist two regions R_1 and R_2 whose boundaries are $b_4b_3ad_3 \cdots d_i(= b_4)$ and $d_j \cdots d_{l-1}cb_{m-1}b_{m-2}(= d_j)$, respectively (see the left of Figure 4.12).

Let S be the set of vertices in $N_1(d) \setminus N_1(b)$ such that all vertices in S lie on the boundary of R_1. Namely, all vertices on the boundary of R_1 which are colored by color 1 other than a and b_4 are in S. We consider the following two cases.

Case (ii)-1. $S = \emptyset$.

Now R_1 is a quadrilateral region. If R_1 is isomorphic to DW_n^- or K_4^-, then we have a contradiction as in the previous case. Thus, there exists at least one face colored by $\{1, 2, 3\}$ inside of R_1 by Lemma 4.12. Moreover, since the boundary of R_2 consists of vertices with colors 1 and 3, there is a face colored by $\{1, 2, 3\}$ in R_2. Therefore, we recolor G similar to the previous case as shown in Figure 4.11 and we obtain a desired 4-coloring f'.

Case (ii)-2. $S \neq \emptyset$.

Let $N_2(S)$ be the set of neighborhoods of vertices in S, which are colored by color 2. Note that R_1 and R_2 both contain a vertex with color 2 in their interior, and in particular, all vertices in $N_2(S) \setminus \{d\}$ lie in the interior of R_1. Thus, as shown in Figure 4.12, we can recolor vertices in $S \cup \{v_1\}$ and ones in $N_2(S) \cup \{b\}$ by colors 2 and 4, respectively, preserving the color of vertices in the interior of R_2, and hence, we obtain a desired 4-coloring f'.

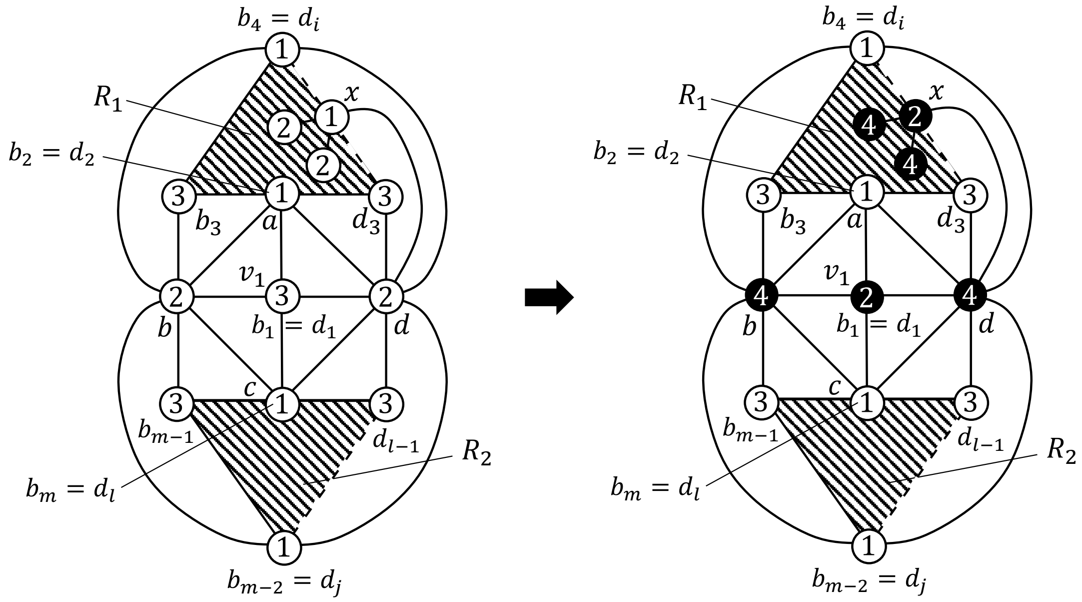

Figure 4.12: Recoloring of G when $l = 0$ with $N_1(b) \subsetneq N_1(d)$ (where $x \in S$)

Case (iii). Otherwise, i.e., $N_1(b) \not\subset N_1(d)$ **and** $N_1(d) \not\subset N_1(b)$.

In this case, there exist $u \in N_1(d) \setminus N_1(b)$ and $w \in N_1(b) \setminus N_1(d)$. Let p and q (resp., r and s) be vertices in $N_3(w)$ (resp., $N_3(u)$) which are on the boundary cycle of a face containing w and b (resp., u and d). Since $\deg(a)$ and $\deg(c)$ are at least 6, there exist such vertices.

Case (iii)-1. There is a vertex with color 2 **other than** b **and** d **which is not in** $N_2(u) \cap N_2(w)$.

Let x be a vertex with color 2 which is not in $(N_2(u) \cap N_2(w)) \cup \{b, d\}$. In this case, we have a desired 4-coloring by recoloring vertices in $N_2(u) \cup \{b\}$ by color 4, and u and v_1 by color 2 as shown in Figure 4.13. (Note that x may not be in $N_2(u) \cup N_2(w)$.)

Case (iii)-2. $N_2(u) \setminus \{d\} = N_2(w) \setminus \{b\}$ **and there is no vertex with color** 2 **other than** $N_2(u) \cup \{b\}$.

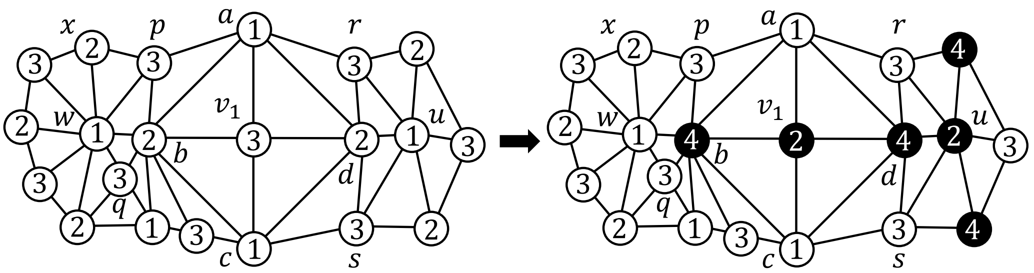

Figure 4.13: Recoloring of G when $l = 0$ (preserving the color of x)

Suppose that at least one of p, q, r and s belongs to $N_3(u) \cap N_3(w)$, say r. Let $b = w_1 \cdots w_k$ be the link of w for $k \geq 4$. If $r = w_l$ for $l \geq 4$, then w_{l-1} is colored by color 2 and not in $N_2(u) \cap N_2(w)$, which is a degenerate case. If $r = w_2$, then there exists a quadrilateral region $rbad$. Since such a region is isomorphic to neither K_4^- nor DW_m^- for any $m \geq 3$ (otherwise, $\mathcal{H}$ has a path of length at least 1), there exists a face inside of the region whose vertices do not coincide with any of r, b, a and d. Since the vertices of the face is colored by color $1, 2$ and 3, there exists a vertex colored by color 2, which is a degenerate case. Therefore, we may assume that none of p, q, r and s belongs to $N_3(u) \cap N_3(w)$. In this situation, we consider the following two cases.

Case (iii)-2-i. $N_3(u) \setminus \{r, s\} \neq N_3(w) \setminus \{p, q\}$.

In this case, there exists the vertex y_w (resp., y_u) belonging to $N_3(w)$ (resp., $N_3(u)$) but not to $N_3(u)$ (resp., $N_3(w)$) and is not any of p, q, r and s such that there is a quadrilateral region R which consists of y_u, y_w and two vertices in $N_2(u) \setminus \{d\}$ as shown in Figure 4.14. By Lemma 4.12, if R is isomorphic to neither K_4^- nor DW_m^- for any $m \geq 3$, then there exists at least one face not touching the boundary of R colored by color $1, 2$ and 3 and hence, we obtain a desired 4-coloring f' as in Figure 4.14. Otherwise, there exists a path $H_i \in \mathcal{H}$ for $i \neq 1$ of length at least 1. (The region cannot be K_4^- similarly to the first paragraph in this subcase.)

Case (iii)-2-ii. $N_3(u) \setminus \{r, s\} = N_3(w) \setminus \{p, q\}$.

If $\deg(u) = \deg(w) \geq 8$, then there exists $H_i \in \mathcal{H}$ for $i \neq 1$ which is a path (colored by colors 2 and 3) and whose length is at least 2, a degenerate case. If $\deg(u) = \deg(w) = 4$, then the degrees of p, q, r and s are at least 6 since there does not exist $H_i \in \mathcal{H}$ for $i \neq 1$ which is a path whose length is at least 1; see Figure 4.15. Thus, there exists a vertex with color 2 which is a neighborhood of p, q, r or s and not in $N_2(u) \cup N_2(w)$ by planarity, which contradicts the condition of the Case (iii)-2.

If $\deg(u) = \deg(w) = 6$, then G is isomorphic to the graph shown in the left of Figure 4.16. In this case, G has a desired 4-coloring by recoloring vertices of G as shown in Figure 4.16. (By the above argument, the degree of each of p, q, r and s is exactly 4 in

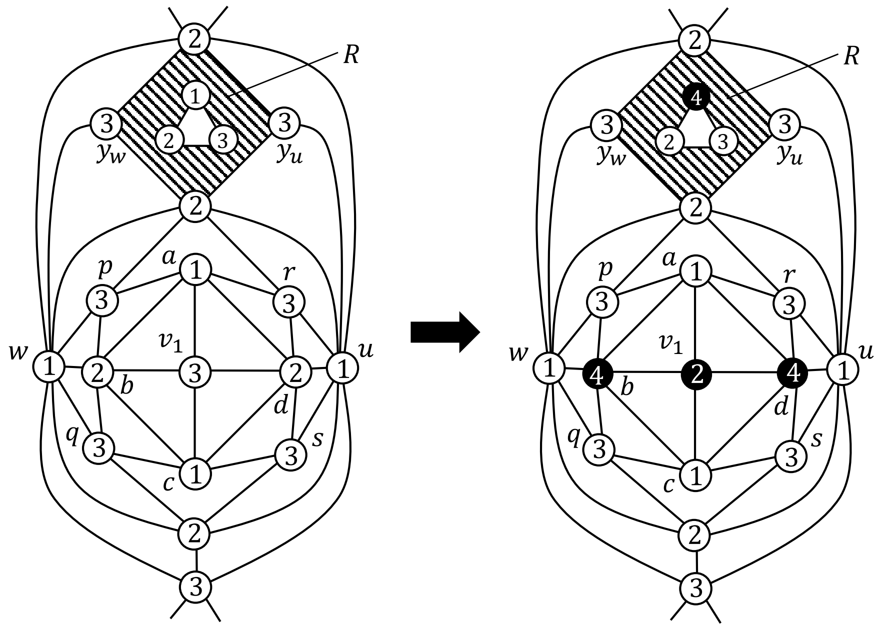

Figure 4.14: Recoloring of G when $N_3(u) \setminus \{r, s\} \neq N_3(w) \setminus \{p, q\}$

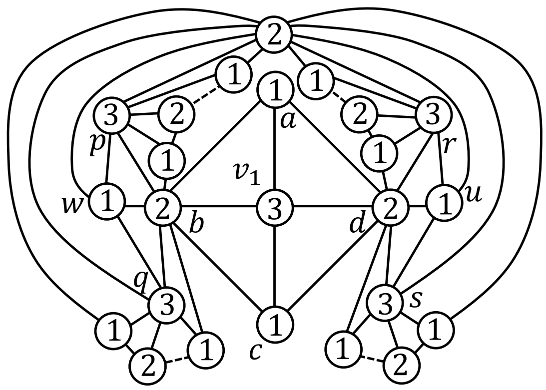

Figure 4.15: A structure of G when $N_3(u)\backslash\{r, s\} = N_3(w)\backslash\{p, q\}$ and $\deg(u) = \deg(w) = 4$

this final case since otherwise we can find a vertex x' with color 2 and $x' \notin N_2(u) \cup N_2(w)$, and so we are done by Case (iii)-1.)

Case 3. $\mathcal{H}$ consists of exactly two triangles.

We suppose that $\mathcal{H} = \{H_1, H_2\}$ and H_1 and H_2 are both triangles. In this case, $G = OC_n$ for some $n \geq 1$ by Lemma 4.13. By the assumption, we have $n \geq 2$. We can color OC_0 such that two triples of colors $\{1, 2, 3\}$ and $\{1, 2, 4\}$ appear. Since we apply the octahedron addition at least two times, we can easily see that at least one of triples $\{1, 3, 4\}$ and $\{2, 3, 4\}$ can be discovered by an octahedron addition and followed by coloring the added three vertices suitably.

In fact, we can color added three vertices by color $\{1, 3, 4\}$ for the first time and

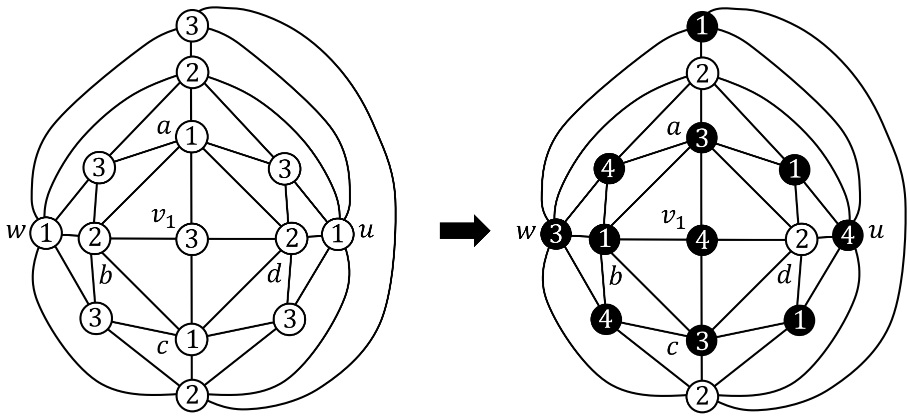

Figure 4.16: Recoloring of G when $l = 0$ with $N(u) \setminus \{d, r, s\} = N(w) \setminus \{b, p, q\}$ and $\deg(u) = \deg(w) = 6$

$\{2, 3, 4\}$ for the second time. The third time or after, by coloring added three vertices by color $\{1, 2, 3\}$, we can obtain a desired 4-coloring f'. $\square$

4.4 Hypergraphs

A *hypergraph* H is a pair (V, E) of disjoint sets, where the elements of E are non-empty subsets of V. An element in V (resp., E) of H is called a *vertex* (resp., an *edge*) the same as a graph. In particular, if every edge of H has k vertices, then H is *k-uniform*. An n-coloring of a hypergraph H is defined as an assignment of n colors to vertices of H such that not all vertices of an edge of H are colored by the same color.

Jucovič and Olejník [50] introduced a complete n-coloring of a hypergraph H as an ordinary n-coloring of H such that for every pair of colors, there exists an edge containing two vertices colored by the two colors, and the achromatic number of H denoted by $\varphi(H)$ as well as that of a graph. Moreover, they gave the upper bound of the achromatic number of hypergraphs, as follows.

Theorem 4.14 (Jucovič and Olejník [50]). *Let H be a k-uniform hypergraph with h edges. Then the inequality $\varphi(H) \leq \xi$ holds, where ξ is the positive solution of the equation $x^2 - x - h(k^2 - k) = 0$.*

Generalizing the above definition, we can define a t-achromatic number of hypergraphs as follows. A *t-complete n-coloring* of a hypergraph H if for any t-element subset X of n colors, there exists at least one edge such that X is a subset of colors assigned to the vertices in the edge. The maximum number of n such that H has a t-complete n-coloring is called the *t-achromatic number* of H and denoted by $\varphi_t(H)$. By this definition, we obtain the following theorem similarly to Theorem 4.14.

Theorem 4.15. *Let H be a k-uniform hypergraph with h edges. Then the inequality $\varphi_t(H) \leq \xi$ holds, where ξ is the positive solution of the equation $x(x-1)(x-2)\dots(x-t+1) - hk(k-1)(k-2)\dots(k-t+1) = 0$.*

Proof. Suppose that $\varphi_t(H) = n$. If H has a t-complete n-coloring, then there exist $\binom{n}{t}$ sets of colors. In one edge of H, $\binom{k}{t}$ different sets of colors can appear. Thus, we obtain that $h \geq \frac{\binom{n}{t}}{\binom{k}{t}}$. □

If a 3-uniform hypergraph H is obtained from a triangulation G on a closed surface by regarding a face of G as an edge containing three vertices in its boundary walk, then we have $\psi_3(G) = \varphi_3(H)$ by the definition of a 3-complete coloring of a hypergraph. Similarly, $\psi_3^p(G)$ is corresponded to the achromatic number of H defined by Dębski et al. [24]. An n-coloring of H is a *rainbow* if all vertices of every edge receive different colors. Dębski et al. [24] defined a complete coloring of a k-uniform hypergraph H as a rainbow coloring of H such that every k-subset of colors appears on at least one edge, and the achromatic number of H is defined in the same way as for simple graphs. Therefore, the study of various complete colorings of 3-uniform hypergraphs may help one of $\psi_3(G)$ and $\psi_3^p(G)$.

4.5 Remarks

In Section 4.2, we show that the more the number of faces which are vertex disjoint of an even triangulation G on the sphere becomes, the larger its proper facial 3-achromatic number is (Theorem 0.7). However, there exists a triangulation on the sphere which has no proper facial 3-complete coloring in general (Propositions 4.7 and 4.8). In particular, there exists a triangulation on the sphere with many faces which are vertex disjoint which has no proper facial 3-complete 4-coloring in general (Theorem 4.10).

Similarly to Theorem 0.7, we can obtain the following theorem for triangulations on closed surfaces other than the sphere. The *heawood number* of F^2, denoted by $h(F^2)$, is $\left\lfloor \frac{7+\sqrt{49-24\varepsilon(F^2)}}{2} \right\rfloor$, where $\varepsilon(F^2)$ is the Euler characteristic of F^2.

Theorem 4.16. *Let G be a proper 3-colorable triangulation on a closed surface F^2 and k be the maximum number of faces which are vertex disjoint of G. If $k \geq h(F^2)\binom{n}{3}$, then $\psi_3^p(G) \geq n$.*

Observe that the exceptions in Theorem 0.8 have at most three faces which are vertex disjoint. Therefore, the following corollary holds.

Corollary 4.17. *Let G be an even triangulation on the sphere and k be the maximum number of faces which are vertex disjoint of G. If $k \geq 4$, then $\psi_3^p(G) \geq 4$.*

In the end, we consider a kind of hereditary property of the (proper) facial achromatic number. Let G be a graph, $k \geq 0$ be an integer and $P(k)$ be some property of graphs depending on k. Then $P(k)$ is *interpolation* if either (i) or (ii) holds:

(i) For all $k \geq 1$, if G satisfies $P(k)$, then it also satisfies $P(k-1)$.

(ii) For all $k \geq 0$, if G satisfies $P(k)$, then it also satisfies $P(k+1)$.

For example, the property that the chromatic number is at most k for $k \geq 1$ has the interpolation property (which satisfies (ii)). Moreover, the property that achromatic number is at least k for $k \geq 1$ has the interpolation property, too (which satisfies (i)) [40]. (Note that the achromatic number cannot go below the chromatic number.) We see that the facial achromatic number has the interpolation property in two senses, that is, if a graph G has a facial t-complete n-coloring, then it has both a facial $(t-1)$-complete n-coloring and a facial t-complete $(n-1)$-coloring. For the proper version, the former similarly holds, however, the latter does not hold in general: Consider the graph shown in Figure 4.17. This graph has a proper facial 3-complete 5-coloring as in the figure. However, the graph has no proper facial 3-complete 4-coloring by Theorem 4.10.

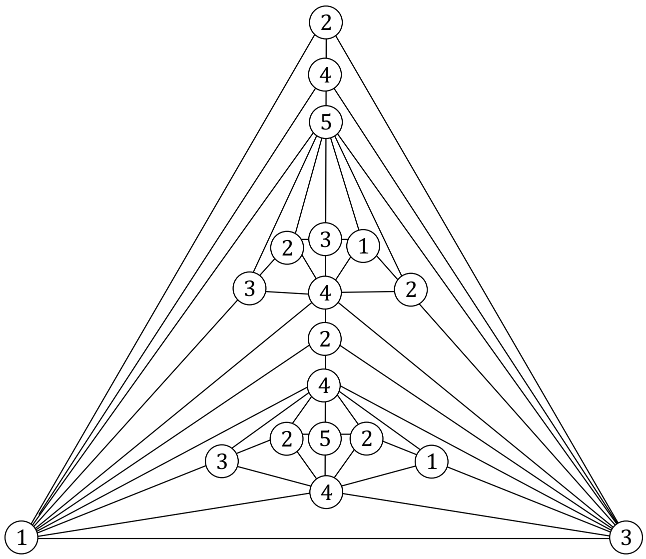

Figure 4.17: A proper facial 3-complete 5-coloring of the graph shown in Figure 4.4

We guess that the reason why some triangulations on the sphere have no proper facial 3-complete 4-coloring concerns Four Color Theorem. Thus, if $n \geq 5$, then the interpolation may hold.

Conjecture 1. *Let G be a triangulation on the sphere. If G has a proper facial 3-complete $(n+1)$-coloring, then G has a proper facial 3-complete n-coloring for $n \geq 5$.*

On the other hand, we have not found an even triangulation on the sphere whose proper facial 3-achromatic number is not interpolation. Therefore, the following conjecture is worth considering.

Conjecture 2. *Let G be an even triangulation on the sphere. For any integer $n \geq 3$, if G has a proper facial 3-complete $(n+1)$-coloring, then G has a proper facial 3-complete n-coloring.*

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