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TOPOLOGICAL TYPES OF FINITELY-C^0-K-DETERMINED MAP-GERMS

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ABSTRACT. In this article, we investigate the following two problems
Problem 1. *Is finite-C^0-K-determinacy a topological invariant among analytic map-germs?*
Problem 2. *Do the topological types of all finitely-C^0-K-determined map-germs have topological moduli, i.e. do they have infinitely many topological types with the cardinal number of continuum?*

Problem 1 is solved affirmatively in the complex case. Problem 2 is solved negatively in the complex case; and affirmatively in the real case.

Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Two map-germs f and $g: (\mathbf{K}^n, 0) \to (\mathbf{K}^p, 0)$ are *topologically equivalent* or *C^0-A-equivalent* if there exist germs of homeomorphisms $h_1: (\mathbf{K}^n, 0) \to (\mathbf{K}^n, 0)$ and $h_2: (\mathbf{K}^p, 0) \to (\mathbf{K}^p, 0)$ such that $g = h_2 \circ f \circ h_1$. A map-germ f is *finitely-C^0-A-determined* or *C^0-A-finite* if there is an integer k such that any germ g with $j^k(g) = j^k(f)$ is C^0-A-equivalent of f. This is the topological version of Mather's A-equivalence and A-determinacy. We can also define C^0-K, C^0-R, C^0-L and C^0-C equivalences and their determinacies in a similar way replacing diffeomorphisms in the C^∞ version by homeomorphisms.

We will give a precise definition of C^0-K-equivalence at the end of the Introduction.

Let $J^k_{\mathbf{K}}(n, p)$ denote the set of all polynomial map-germs: $(\mathbf{K}^n, 0) \to (\mathbf{K}^p, 0)$ with degree $\leq k$ and let $J^k_{\mathbf{K}}(n, p)_{C^0\text{-}K}$ denote the set of all finitely-C^0-K-determined elements of $J^k_{\mathbf{K}}(n, p)$. Let $J^k_{\mathbf{K}}(n, p)_{C^0\text{-}K}/C^0\text{-}A$ denote the set of topological equivalence classes of elements of $J^k_{\mathbf{K}}(n, p)_{C^0\text{-}K}$. Then our main results are

Theorem 1. *Let $f, g: (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ be holomorphic map-germs satisfying the following:*

(1) *the map-germ f is C^0-K-finite,*
(2) *the map-germs f and g are C^0-A-equivalent.*

Then g is also C^0-K-finite.

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Theorem 2. *The set* $J^k_{\mathbf{C}}(n, p)_{C^0\text{-}K}/C^0\text{-}A$ *is a finite set for any positive integers* n, p, k.

Theorem 3. (1) *The set* $J^k_{\mathbf{R}}(n, p)_{C^0\text{-}K}/C^0\text{-}A$ *is a finite set for* $p = 1, 2$, *any positive integers* n, k.

(2) *The set* $J^k_{\mathbf{R}}(n, p)_{C^0\text{-}K}/C^0\text{-}A$ *is an infinite set if* $n \geq 4$, $p \geq 4$, $k \geq 12$. *In fact they have topological moduli.*

When we compare our Theorem 2 and Theorem 3(2) with the results in [3, 2 and 10], it is interesting that there is a difference of cardinal numbers between the real case and the complex case. Our Theorems 1 and 2, combined with the fact that C^0-K-finiteness is a generic property, tell us that finitely-C^0-K-determined holomorphic map-germs are fascinating objects to study from the toplogical viewpoint.

Theorem 1 is deduced from Theorem 4 following.

Theorem 4. *Let* $f, g: (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ *be holomorphic map-germs satisfying the following:*

(1) *the map-germ* f *is nonsingular,*
(2) *the map-germs* f *and* g *are* C^0-A*-equivalent.*

Then g *is also nonsingular.*

Definition 1. Two map-germs f and $g: (\mathbf{K}^n, 0) \to (\mathbf{K}^p, 0)$ are C^0-K-*equivalent* if there exist germs of homeomorphisms

$$h: (\mathbf{K}^n, 0) \to (\mathbf{K}^n, 0) \quad \text{and} \quad H: (\mathbf{K}^n \times \mathbf{K}^p, 0) \to (\mathbf{K}^n \times \mathbf{K}^p, 0)$$

such that the following diagram commutes:

$$\begin{array}{ccccc}
(\mathbf{K}^n, 0) & \xrightarrow[(i,f)]{} & \begin{array}{c}(\mathbf{K}^n \times \mathbf{K}^p, 0)\\ \cap \\ (\mathbf{K}^n \times \mathbf{K}^p, \mathbf{K}^n \times \{0\})\end{array} & \xrightarrow[\pi_1]{} & (\mathbf{K}^n, 0) \\
\downarrow h & & \downarrow H & & \downarrow h \\
(\mathbf{K}^n, 0) & \xrightarrow{(i,f)} & \begin{array}{c}(\mathbf{K}^n \times \mathbf{K}^p, \mathbf{K}^n \times \{0\})\\ \cup \\ (\mathbf{K}^n \times \mathbf{K}^p, 0)\end{array} & \xrightarrow{\pi_i} & (\mathbf{K}^n, 0)
\end{array}$$

where $(i, f)(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x}))$ and $\pi_1(\mathbf{x}, \mathbf{y}) = \mathbf{x}$.

In §§3 and 6, we recall quickly some concepts and results; Milnor fibration and geometric characterizations (§3) and Thom's example having topological moduli (§6), which are used in our proof of Theorem 1 and 3(2). Theorem 1 and Theorem 4 are proved in §4, Theorem 2 and Theorem 3(1) are proved in §5 and Theorem 3(2) is proved in §7.

The author would like to express his sincere gratitude to the referee who was kind enough to give a clearer proof of Theorem 1 and to make valuable suggestions. He also wishes to thank T. Fukuda for his kind advice.

2. Remarks/Related topics

(A) The following simple example shows that the real version of Theorem 1 does not hold.

Example 1. $f(x, y) = xy$, $g(x, y) = x^3 y$.

Function f is finitely C^0-K-determined but g is not, although f and g are topologically equivalent as real functions.

(B) Let G be any of A, K, R, L and C. Then the following questions are more natural to be asked than our Problems 1 and 2.

Problem 3. Is finite-C^0-G-determinacy a C^0-G-invariant among analytic map-germs?

Problem 4. Is $J^k_{\mathbf{K}}(n, p)_{C^0\text{-}G}/C^0\text{-}G$ a finite set for any positive integers n, p, k?

We easily have the following answers to these problems.

Problem (3)		$G=A$	K	R	L	C
Answer	$\mathbf{K}=\mathbf{R}$	(3-1) No	(3-2) No	(3-3) No	$(3-5)$ No	(3-6) Yes
	$\mathbf{K}=\mathbf{C}$	?	?	(3-4) Yes	?	

Problem (4)		$G=A$	K	R	L	C
Answer	$\mathbf{K}=\mathbf{R}$	(4-1) finite	(4-3) finite	(4-4) finite	(4-5) finite	(4-6) finite
	$\mathbf{K}=\mathbf{C}$	(4-2) finite			?	

(3-1) Example 1.

(3-2) Example 1.

(3-3) Example 1.

(3-4) This is a corollary of Theorem 1 (see the end of §4).

(3-5) Consider the following two map-germs; $f, g: (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$, $f(x, y) = (x^3, y^3, x^2 + y^2)$, $g(x, y) = (x^2, y, x^3)$.

(3-6) Recall the geometric characterization of C^0-C-finiteness of analytic map-germs (Proposition 2(c) in §3).

(4-1) This is a Thom's result [12].

(4-2) This is a corollary of Theorem 2.

(4-3) Recall the geometric characterization of C^0- K-finitenes (Proposition 2(b) in §3) and use the local version of Thom's second isotopy lemma for the sequence

$$\mathrm{Image}(F) \xrightarrow{\pi_1} \mathbf{K}^n \times J^k_{\mathbf{K}}(n\,,p)_{C^0\text{-}K} \xrightarrow{\pi_2} J^k_{\mathbf{K}}(n\,,p)_{C^0\text{-}K}\,,$$

where $F: \mathbf{K}^n \times J^k_{\mathbf{K}}(n\,,p)_{C^0\text{-}K} \to \mathbf{K}^n \times \mathbf{K}^p \times J^k_{\mathbf{K}}(n\,,p)_{C^0\text{-}K}$ is the mapping defined by $F(x\,,f) = (x\,,f(x)\,,f)$ and $\pi_1\,,\pi_2$ are canonical projections.

(4-4) Recall the geometric characterization of C^0- R-finiteness (Proposition 2(a) in §3). Then we see that (4-4) is essentially due to King [6 or 7].

(4-5) Recall the geometric characterization of C^0- L-finiteness (Proposition 2(b) in §3) and use the local version of Thom's first isotopy lemma for the stratified mapping

$$\mathrm{Image}(G) \xrightarrow{\pi} \mathbf{R}^n \times J^k_{\mathbf{R}}(n\,,p)_{C^0\text{-}L}\,,$$

where $G: \mathbf{R}^n \times J^k_{\mathbf{R}}(n\,,p)_{C^0\text{-}L} \to \mathbf{R}^p \times J^k_{\mathbf{R}}(n\,,p)_{C^0\text{-}L}$ is the mapping defined by $G(x\,,f) = (f(x)\,,f)$ and π is the mapping defined by $\pi(f(x)\,,f) = (x\,,f)$.

(4-6) Recall the geometric characterization of C^0- C-finiteness (Proposition 2(c) in §3) and use the local version of Thom's second isotopy lemma for the sequence

$$\mathrm{Image}(F) \xrightarrow{\pi_1} \mathbf{K}^n \times J^k_{\mathbf{K}}(n\,,p)_{C^0\text{-}C} \xrightarrow{\pi_2} J^k_{\mathbf{K}}(n\,,p)_{C^0\text{-}C}\,,$$

where F, π_1 and π_2 are the same as those of (4-3).

(C) In [11] where his second isotopy lemma and his condition a_f were announced for the first time, Thom gave an example of a family of polynomial mappings of $\mathbf{R}^3$ into $\mathbf{R}^3$ which contains continuously many topological types. Fukuda [3] and Aoki [2] showed that every family of polynomial functions of several variables or of polynomial map-germs of $\mathbf{R}^2$ into $\mathbf{R}^2$ (or $\mathbf{C}^2$ into $\mathbf{C}^2$) has only finitely many topological types. Nakai gave examples of families of polynomial map-germs of $\mathbf{R}^n$ into $\mathbf{R}^p$ (or $\mathbf{C}^n$ into $\mathbf{C}^p$) of degree k with $n,\ p,\ k \geq 3$ or $n \geq 3,\ p \geq 2,\ k \geq 4$ which contain continuously many topological types.

The examples of Thom and Nakai motivate us to consider what will happen if we restrict objects of study within better map-germs, for example finitely- C^0-K-determined ones? Thus our Problems 1 and 2 arise.

(D) In the complex case, Wall proved that C^0- G-finiteness implied G-finiteness (except possibly if $G = L$, $p \geq 2n-1$ and $G = A$) [14, p. 532]. But the notion of C^0- G-finiteness is topologically better than the one of G-finiteness in the real case and since we are interested to see whether or not variously topological phenomena in the real case contrast with the ones in the complex case, we adopt the notion of C^0- G-finiteness as one of our restricted objects.

3. Milnor fibration and geometric characterization

(A) Let $f: (\mathbf{C}^n\,,\mathbf{z}^0) \to (\mathbf{C}^p\,,f(\mathbf{z}^0))$ be a nonconstant holomorphic function-germ. There exists a positive number ε such that for any positive number r

$(r < \varepsilon)$ the space $rS^{2n-1}(\mathbf{z}^0) - f^{-1}(f(\mathbf{z}^0))$ is a smooth fibration over S^1, with projection mapping

$$\phi(\mathbf{z}-\mathbf{z}^0) = f(\mathbf{z}-\mathbf{z}^0)/\|f(\mathbf{z}-\mathbf{z}^0)\|,$$

where $rS^{2n-1}(\mathbf{z}^0) = \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbf{C}^n \mid \|\mathbf{z}-\mathbf{z}^0\| = r^2\}$ and S^1 is a unit circle. We call this fibration the *Milnor fibration of* f *at* $\mathbf{z}^0$. Moreover the fiber of the Milnor fibration (which is called *the Milnor fiber*) of f at $\mathbf{z}^0$ is diffeomorphic to a smooth manifold which is the intersection of the open r-disk centered at $\mathbf{z}^0$ and the level set $f^{-1}(c)$ where c $(\neq f(\mathbf{z}^0))$ is sufficiently close to $f(\mathbf{z}^0)$ (see [9]).

The following result due to A′Campo plays an essential role in our proof of Theorem 1.

Proposition 1 [1]. *Let* $f\colon (\mathbf{C}^n, \mathbf{z}^0) \to (\mathbf{C}, f(\mathbf{z}^0))$ *be a nonconstant holomorphic function-germ. Then the fiber of Milnor fibration of* f *at* $\mathbf{z}^0$ *has the homology of a point if and only if* $\mathbf{z}^0$ *is a regular point of* f.

(B) Let $f\colon (\mathbf{K}^n, 0) \to (\mathbf{K}^p, 0)$ be an analytic map-germ ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$). Then we have the following proposition, which is called the geometric characterization (see [14]).

Proposition 2. (a) *The map-germ* f *is* C^0-R*-finite if and only if* $\mathrm{Sing}(f) - \{0\} = \phi$ *as germs, where* $\mathrm{Sing}(f) = \{\mathbf{x} \in \mathbf{K}^n \mid$ *the differential* df *at* $\mathbf{x}$ *is not surjective*$\}$.

(b) *The map-germ* f *is* C^0-K*-finite if and only if* $\mathrm{Sing}(f) \cap f^{-1}(0) - \{0\} = \phi$ *as germs.*

(c) *The map-germ* f *is* C^0-C*-finite if and only if* $f^{-1}(0) = \{0\}$ *as germs.*

(d) *In the real case, the map-germ* f *is* C^0-L*-finite if and only if* f *is an embedding except at* 0 *as a germ.*

Proposition 2 is very useful, and one of the essential tools in this article. For details on C^0-G-finiteness, see Wall's prominent survey [14].

4. Proofs of Theorem 1 and Theorem 4

Proof of Theorem 4. The author's original proof is needlessly complicated. The following proof, due to the referee, is much clear and simpler.

By the definition of singular points, the hypothesis (1) implies that $n \geq p$. By hypothesis (2), we can put

$$(*) \qquad \psi \circ g \circ \phi(\mathbf{x}, \mathbf{y}) = \mathbf{y},$$

where $\phi\colon (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ and $\psi\colon (\mathbf{C}^p, 0) \to (\mathbf{C}^p, 0)$ are germs of homeomorphisms. Put $h(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{x}, \psi(\mathbf{y}))$. By $(*)$, $\psi \circ g \circ h(\mathbf{x}, \mathbf{y}) = \psi \circ g \circ \phi(\mathbf{x}, \psi(\mathbf{y})) = \psi(\mathbf{y})$. Since ψ is a bijective germ, we have $g \circ h(\mathbf{x}, \mathbf{y}) = \mathbf{y}$. Since h is a germ of homeomorphism g is C^0-R-equivalent to a projection germ.

Now suppose g is singular. Then the differential dg at 0 is not surjective. Therefore there is a linear function $\rho\colon \mathbf{C}^p \to \mathbf{C}$ such that $\rho \circ g$ is singular.

Since g is C^0-R-equivalent to a projection germ, $\rho \circ g$ is C^0-R-equivalent to ρ composed with a projection germ. However, ρ composed with a projection germ is of course nonsingular. Hence we have reduced the problem to the case $p = 1$.

From now on, assume that $p = 1$ and $g(\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ is singular. Pick one representative $\tilde{g}$ of g. Let εD^{2n} stand for the closed disc in $\mathbf{C}^n$ of radius ε centered at the origin. Let $h: (\varepsilon D^{2n-2} \times \varepsilon D^2, 0 \times 0) \to (\mathbf{C}^n, 0)$ be a topological embedding so that $\tilde{g} \circ h(\mathbf{x}, \mathbf{y}) = \mathbf{y}$ for all $(\mathbf{x}, \mathbf{y}) \in \varepsilon D^{2n-2} \times \varepsilon D^2$.

Pick $\delta > 0$ so small that $\tilde{g}$ is nonsingular at all points of $\delta D^{2n} - \tilde{g}^{-1}(0)$ and $\delta D^{2n} \subset \mathrm{Int}(h(\varepsilon D^{2n-2} \times \varepsilon D^2))$, where Int stands for the interior. Pick $\varepsilon' > 0$ so small that $h(\varepsilon' D^{2n-2} \times \varepsilon' D^2) \subset \mathrm{Int}(\delta D^{2n})$. Pick $\delta' > 0$ so small that $\delta' D^{2n} \subset \mathrm{Int}(h(\varepsilon' D^{2n-2} \times \varepsilon' D^2))$. Let $r: \mathbf{C}^n \to \mathbf{R}$ be the function of the form $r(\mathbf{z}) = \|\mathbf{z}\|^2$. Then by [9] if we pick $\mathbf{z}_0$ small enough, then r restricted to $\tilde{g}^{-1}(\mathbf{z}_0)$ has no singular points on $\delta D^{2n} - (\delta'/2) D^{2n}$.

So pick a small enough $\mathbf{z}_0 \neq 0$ with $\|\mathbf{z}_0\| < \varepsilon'$. Let $W_0 = \tilde{g}^{-1}(\mathbf{z}_0) \cap \delta' D^{2n}$, let $W_1 = \tilde{g}^{-1}(\mathbf{z}_0) \cap h(\varepsilon' D^{2n-2} \times \varepsilon' D^2) = h(\varepsilon' D^{2n-2} \times \mathbf{z}_0)$, and let $W_2 = \tilde{g}^{-1}(\mathbf{z}_0) \cap \delta D^{2n}$. Then we have that $W_0 \subset W_1 \subset W_2$ and W_0 and W_2 are Milnor fibers of $\tilde{g}$. Let X_{ij} denote $W_i - \mathrm{Int}(W_j)$ for $i > j$. Since the function r restricted to X_{20} has no singular points, X_{20} is diffeomorphic to $\partial W_0 \times [0, 1]$. Therefore W_0 is a deformation retract of W_2. Hence the inclusion mapping $W_1 \hookrightarrow W_2$ induces surjective homomorphisms of homology. However W_1 is a topological disk. Hence W_2 has the homology of a point. So [1] implies that $\tilde{g}$ is nonsingular and we have our conclusion. Q.E.D.

Proof of Theorem 1. If $n < p$, all points $\mathbf{x} \in \mathbf{C}^n$ are singular points of f. If f is C^0-K-finite $f^{-1}(0) = \{0\}$ as germs at 0 by Proposition 2(b). Hence by the hypothesis (2), $g^{-1}(0)$ is also $\{0\}$ as a germ at 0. Therefore g is C^0-K-finite by Proposition 2(b).

Hence our interest is essentially in the case $n \geq p$. By the hypothesis (2), we can put

$$(**) \qquad \tilde{g} = (h')^{-1} \circ \tilde{f} \circ h$$

on a certain neighborhood of the origin in $\mathbf{C}^n$, where $\tilde{f}$ (resp. $\tilde{g}$) is a representative of f (resp. g) and h (resp. h') is a homeomorphism between neighborhoods of the origin in $\mathbf{C}^n$ (resp. $\mathbf{C}^p$).

Suppose that g is not C^0-K-finite. Then by Proposition 2(b),

$$\mathrm{Sing}(\tilde{g}) \cap \tilde{g}^{-1}(0) - \{0\} \neq \phi,$$

where $\mathrm{Sing}(\tilde{g}) = \{z \in \mathbf{C}^n : z \text{ is a singular point of } \tilde{g}\}$. By the hypothesis (1) and $(**)$, for any point $\mathbf{z}_0 \in \mathrm{Sing}(\tilde{g}) \cap \tilde{g}^{-1}(0) - \{0\}$, $h(\mathbf{z}_0)$ is a regular point of $\tilde{f}$ and $\tilde{f}(h(\mathbf{z}_0)) = 0$. Therefore our situation is as follows:

(1) *the map-germ* $\tilde{f}: (\mathbf{C}^n, h(\mathbf{z}_0)) \to (\mathbf{C}^p, 0)$ *is nonsingular,*

(2) *two map-germs* $\tilde{f}: (\mathbf{C}^n, h(\mathbf{z}_0)) \to (\mathbf{C}^p, 0)$ *and* $\tilde{g}: (\mathbf{C}^n, \mathbf{z}_0) \to (\mathbf{C}^p, 0)$ *are* C^0- *A-equivalent,*

(3) *the map-germ* $\tilde{g}: (\mathbf{C}^n, \mathbf{z}_0) \to (\mathbf{C}^p, 0)$ *is singular.*

This contradicts Theorem 4. Therefore the map-germ g must be C^0-K-finite. Q.E.D.

Corollary 1. *Let* $f, g: (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ *be holomorphic map-germs satisfying the following:*

(1) f *is* C^0- *R-finite,*

(2) f *and* g *are* C^0- *R-equivalent.*

Then g *is also* C^0- *R-finite.*

Proof of Corollary 1. By Proposition 2(a) and the definition of singular points, the hypothesis (1) implies that $n \geq p$. By Theorem 4, we may assume that the origin of $\mathbf{C}^n$ is a singular point of f. Then by the same argument as in the proof of Theorem 1, we have that the origin is an isolated singular point for any representative of g. Hence by Proposition 2(a), g must be C^0-R-finite. Q.E.D.

5. Proofs of Theorem 2 and Theorem 3(1)

We identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$. We also identify $J^k_{\mathbf{K}}(n, p)$ not only with the set of all polynomial mappings of $(\mathbf{K}^n, 0)$ into $(\mathbf{K}^p, 0)$, with degree $\leq k$, but also with an Euclidean space $\mathbf{R}^{\varepsilon pN}$ of a suitable dimension εpN ($\varepsilon = 1$ if $\mathbf{K} = \mathbf{R}$ and $\varepsilon = 2$ if $\mathbf{K} = \mathbf{C}$) as usual.

Under these identifications, the mapping

$$F: J^k_{\mathbf{K}}(n, p) \times \mathbf{R}^{\varepsilon n} \to J^k_{\mathbf{K}}(n, p) \times \mathbf{R}^{\varepsilon p}$$

defined by $F(f, \mathbf{x}) = (f, f(\mathbf{x}))$ can be considered as a real polynomial mapping, where $f \in J^k_{\mathbf{K}}(n, p)$, $\mathbf{x} \in \mathbf{R}^{\varepsilon n}$ and $\varepsilon = 1$ if $\mathbf{K} = \mathbf{R}$ and $\varepsilon = 2$ if $\mathbf{K} = \mathbf{C}$.

Lemma 1. *The set* $J^k_{\mathbf{K}}(n, p)_{C^0\text{-}K}$ *is a semialgebraic subset in* $J^k_{\mathbf{K}}(n, p) = \mathbf{R}^{\varepsilon pN}$ *for any positive integers* n, p *and* k.

Proof of Lemma 1. By Proposition 2(b), for each polynomial mapping f in $J^k_{\mathbf{K}}(n, p)$, f is contained in $J^k_{\mathbf{K}}(n, p)_{C^0\text{-}K}$ if and only if there exists a neighborhood V of 0 in $\mathbf{K}^n$ such that $V \cap \mathrm{Sing}(f) \cap f^{-1}(0) - \{0\} = \phi$, which is equivalent to the statement that there exists a neighborhood V of 0 in $\mathbf{R}^{\varepsilon n}$ such that

$$(\{f\} \times V) \cap \mathrm{Sing}(F) \cap F^{-1}(J^k_{\mathbf{K}}(n, p) \times \{0\}) - \{f \times 0\} = \phi.$$

Clearly $A \subset \mathbf{R} \times \mathbf{R}^{\varepsilon n} \times J^k_{\mathbf{K}}(n, p) \times \mathbf{R}^{\varepsilon n}$, comprised of all quadruples $(t, \mathbf{y}, f, \mathbf{x})$ with $(f, \mathbf{x}) \in F^{-1}(0) \cap \mathrm{Sing}(F) - J^k_{\mathbf{K}}(n, p) \times \{0\}$ and $\|x - y\| < t$, is semialgebraic. Consider the following polynomial projections:

$$(\mathbf{R} \times \mathbf{R}^{\varepsilon n} \times J^k_{\mathbf{K}}(n, p)) \times \mathbf{R}^{\varepsilon n} \xrightarrow{p_1} \mathbf{R} \times \mathbf{R}^{\varepsilon n} \times J^k_{\mathbf{K}}(n, p)$$
$$\xrightarrow{p_2} \mathbf{R}^{\varepsilon n} \times J^k_{\mathbf{K}}(n, p) \xrightarrow{p_3} J^k_{\mathbf{K}}(n, p).$$

The Tarski-Seidenberg theorem implies

$$(\mathbf{R}^{\varepsilon n} \times J^k_{\mathbf{K}}(n, p) - p_2(\mathbf{R} \times \mathbf{R}^{\varepsilon n} \times J^k_{\mathbf{K}}(n, p) - p_1(A))) \cap (\{0\} \times J^k_{\mathbf{K}}(n, p))$$

is semialgebraic. This set is denoted by B.

A minor computation verifies that

$$J^k_{\mathbf{K}}(n, p)_{C^0\text{-}K} = J^k_{\mathbf{K}}(n, p) - p_3(B),$$

which is also semialgebraic. Q.E.D.

Now we consider the following sequence:

$$J^k_{\mathbf{K}}(n, p) \times \mathbf{R}^{\varepsilon n} \xrightarrow{F} J^k_{\mathbf{K}}(n, p) \times \mathbf{R}^{\varepsilon p} \xrightarrow{\pi} J^k_{\mathbf{K}}(n, p),$$

where π is the canonical projection. Since F and π are polynomial mappings, by Lemma 1 there exists semialgebraic stratifications $S(J^k_{\mathbf{K}}(n, p) \times \mathbf{R}^{\varepsilon n})$, $S(J^k_{\mathbf{K}}(n, p) \times \mathbf{R}^{\varepsilon p})$ and $S(J^k_{\mathbf{K}}(n, p))$ with which F and π are stratified mappings and $J^k_{\mathbf{K}}(n, p) \times \{0\}$, $J^k_{\mathbf{K}}(n, p) \times \{0\}$ and $J^k_{\mathbf{K}}(n, p)_{C^0\text{-}K}$ are stratified subsets of $J^k_{\mathbf{K}}(n, p) \times \mathbf{R}^{\varepsilon n}$, $J^k_{\mathbf{K}}(n, p) \times \mathbf{R}^{\varepsilon p}$ and $J^k_{\mathbf{K}}(n, p)$ respectively (see [3]).

Note that $\mathrm{Sing}(F)$ is a stratified subset of the set $J^k_{\mathbf{K}}(n, p) \times \mathbf{R}^{\varepsilon n}$. Then for each stratum Z of $S(J^k_{\mathbf{K}}(n, p)_{C^0\text{-}K})$, the sequence of restricted mappings,

$$(***) \qquad Z \times \mathbf{R}^{\varepsilon n} \xrightarrow{F} Z \times \mathbf{R}^{\varepsilon p} \xrightarrow{\pi} Z$$

is also a sequence of stratified maps with the canonically induced semialgebraic stratifications $S(Z \times \mathbf{R}^{\varepsilon n})$, $S(Z \times \mathbf{R}^{\varepsilon p})$ and $\{Z\}$ from $S(J^k_{\mathbf{K}}(n, p) \times \mathbf{R}^{\varepsilon n})$, $S(J^k_{\mathbf{K}}(n, p) \times \mathbf{R}^{\varepsilon p})$ and $S(J^k_{\mathbf{K}}(n, p))$ respectively, where F and π in $(***)$ stand for $F|_{Z \times \mathbf{R}^{\varepsilon n}}$ and $\pi|_{Z \times \mathbf{R}^{\varepsilon p}}$ respectively.

We use this sequence $(***)$ to prove Theorem 2 and Theorem 3(1).

Proof of Theorem 2. We consider the stratified sequence $(***)$. We want to state that for each stratum Z of $S(J^k_{\mathbf{C}}(n, p)_{C^0\text{-}K})$ there exists a semialgebraic stratification $S'(Z)$ of Z such that for each stratum W of $S'(Z)$ there exists a semialgebraic neighborhood U_W of $W \times \{0\}$ in $W \times \mathbf{R}^{2n}$ and the restricted mapping

$$U_W \xrightarrow{F} W \times \mathbf{R}^{2p}$$

is a Thom mapping with respect to the canonically induced semialgebraic stratifications $S((W \times \mathbf{R}^{2n}) \cap U_W)$ and $S(W \times \mathbf{R}^{2p})$.

By Proposition 2(b), any polynomial mapping $f \in Z$ has the condition that $\mathrm{Sing}(f) \cap f^{-1}(0) - \{0\} = \phi$ as germs. It is well known that if $\mathrm{Sing}(f) \cap f^{-1}(0) - \{0\} = \phi$ as germs then there exists a neighborhood U of 0 in $\mathbf{C}^n$ such that the restriction

$$f|_{U \cap \mathrm{Sing}(f)} : U \cap \mathrm{Sing}(f) \to \mathbf{C}^p$$

is proper and finite to one (for example see [4, p. 493])). As $\mathrm{Sing}(F) = \{(f, \mathrm{Sing}(f)) : f \in J^k_{\mathbf{K}}(n, p)\}$, we can deduce that there exists a semialgebraic stratification $S'(Z)$ of Z such that for any stratum W of $S'(Z)$ there exists a semialgebraic neighborhood U_W of $W \times \{0\}$ in $W \times \mathbf{R}^{2n}$ and the restricted mapping

$$U_W \cap \mathrm{Sing}(F) \xrightarrow{F} W \times \mathbf{R}^{2p}$$

is proper and finite to one. Also the restricted mapping

$$U_W \xrightarrow{F} W \times \mathbf{R}^{2p}$$

is a stratified mapping with respect to the canonically induced semialgebraic stratifications $S((W \times \mathbf{R}^{2n}) \cap U_W)$, $S(W \times \mathbf{R}^{2p})$ and $U_W \cap \mathrm{Sing}(F)$ is a stratified subset of $(W \times \mathbf{R}^{2n}) \cap U_W$.

For any point $(f, \mathbf{x}) \in U_W \cap \mathrm{Sing}(F)$, as the restricted mapping $U_W \cap \mathrm{Sing}(F) \to W \times \mathbf{R}^{2p}$ is proper and finite to one, $\ker(d(F|_X)_{(f, \mathbf{x})}) = 0$. Here X is the stratum of the stratification $S((W \times \mathbf{R}^{2n}) \cap U_W)$ which contains $(f, \mathbf{x})$. For any pair of nonsingular strata (X, Y) such that $X, Y \in S(W \times \mathbf{R}^{2n}) \cap U_W)$ and $\overline{X} \supset Y$, the pair (X, Y) always satisfies condition a_F. Here nonsingular means that for any point $(f, \mathbf{x}) \in Y$ $(f, \mathbf{x}) \notin U_W \cap \mathrm{Sing}(F)$.

These observations show that the restricted mapping

$$U_W \xrightarrow{F} W \times \mathbf{R}^{2p}$$

is a Thom mapping with respect to the canonically induced semialgebraic stratifications $S((W \times \mathbf{R}^{2n}) \cap U_W)$ and $S(W \times \mathbf{R}^{2p})$. Now Theorem 2 follows from a local version of Thom's second isotopy lemma (see [4 or 11]). Q.E.D.

Proof of Theorem 3(1). In the function case, that is $p = 1$, for any positive integers n and k, our theorem is contained in the local case of Fukuda's theorem [3]. Hence we prove our theorem only in the case $p = 2$.

Consider the stratified sequence $(***)$. Let X, Y be strata of $S(Z \times \mathbf{R}^n)$ such that $\overline{X} - X \supset Z \times \{0\}$, $\overline{Y} - Y \supset Z \times \{0\}$ and $\overline{X} \supset Y$, where $\overline{X}$ denotes the closure of X in $Z \times \mathbf{R}^n$. Let $\widetilde{X}, \widetilde{Y}$ be strata of $S(Z \times \mathbf{R}^2)$ such that $F(X) \subset \widetilde{X}$ and $F(Y) \subset \widetilde{Y}$. In the case $\widetilde{X} = \widetilde{Y}$, the existence theorem of tubular neighborhoods of strata shows that the pair (X, Y) satisfies condition a_F (see [8]).

There are three possibilities of dimensions of a pair of strata $(\widetilde{X}, \widetilde{Y})$ when $\widetilde{X} \neq \widetilde{Y}$ and $\overline{\widetilde{X}} \supset \widetilde{Y}$ as follows, where $\overline{\widetilde{X}}$ denotes the closure of $\widetilde{X}$ in $Z \times \mathbf{R}^2$.

$\dim \widetilde{X}$	$\dim \widetilde{Y}$
$2 + \dim Z$	$1 + \dim Z$
$2 + \dim Z$	$0 + \dim Z$
$1 + \dim Z$	$0 + \dim Z$

(1) The case $(\dim \widetilde{X}, \dim \widetilde{Y}) = (2 + \dim Z, 0 + \dim Z)$ or $(1 + \dim Z, 0 + \dim Z)$.

In this case, by Proposition 2(b) and

$$\mathrm{Sing}(F) = \{(f, \mathrm{Sing}(F)) : f \in J^k_{P\mathbf{R}}(n, p)\},$$

there exists a semialgebraic neighborhood U_Z of $Z \times \{0\}$ in $Z \times \mathbf{R}^n$ such that the pair $(X \cap U_Z, Y \cap U_Z)$ is a nonsingular pair. Hence the pair $(X \cap U_Z, Y \cap U_Z)$ satisfies condition a_F.

(2) The case $(\dim \widetilde{X}, \dim \widetilde{Y}) = (2 + \dim Z, 1 + \dim Z)$.

It is sufficient to consider only the case $Y \subset \mathrm{Sing}(F)$. In this case there exists a semialgebraic neighborhood U_Z of $Z \times \{0\}$ in $Z \times \mathbf{R}^n$ such that for each point $(f^0, \mathbf{x}^0) \in Y \cap U_Z$, $\mathrm{rank}\, F$ at $(f^0, \mathbf{x}^0)$ is $1 + \dim J^k_{\mathbf{R}}(n, 2)$. By a suitable analytic coordinate transformation we can assume that $F(f, \mathbf{x}) = (f, x_1, g(f, x_1, \dots, x_n))$ in a sufficiently small neighborhood $V_{(f^0, \mathbf{x}^0)}$ of $(f^0, \mathbf{x}^0)$ in U_Z, where $\mathbf{x} = (x_1, \dots, x_n)$ and $g: V_{(f^0, \mathbf{x}^0)} \to \mathbf{R}$ is an analytic function.

We set $\mathbf{x}^0 = (x_1^0, \dots, x_n^0)$ under this coordinate chart. We also set

$$D = \{(f, \mathbf{x}) \in V_{(f^0, \mathbf{x}^0)} : x_1 = x_1^0, f = f^0\},$$

$$D' = \{(f, \mathbf{x}) \in V_{(f^0, \mathbf{x}^0)} : x_1 = x_1^0\} \quad \text{and}$$

$$\widetilde{D} = \{(f, y_1, y_2) \in Z \times \mathbf{R}^2 : (f, y_1, y_2) \in F(V_{(f^0, \mathbf{x}^0)}), y_1 = x_1^0\}.$$

We may assume that $g^{-1}(g(f^0, \mathbf{x}^0)) \cap V_{(f^0, \mathbf{x}^0)} = Y \cap D'$. In the sufficiently small neighborhood $V_{(f^0, \mathbf{x}^0)}$ of $(f^0, \mathbf{x}^0)$ in U_Z, we can assume that the stratum Y is transversal to the submanifold D. Since the mapping $g: V_{(f^0, \mathbf{x}^0)} \to \mathbf{R}$ is a function, the existence theorem of a good stratification implies that there exists a stratification $S(Y \cap D)$ such that the restricted function

$$g|_{(X \cup Y) \cap D} : (X \cup Y) \cap D \to (\widetilde{X} \cup \widetilde{Y}) \cap \widetilde{D}$$

is a Thom mapping with respect to the stratifications $\{X \cap D, S(Y \cap D)\}$ and $\{\widetilde{X} \cap \widetilde{D}, \widetilde{Y} \cap \widetilde{D}\}$ (see [3 or 5]).

We also see that in the sufficiently small neighborhood $V_{(f^0, \mathbf{x}^0)}$ of $(f^0, \mathbf{x}^0)$ in U_Z, the restricted mapping

$$F|_{X \cup Y} : X \cup Y \to \widetilde{X} \cup \widetilde{Y}$$

is considered as an analytically trivial unfolding of the restricted function

$$g|_{(X\cup Y)\cap D}\colon (X\cup Y)\cap D \to (\widetilde{X}\cup\widetilde{Y})\cap\widetilde{D}.$$

Therefore the restricted mapping

$$F|_{(X\cup Y)\cap V_{(f^0,x^0)}}\colon (X\cup Y)\cap V_{(f^0,x^0)} \to (\widetilde{X}\cup\widetilde{Y})$$

is a Thom mapping with respect to the canonically extended stratifications from $\{X\cap D, S(Y\cap D)\}$ and $\{\widetilde{X}\cap\widetilde{D}, \widetilde{Y}\cap\widetilde{D}\}$.

By (1) and (2) above. We see that for each stratum Z of $S(J^k_{\mathbf{R}}(n,2)_{C^0\text{-}K})$ there exist a neighborhood U_Z of 0 in $Z\times\mathbf{R}^n$ and stratifications $S''(Z\times\mathbf{R}^n)$, $S''(Z\times\mathbf{R}^2)$ such that the restricted mapping

$$F|_{U_Z}\colon U_Z \to Z\times\mathbf{R}^2$$

is a Thom mapping with respect to the canonically induced stratifications $S''((Z\times\mathbf{R}^n)\cap U_Z)$, $S''(Z\times\mathbf{R}^2)$. Now Theorem 3(2) follows from a local version of Thom's second isotopy lemma (see [4 or 11]). Q.E.D.

6. Thom's example

Let $\mathbf{K}=\mathbf{R}$ or $\mathbf{C}$ and let $f, g\colon \mathbf{K}^n\to\mathbf{K}^p$ be C^∞ (for $\mathbf{K}=\mathbf{R}$) or holomorphic ($\mathbf{K}=\mathbf{C}$) mappings. We say f and g are *topologically equivalent* if there are homeomorphisms $h\colon\mathbf{K}^n\to\mathbf{K}^n$ and $h'\colon\mathbf{K}^p\to\mathbf{K}^p$ such that $f=(h')^{-1}\circ g\circ h$.

In [11], Thom considered the following one-parameter real polynomial mapping family $P(k)\colon\mathbf{R}^3\to\mathbf{R}^3$, where k is a real parameter, and he proved that if any two fixed nonzero real numbers k_1, k_2 are not equal then $P(k_1)$ and $P(k_2)$ are not topologically equivalent.

$$P(k)\colon\begin{cases} X=[x(x^2+y^2-a^2)-2ayz]^2[(x+ky)(x^2+y^2-a^2)-2a(y-kx)z]^2,\\ Y=x^2+y^2-a^2,\\ Z=z,\end{cases}$$

where (x,y,z), (X,Y,Z) are coordinates of the source space and the target space respectively, a is a nonzero fixed real number and k is a real parameter.

In this section, we recall quickly Thom's idea of proof, which is used in our proof of Theorem 3(2).

Thom's Idea of Proof. Let k_0 be a fixed real number. We consider the following surface $H(k_0)$ and circle $C(k_0)$:

$$\begin{aligned} H(k_0)=\{(x,y,z)\in\mathbf{R}^3\colon &[x(x^2+y^2-a^2)-2ayz]^2\\ &\times[(x+k_0y)(x^2+y^2-a^2)-2a(y-k_0x)z]^2=0\},\\ C(k_0)=\{(x,y,0)\in\mathbf{R}^3\colon &x^2+y^2-a^2=0\}.\end{aligned}$$

Then $C(k_0)\subset H(k_0)$ and $C(k_0)\subset \operatorname{Sing}(P(k_0))$.

We also consider the following two surfaces $H_1(k_0)$ and $H_2(k_0)$:

$$H_1(k_0) = \{(x, y, z) \in \mathbf{R}^3 : x(x^2 + y^2 - a^2) - 2ayz = 0\},$$
$$H_2(k_0) = \{(x, y, z) \in \mathbf{R}^3 : (x + k_0y)(x^2 + y^2 - a^2) - 2a(y - k_0x)z = 0\}.$$

We see that these two surfaces $H_1(k_0)$ and $H_2(k_0)$ are irreducible components of $\mathrm{Sing}(P(k_0))$, $H(k_0) = H_1(k_0) \cup H_2(k_0)$ and $H_1(k_0) \cap H_2(k_0) = C(k_0) \cup \{(0, 0, z) \in \mathbf{R}^3\}$ (provided that $k_0 \neq 0$). Furthermore we have

$$P(k_0)(H_1(k_0) \cap \{(x, y, z) \in \mathbf{R}^3 : lx + my = 0\})$$
$$= \{(0, Y, Z) \in \mathbf{R}^3 : mY + 2alZ = 0\}$$

and

$$P(k_0)(H_2(k_0) \cap \{(x, y, z) \in \mathbf{R}^3 : lx + my = 0\})$$
$$= \{(0, Y, Z) \in \mathbf{R}^3 : (m - k_0l)Y + 2a(l + k_0m)Z = 0\}$$

for any two real numbers l, m such that $l^2 + m^2 \neq 0$.

Now if there exist homeomorphisms $h, h' : \mathbf{R}^3 \to \mathbf{R}^3$ such that $P(k_0) = (h')^{-1} \circ P(k_1) \circ h$ for any two fixed nonzero real numbers k_0, k_1 $(k_0 \neq k_1)$, then we have the following:

Lemma 2. (1) $h(H(k_0)) = H(k_1)$,

(2) $h(C(k_0)) = C(k_1)$ *and*

(3) *for any germ of a continuous curve* $\mathbf{q}(t)$ *at any point* $\mathbf{p} \in C(k_0)$ (*resp.* $C(k_1)$) *in* $H(k_0)$ (*resp.* $H(k_1)$), $P(k_0)$ (*resp.* $P(k_1)$) *maps* $\mathbf{q}(t)$ *to a germ of a continuous curve at* $(0, 0, 0) \in \{(0, Y, Z) \in \mathbf{R}^3\}$ *in* $\{(0, Y, Z) \in \mathbf{R}^3\}$ *and this germ of a curve has a tangent line at* $(0, 0, 0)$.

By Lemma 2, if k_0, k_1 are both nonzero, then the restricted homeomorphism $h|_{C(k_0)} : C(k_0) \to C(k_1)$ must have the property that for any two points $\mathbf{x}, \mathbf{y} \in C(k_0)$ such that angle $\angle\widehat{\mathbf{x}\mathbf{y}} = \mathrm{Tan}^{-1}(k_0)$ angle $\angle\widehat{h(\mathbf{x})h(\mathbf{y})} = \mathrm{Tan}^{-1}(k_1)$. But this contradicts Van Kampen's theorem in [13].

Remark 1. It is easily seen that if we change the one-parameter real polynomial mapping family $P(k) : \mathbf{R}^3 \to \mathbf{R}^3$ to $\widetilde{P}(k) : \mathbf{R}^3 \to \mathbf{R}^3$ as follows, then we also have the property that if $k_0 \neq k_1$ $(k_i \neq 0)$ then $\widetilde{P}(k_0)$ and $\widetilde{P}(k_1)$ are not topologically equivalent;

$$\widetilde{P}(k) : \begin{cases} X = [x(x^2 + y^2 - a^2) - yz]^2[(x + ky)(x^2 + y^2 - a^2) - (y - kx)z]^2, \\ Y = x^2 + y^2 - a^2, \\ Z = z. \end{cases}$$

7. Proof of Theorem 3(2)

Our proof splits into the following three cases.

(*Case* 1) $n = 4, \ p \geq 4$,
(*Case* 2) $n > 4, \ n \leq p$,
(*Case* 3) $n > p, \ p \geq 4$.

Proof in the Case 1. Let $\tilde{Q}(k): (\mathbf{R}^4, 0) \to (\mathbf{R}^4, 0)$ be a one-parameter polynomial map-germ family defined as follows:

$$\tilde{Q}(k): \begin{cases} X = [x(x^2+y^2-u^2) - yz]^2[(x+ky)(x^2+y^2-u^2) - (y-kx)z]^2, \\ Y = x^2+y^2-u^2, \\ Z = z, \\ U = u^2, \end{cases}$$

where (x, y, z, u), (X, Y, Z, U) are coordinates of the source and the target spaces respectively and k is a real parameter. Let $P'(k)$ be a one-parameter polynomial map-germ family defined as follows:

$$P'(k): (\mathbf{R}^4, 0) \to (\mathbf{R}^p, 0), \qquad P'(k)(x, y, z, u) = (\tilde{Q}(k), 0).$$

For any fixed k_0, $P'(k_0)^{-1}(\{0\}) = \{0\}$. Hence $P'(k_0)$ is a C^0-K-finite polynomial map-germ by Proposition 2(b).

Let $H_1'(k_0)$, $H_2'(k_0)$, $H'(k_0)$ and $C'(k_0)$ be as follows:

$$H_1'(k_0) = \{(x, y, z, u) \in \mathbf{R}^4 : x(x^2+y^2-u^2) - yz = 0\},$$
$$H_2'(k_0) = \{(x, y, z, u) \in \mathbf{R}^4 : (x+k_0y)(x^2+y^2-u^2) - (y-k_0x)z = 0\},$$
$$H'(k_0) = H_1'(k_0) \cup H_2'(k_0),$$
$$C'(k_0) = \{(x, y, 0, u) \in \mathbf{R}^4 : x^2+y^2-u^2 = 0\}.$$

Then we see that $H_1'(k_0)$ and $H_2(k_0)$ are both irreducible components of $\mathrm{Sing}(\tilde{Q}(k_0))$ and $H_1'(k_0) \cap H_2'(k_0) = C'(k_0) \cup \{(0, 0, z, u) \in \mathbf{R}^4\}$ (provided that $k_0 \neq 0$). We also have

$$\begin{aligned} &P'(k_0)(H_1'(k_0) \cap \{(x, y, z, u) \in \mathbf{R}^4 : lx + my = 0\}) \\ &\quad = \{(0, Y, Z, U, \mathbf{0}) \in \mathbf{R}^p : mY + lZ = 0\}, \\ &P'(k_0)(H_2'(k_0) \cap \{(x, y, z, u) \in \mathbf{R}^4 : lx + mu = 0) \\ &\quad = \{(0, Y, Z, U, \mathbf{0}) \in \mathbf{R}^p : (m - k_0l)Y + (l + k_0m)Z = 0\}, \end{aligned}$$

for any two real numbers l, m such that $l^2 + m^2 \neq 0$.

If there are germs of homeomorphisms $h: (\mathbf{R}^4, 0) \to (\mathbf{R}^4, 0)$, $h': (\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ such that $P'(k_0) = (h')^{-1} \circ P'(k_1) \circ h$ as germs at 0 for two fixed nonzero real numbers k_0, k_1 $(k_0 \neq k_1)$, then we have the following lemma like Lemma 2 in §6.

Lemma 3. (1) $h(H'(k_0)) = H'(k_1)$ *as germs at* 0,

(2) $h(C'(k_0)) = C'(k_1)$ *as germs at* 0,

(3)

$$h(C'(k_0) \cap P'(k_0)^{-1}((0,0,0,u_0,\mathbf{0}))) \\ = C'(k_1) \cap P'(k_1)^{-1}((0,0,0,h'_4((0,0,0,u_0,\mathbf{0})),\mathbf{0}))$$

as germs at 0 *for any real number* u_0 *close to zero and* h'_4 *is the fourth component function of* h' *(see Figure* 1),

(4) *for any germ of a continuous curve* $\mathbf{q}(t)$ *at any point* $\mathbf{p} = (x, y, 0, u) \in C'(k_0)$ *(resp.* $C'(k_1)$*) in* $H'(k_0)$ *(resp.* $H'(k_1)$*),* $P'(k_0)$ *(resp.* $P'(k_1)$*) maps* $\mathbf{q}(t)$ *to a germ of a continuous curve at* $(0,0,0,u^2,\mathbf{0}) \in \mathbf{R}^p$ *in* $\{(0, Y, Z, U, \mathbf{0}) \in \mathbf{R}^p\}$ *and* $\pi \circ P'(k_0)(\mathbf{q}(t))$ *(resp.* $\pi \circ P'(k_1)(\mathbf{q}(t))$*) is a germ of a continuous curve at* $(0,0,0) \in \mathbf{R}^3$ *in* $\{(0, Y, Z) \in \mathbf{R}^3\}$ *and this germ of a curve has a tangent line at* $(0,0,0)$*, where* $\pi: \mathbf{R}^p \to \mathbf{R}^3$ *is a natural projection*

$$(X, Y, Z, U, V_1, \dots, V_{p-4}) \mapsto (X, Y, Z).$$

Our proof of Lemma 3 is analogous to that of Lemma 2 and we omit it. By this lemma, we have a contradiction to Van Kampen's theorem similar to Thom's proof. Q.E.D.

Proof in the Case 2. Let $P''(k)$ be a one-parameter polynomial map-germ family as follows:

$$P''(k): (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0),$$
$$P''(k)(x, y, z, u, v_1, \dots, v_{n-4}) = (\widetilde{Q}(k)(x, y, z, u), v_1, \dots, v_{n-4}, \mathbf{0}),$$

where $(x, y, z, u, v_1, \dots, v_{n-4})$ is a coordinate of the source space and $\widetilde{Q}(k)$ is as before.

For any fixed k_0, $P''(k_0)^{-1}(\{0\}) = \{0\}$. Hence $P''(k_0)$ is a C^0-K-finite polynomial map-germ by Proposition 2(b).

Let $H''(k_0)$ and $C''(k_0)$ be as follows:

$$H''(k_0) = \{(x, y, z, u, v_1, \dots, v_{n-4}) \in \mathbf{R}^n : \\ [x(x^2+y^2-u^2) - yz][(x+k_0y)(x^2+y^2-u^2) - (y-k_0x)z] = 0\},$$
$$C''(k_0) = \{(x, y, 0, u, v_1, \dots, v_{n-4}) \in \mathbf{R}^n : x^2+y^2-u^2 = 0\}.$$

If there are germs of homeomorphism $h: (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and $h': (\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ such that $P''(k_0) = (h')^{-1} \circ P''(k_1) \circ h$ as germs at 0 for any fixed real numbers k_0, k_1 $(k_0 \neq k_1)$, then we have the following lemma, which is analogous to Lemma 3.

Lemma 4. (1) $h(H''(k_0)) = H''(k_1)$ *as germs at* 0,

(2) $h(C''(k_0)) = C''(k_1)$ *as germs at* 0,

(3)

$$h(C''(k_0) \cap P''(k_0)^{-1}((0,0,0,u^0,v_1^0,\dots,v_{-4}^0,\mathbf{0}))) \\ = C''(k_1) \cap P''(k_1)^{-1}(h'((0,0,0,u^0,v_1^0,\dots,v_{n-4}^0,\mathbf{0})))$$

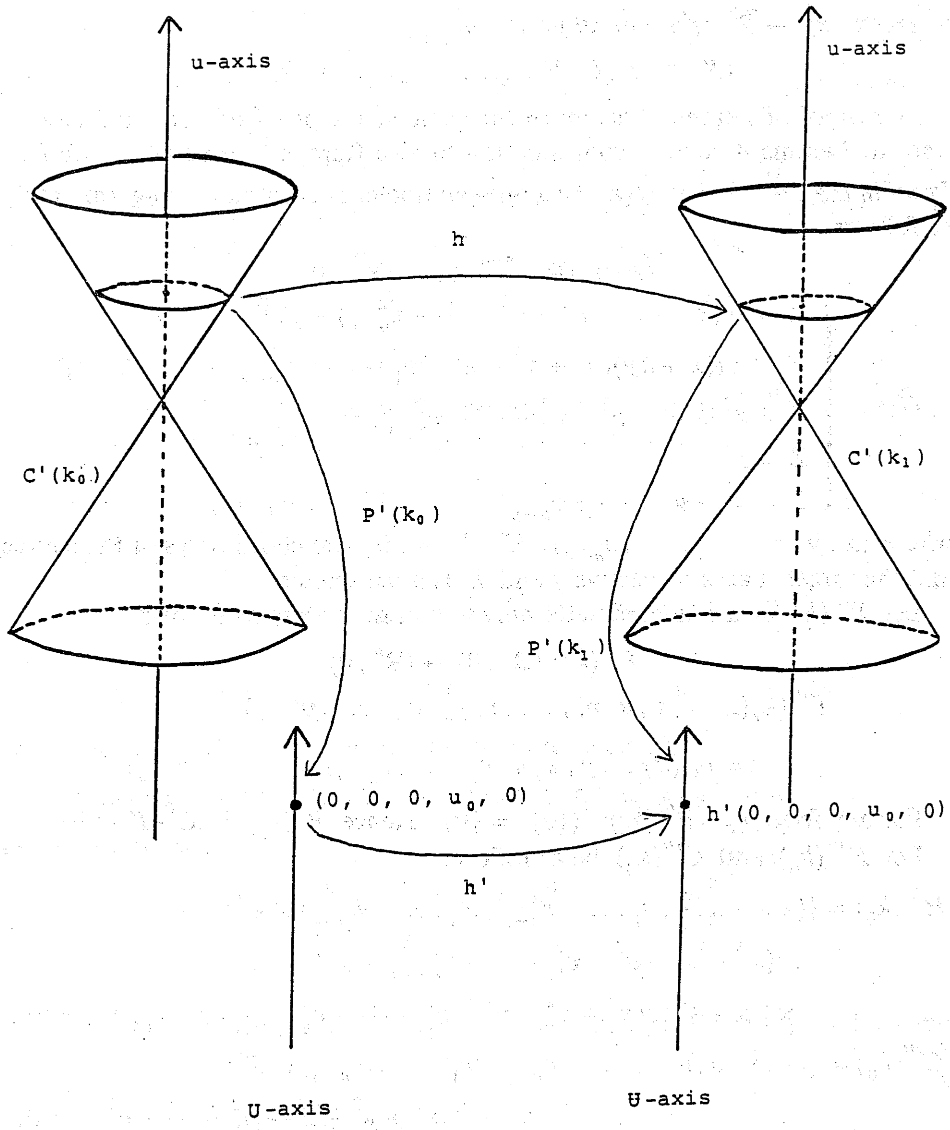

FIGURE 1

as germs at 0 *for any real numbers* u^0 (≥ 0), $v_1^0, \dots, v_{n-4}^0$ *sufficiently close to zero.*

(4) *for any germ of a continuous curve* $\mathbf{q}(t)$ *at any point* $\mathbf{p} = (x, y, 0, u, v_1, \dots, v_{n-4}) \in C''(k_0)$ *(resp.* $C''(k_1)$*) in* $H''(k_0)$ *(resp.* $H''(k_1)$*),* $P''(k_0)$ *(resp.* $P''(k_1)$*) maps* $\mathbf{q}(t)$ *to a germ of a continuous curve at* $(0, 0, 0, u^2, v_1, \dots, v_{n-4}, \mathbf{0}) \in \mathbf{R}^p$ *in* $\{(0, Y, Z, U, V_1, \dots, V_{n-4}, \mathbf{0}) \in \mathbf{R}^p\}$ *and* $\pi \circ P''(k_0)(\mathbf{q}(t))$ *(resp.* $\pi \circ P''(k_1)(\mathbf{q}(t))$*) is a germ of a continuous curve at* $(0, 0, 0) \in \mathbf{R}^3$ *in* $\{(0, Y, Z) \in \mathbf{R}^3\}$ *and this germ of curve has a tangent line at* $(0, 0, 0)$,

where $\pi\colon \mathbf{R}^p \to \mathbf{R}^3$ *is a natural projection*

$$(X, Y, Z, U, V_1, \dots, V_{p-4}) \mapsto (X, Y, Z).$$

Our proof of Lemma 4 is almost the same as the proof of Lemma 3 and we omit it. Lemma 4 yields a contradiction to Van Kampen's theorem. Q.E.D.

Proof in the Case 3. Let $\widehat{Q}(k)$ be a one-parameter polynomial map-germ family as follows:

$$\widehat{Q}(k)\colon (\mathbf{R}^{n-p+4}, 0) \to (\mathbf{R}^4, 0),$$

$$\widehat{Q}(k)\colon \begin{cases} X = [x(x^2+y^2-u^2-v_1^2-\cdots-v_{n-p}^2) - yz]^2 \\ \qquad \times [(x+ky)(x^2+y^2-u^2-v_1^2-\cdots-v_{n-p}^2) - (y-kx)z]^2, \\ Y = x^2+y^2-u^2-v_1^2-\cdots-v_{n-p}^2, \\ Z = z, \\ U = u^2+v_1^2+\cdots+v_{n-p}^2, \end{cases}$$

where $(x, y, z, u, v_1, \dots, v_{n-p})$, (X, Y, Z, U) are coordinates of the source and the target spaces respectively and k is a parameter.

Let $P'''(k)$ be a one-parameter polynomial map-germ as follows:

$$P'''(k)\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0),$$

$$P'''(k)(x, y, z, u, v_1, \dots, v_{n-p}, w_1, \dots, w_{p-4})$$
$$= (\widehat{Q}(k)(x, y, z, u, v_1, \dots, v_{n-p}), w_1, \dots, w_{p-4}).$$

For any fixed k_0, $P'''(k_0)^{-1}(\{0\}) = \{0\}$. Hence $P''(k_0)$ is C^0-K-finite.

Let $H'''(k_0)$ and $C'''(k_0)$ be as follows:

$$H'''(k_0) = \{(x, y, z, u, v_1, \dots, v_{n-p}, w_1, \dots, w_{p-4}) \in \mathbf{R}^n :$$
$$[x(x^2+y^2-u^2-v_1^2-\cdots-v_{n-p}^2) - yz]$$
$$\times [(x+k_0y)(x^2+y^2-u^2-v_1^2-\cdots-v_{n-p}^2) - (y-k_0x)z] = 0\},$$

$$C'''(k_0) = \{(x, y, 0, u, v_1, \dots, v_{n-p}, w_1, \dots, w_{p-4}) \in \mathbf{R}^n :$$
$$x^2+y^2-u^2-v_1^2-\cdots-v_{n-p}^2 = 0\}.$$

If there are germs of homeomorphisms $h\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and $h'\colon (\mathbf{R}^p, 0) \to (\mathbf{R}^p, 0)$ such that $P'''(k_0) = (h')^{-1} \circ P'''(k_1) \circ h$ as germs at 0 for any two fixed nonzero real numbers k_0, k_1, $(k_0 \neq k_1)$, then we have the following lemma, which is analogous to Lemma 3 and Lemma 4; hence no proof is given.

Lemma 5. (1) *$h(H'''(k_0)) = H'''(k_1)$ as germs at 0,*

(2) *$h(C'''(k_0)) = C'''(k_1)$ as germs at 0,*

(3) *for any real numbers u^0 (≥ 0), $w_1^0, \dots, w_{n-p}^0$ sufficiently close to zero,*

$$h(C'''(k_0) \cap P'''(k_0)^{-1}((0, 0, 0, u^0, w_1^0, \dots, w_{p-4}^0)))$$
$$= C'''(k_1) \cap P'''(k_1)^{-1}(h'(0, 0, 0, u^0, w_1^0, \dots, w_{p-4}^0))$$

as germs at 0,

(4) *for any germ of a continuous curve* $\mathbf{q}(t)$ *at any point* $\mathbf{p} \doteq (x, y, 0, u, v_1, \dots, v_{n-p}, w_1, \dots, w_{p-4}) \in C'''(k_0)$ *(resp.* $C'''(k_1)$*) in* $H'''(k_0)$ *(resp.* $H'''(k_1)$*),* $P'''(k_0)$ *(resp.* $P'''(k_1)$*) maps* $\mathbf{q}(t)$ *to a germ of a continuous curve at* $(0, 0, 0, u^2 + \sum v_i^2, w_1, \dots, w_{p-4}) \in \mathbf{R}^p$ *in* $\{(0, Y, Z, U, W_1, \dots, W_{p-4})\} \subset \mathbf{R}^p$ *and* $\pi \circ P'''(k_0)(\mathbf{q}(t))$ *(resp.* $\pi \circ P'''(k_1)(\mathbf{q}(t))$*) is a germ of a continuous curve at* $(0, 0, 0) \in \mathbf{R}^3$ *in* $\{(0, Y, Z) \in \mathbf{R}^3\}$ *and this germ of a curve has a tangent line at* $(0, 0, 0)$*, where* $\pi: \mathbf{R}^p \to \mathbf{R}^3$ *is a natural projection*

$$(X, Y, Z, U, W_1, \dots, W_{p-4}) \mapsto (X, Y, Z).$$

As $C'''(k_0) \cap P'''(k_0)^{-1}(0, 0, 0, u^0, w_1^0, \dots, w_{p-4}^0)$ is a space $\sqrt{u^0}S^1 \times \sqrt{u^0} S^{n-p}$ if $u^0 > 0$, the restriction of the homeomorphism h to $\sqrt{u^0}S^1 \times \sqrt{u^0}S^{n-p}$ maps it to $\sqrt{\tilde{u}^0}S^1 \times \sqrt{\tilde{u}^0}S^{n-p}$, where $\tilde{u}^0 = h_4'(0, 0, 0, u^0, w_1^0, \dots, w_{p-4}^0)$.

Definition 2. In the space $\sqrt{c}S^1 \times \sqrt{c}S^{n-p} = \{(x, y, u, v_1, \dots, v_{n-p}) \in \mathbf{R}^{n-p+3} | x^2 + y^2 = u^2 + \sum v_i^2 = \text{const}\, c\}$ the spaces

$$\{(x, y, u, v_1, \dots, v_{n-p}) \in \mathbf{R}^{n-p+3} | x = \text{const}, y = \text{const}\}$$

and

$$\{(x, y, u, v_1, \dots, v_{n-p}) \in \mathbf{R}^{n-p+3} | u, v_1, \dots, v_{n-p}: \text{const}\}$$

are called *longitude spheres, meridian circles* respectively.

To conclude our proof in Case 3, we need the following lemma.

Lemma 6. *In each*

$$C'''(k_0) \cap P'''(k_0)^{-1}((0, 0, 0, u^0, w_1^0, \dots, w_{p-4}^0)) = \sqrt{u^0}S^1 \times \sqrt{u^0}S^{n-p}$$

for u^0, (> 0) *close to zero, each longitude sphere is mapped to a longitude sphere in*

$$C'''(k_1) \cap P'''(k_1)^{-1}(h'((0, 0, 0, u^0, w_1^0, \dots, w_{p-4}^0)) = \sqrt{\tilde{u}^0}S^1 \times \sqrt{\tilde{u}^0}S^{n-p}$$

by the restriction of the homeomorphism h *of the source space.*

Proof of Lemma 6. We take any germ of a continuous curve $\mathbf{q}(t)$ at $(0, 0, 0, u^0, w_1^0, \dots, w_{p-4}^0)$ in $\{(0, Y, Z, u^0, w_1^0, \dots, w_{p-4}^0) \in \mathbf{R}^p\}$ which has a tangent line at $(0, 0, 0, u^0, w_1^0, \dots, w_{p-4}^0)$. Then $P'''(k_0)^{-1}(\mathbf{q}(t))$ is homeomorphic to $S^{n-p} \times I$ with a certain longitude sphere in $H'''(k_0)$ as its center, where I is an open interval. If the inverse image of this longitude sphere by the homeomorphism h of the source space is not a longitude sphere, then

$$P'''(k_1)(h^{-1}(P'''(k_0)^{-1}(\mathbf{q}(t))))$$

is not a germ of continuous curve at $h'(0,0,0,u^0,w_1^0,\ldots,w_{p-4}^0)$. This is a contradiction to the commutativity $P'''(k_0) = (h')^{-1} \circ P'''(k_1) \circ h$ with homeomorphisms h, h'. Q.E.D.

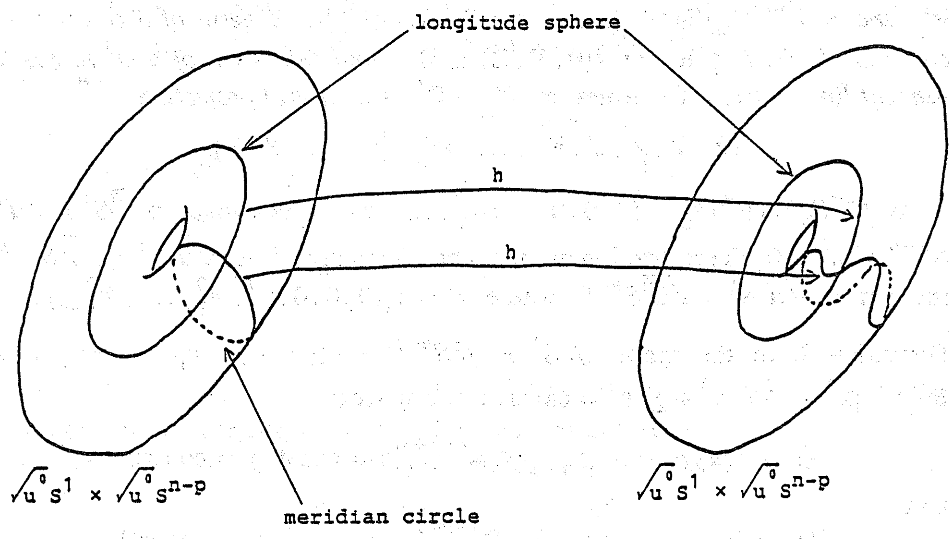

FIGURE 2

By Lemma 6, we have the following:

Lemma 7. *For any*

$$C'''(k_0) \cap P'''(k_0)^{-1}((0,0,0,u^0,w_1^0,\ldots,w_{p-4}^0)) = \sqrt{u^0}S^1 \times \sqrt{u^0}S^{n-p}$$

for any positive number u^0 *close to zero, the image of any meridian circle by the restriction of homeomorphism* h *is isotopic to any meridian circle in*

$$C'''(k_1) \cap P'''(k_1)^{-1}(h'(0,0,0,u^0,w_1^0,\ldots,w_{p-4}^0))$$

by an isotopy with (x, y)*-coordinates preserving.*

Now Lemma 5 and Lemma 7 yield a contradiction to Van Kampen's theorem as same as we see in Cases 1 and 2. Q.E.D.

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