ON THE IMBEDDING OF POLYHEDRA IN MANIFOLDS

By

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This paper deals with the imbedding of finite k-polyhedra in finite combinatorial n-manifolds, $2k+2 \leqq n$, and the main theorem is:

Theorem. *Let M^n, $\tilde{M}^n$ and $\tilde{P}^k$ be two finite combinatorial n-manifolds and a finite k-polyhedron such that $\tilde{M}^n$ is topologically imbedded in M^n, $\tilde{P}^k$ is piecewise linearly imbedded in int $(\tilde{M}^n)$ and $2k+2 \leqq n$. Then for any $\varepsilon>0$ there is an ε-homeomorphism F of M^n onto M^n such that*

$$F|\tilde{M}^n\text{-}U\varepsilon(\tilde{P}^k)=1 \quad \text{and}$$
$$F|\tilde{P}^k \text{ is piecewise linear.}$$

1. V. K. A. M. Gugenheim proved the following:

Lemma 1.[(1)] *Let P^k be a finite k-polyhedron which is piecewise linearly imbedded in a euclidean n-space E^n, $2k+2 \leqq n$, and let f be a piecewise linear homeomorphism of P^k into E^n. Then there is a piecewise linear homeomorphism F of E^n onto E^n such that*

$$F|P^k = f,$$

Moreover it is easy to see that his proof induces the more general theorem as follows:

Proposition 1. *Under the same conditions as in Lemma 1, if L is a subpolyhedron of P^k such that $f|L=1$, then for any $\varepsilon>0$ there is a piecewise linear homeomorphism F of E^n onto E^n such that*

$$d(F, 1) < d(f, 1) + \varepsilon$$
$$F|E^n - U\varepsilon(\bigcup_{t=0}^{1} f_t(P^k - L)) = 1$$

where $f_t(x) = (1-t)x + tf(x)$, $0 \leqq t \leqq 1$.

Let M^n be a finite combinatorial n-manifold. Then we can choose a positive number η and two sets of combinatorial n-balls, $\{B_1, \cdots\cdots, B_j\}$ $\{B'_1, \cdots\cdots, B'_j\}$, which are piecewise linearly imbedded in M^n and satisfy

$$B_1 \cup \cdots\cdots \cup B_j = M^n$$
$$B_i' \supset U_{3\eta}(B_i) \qquad i = 1, \cdots\cdots\cdots, j.$$

For any $i = 1, \cdots\cdots\cdots, j$ we take a continuous function $\varphi_i(x)$, $x \varepsilon M^n$, such that

$$0 \leqq \varphi_i(x) \leqq 1$$
$$\varphi_i(x) = 1 \quad \text{for } x \varepsilon B_i$$
$$= 0 \quad \text{for } x \varepsilon M^n - U_\eta(B_i)$$

Then we define a set of continuous function on M_n, $\{\chi_o(x), \cdots\cdots, \chi_j(x)\}$ by the formula

$$\chi_o(x) = 0$$
$$\chi_i(x) = Max\{\varphi_1(x), \cdots\cdots\cdots, \varphi_i(x)\}$$

Then it is clear that $0 = \chi_o(x) \leqq \chi_1(x) \leqq \cdots\cdots \leqq \chi_j(x) = 1$ and $\chi_{i-1}(x) < \chi_i(x)$ implies $x \varepsilon U_\eta(B_i)$.

Definition. Let P^k and L be a finite polyhedron piecewise linearly imbedded in M^n and a subpolyhedron of P^k such that

$$cl(P^k - L) \subset \text{int } M^n.$$

Then we call (P^k, L) *a pair of subpolyhedra of* M^n. A piecewise linear homeomorphism f of P^k into M^n is called *a piecewise linear homeomorphism of* (P^k, L), if f satisfies

$$f|L = 1$$
$$f(cl(P^k - L)) \subset \text{int } M^n.$$

Proposition 2. *For any $\varepsilon > 0$ there is a positive number $\gamma = \gamma(M^n, \varepsilon)$ such that for any piecewise linear γ-homeomorphism f of any pair (P^k, L) of subpolyhedra of M^n, $2k+2 \leqq n$, there exists a sequence $\{f_o, \cdots\cdots\cdots, f_j\}$ of piecewise linear ε-homeomorphisms of (P^k, L) satisfying*

$$f_o = 1, \qquad f_j = f$$
$$f_i | P^k \cap (M^n - U_\eta(B_i)) = f_{i-1} | P^k \cap (M^n - U_\eta(B_i))$$

and $f_i f_{i-1}^{-1}$ is a piecewise linear ε-homeomorphism of $(f_{i-1}(P^k), L)$, $i = 1, \cdots\cdots\cdots, j$.

Proof. Since M^n is uniformly i-connected, $i = 0, \cdots\cdots\cdots, k$, there is a $\gamma > 0$ such that for any piecewise linear γ-homeomorphism f *of* any pair (P^k, L) there exists an ε-homotopy $\{g_t\}$, $0 \leqq t \leqq 1$, of (P^k, L) in M^n satisfying

$$g_o = 1 \quad g_1 = f \quad g_t | L = 1$$
$$g_t(cl(P^k - L)) \subset \text{int } M^n.$$

We put $f'_i(x) = g_{\chi_i(x)}(x)$, $x \varepsilon P^k$ and $i = 0, \cdots\cdots, j$. Then it is clear that $\{f'_o(x), \cdots\cdots, f'_j(x)\}$ satisfy all conditions of Proposion *2* except that of piecewise linearity. Since $2k+2 \leqq n$, we approximate $\{f'_o(x), \cdots\cdots, f'_j(x)\}$ by piecewise linear ε-homeomorphisms $\{f_o(x), \cdots\cdots\cdots, f_j(x)\}$ of (P^k, L) which sasisfy all the conditions of Proposition *2*.

The main theorem of Part 1 is;

Lemma 2. *For any $\varepsilon > 0$ there is a $\delta = \delta(M^n, \varepsilon) > 0$ such that any piecewise linear δ-homeomorphism f of any pair (P^k, L) of subpolyhedra of M^n, $2k+2 \leqq n$, can be extended to* *a piecewise linear ε-homeomorphism F of M^n onto M^n satisfying*

$$F|P^k = f$$
$$F|M^n - U\varepsilon(P^k - L) = 1$$
$$F|bdry\ M^n = 1.$$

Proposition 1 can be restated as follows:

Proposition 1′. *If M^n is a combinatorial n-ball, Lemma 2 is true.*

Proof of Lemma 2. We take the positive number $\varepsilon' = Min(\frac{\varepsilon}{j}, \eta)$ and by Proposition 1′ we get $\{\delta(B'_1, \varepsilon'), \cdots\cdots, \delta(B'_j, \varepsilon')\}$. Moreover by Proposition *2* we get the number

$$\delta = \gamma(M^n, \delta')$$

where $\delta' = \underset{i=1}{\overset{j}{Min}}\ \{\delta(B'_i, \varepsilon')\}$. Let f be any piecewise linear δ-homeomorphism of any pair (P^k, L) of subpolyhedra of M^n, $2k+2 \leq n$. Then by Proposition *2* there is a sequence of piecewise linear δ′-homeomorphisms, $\{f_o, \cdots\cdots, f_j\}$ of M^n such that

$$f_o = 1 \qquad f_j = f$$
$$f_i|P^k \cap (M^n - U_\eta(B_i)) = f_{i-1}|P^k \cap (M^n - U_\eta(B_i))$$

and $f_i f_{i-1}^{-1}$ is a piecewise linear δ′-homeomorphism of $(f_{i-1}(P^k), L)$. Since $\delta' \leqq \eta$, we have

$$f_i f_{i-1}^{-1}|f_{i-1}(P^k) \cap (M^n - U_{2\eta}(B_i)) = 1, \text{ and}$$

$$f_i f_{i-1}^{-1}|L = 1.$$

We can choose a subpolyhedron L_i of $f_{i-1}(P^k)$ such that $f_i f_{i-1}^{-1}|L_i = 1$ and $L_i \supset ((f_{i-1}(P^k) \cap (M^n - U_{2\eta}(B_i))) \cup L$. Then we have

$$cl(f_{i-1}(P^k) - L_i) \subset \text{int } B'_i$$
$$f_i f_{i-1}^{-1}\ cl(f_{i-1}(P^k) - L_i) \subset \text{int } B'_i.$$

Therefore by Proposition 1′ there is a sequence of piecewise linear ε′-hoemorphisms $\{F_1, F_2, \cdots\cdots\cdots, F_j\}$ of M^n onto M^n such that

$$F_i f_{i-1} = f_i$$
$$F_i|M^n - U\varepsilon'(f_{i-1}(P^k) - L_i) = 1$$
$$F_i|cl(M^n - B'_i) = 1.$$

Since $\varepsilon' \leqq \frac{\varepsilon}{j}$, it is clear that $F = F_j F_{j-1} \cdots\cdots\cdots F_1$ is the required piecewise linear ε-homeomorphism of M^n onto M^n and we have proved Lemma *2*.

Corollary. *Let f be a piecewise linear homeomorphism of a subpolyhedron P^k of a finite combinatorial n-manifold M^n into M^n such that $P^k \subset int\ M^n$, $f(P^k) \subset int\ M^n$, f is homotopic to the identity and $2k+2 \leqq n$. Then f can be extended to a piecewise linear homeomorphism of M^n onto M^n.*

Proof. Since f is homotopic to 1 and $2k+2 \leqq n$, we can choose a sequence of piecewise linear homeomorphisms $\{f_1, \cdots\cdots, f_l\}$ of P^k into $\mathrm{int}(M^n)$ such that $f_0 = 1$, $f_o = f$ and $d(f_i, f_{i-1}) < \delta(M^n, 1)$. By Lemma 2 we have a sequence of piecewise linear homeomorphisms $\{F_1, \cdots\cdots\cdots, F_l\}$ of M^n onto M^n such that

$$F_i f_{i-1} = f_i \qquad i = 1, \cdots\cdots\cdots, l.$$

Then $F = F_l \cdots\cdots F_1$ is the required piecewise linear homeomorphism of M^n onto M^n.

2. We shall prove the Theorem. We have two combinatorial n-manifolds M^n and $\tilde{M}^n$ and then two different metrics and two different piecewise linearities. Therefore the notations without $\sim$ or with $\sim$ relate to M^n or $\tilde{M}^n$ respectively.

Proof of Theorem. At first we choose a piecewise linear $\delta(M^n, \frac{\varepsilon}{4})$-hemeomorphism f of $\tilde{P}^k$ into int M^n and denote $f(\tilde{P}^k)$ by P^k. Since $\tilde{M}^n$ is a compact metric space in M^n, we can choose a positive number $\tilde{\varepsilon}$ such that for any two points x, y of $\tilde{M}^n$, $\tilde{d}(x, y) < \frac{\tilde{\varepsilon}}{2}$ implies $d(x, y) < \frac{\varepsilon}{2}$. We shall contruct two sequences $\{F_o, F_1, \cdots\}$ and $\{\tilde{F}_o, \tilde{F}_1, \cdots\cdots\}$ of piecewise linear homeomorphisms of M^n and of $\tilde{M}^n$ respectively, two sequences $\{\varepsilon_o, \varepsilon_1, \cdots\cdots\cdots\}$ and $\{\tilde{\varepsilon}_o, \tilde{\varepsilon}_1, \cdots\cdots\cdots\}$ of positive numbers and two sequences $\{P_o, P_1, \cdots\cdots\}$ and $\{\tilde{P}_o, \tilde{P}_1, \cdots\cdots\}$ of simplicial subdivisions of P^k and $\tilde{P}^k$ respectively such that

(0) $\tilde{d}(F_i | P^k, \tilde{F}_{i-1} f^{-1}) < \delta(\tilde{M}^n, \tilde{\varepsilon}_{i-1})$

($\tilde{0}$) $d(\tilde{F}_i | \tilde{P}^k, F_i f) < \delta(M^n, \varepsilon_i)$

(1) $d(F_i, F_{i-1}) < \varepsilon_{i-1}$ $\qquad F_0 = 1$

($\tilde{1}$) $\tilde{d}(\tilde{F}_i \tilde{F}_{i-1}) < \tilde{\varepsilon}_{i-1}$ $\qquad \tilde{F}_0 = 1$

(2) $F_i F_{i-1}^{-1} | M^n - U_{\varepsilon i-1}(F_{i-1}(P^k)) = 1$

($\tilde{2}$) $\tilde{F}_i \tilde{F}_{i-1}^{-1} | \tilde{M}^n - \tilde{U}_{\tilde{\varepsilon} i-1}(\tilde{F}_{i-1}(\tilde{P}^k)) = 1$

(3) P_i is a simplicial subdivision of P_{i-1} $\quad$ mesh $(P_i) < \frac{1}{i}$

($\tilde{3}$) $\tilde{P}_i$ is a simplicial subdivision of $\tilde{P}_{i-1}$ $\quad$ mesh $(\tilde{P}_i) < \frac{1}{i}$

(4) F_i is simplicial on P_i

($\tilde{4}$) $\tilde{F}_i$ is simplicial on $\tilde{P}_i$

(5) $2\varepsilon_i < \varepsilon_{i-1}$ $\qquad \varepsilon_0 = \frac{\varepsilon}{4}$

($\tilde{5}$) $2\tilde{\varepsilon}_i < \tilde{\varepsilon}_{i-1}$ $\qquad \tilde{\varepsilon}_0 = \frac{\tilde{\varepsilon}}{4}$

(6) $4\varepsilon_i < d(F_i(\sigma), F_i(\sigma'))$ for any disjoint simplexes σ, σ' of P_i

($\tilde{6}$) $4\tilde{\varepsilon}_i < \tilde{d}(\tilde{F}_i(\tilde{\sigma}), \tilde{F}_i(\tilde{\sigma}'))$ for any disjoint simplexes $\tilde{\sigma}$, $\tilde{\sigma}'$ of $\tilde{P}_i$

(7) $U\varepsilon_i\,(F_i(P^k)) \subset F_i(U_{\frac{1}{i}}(P^k))$

($\tilde{7}$) $\tilde{U}\tilde{\varepsilon}_i\,(\tilde{F}_i(\tilde{P}^k)) \subset \tilde{F}_i(\tilde{U}_{\frac{1}{i}}(\tilde{P}^k))$

First of all we take $F_0 = 1$, $\tilde{F}_0 = 1$, $\varepsilon_0 = \frac{\varepsilon}{4}$, $\tilde{\varepsilon}_0 = \frac{\tilde{\varepsilon}}{4}$ and any simplicial subdivisions P_0 and $\tilde{P}_0$ of P^k and $\tilde{P}^k$ respectively.

Step 1. Since $d(f^{-1}, 1) < \delta(M^n, \varepsilon_0)$, we can choose a piecewise linear $\delta(M^n, \varepsilon_0)$-homeomorphism f_1 of P^k into int M^n such that

$$\tilde{d}(f_1, f^{-1}) < \delta(\tilde{M}^n, \tilde{\varepsilon}_0)$$

By Lemma 2 there is a piecewise linear ε_0-homeomorphism F_1 of M^n onto M^n satisfying the conditions (0), (1), (2). Then we can choose a simplicial subdivision P_1 of P_0 satisfying the conditions (3), (4) and a positive number ε_1 satisfying (5), (6), (7).

Step $\tilde{1}$. Since $\tilde{d}(F_1|P^k, \tilde{F}_0 f^{-1}) < \delta(\tilde{M}^n, \tilde{\varepsilon}_0)$, there is a piecewise linear $\delta(\tilde{M}^n, \tilde{\varepsilon}_0)$-homeomorphism $\tilde{f}_1$ of $\tilde{P}^k$ into int$(\tilde{M}^n)$ such that

$$d(\tilde{f}_1, F_1 f) < \delta(M^n, \varepsilon_1).$$

Then by Lemma 2 there is a piecewise linear $\tilde{\varepsilon}_0$-homeomerphism $\tilde{F}_1$ of $\tilde{M}^n$ onto $\tilde{M}^n$ satisfying ($\tilde{0}$), ($\tilde{1}$), ($\tilde{2}$). We choose $\tilde{P}_1$ satisfying ($\tilde{3}$), ($\tilde{4}$) and $\tilde{\varepsilon}_1$ satisfying ($\tilde{5}$), ($\tilde{6}$), ($\tilde{7}$).

Step 2 is the same as Step 1 and we get F_2, P_2, ε_2 satisfying (0),$\cdots$(7). And then we get all the sequences inductively.

By the conditions (1), ($\tilde{1}$), (5) and ($\tilde{5}$) it is clear that $\{F_0, F_1, \cdots\cdots\}$ and $\{\tilde{F}_0, \tilde{F}_1, \cdots\}$ converge to a continuous mapping G and $\tilde{G}$ of M^n onto M^n and of $\tilde{M}^n$ onto $\tilde{M}^n$ respectively. Furthermore G and $\tilde{G}$ are homeomorphisms. In fact for any different points p and p' of P^k there is an integer $i > 0$ and two disjoint simplexes σ and σ' of P_i such that

$$p \in \sigma \quad \text{and} \quad p' \in \sigma'.$$

Then by the conditions (1), (5), (6) we have

$$\begin{aligned} & d(G(p), G(p')) \\ & \quad \geqq d(F_i(p), F_i(p')) - d(F_i(p), G(p)) - d(F_i(p'), G(p')) \\ & \quad > d(F_i(\sigma), F_i(\sigma')) - 2\varepsilon_i - 2\varepsilon_i \\ & \quad > 4\varepsilon_i - 4\varepsilon_i = 0. \end{aligned}$$

Hence $G(p) \neq G(p')$ and then $G|P^k$ is one to one. Therfore by (2), (7) G is a homeomorphism. Similarly $\tilde{G}$ is a homeomorphism. Furthermore $\tilde{G}$ can be extended to a homeomorphism. $\tilde{G}'$ of M^n onto M^n such that

$$\tilde{G}' | M^n - \mathring{M}^n = 1$$

By (0) we have $G^{-1}\tilde{G}' | \tilde{P}^k = f$. We denote the homeomorphism $G^{-1}\tilde{G}$ by F. Then by the conditions (1), ($\tilde{1}$), (2), ($\tilde{2}$), (5), ($\tilde{5}$), F is the ε-homeomorphism of M^n onto M^n such that

$$F | \tilde{P}^k = f$$

$$F | M^n - U_\varepsilon(\tilde{P}^k) = 1$$

Since f is a piecewise linear homeomorphism of $\tilde{P}^k$ into M^n, we have proved the Theorem.

References

[1] V. K. A.M. Gugenheim "Piecewise Linear Isotopy and Embedding of Elements and Spheres I,,. Proc. London Math. Soc. 3(1952), 29-53.