SUFFICIENTLY DECOMPOSABLE SURFACES IN THE 3-SPHERE

By

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(Received October 20, 1993)

1. Introduction

Waldhausen \([W]\) showed that any Heegaard surface \(H\) of the 3-dimensional sphere \(S^3\) is decomposed into a connected sum of unknotted tori. As its generalization, Tsukui \([T1]\) and Suzuki \([Su1]\) formulated a prime decomposition theorem for any pair \((F \subset S^3)\) of a connected, closed (=compact, without boundary), oriented surface \(F\) in \(S^3\) with a fixed orientation, and discuss among others that whether such prime decompositions are unique. We refer the reader to Tsukui \([T2]\), Suzuki \([Su2]\), \([Su3]\), and Motoo \([M]\) for some related topics.

In this paper, we give an affirmative answer to a question raised by Tsukui \([T1, Conjecture (7.2)]\); that is,

**Theorem.** Let \((F \subset S^3)\) be a pair of a connected, closed, oriented surface \(F\) of genus \(g(F) = n\) in \(S^3\), and \(V_F\) and \(W_F\) be the closures of components of \(S^3\)–\(F\). If both \(V_F\) and \(W_F\) have \(\partial\)-prime decompositions with \(n\) factors, then \((F \subset S^3)\) is not prime.

In \([T2]\), Tsukui gave the proof of this theorem for the case \(n=2\). As a corollary to Theorem, we have the following by induction on \(n\).

**Corollary 1.** Under the hypothesis of Theorem, \((F \subset S^3)\) has a prime decomposition with \(n\) factors.

Combining this with the knot complement theorem due to Gordon and Luecke \([GL]\) and the uniqueness theorem of \(\partial\)-prime decomposition for a compact, orientable 3-manifold with connected boundary \([G1, Sw]\), we have the following.

**Corollary 2.** Under the hypothesis of Theorem, the prime decomposition for \((F \subset S^3)\) is unique, and the knot type of \((F \subset S^3)\) is determined by its complement

1991 Mathematics Subject Classification: Primary 57M99, 57M25.
Key words and phrases: surface, prime decomposition.
(\(V_F, W_F\)).

Throughout the paper, we shall only concern with the combinatorial category, consisting of simplicial complexes and piecewise-linear maps.

After establishing two systems of proper disks in 3-manifolds in §1, we prove [Theorem] in §2.

2. Preliminaries

We will make free use of notation and definitions which were introduced in the paper [Su1].

We use a complete disk system for a compact, connected and orientable 3-manifold with nonvoid connected boundary, which is a generalization of a complete disk system for a compression body (see Casson and Gordon [CG]).

**Definition 1.1.** Let \(E_1, \ldots, E_k\) be exteriors of non-trivial knots in \(S^3\), and \(E_{k+1}, \ldots, E_n\) be solid tori \((\cong D^2 \times S^1)\).

(1) Let \(M\) be a 3-manifold homeomorphic to the disk-sum \(E_1 \cup \cdots \cup E_n\). A disjoint union \(D = D_1 \cup \cdots \cup D_{n-1}\) of proper disks in \(M\) is said to be a *decomposition disk system* for \(M\) iff \(\text{cl}(M - N(D; M)) = E_1 \cup \cdots \cup E_k\), provided that \(k \geq 1\). If \(k = 0\), then \(M\) is a handlebody \(n(D^2 \times S^1)\) of genus \(n\), and a disjoint union \(D = D_1 \cup \cdots \cup D_n\) of proper disks in \(M\) is said to be a *decomposition disk system* iff \(\text{cl}(M - N(D; M)) \cong D^3\), which will be sometimes called a *complete meridian-disk system*.

(2) Let \(F\) be a connected, closed and orientable surface, and let \(d_1, \ldots, d_n\) be mutually disjoint disks in one boundary component \(F \times 1\) of \(F \times I\). Let \(d_i'\) be a disk in \(\partial E_i\) for \(i = 1, \ldots, n\). Now let \(M\) be a 3-manifold obtained from \(F \times I\) and \(E_1 \cup \cdots \cup E_n\) by identifying \(d_i\) and \(d_i'\) for \(i = 1, \ldots, n\). Then, \(\partial M\) consists of two components, and we denote one which corresponds to \(F \times 0\) by \(\partial_- M\) and the other by \(\partial_+ M\). \(\partial_+ M\) is a closed orientable surface of genus \(g(F) + n\).

A disjoint union \(D = D_1 \cup \cdots \cup D_n\) of \(n\) proper disks in \(M\) is said to be a *complete disk system* for \(M\) iff \(\partial D \subset \partial M\) and \(\text{cl}(M - N(D; M)) = (F \times I) \cup E_1 \cup \cdots \cup E_n\). If \(k = 0\), then \(M\) is a compression body and \(D\) is a *complete disk system* for \(M\) in a sense of Casson and Gordon [CG].

3. Proof of Theorem

Let \(V_F \cong A_1 \cup \cdots \cup A_n\) and \(W_F \cong B_1 \cup \cdots \cup B_n\) be \(\partial\)-prime decompositions for \(V_F\) and \(W_F\), respectively. It will be noted that each \(A_i\) and each \(B_i\) are exteriors of knots. If both \(V_F\) and \(W_F\) are handlebodies (i.e. \(A_i \cong B_i \cong D^2 \times S^1\) for \(i = 1, \ldots, n\)), [Theorem] is true by Waldhausen [W] (see also [T1] and [Su2]). Thus,
we may assume that $V_F$ is not a handlebody and thus that $A_i \neq D^2 \times S^1$ for $i=1, \ldots, r$, and $A_j \cong D^2 \times S^1$ for $j=r+1, \ldots, n$, and $r \geq 1$.

Let $D_V$ be a decomposition disk system for $V_F$, and let $cI(V_F-N(D_V; V_F)) = V_1 \cup \cdots \cup V_r$ with $V_i \cong A_i$ for $i=1, \ldots, r$. Now let $U = N(\partial V_1; V_1) \cup N(D_V; V_F) \cup V_2 \cup \cdots \cup V_r$. It will be noticed that $D_V$ is a complete disk system for $U$ and $\partial U = F$. We can easily see that $W_F \cup U = (S^3 - *V_1) \cup N(\partial V_1; V_1) \cong S^3 - *V_1$ is a solid torus by [F] or [Ho]. (See Figure 1.)

We have the following claim:

Claim 1. There exists a meridian disk $D$ for $W_F \cup U$ with $D \cap (V_2 \cup \cdots \cup V_r) = \emptyset$.

Proof. Let $W = W_F \cup U$. We may show that

(*) $W - *(V_2 \cup \cdots \cup V_r) \cong r(D^2 \times S^1)$.

If (*) is true, then we see that the homomorphism $\pi_1(\partial W) \to \pi_1(W - *(V_2 \cup \cdots \cup V_r))$ of fundamental groups induced by the inclusion is not injective, and thus that there exists a simple essential loop $\alpha$ in $\partial W$ such that $\alpha$ bounds a disk $D$ in $W - *(V_2 \cup \cdots \cup V_r)$ by the loop theorem, and $D$ is a desired meridian disk.

We now show (*) by induction on $r$. If $r=1$, then there is nothing to prove. So, we assume that $r \geq 2$. We know that $W - *(V_2 \cup \cdots \cup V_{r-1}) \cong (r-1)(D^2 \times S^1)$ by the induction hypothesis, and that $V_r$ is contained in $*(W - *(V_2 \cup \cdots \cup V_{r-1}))$. Let us consider the following diagram of homomorphisms of fundamental groups induced by inclusions.
The map $i_{1}$ is injective because $V_{r}$ is an exterior of a nontrivial knot. If $i_{2}$ is injective, then both $i_{3}$ and $i_{4}$ are injective by Van Kampen's theorem, and then we have an injection from $Z \oplus Z$ to the free group of rank $r-1$, which is a contradiction. Therefore, $i_{2}$ is not injective. By the loop theorem, we have a simple essential loop $\beta$ in $\partial V_{r}$ such that $\beta$ bounds a disk $E$ in $W-\ast(V_{1}\cup \cdots \cup V_{r-1})$. Let $S=N(\partial V_{r}\cup E; W-\ast(V_{1}\cup \cdots \cup V_{r-1}))$. Then, we can easily see that $S$ is homeomorphic to a one-punctured solid torus and so $(W-\ast(V_{1}\cup \cdots \cup V_{r-1}))-\ast S$ is homeomorphic to a one-punctured $(r-1)$ $(D^{2}\times S^{1})$. Now we can conclude that $W-\ast(V_{1}\cup \cdots \cup V_{r}) \cong r(D^{2}\times S^{1})$, and completing the proof of Claim 1. \( \square \)

By Claim 1 we can choose a meridian disk $D$ for the solid torus $W_{F}\cup U$ such that $D\cap U$ consists of some disks and an annulus $A$. We may assume that $D\cap U$ has the minimum number of disks among all such meridian disks. It follows from the choice of $D$ that the connected planar surface $P=D\cap W_{F}$ is incompressible in $W_{F}$.

The proof of Theorem is divided into two cases.

**Case 1.** $W_{F}$ is a handlebody (i.e. $B_{i}\cong D^{3}\times S^{1}$ for each $i$): In this case, we have a similar result to Haken's lemma (Casson and Gordon [CG], Lemma 1.1) for $D\subset W_{F}\cup U$. This result enables us to construct a 2-sphere in $S^{3}$ which gives a non-trivial decomposition for $(F\subset S^{3})$.

We have the following claim:

**Claim 2.** There exists a complete meridian-disk system $D_{w}$ for $W_{F}$ such that $P\cap D_{w}=\emptyset$.

**Proof.** Let $D_{w}=D_{1}\cup \cdots \cup D_{n}$ be a complete meridian-disk system for $W_{F}$, and we assume that $P\cap D_{w}$ has the minimum number of components among all such meridian-disk systems. By incompressibility of $P$ and the standard innermost circle argument, we may assume that $P\cap D_{w}$ consists of simple proper arcs in $P$.

We suppose that there exist arcs $\alpha_{i}$ in $P\cap D_{w}$ which are inessential in $P$, and let $\nabla_{i}$ be the disks on $P$ cut off by $\alpha_{i}$. We choose an innermost arc, say $\alpha_{i}$, so that $\nabla_{i}$ does not contain any other $\alpha_{i}$. We assume that $\alpha_{i}\subset P\cap D_{i}$, and $\alpha_{i}$ divides $D_{i}$ into two subdisks, say $d_{i}^{'}$ and $d_{i}^{''}$. Then, we have proper disks $D_{i}^{'}=\nabla_{i}\cup d_{i}^{'}$ and $D_{i}^{''}=\nabla_{i}\cup d_{i}^{''}$ in $W_{F}$. We can deform $D_{i}^{'}\cup D_{i}^{''}$ into general position in $W_{F}$, so that

$$P\cap(D_{i}^{'}\cup D_{i}^{''}\cup D_{2}\cup \cdots \cup D_{n})=P\cap D_{w}-\alpha_{1},$$

and

$$(D_{i}^{'}\cup D_{i}^{''})\cap D_{w}=\emptyset.$$
(In fact, $D_i' \cup D_i''$ is obtained from $D_i$ by a modification $\nabla$ along $\nabla_i$ in the sense of [Sul, Def. 3.1]. See Figure 2.) Since $D_i' \cup D_i''$ is contained in the 3-ball $B^3 = c(B(F - N(D_i W ; F)))$, both $\partial D_i'$ and $\partial D_i''$ bound disks on $\partial B^3$. It is easily checked that one of $D' = D_i' \cup D_1 \cup \cdots \cup D_n$ and $D'' = D_i'' \cup D_1 \cup \cdots \cup D_n$ is a complete meridian-disk system for $F$. This contradicts to the minimality of $P \cap D_i W$, and so $P \cap D_i$ does not contain inessential arcs.

![Figure 2](image_url)

We now suppose that each component $\beta_i$ of $\beta = P \cap D_i W$ is an essential arc in $P$. Let $D \cap F = C_1 \cup \cdots \cup C_m$ be simple loops. Let $Q$ be the planar surface obtained from $P$ by cutting along the arcs $\beta$, that is, $Q = c(P - N(P \cap D_i W ; P))$, which is properly embedded in the 3-ball $B^3 = c(B(F - N(D_i W ; F)))$. Since $P$ is incompressible in $F$, $Q$ is incompressible in $B^3$. Therefore we see that each component of $Q$ is a disk and thus that $C_i \cap \beta = \emptyset$ for each $i$.

Now we say that an arc $\beta_i$ is of type I (resp. of type II) if the two points $\partial \beta_i$ contains a single component of $D \cap F$ (resp. two distinct components of $D \cap F$). Then, from the proof of Lemmas 1 and 2 of Ochiai [OC], there exists a disk $C_i$ such that each arc in $\beta$ which meets $C_i$ is of type II, and some sequence of isotopies of type A at these arcs (see Jaco [J] p. 24) has been to reduce the number of disks in $D \cap U$. This contradicts to the minimality of $D \cap U$ (and $P \cap D_i W$), and completing the proof of Claim 2.

Since $P$ is incompressible in $F$, by Claim 2, we conclude that $P$ is a disk, and so $D \cap U$ consists of the annulus $A$. Now we have the following.

**Claim 3.** There exists a complete disk system $D_v \ast$ for $U$ with $D_v \ast \cap A = \emptyset$.

**Proof.** We remember that $D_v$ is a complete disk system for $U$, and let $D_v = D_1 \cup \cdots \cup D_n$. If $D_v \cap A = \emptyset$, then $D_v$ is a required system for $U$. Thus, we may suppose that $D_v \cap A \neq \emptyset$ and that each component of $D_v \cap A$ is an arc since $A$ is incompressible in $U$. It will be noticed that each arc in $D_v \cap A$ is inessential in $A$, since its both endpoints are contained in one boundary component $\partial A \cap \partial U = \partial A \cap F$ of $A$. Let $\alpha$ be an arc in $D_v \cap A$ which is innermost.
on $A$, and let $\nabla$ be the disk on $A$ cut off by $\alpha$. We assume that $\alpha \subset D_i \cap A$, and $\alpha$ divides $D_i$ into two subdisks, say $d_i'$ and $d_i''$. Then, we have proper two disks $D_i' = \nabla \cup d_i'$ and $D_i'' = \nabla \cup d_i''$ in $U$. We can deform $D_i' \cup D_i''$ into general position in $U$, so that

$$(D_i' \cup D_i'' \cup D_3 \cup \cdots \cup D_{n-1}) \cap A = D_v \cap A - \alpha,$$

and

$$(D_i' \cup D_i'') \cap D_v = \emptyset.$$

We may assume that $D_i' \cup D_i''$ is contained in one of $N(\partial V; V), V_1, \ldots, V_r$.

If $D_i' \cup D_i'' \subset N(\partial V; V)$ (resp. $D_i' \cup D_i'' \subset V$ for some $i$), then both $D_i'$ and $D_i''$ bound disks on $\partial N(\partial V; V)$ (resp. $\partial V$) and cut off 3-balls from $N(\partial V; V)$ (resp. $V$), since both $N(\partial V; V)$ and $V$ are $\partial$-irreducible and irreducible. By a similar way to the proof of Claim 2, it is easily checked that one of $D_v' = D_i' \cup D_i'' \cdots \cup D_{n-1}$ and $D_v'' = D_i' \cup D_i'' \cdots \cup D_{n-1}$ is a complete disk system for $U$.

By the repetition of the procedure, we can get rid of all arcs in $D_v \cap A$, and we have a required complete disk system $D_v^*$. $\square$

Now let $W^* = cl(U - N(D_v^*; U))$, and let $N^*$ be the component of $W^*$ which corresponds to $F \times I$ in Definition 1.1 (2). Then, we can see that only one component of $\partial N^*$ contains some disks in $cl(\partial N(D_v^*; U) - \partial U)$. Now we denote this component by $\partial N^*$. Since only one component of $\partial A$ is contained in $\partial N^*$ and $\partial A$ does not separate $\partial N^*$, we can take a simple loop $\gamma$ in $\partial N^*$ such that $\gamma \cap \partial A$ consists of one point and $\gamma \cap N(D_v^*; U) = \emptyset$. Let

$$\Delta = cl(\partial N(P \cup \gamma; W_F) - F),$$

where $P = D_v \cap W_F$ is a disk. Then, $\Delta$ is a proper disk in $W_F$, and $\partial \Delta$ bounds a disk in $\partial N^*$. Hence $\partial \Delta$ bounds a proper disk, say $\Delta^*$ in $N^*$ and thus in $U = W^* \cup N(D_v^*; U) \subset V_F$.

Let $\Sigma = \Delta \cup \Delta^*$. Then $\Sigma$ is a 2-sphere which gives a decomposition for $(F \subset S^3)$ into a surface of genus 1 and a surface of genus $n - 1$. This completes the proof of Case I. $\square$

**Case II.** $W_F$ is not a handlebody: In this case, we may assume that $B_i \neq D^* \times S^1$ for $i = 1, \ldots, s$, $B_j \cong D^* \times S^1$ for $j = s+1, \ldots, n$ and $s \geq 1$.

If $D \cap U$ has no disks, that is $D \cap U$ is an annulus, then we can construct a 2-sphere which gives a decomposition for $(F \subset S^3)$ as in Case I. Therefore, we may suppose that $D \cap U$ has some disks. We have the following claim by similar arguments to the proofs of Claim 2 and Claim 3.

**Claim 4.** There exists a complete disk system $D_w$ for $W_F$ such that each component of $P \cap D_w$ is an essential arc in $P$. $\square$
Let $W_i, \ldots, W_s$ be the components of $\text{cl}(W_P-\text{N}(D_w;W_F))$ with $W_i \cong B_i$ for $i=1, \ldots, s$.

We suppose that $P \cap D_w \neq \emptyset$, and let $d_1, \ldots, d_m$ be the disks of $D \cap U$ and let $C_i=\partial d_i$ for $i=1, \ldots, m$. Then we have the following claim.

**Claim 5.** There exists $C_i$ with $C_i \cap (P \cap D_w) = \emptyset$.

**Proof.** We suppose that Claim 5 is false. Then for every $i$, there exists an arc in $P \cap D_w$ that meets $C_i$. By the technique of Ochiai [OC], there exists $C_j$ such that each arc in $P \cap D_w$ that meets $C_j$ is of type II, and some sequence of isotopies of type A reduces the number of disks in $D \cap U$, and contradicting minimality of $D \cap U$. \[\square\]

Let $C_1$ be a loop with $C_1 \cap (P \cap D_w) = \emptyset$, and let $Q$ be the component of planar surface $\text{cl}(P-\text{N}(P \cap D_w;P))$ with $Q \supset C_1$. It will be noticed that $Q$ is a planar surface properly embedded in some $W_j$, and $Q$ is incompressible in $W_j$ since $P$ is incompressible in $W_F$. Hence $C_1$ is essential in $\partial W_j$ and bounds the disk $d_1$ in $U \subset V_F$. Since $C_1$ does not separate $\partial W_j$, we can take a simple loop, say $\gamma$, on $\partial W_j$ such that $\gamma \cap C_1$ consists of one point and $\gamma \cap \text{N}(D_w;W_F) = \emptyset$. Now let

$$\Delta = \text{cl}(\partial \text{N}(d_1 \cup \gamma;V_F)-\partial W_j).$$

Then $\Delta$ is a proper disk in $V_F$ and $\partial \Delta$ bounds a disk in $\partial W_j$. Hence $\partial \Delta$ bounds a proper disk, say $\Delta'$, in $W_j \subset W_F$.

Then the 2-sphere $\Sigma = \Delta \cup \Delta'$ gives a decomposition for $(F \subset S^3)$ into a surface of genus 1 and a surface of genus $n-1$.

If $P \cap D_w = \emptyset$, then we may assume that $P = D \cap W_F$ is contained in some $W_j$, and we have a 2-sphere which gives a decomposition for $(F \subset S^3)$ by the same argument as above (provided that $P$ is substituted for $Q$).

This completes the proof of Case II, and we complete the proof of Theorem. \[\square\]

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