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EXTREME AND EXPOSED POINTS IN QUOTIENTS OF DOUGLAS ALGEBRAS BY H^{∞} OR $H^{\infty}+C$

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ABSTRACT. For a Douglas algebra B, we study extreme and exposed points of the unit ball of B/H^{∞} or $B/H^{\infty}+C$. Characterizations of extreme and exposed points in B/H^{∞} are given. And we give conditions on B that the unit ball of $B/H^{\infty}+C$ has extreme points or no extreme points.

1. Introduction

Let H^{∞} be the set of boundary values of bounded analytic functions in the unit disk D of the complex number plane. Then H^{∞} is the (essentially) uniformly closed subalgebra of L^{∞}, bounded measurable functions on ∂D with respect to the nomalized Lebesgue measure m. A uniformly closed subalgebra B between H^{∞} and L^{∞} is called a Douglas algebra. We denote by $M(B)$ the maximal ideal space of B. We put $X=M(L^{\infty})$. Let $\hat{m}$ be the lifting measure of m onto X. Let C be the space of continuous functions on ∂D, then $H^{\infty}+C$ is the smallest Douglas algebra containing H^{∞} properly. Basic properties for Douglas algebras and H^{∞} can be found in [7] and for uniform algebras in [6].

We put

$$QC=(H^{\infty}+C)\cap\overline{(H^{\infty}+C)} \qquad \text{and} \qquad QA=H^{\infty}\cap QC\,.$$

In [17], Wolff showed the following excellent theorem.

Wolff's theorem. *If f is a function in L^{∞}, then there is an outer function q in QA such that $qf\in QC$.*

Wolff's theorem gives us many informations about the behaviors of L^{∞} functions on X (see [17]). Here we use it some times.

In [1], Amar and Lederer showed that if E is a closed subset of X with $\hat{m}(E)=0$, then there is a peak set P for H^{∞} with $E\subset P\subsetneqq X$. In Section 2, we will show that P can be taken as a peak set for QA (Theorem 1). If we use both Amar and Lederer, and Wolff's theorem, it is easy to show Theorem 1.

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For, by Amar and Lederer's theorem there is a peaking function f in H^∞ for some peak set P with $E\subset P\subsetneq X$, then by Wolff's theorem there is an outer function q in QA such that $(1-f)q\in H^\infty\cap QC=QA$. By [17, Lemma 2.3], $\{x\in X; (1-f)q(x)=0\}$ is a desired peak set for QA. We will give the proof of Theorem 1 using only Wolff's theorem.

The main subject in this paper is to study extreme and exposed points in quotient spaces of Douglas algebras. For a Banach space Y, we denote by ball(Y) the closed unit ball of Y. A point $x\in$ ball(Y) is called extreme if $x=(x_1+x_2)/2$ for $x_1, x_2\in$ ball(Y) implies $x=x_1=x_2$. A point $x\in$ ball(Y) is called exposed if there is a linear functional L in the dual space Y^* such that $\|L\|=L(x)=1$ and $L(y)\neq 1$ for every $y\in$ ball(Y) with $y\neq x$. We note that exposed points are extreme points. A characterization of extreme points of ball(L^∞/H^∞) is given by Koosis ([14]), and a characterization of exposed points of ball(L^∞/H^∞) is given by Izuchi and Younis ([13]). Axler, Berg, Jewell and Shields ([3]) showed that ball$(L^\infty/H^\infty+C)$ does not have extreme points. For a general Douglas algebra B, extreme and exposed points of ball(L^∞/B) are studied in [10, 11 and 13] (also see these references). Our problem here is to study the case that L^∞ is replaced by a Douglas algebra B. Our questions are:

Question 1. *Give characterizations of extreme and exposed points of* ball(B/H^∞).

Question 2. *For which Douglas algebra B, does* ball$(B/H^\infty+C)$ *have extreme points?*

Answers for Question 1 will be given in Theorems 2 and 3 (in Section 3). But we can not give a complete answer for Question 2. We will give partial answers for Question 2 in Theorems 4 and 5 (in Section 4).

2. Peak sets for QA

For a point $x\in M(H^\infty)$, we denote by μ_x the unique representing measure on X for x. A closed subset E of X is called a support set if there is $x\in M(H^\infty+C)\backslash X$ such that $E=\operatorname{supp}\mu_x$. In [16], Sarason gave the following characterization of QC.

Lemma 1. *$QC=\{f\in L^\infty$; f is constant on each support set$\}$.*

Since QC is the C^*-subalgebra of L^∞, $M(QC)$ is a quotient space of X by considering that each QC-level set is one point. Here, for a point x_0 in X, $\{x\in X; f(x)=f(x_0)$ for every $f\in QC\}$ is called a QC-level set. Thus there is a natural projection π_0 from X onto $M(QC)$. Let $\hat{m}_0$ be the lifting measure of m onto $M(QC)$. That is, $\hat{m}_0$ is the probability measure on $M(QC)$ such that

$$\int_{\partial D} f\,dm = \int_{M(QC)} f\,d\hat{m}_0 \quad \text{for every} \quad f \in QC\,.$$

Our theorem is a generalization of Amar and Lederer's H^∞ peak set theorem ([1]).

Theorem 1. *If E is a closed subset of X such that $\hat{m}(E)=0$, then $\hat{m}_0(\pi_0(E))=0$ and there is a peak set P for QA such that $E\subset P\subsetneqq X$.*

To show Theorem 1, we need some lemmas. Wolff gave the following lemma in [17, Lemma 2.3].

Lemma 2. *A closed G_δ-set S of $M(QC)$ with $\hat{m}_0(S)=0$ is a peak interpolation set for QA.*

The key point to prove Theorem 1 is how to use Wolff's theorem to show $\hat{m}_0(\pi_0(E))=0$. For a subset F of L^∞, we denote by $[F]$ the closed subalgebra generated by F.

Lemma 3. *For a sequence $\{f_n\}_{n=1}^\infty$ in L^∞, we put $B=[H^\infty, f_n; n=1, 2, \cdots]$. Then there is an outer function $q\in QA$ such that $qB\subset H^\infty+C$.*

Proof. By Lemma 2.2 in [12], there is a Blaschke product b such that $bB\subset H^\infty+C$. By Wolff's theorem, there is an outer function $q\in QA$ such that $qb\in QC$. Then

$$qB=qb\cdot bB\subset QC(H^\infty+C)\subset H^\infty+C\,.$$

Lemma 4. *For a sequence $\{f_n\}_{n=1}^\infty$ in L^∞, there is an outer function $q\in QA$ such that $qf_n\in QC$ for every n.*

Proof. We put $B=[H^\infty, f_n, \bar{f}_n; n=1, 2, \cdots]$. Then by Lemma 3, there is an outer function $q\in QA$ such that $qB\subset H^\infty+C$. Thus we get

$$qf_n,\ q\bar{f}_n\in H^\infty+C \quad \text{for every} \quad n\,.$$

Let E be a support set such that $q\neq 0$ on E. Then q is non-zero constant on E by Lemma 1. Also we get

$$\operatorname{Re} f_n|_E\in (H^\infty+C)|_E=H^\infty|_E \quad \text{and} \quad \operatorname{Im} f_n|_E\in H^\infty|_E\,.$$

This shows that f_n is constant on E, because E is a set of antisymmetry for H^∞. Hence qf_n is constant on E and this means that $qf_n\in QC$ by Lemma 1.

Proof of Theorem 1. We can take a decreasing sequence $\{U_n\}_{n=1}^{\infty}$ of open-closed subsets of X such that $E\subset U_n$ $(n=1, 2, \cdots)$ and $\hat{m}(\bigcap_n U_n)=0$. Then clearly we get $\hat{m}(U_n)\to 0$ as $n\to\infty$. Let χ_n be the characteristic function of U_n. Then by Lemma 4, there is an outer function $q\in QA$ such that $q\chi_n\in QC$ for every n. We put

$$V_n=\{x\in X;\ (q\chi_n)(x)\neq 0\} \quad \text{and} \quad V_0=\{x\in X;\ q(x)=0\}\ .$$

We note that $V_n\subset U_n$. Since q and $q\chi_n$ are contained in QC, we get

$$V_n=\pi_0^{-1}(\pi_0(V_n)) \quad \text{and} \quad V_0=\pi_0^{-1}(\pi_0(V_0))\ .$$

If we put $W_n=V_n\cup V_0$, then $W_n=\pi_0^{-1}(\pi_0(W_n))$. Since $\bar{V}_n\subset U_n$, q vanishes on $\bar{V}_n\setminus V_n$. This implies that $\bar{V}_n\setminus V_n\subset V_0$. Hence W_n is closed and $W_n\supset U_n$. We note that $\hat{m}_0(G)=\hat{m}(\pi_0^{-1}(G))$ for any closed subset G of $M(QC)$. Since q is outer, we have $\hat{m}(V_0)=0$ and thus $\hat{m}(W_n)=\hat{m}(U_n)$. If we put $K=\bigcap_n W_n$, then $E\subset\bigcap_n U_n\subset K$ and $K=\pi_0^{-1}(\pi_0(K))$. Since

$$\hat{m}(K)\leqq\hat{m}(W_n)=\hat{m}(U_n)\to 0 \quad \text{as} \quad n\to\infty\ ,$$

we have $\hat{m}_0(\pi_0(E))=0$. Then there exists a closed G_δ-set P_0 of $M(QC)$ such that $\pi_0(E)\subset P_0$ and $\hat{m}_0(P_0)=0$. If we put $P=\pi_0^{-1}(P_0)$, then P is a peak set for QA such that $E\subset P\subsetneqq X$ by Lemma 2.

Corollary 1. *For a closed subset E of X, $\hat{m}(E)=0$ if and only if $\hat{m}_0(\pi_0(E))=0$.*

Using Lemma 4, we get the following proposition by the same way as the proof of Theorem 2.1 of [12].

Proposition 1. *Let B be a Douglas algebra with $B\supset H^\infty+C$ and let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of annihilating measure on X for B, that is, $\mu_n\in B^\perp$ for every n. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of measures on X such that λ_n is absolutely continuous with respect to μ_n for every n. Then there exists an outer function $q\in QA$ such that $q\lambda_n\in B^\perp$ for every n.*

In [12], we showed the following corollary using M-ideal's theorem. Here we give another proof using Proposition 1.

Corollary 2. *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of measures on X such that $\mu_n\in(H^\infty+C)^\perp$ for every n. If we put E the closure of $\bigcup\{\text{supp}\,\mu_n;\ n=1, 2, \cdots\}$ in X, then $\hat{m}(E)=0$.*

Proof. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of measures on X such that $\mu_n\in(H^\infty+C)^\perp$. By Proposition 1, there is an outer function $q\in QA$ such that $q|\mu_n|\in(H^\infty+C)^\perp$ for every n. Then we get

$$\int |q|^2 d|\mu_n|=0 \quad \text{for every} \quad n\,,$$

because $\bar{q}\in H^\infty+C$. Thus $q=0$ on the closure of $\bigcup\{\operatorname{supp}\mu_n;\, n=1,2,\cdots\}$. Since q is outer, we get our assertion.

3. Extreme and exposed points of ball(B/H^∞)

Throughout of this and next sections, let B be a Douglas algebra with $B\supset H^\infty+C$ and let Γ be the essential set for B, that is, Γ is the smallest closed subset of X for which $f\in L^\infty$ vanishing on Γ implies $f\in B$. In this section, we give a complete answer for Question 1.

Theorem 2. *Let $f\in B$ with $\|f+H^\infty\|=1$. Then $f+H^\infty$ is an extreme point of* ball(B/H^∞) *if and only if $f+H^\infty$ is an extreme point of* ball(L^∞/H^∞).

Proof. Assume that $f+H^\infty$ is an extreme point of ball(B/H^∞). Since H^∞ has the best approximation property, we may assume that $\|f\|=1$. Moreover suppose that

$$|f+h|\leqq 1 \quad \text{and} \quad |f+h|\neq 1 \quad \text{on } X \quad \text{for some} \quad h\in H^\infty\,.$$

Since $|H^\infty+C|=|L^\infty|$ by [2], there is $g\in H^\infty+C$ such that

$$g\neq 0\,, \quad g\notin H^\infty \quad \text{and} \quad |f+h\pm g|\leqq 1 \quad \text{on} \quad X\,.$$

Then we have $\|f+H^\infty\pm(g+H^\infty)\|\leqq 1$ and $g+H^\infty\neq H^\infty$. Since $g\in B$, this implies that $f+H^\infty$ is not an extreme point of ball(B/H^∞). So that we get

$$|f+h|=1 \quad \text{on } X \quad \text{for every} \quad h\in H^\infty \quad \text{with} \quad \|f+h\|=1\,.$$

This shows us f has a unique best approximation 0 in H^∞ and $|f|=1$ on X, because if $h\neq 0$, consider $f+h/2$. By Koosis' theorem ([14]), $f+H^\infty$ is an extreme point of ball(L^∞/H^∞).

The converse is trivial.

Theorem 3. *Let $f\in B$ with $\|f+H^\infty\|=1$. Then $f+H^\infty$ is an exposed point of* ball(B/H^∞) *if and only if $f+H^\infty$ is an exposed point of* ball(L^∞/H^∞).

Proof. Assume that $f+H^\infty$ is an exposed point of ball(B/H^∞) and $\|f\|=1$. Then there is a measure μ on X such that

$$\|\mu\|=1\,, \qquad \mu\perp H^\infty\,, \qquad \int f d\mu=1 \quad \text{and} \quad \int g d\mu\neq 1$$

$$\text{for every} \quad g\in B \quad \text{with} \quad \|g+H^\infty\|=1 \quad \text{and} \quad g+H^\infty\neq f+H^\infty\,.$$

We put $\mu=\mu_a+\mu_s$, where $\mu_a \ll \hat{m}$ and $\mu_s \perp \hat{m}$. To show $\mu_a \neq 0$, suppose that $\mu_a=0$. Then we get $\mu \perp H^\infty+C$. By Corollary 2, we have $\hat{m}(\operatorname{supp}\mu)=0$. By Amar and Lederer's therorem (or Theorem 1), there is a non-constant function h_1 in H^∞ such that $\|h_1\|=1$ and

$$\{x \in X;\ h_1(x)=1\}=\{x \in X;\ |h_1(x)|=1\} \supset \operatorname{supp}\mu .$$

By [11, Corollary 2], we have $\{x \in X;\ h_1(x)=1\} \supsetneqq \operatorname{supp}\mu$, so we can take a non-zero function $h_2 \in H^\infty+C$ such that

$$\|h_1+h_2\|=1 \quad \text{and} \quad \operatorname{supp} h_2 \cap \operatorname{supp}\mu=\emptyset .$$

Since non-trivial peak set, $\{x \in X;\ h_1(x)=1\}$, has $\hat{m}$-measure zero, we have $\operatorname{supp} f \not\subset \{x \in X;\ h_1(x)=1\}$. So we may asuume that $h_2 f \neq 0$. Since $\operatorname{supp} h_2 \neq X$, we note that $h_2 f \notin H^\infty$. Then we have $h_1 f,\ (h_1+h_2)f \in B$, $\|h_1 f\|=\|(h_1+h_2)f\|=1$ and

$$\int h_1 f\,d\mu=\int (h_1+h_2) f\,d\mu=1 .$$

This shows $\|h_1 f+H^\infty\|=\|(h_1+h_2)f+H^\infty\|=1$. Since $f+H^\infty$ is exposed, we get $h_1 f+H^\infty=(h_1+h_2)f+H^\infty=f+H^\infty$. Thus we get a contradiction $h_2 f \in H^\infty$. This contradiction gives us $\mu_a \neq 0$. Since $\|f\|=1$, $\|\mu\|=1$ and $\int f\,d\mu=1$, we have $\int f\,d\mu_a=\|\mu_a\|$. Since $\mu_a \perp H^\infty$, there is a function F in H_0^1 such that $\int_{\partial D} fF\,dm=\|F\|_1$. Thus we get $fF \geqq 0$. By Izuchi and Younis' characterization theorem of exposed points of ball (L^∞/H^∞) ([13]), $f+H^\infty$ is an exposed points of ball (L^∞/H^∞).

The converse is trivial.

Using Theorems 2 and 3, we can study extreme and exposed points of other quotient spaces. Chang ([5]) showed that $B=H^\infty+C_B$, where C_B is the C^*-subalgebra generated by inner functions I with $\bar{I} \in B$. Also she showed that $\|f+H^\infty\|=\|f+H^\infty \cap C_B\|$ for $f \in C_B$. By this fact, we can consider that

$$B/H^\infty=(H^\infty+C_B)/H^\infty=C_B/H^\infty \cap C_B .$$

Corollary 3. *Let $f \in C_B$ with $\|f+H^\infty \cap C_B\|=1$. Then $f+H^\infty \cap C_B$ is an extreme (exposed) point of* ball $(C_B/H^\infty \cap C_B)$ *if and only if $f+H^\infty$ is an extreme (exposed) point of* ball (L^∞/H^∞).

For each f in C with $\|f+H^\infty\|=1$, there exist unique $g \in H^\infty$ and $F \in H_0^1$ such that $\|f+g\|=1$ and $(f+g)F \geqq 0$ ([7, p. 137]). By Izuchi and Younis' theorem [13], $f+H^\infty$ is an exposed point of ball (L^∞/H^∞). Thus we get

Corollary 4. *Every boundary point of* ball $(H^\infty+C/H^\infty)$, ball $(C/H^\infty \cap C)$ *and* ball (QC/QA) *is an exposed point of respective space.*

Proof. By Wolff ([17, Lemma 2.1]), $QC=QA+C$. So that $H^\infty+C/H^\infty=C/H^\infty\cap C=QC/QA$.

We note that $H^\infty\cap C$ is called a disk algebra usually.

4. Extreme points of ball $(B/H^\infty+C)$

In this section, we study Question 2 and give two partial answers.

Theorem 4. *If $qB\not\subset H^\infty+C$ for every outer function $q\in QA$, then* ball $(B/H^\infty+C)$ *does not have extreme points.*

To show Theorem 4, we need the following two lemmas.

Lemma 5 ([3]). *$H^\infty+C$ has the best approximation property.*

Lemma 6 ([16]). *For $f\in L^\infty$, $f\in H^\infty+C$ if and only if $f|_E\in H^\infty|_E$ for every support set E.*

Proof of Theorem 4. Let $f+H^\infty+C\in$ ball $(B/H^\infty+C)$ with $\|f+H^\infty+C\|=1$. By Lemma 5, we may assume $\|f\|=1$. By Wolff's theorem, there is an outer function $q\in QA$ such that $qf\in QC$. We may assume $\|q\|=1$. By Lemma 1, we have

$$(1)\qquad |q|f\in QC\,.$$

By our condition, there is $F\in B$ such that

$$(2)\qquad qF\notin H^\infty+C \quad\text{and}\quad \|F\|=1\,.$$

By Lemmas 1 and 6, we have $|q|F\notin H^\infty+C$. We note that $|q|F\in B$. Then we have

$$\begin{aligned}\|f+H^\infty+C\pm(|q|F+H^\infty+C)\|&=\|(1-|q|)f+|q|f\pm qF+H^\infty+C\|\\&\leqq\|(1-|q|)f\pm qF\|\quad\text{by (1)}\\&\leqq\|1-|q|+|q|\|\quad\text{by}\quad\|f\|=\|q\|=\|F\|=1\\&=1\,.\end{aligned}$$

This shows that $f+H^\infty+C$ is not an extreme point of ball $(B/H^\infty+C)$.

For a Douglas algebra B, we put

$$N(B)=\text{the closure of }\bigcup\{\operatorname{supp}\mu_x;\ x\in M(H^\infty+C)\setminus M(B)\}\,.$$

Corollary 6. *If $\hat{m}(N(B))>0$, then* ball $(B/H^\infty+C)$ *does not have extreme points.*

Proof. By Corollary 1, $\hat{m}(N(B))>0$ if and only if $\hat{m}_0(\pi_0(N(B)))>0$. Here we will show that $\hat{m}_0(\pi_0(N(B)))>0$ if and only if $qB\not\subset H^\infty+C$ for every outer function $q\in QA$.

Suppose that $\hat{m}_0(\pi_0(N(B)))>0$ and $q \in QA$ is an outer function. Since $\hat{m}_0(\{x \in M(QC);\ q(x)=0\})=0$, there is $x_0 \in M(H^\infty+C)\setminus M(B)$ such that $q\neq 0$ on supp μ_{x_0}. Then $q(x_0)\neq 0$. By Chang-Marshall's theorem ([4], [15]),

$$M(B)=\{x \in M(H^\infty+C);\ B|_{\mathrm{supp}\,\mu_x}=H^\infty|_{\mathrm{supp}\,\mu_x}\}\ .$$

Then there is $F \in B$ such that $F|_{\mathrm{supp}\,\mu_{x_0}} \notin H^\infty|_{\mathrm{supp}\,\mu_{x_0}}$. Thus we get $qF \notin H^\infty+C$ by Lemma 6.

Suppose that $\hat{m}_0(\pi_0(N(B)))=0$. Then $\pi_0(N(B))$ is contained in a proper peak set for QA by Lemma 2. Hence there is an outer function $q \in QA$ such that $q=0$ on $\pi_0(N(B))$, and then $qB\subset H^\infty+C$.

Corollary 7. *If* $\hat{m}(\Gamma)<1$, *then* ball $(B/H^\infty+C)$ *does not have extreme points.*

Proof. If $\hat{m}(\Gamma)<1$, then $N(B)\supset \Gamma^c$ and $\hat{m}(N(B))>0$.

When $\hat{m}(N(B))=0$, Theorem 4 does not work for Question 2. The last part of this paper, we will give a Douglas algebra B such that ball $(B/H^\infty+C)$ has an extreme point.

A sequence $\{z_n\}_{n=1}^\infty$ in D is called interpolating if for each bounded sequence $\{a_n\}_{n=1}^\infty$ there is $h \in H^\infty$ such that $h(z_n)=a_n$ for $n=1, 2, \cdots$. A Blaschke product with zeros $\{z_n\}_{n=1}^\infty$ is called interpolating if $\{z_n\}_{n=1}^\infty$ is interpolating.

Theorem 5. *Let b be an interpolating Blaschke products and* $B=[H^\infty, \bar{b}]$. *Then* $\bar{b}+H^\infty+C$ *is an extreme point of* ball $(B/H^\infty+C)$.

To show this, we need two lemmas. For $f \in H^\infty+C$, we put

$$Z(f)=\{x \in M(H^\infty+C);\ f(x)=0\}\ .$$

The following is a special case of [8, Theorem 1].

Lemma 7. *If* $f \in H^\infty+C$ *and b is an interpolating Blaschke products with* $Z(f)\supset Z(b)$, *then* $fb \in H^\infty+C$.

Lemma 8 ([9, p. 176]). *Let* $f \in H^\infty+C$ *and I is an inner function. If f vanishes on* $\{x \in M(H^\infty+C);\ |I(x)|<1\}$, *then* $f\bar{I}^n \in H^\infty+C$ *for every n.*

Proof of Theorem 5. First we note that $\|\bar{b}+H^\infty+C\|=1$. Suppose that

$$\bar{b}+H^\infty+C=\frac{1}{2}(g_1+H^\infty+C)+\frac{1}{2}(g_2+H^\infty+C)$$

with $\|g_i+H^\infty+C\|=1$ and $g_i \in B$ $(i=1, 2)$. By Lemma 5, there are h_i $(i=1, 2)$ in $H^\infty+C$ such that $\|g_i+h_i\|=1$. Then there is h in $H^\infty+C$ such that

$$\bar{b}+h=(g_1+h_1+g_2+h_2)/2 \quad \text{and} \quad \|\bar{b}+h\|=1\ . \tag{1}$$

Here our claim is

Claim. $$h=0 \quad \text{on} \quad N(B).$$

Suppose that the above claim is true. Since $|\bar{b}|=1$ on $N(B)$ and $\|g_i+h_i\|=1$, by (1) and our claim, we get

$$\bar{b}=g_1+h_1=g_2+h_2 \quad \text{on} \quad N(B).$$

Then $\bar{b}-g_1-h_1=0$ on $N(B)$ and $\bar{b}-g_1-h_1 \in B$. Since $B|_{\mathrm{supp}\,\mu_y}=H^\infty|_{\mathrm{supp}_y}$ for every $y \in M(H^\infty+C)$ with $|b(y)|=1$, we get

$$(\bar{b}-g_1-h_1)|_{\mathrm{supp}\,\mu_z} \in H^\infty|_{\mathrm{supp}\,\mu_z} \quad \text{for every} \quad z \in M(H^\infty+C).$$

By Lemma 6, we have $\bar{b}-g_1-h_1 \in H^\infty+C$. Thus $\bar{b}+H^\infty+C=g_1+H^\infty+C$. This implies that $\bar{b}+H^\infty+C$ is an extreme point of ball$(B/H^\infty+C)$.

Proof of Claim. To show our claim, we need Lemmas 7 and 8. Since $M(B)=\{x \in M(H^\infty+C); |b(x)|=1\}$, $N(B)$ coincides with the closure of $\bigcup\{\mathrm{supp}\,\mu_x; x \in M(H^\infty+C), |b(x)|<1\}$. Let $\varphi \in Z(b)$. Since $\|1+bh\|=1$ and $1=\int(1+bh)d\mu_\varphi$, we have $1+bh=1$ on supp μ_φ. Thus we get

(2) $$h=0 \quad \text{on} \quad \mathrm{supp}\,\mu_\varphi \quad \text{for every} \quad \varphi \in Z(b).$$

This means that $h=0$ on $Z(b)$. By Lemma 7, we have $h\bar{b} \in H^\infty+C$. By (2), we have $h\bar{b}=0$ on $Z(b)$. Again we get $h\bar{b}^2 \in H^\infty+C$. Continuing this argument, we get

(3) $$h\bar{b}^n \in H^\infty+C \quad \text{for every} \quad n=1, 2, \cdots.$$

By Lemma 8, we have

$$h=0 \quad \text{on} \quad \{x \in M(H^\infty+C); |b(x)|<1\}.$$

By the same way as the first part, we get

$$h=0 \quad \text{on} \quad \mathrm{supp}\,\mu_x \quad \text{for every} \quad x \in M(H^\infty+C) \quad \text{with} \quad |b(x)|<1.$$

Thus we get our claim.

This work was done while the both authors were visiting scholars at the University of California, Berkeley.

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