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ON SINGULAR POINTS OF ELECTRICAL CIRCUITS

By

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1. Introduction.

A state of an electrical circuit with b elements is specified by a current vector $\boldsymbol{i}=(i_1, \cdots, i_b) \in \mathbb{R}^b$ and a voltage vector $\boldsymbol{v}=(v_1, \cdots, v_b) \in \mathbb{R}^b$. Let G be the oriented graph of the circuit, and we can regard naturally $\boldsymbol{v}$ and $\boldsymbol{i}$ as a real 1-chain and 1-cochain of G, i.e., $\boldsymbol{i} \in C_1(G)$, $\boldsymbol{v} \in C^1(G)$. Kirchhoff laws restricts the possible states to a b-dimensional subspace $K=\operatorname{Ker} \partial \times \operatorname{Im} \partial^* \subset C_1(G) \times C^1(G)$, where ∂: $C_1(G) \to C_0(G)$ (∂^*: $C^0(G) \to C^1(G)$) is the boundary (coboundary) operator. The characteristics of resistors (possibly with couplings) of the circuit give the restraint that $(\boldsymbol{i}_R, \boldsymbol{v}_R)$ to be in an n_R-dimensional submanifold $\Lambda_R \subset C_1(G_R) \times C^1(G_R)$, where $(\boldsymbol{i}_R, \boldsymbol{v}_R)$ denotes the currents and voltages of resistive elements, n_R the number of resistive elements in the circuit, G_R the subgraph of G consisting of all resistive elements.

Combining Kirchhoff laws and the restraint of the characteristics of resistors, we have a space $\Sigma=K \cap \Lambda \subset C_1(G) \times C^1(G)$, where $\Lambda=\{(\boldsymbol{i}, \boldsymbol{v})=(\boldsymbol{i}_R, \boldsymbol{v}_R, \boldsymbol{i}_L, \boldsymbol{v}_L, \boldsymbol{i}_C, \boldsymbol{v}_C);$ $(\boldsymbol{i}_R, \boldsymbol{v}_R) \in \Lambda_R\}$, on which the dynamics of the circuit takes place. Now, we assume the transversality of K and hence Σ is $(b-n_R)$-dimensional submanifold of $C_1(G) \times C^1(G)$.

The dynamics is described by the following form ([6], [4]). Let

$$J=\Sigma C_{mn}(\boldsymbol{v}_C) dv_{C_m} \otimes dv_{C_n} - \Sigma L_{mn}(\boldsymbol{i}_L) di_{L_m} \otimes di_{L_n}$$

be a 2-tensor on $C_1(G) \times C^1(G)$, where $C_{mn}(\boldsymbol{v}_C)$ ($L_{mn}(\boldsymbol{i}_L)$) is incremental capacitance (inductance) matrix and is assumed symmetric and positive definite ([4]).

The vector field X on Σ which describes the dynamics satisfies the following:

$$(\pi^* J)_{(i,v)}(X_{(i,v)}, \xi)=(\iota^* \eta)_{(i,v)}(\xi)\ , \qquad \text{for} \quad \xi \in T_{(i,v)}(\Sigma)\ ,$$

where η is a certain 1-form and π is the projection to the components of inductor currents and capacitor voltages,

$$\pi': C_1(G) \times C^1(G) \to C_1(G_L) \times C^1(G_C)\ ,$$

with its domain restricted to Σ, and

$$\iota: \Sigma \to C_1(G) \times C^1(G)$$

is the natural inclusion.

If $\pi\colon \Sigma \to C_1(G_L)\times C^1(G_C)$ is regular at $(\boldsymbol{i}, \boldsymbol{v})$, i.e., the differential of π at $(\boldsymbol{i}, \boldsymbol{v})$, $D\pi(\boldsymbol{i}, \boldsymbol{v})$ has full rank $(b-n_R)$, then $X_{(i,v)}$ is uniquely determined by the above equation, for J is non-degenerate bi-linear from at every point. A point $(\boldsymbol{i}, \boldsymbol{v})\in\Sigma$ is called *singular point* iff π is not regular at $(\boldsymbol{i}, \boldsymbol{v})\in\Sigma$. Since π^*J is degenerate at the singular point $(\boldsymbol{i}, \boldsymbol{v})\in\Sigma$, X is not determined at $(\boldsymbol{i}, \boldsymbol{v})$. In fact, there is a case in which we cannot define $X_{(i,v)}$ at some singular points consistently with other regular points governed by the above equation. In most cases, however, we can remove singular points by adding arbitrarily small capacitors and inductors appropriately to the original circuits. This procedure is called "*regularization*" and justified by the fact that it corresponds "*to take account of parasitive elements*" in circuit theory ([6], [1]). But at least theoretically there is a circuit which is not regularizable, and even in regularizable cases the regularized circuit have more reactive elements than the original one ([3]). The purpose of this paper is to point out that singular points are derived from conflictions of Kirchhoff laws and the restraints of resistive characteristics, therefore in general at singular points the solution jumps to another branch of the characteristic submanifold. This process is just a kind of "*catastrophe*". Of course, this phenomenon is already known by circuit theorists, for example, as "*relaxed oscillation*" or "*discontinuous oscillation*" ([1]).

2. Statement of results.

A tree T is called *proper* iff T contains all the capacitance branches and contains no inductance branch. The complements of T in G is called *the link of* T and is denoted by L. If the graph of the circuit has no proper tree, the map $\pi\colon \Sigma\to C_1(G_L)\times C^1(G_C)$ is singular at any point $(\boldsymbol{i}, \boldsymbol{v})\in\Sigma$, for the projection $\pi'|K\colon K\to C_1(G_L)\times C^1(G_C)$ is already singular. This situation is called "*forced degeneracy*" ([6], [4]). Excluding the forced degeneracy, we assume the existence of proper tree.

Let B and Q are the fundamental loop matrix and the fundamental cutset matrix with respect to a proper tree. (For definition of B and Q, see [5], [2].) And Kirchhoff space K is the image of the following into-isomorphism:

$$\begin{bmatrix} B^t, & 0 \\ 0, & Q^t \end{bmatrix}\colon C_1(L)\times C^1(T)\to C_1(G)\times C^1(G)\,,$$

where L is the link of T in G. Let $K(\boldsymbol{i}_L, \boldsymbol{v}_C)$ be the affine subspace of K determined by fixing the currents of inductors and the voltages of capacitors, this is possible because vector $(\boldsymbol{i}_L, \boldsymbol{v}_C)$ is subvector of $(\boldsymbol{i}_L, \boldsymbol{v}_T)\in C_1(L)\times C^1(T)$. Clearly the space $K(\boldsymbol{i}_L, \boldsymbol{v}_C)$ is the parallel translation in K of $K(0,0)$ to the point

$$b(\boldsymbol{i}_L, \boldsymbol{v}_C)=(\boldsymbol{i}, \boldsymbol{v})=\begin{bmatrix} B^t, & 0 \\ 0, & Q^t \end{bmatrix}\begin{bmatrix} \boldsymbol{i}_L \\ 0 \\ 0 \\ \boldsymbol{v}_C \end{bmatrix},$$

here we assume the numbering of the elements is appropriately arranged.

Let

$$\pi'_R: C_1(G)\times C^1(G)\to C_1(G_R)\times C^1(G_R)$$

be the natural projection to the currents and voltages of resistors, and $\pi'_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C))=K_0+(\boldsymbol{i}_R, \boldsymbol{v}_R)$ where $K_0=K(0,0)\subset C_1(G_R)\times C^1(G_R)$ and $(\boldsymbol{i}_R, \boldsymbol{v}_R)=\pi'_R(b(\boldsymbol{i}_L, \boldsymbol{v}_C))$.

Now we can state our result.

Theorem. *Let C be a circuit whose graph has a proper tree. Suppose Λ and Σ are transversal. Then, a point $(\boldsymbol{i}, \boldsymbol{v})=(\boldsymbol{i}_L, \boldsymbol{i}_C, \boldsymbol{i}_R, \boldsymbol{v}_L, \boldsymbol{v}_C, \boldsymbol{v}_R)$ is singular point if and only if the characteristic submanifold Λ_R and the affine subspace $K_0(\boldsymbol{i}_R, \boldsymbol{v}_R)$ are not transverse at $(\boldsymbol{i}_R, \boldsymbol{v}_R)$ in $C_1(G_R)\times C^1(G_R)$.*

3. Proof of Theorem.

Let $\boldsymbol{i}(L)$, $\boldsymbol{v}(T)$, $\boldsymbol{i}(R(L))$, $\boldsymbol{v}(R(T))$ denote the currents of link branches, voltages of tree branches, currents of link resistors, and voltages of tree resistors, respectively. For $(\boldsymbol{i}_L, \boldsymbol{v}_C)\in C_1(G_L)\times C^1(G_C)$, we define the map $k_{(i_L, v_C)}: C_1(G_{R(L)})\times C^1(G_{R(T)})\to K(\boldsymbol{i}_L, \boldsymbol{v}_C)$ by the following:

$$k_{(i_L, v_C)}(\boldsymbol{i}_{R(L)}, \boldsymbol{v}_{R(T)})=\begin{bmatrix} B^t, & 0 \\ 0, & Q^t \end{bmatrix}\begin{bmatrix} \boldsymbol{i}_L \\ \boldsymbol{i}_{R(L)} \\ \boldsymbol{v}_C \\ \boldsymbol{v}_{R(T)} \end{bmatrix}.$$

Then $k_{(i_L, v_C)}$ is an isomorphism with its inverse:

$$\pi'_{R(L),R(T)}\circ(\pi'_R|K(\boldsymbol{i}_L, \boldsymbol{v}_C)): K(\boldsymbol{i}_L, \boldsymbol{v}_C)\to C_1(G_{R(L)})\times C^1(G_{R(T)}),$$

for $\pi'_{R(L),R(T)}\circ k_{(i_L, v_C)}=id_{C_1(G_{R(L)})\times C^1(G_{R(T)})}$ and $\dim K(\boldsymbol{i}_L, \boldsymbol{v}_C)=n_R=\dim C_1(G_{R(L)})\times C^1(G_{R(T)})$.

And hence, the projection π'_R with its domain and range restricted as follows:

$$\pi'_R|K(\boldsymbol{i}_L, \boldsymbol{v}_C): K(\boldsymbol{i}_L, \boldsymbol{v}_C)\to\pi'_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C))=K_0+(\boldsymbol{i}_R, \boldsymbol{v}_R)$$

is also an isomorphism with inverse:

$$k_{(i_L, v_C)}\circ\pi'_{R(L),R(T)}: K_0+(\boldsymbol{i}_R, \boldsymbol{v}_R)\to K(\boldsymbol{i}_L, \boldsymbol{v}_C).$$

Now, we prove the theorem. Suppose $p=(\boldsymbol{i}, \boldsymbol{v}) \in \Sigma$ is a singular point, i.e., $D\pi(p)$: $T_p(\Sigma) \to T_{\pi(p)}(C_1(G_L) \times C^1(G_C))$ is singular. Then $\mathrm{Ker}\,(D\pi'(p) \cap T_p(\Sigma)) \neq \{0\}$, by projecting this to the space $C_1(G_R) \times C^1(G_R)$, we obtain:

$$\pi'_R(\mathrm{Ker}\, D\pi'(p) \cap T_p(\Sigma)) \neq 0 .$$

Since

$$T_p(\Sigma) = T_p(\Lambda) \cap K = (\pi'_R)^{-1}(T\pi_{R(p)}(\Lambda_R)) \cap K,$$

the above equation implies:

$$\pi'_R(K \cap \mathrm{Ker}\, D\pi'(p)) \cap T\pi_{R(p)}(\Lambda_R) \neq 0 .$$

But,

$$\pi'_R(K \cap \mathrm{Ker}\, D\pi'(p)) = \pi'_R(K \cap \pi'^{-1}(\boldsymbol{i}_L, \boldsymbol{v}_C)) = \pi'_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C)) = K_0 + (\boldsymbol{i}_R, \boldsymbol{v}_R) ,$$

this shows that $K_0 + (\boldsymbol{i}_R, \boldsymbol{v}_C)$ and Λ_R are not transverse at $(\boldsymbol{i}_R, \boldsymbol{v}_R)$. This proves the sufficiency of the theorem.

Conversely, if $K_0 + (\boldsymbol{i}_R, \boldsymbol{v}_R)$ and Λ_R are not transverse at $(\boldsymbol{i}_R, \boldsymbol{v}_R)$ with $p = (\boldsymbol{i}_R, \boldsymbol{i}_L, \boldsymbol{i}_C, \boldsymbol{v}_R, \boldsymbol{v}_L, \boldsymbol{v}_C) \in \Sigma$, then

$$\pi'_R(K \cap \mathrm{Ker}\, D\pi'(p)) \cap T\pi_{R(p)}(\Lambda_R) \neq 0 .$$

Since $\pi'_R | K(\boldsymbol{i}_L, \boldsymbol{v}_C)$ is isomorphism,

$$(K \cap \mathrm{Ker}\, D\pi'(p)) \cap (\pi'_R | K(\boldsymbol{i}_L, \boldsymbol{v}_C))^{-1}(T\pi_{R(p)}(\Lambda_R)) \neq 0 ,$$

this means:

$$\mathrm{Ker}\, D\pi'(p) \cap T_p(\Sigma) \neq \{0\} .$$

This proves the necessity of the theorem.

Remark. In terms of B and Q, the space $K_0 \subset C_1(G_R) \times C^1(G_R)$ is given as follows. Let us decompose the matrices B and Q into the following forms:

$$B = \begin{array}{cc} \begin{array}{cccc} R(L) & L & R(T) & C \end{array} & \\ \begin{bmatrix} 1 & 0 & A_{RT} & A_{RC} \\ 0 & 1 & A_{LT} & A_{LC} \end{bmatrix} & \begin{array}{l} R(L) \\ L \end{array} \end{array}, \qquad Q = \begin{array}{cc} \begin{array}{cccc} R(L) & L & R(T) & C \end{array} & \\ \begin{bmatrix} -A^t_{RT} & -A^t_{LT} & 1 & 0 \\ -A^t_{RC} & -A^t_{LC} & 0 & 1 \end{bmatrix} & \begin{array}{l} R(T) \\ C \end{array} \end{array} .$$

Then, it is easily seen that

$$\begin{aligned} K_0 &= \pi'_R(K(0, 0)) \\ &= \{(\boldsymbol{i}_R, \boldsymbol{v}_R) = (\boldsymbol{i}_{R(L)}, \boldsymbol{i}_{R(T)}, \boldsymbol{v}_{R(L)}, \boldsymbol{v}_{R(T)}) | \boldsymbol{i}_{R(T)} = A^t_{RT} \boldsymbol{i}_{R(L)}, \boldsymbol{v}_{R(L)} = -A_{RT} \boldsymbol{v}_{R(T)}, \\ &\qquad (\boldsymbol{i}_{R(T)}, \boldsymbol{v}_{R(T)}) \in C_1(G_{R(L)}) \times C^1(G_{R(T)})\} . \end{aligned}$$

This means that the space K_0 is just Kirchhoff space of the resistive circuit obtained from the given one by open-circuitting all inductance branches and short-circuitting all capacitance branches.

Examples 1. (Example 5 in [6].) Consider a circuit of Fig. 1 consisting of one non-linear resistor with characteristic of Fig. 2, one capacitor and one inductor. By Remark, K_0 is Kirchhoff space of Fig. 3, i.e., K_0 is just v-axis in Fig. 2. Therefore at p_i the solution must jump into p_i'.

2. (Example 6 in [6], A regularization of the above example.) To regularize the above example, we add a parasitic element C' in parallel to L as in Fig. 4.

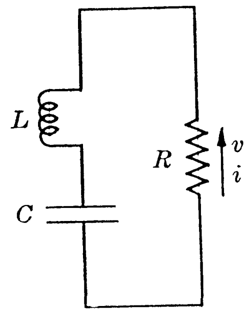

Fig. 1.

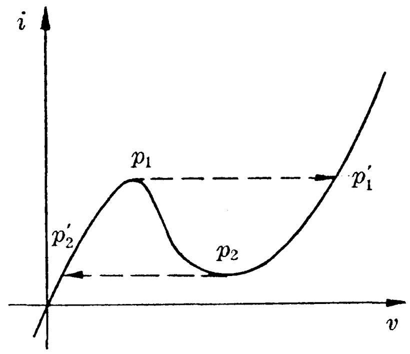

Fig. 2.

Fig. 3.

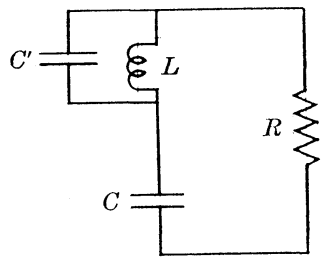

Fig. 4.

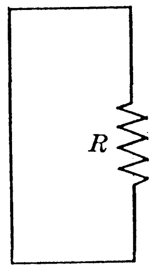

Fig. 5.

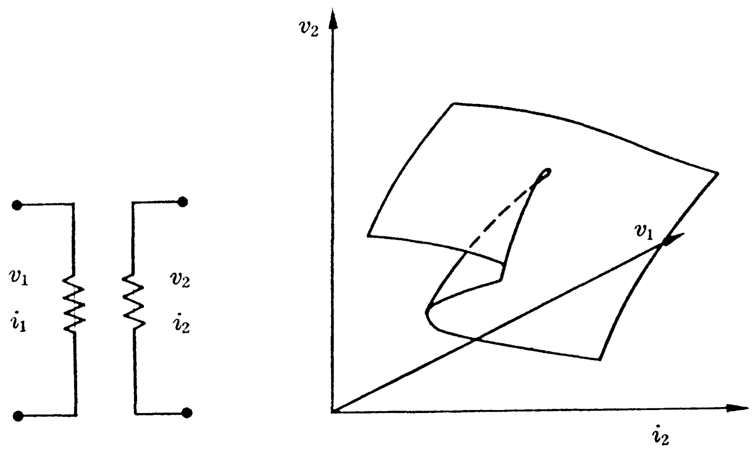

Fig. 6.

Then, K_0 is Kirchhoff space of Fig. 5, i.e., K_0 is just $\boldsymbol{i}$-axis. Therefore $K_0+(\boldsymbol{v}_R, \boldsymbol{i}_R)$ is always transverse to Λ_R, and hence the circuit of Fig. 4 has no singular point at all.

Finally, we propose an engineering problem concerning the catastrophy theory.

Problem. Is it possible to make a device (coupled resistors) with its characteristic "*cusp type singularity*", as is shown in Fig. 6?

Certainly, Esaki diode has a characteristic of "*fold type singularity*". The "*cusp type singularity*" is the simplest singularity next to the "*fold type singularity*". The cusp type device may be very usefull as was the case in Esaki diodes.

Addendum. Professor H. Kawakami informed the author that the coupled resistors with "*cusp type*" characteristics ($y=-3bx+x^3$) could be constructed from operational amplifiers and nonlinear analog elements [7].

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