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NON-UNIFORM ESTIMATES IN THE CENTRAL LIMIT THEOREM

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1. Let $\{X_k, k=1, 2, \cdots\}$ be a sequence of independent random variables with $EX_k=0$, $EX_k^2=\sigma_k^2<\infty$ and with distribution $F_k(x)$. Write $S_n=\sum_{k=1}^{n} X_k$, $s_n^2=\sum_{k=1}^{n} \sigma_k^2$, $\bar{F}_n(x)=P(S_n \leqq s_n x)$ and $\Delta_n(x)=|\bar{F}_n(x)-\Phi(x)|$, where $\Phi(x)$ is the standard normal distribution. When $\{X_k\}$ is a sequence of independent, identically distributed (i.i.d.) random variables, we put $EX_1^2=1$ and write $F(x)=F_k(x)$.

In this paper, two types of non-uniform convergence of $\Delta_n(x)$ will be considered; one is

(1.1) $$\sup_x (1+|x|)^{\alpha} \Delta_n(x) \to 0 \quad \text{as} \quad n\to\infty \quad \text{for some} \quad \alpha>0\,,$$

and the other is

$$\int_{-\infty}^{\infty} (1+|x|)^{\beta} \Delta_n(x) dx \to 0 \quad \text{as} \quad n\to\infty \quad \text{for some} \quad \beta>0\,.$$

In the last section, we shall also consider non-uniform estimates in asymptotic expansions in the central limit theorem.

2. In this section, we first state some results concerned with the type (1.1). In what follows, C denotes an absolute positive constant which may differ from one expression to another.

The following theorem is given by Bikelis [3].

Theorem 1 ([3]). *Put*

$$R_k(z)=\int_{|u|>z} u^2 dF_k(u)$$

and

$$Q_k(z)=\left|\int_{|u|\leqq z} u^3 dF_k(u)\right|.$$

If

$$\rho_k=\sup_{0<z\leqq s_n(1+|x|)} (Q_k(z)+zR_k(z))<\infty\,,$$

then

$$\Delta_n(x) \leqq \frac{C}{s_n^3(1+|x|)^3} \sum_{k=1}^n \rho_k .$$

From this result, we easily derive an extended Berry-Esseen theorem which includes Theorem 1 itself as a special case.

Let us denote two classes of functions $g(z)$ as follows:

$G=\{g(z) \mid g(z)$ is even on the real line with $g(z) \geqq 0$, $g(z) \uparrow \infty$ as $0 \leqq z \uparrow \infty$ and $z/g(z)$ is well-defined and non-decreasing on $(0, \infty)\}$,

$G_0=\{g(z) \mid g(z) \in G$, and in addition, $z^q/g(z)$ is non-decreasing on $(0, \infty)$ for some q with $0<q<1\}$.

Theorem 2. *Let* $g(z) \in G$. *If*

$$\rho_k(g) = \sup_{0<z \leqq s_n(1+|x|)} \left(\frac{g(z)}{z} Q_k(z) + g(z) R_k(z)\right) < \infty ,$$

then

$$\Delta_n(x) \leqq \frac{C}{s_n^2(1+|x|)^2 g(s_n(1+|x|))} \sum_{k=1}^n \rho_k(g) .$$

When $g(z)=|z|$, Theorem 2 is no more than Theorem 1.

Proof of Theorem 2. Using that $z/g(z)$ is non-decreasing, we have

$$\begin{aligned}
\Delta_n(x) &\leqq \frac{C}{s_n^3(1+|x|)^3} \sum_{k=1}^n \sup_{0<z \leqq s_n(1+|x|)} (Q_k(z) + z R_k(z)) \\
&= \frac{C}{s_n^3(1+|x|)^3} \sum_{k=1}^n \sup_{0<z \leqq s_n(1+|x|)} \frac{z}{g(z)} \left(\frac{g(z)}{z} Q_k(z) + g(z) R_k(z)\right) \\
&\leqq \frac{C}{s_n^2(1+|x|)^2 g(s_n(1+|x|))} \sum_{k=1}^n \sup_{0<z \leqq s_n(1+|x|)} \left(\frac{g(z)}{z} Q_k(z) + g(z) R_k(z)\right) .
\end{aligned}$$

Theorem 3. *Let* $g(z) \in G_0$. *If*

$$\lambda_k(g) = \sup_{0<z \leqq s_n(1+|x|)} g(z) R_k(z) < \infty ,$$

then

$$\Delta_n(x) \leqq \frac{C}{s_n^2(1+|x|)^2 g(s_n(1+|x|))} \sum_{k=1}^n \lambda_k(g) .$$

Proof. We note that $Q_k(z) \leqq \int_0^z R_k(u) du$. Since $g(z) \in G_0$, $z^q/g(z)$ is non-decreasing for some q with $0<q<1$, so that we see that, for $0<z \leqq s_n(1+|x|)$,

$$\frac{g(z)}{z} Q_k(z) \leqq \frac{g(z)}{z} \int_0^z R_k(u) du$$

$$=\frac{g(z)}{z}\int_0^z \frac{u^q}{g(u)}\frac{g(u)}{u^q}R_k(u)du$$
$$\leqq \lambda_k(g)\frac{g(z)}{z}\frac{z^q}{g(z)}\int_0^z \frac{1}{u^q}du$$
$$=\frac{1}{(1-q)}\lambda_k(g)\ .$$

Therefore, Theorem 3 follows from Theorem 2.

In particular, if we put $g(z)=|z|^\delta$ $(0<\delta<1)$, we have a result given by Gafurov [4].

Furthermore, as to the i.i.d. case, we have the following theorem, which is an extension of results due to Bikelis [2], [3] and also extends partially a result of Michel ([11], the case $0<c\leqq 1$ in his theorem).

Theorem 4. *Let $\{X_k\}$ be a sequence of i.i.d. random variables. Let $g(z)\in G$. Then*

(2.1) $$\sup_x (1+|x|)^2 g(\sqrt{n}(1+|x|))\Delta_n(x)=O(1)$$

if and only if

(2.2) $$g(z)R(z)=O(1)\ , \qquad as \quad z\to\infty$$

and

$$\frac{g(z)}{z}Q(z)=O(1)\ , \qquad as \quad z\to\infty\ ,$$

where $R(z)$ and $Q(z)$ are defined for $F(u)$ similarly to $R_k(z)$ and $Q_k(z)$ for $F_k(u)$. When $g(z)\in G_0$, (2.1) *is equivalent to the single condition* (2.2).

Proof. Sufficiency part follows from Theorems 2 and 3 as a special case. Necessity part is shown as a direct consequence of the following theorem given by Rosovskii [15]: *Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n\downarrow 0$ and $\varepsilon_n\geqq\tau^*/\sqrt{n}$, where $\tau^*=\min\left\{\tau;\ \frac{1}{\tau^2}\int_0^\tau zR(z)dz\leqq\frac{1}{24}\right\}$. Then, $\sup_x \Delta_n(x)=O(\varepsilon_n)$ if and only if $\Psi_n=O(\varepsilon_n)$, where*

$$\Psi_n=R(\sqrt{n})+\frac{1}{\sqrt{n}}Q(\sqrt{n})+\frac{1}{n}\int_{|u|\leqq\sqrt{n}}u^4dF(u)\ .$$

This completes the proof.

Now, we state here the following result on the different type of convergence given by Heyde [8].

Theorem 5 ([8]). *Let $\{X_k\}$ be a sequence of i.i.d. random variables, and let $0\leqq\delta<1$. In order that*

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x (1+|x|)^2 \Delta_n(x) < \infty ,$$

it is necessary and sufficient that

$$E|X_1|^{2+\delta} < \infty , \qquad \textit{if} \quad 0<\delta<1 ,$$
$$EX_1^2 \log(1+|X_1|) < \infty , \qquad \textit{if} \quad \delta=0 .$$

At the end of this section, we mention that the possible value of the power of $(1+|x|)$ in (1.1) is restricted by the moment condition or the condition on the tail of the distribution. For simplicity, we consider the case of i.i.d. random variables.

Theorem 6. *Under the condition that* $E|X_1|^{2+\gamma}<\infty$ *or* $\int_{|u|>z} u^2 dF(u)=O(z^{-\gamma})$ *for some* $\gamma>0$, α *in* (1.1) *must be less than or equal to* γ.

Proof. If the distribution $F(x)$ satisfies that $1-F(x)+F(-x)=O(x^{-p})$, then $E|X_1|^q<\infty$ for any $q<p$. Suppose that $E|X_1|^\gamma<\infty$ but $E|X_1|^r=\infty$ for all $r>\gamma$. If (1.1) holds for some $\alpha>\gamma$, then $1-\bar{F}_m(x)+\bar{F}_m(-x)=O(x^{-\alpha})$ for $m=n$, $n+1$. Therefore $E|S_m|^{(\alpha+\gamma)/2}<\infty$ for $m=n$, $n+1$, so that $E|X_1|^{(\alpha+\gamma)/2}<\infty$, which is a contradiction. This theorem is thus shown. Michel [12] has also pointed out this fact.

We conclude from Theorem 6 that we connot improve the non-uniformity of $\sup_x (1+|x|)^\gamma \Delta_n(x)\to 0$ under $E|X_1|^\gamma<\infty$, in the sense that the power of $(1+|x|)$ cannot be replaced by a higher one.

In the next section, however, we shall consider the possibility of the validity of $\int (1+|x|)^{\gamma-1}\Delta_n(x)dx\to 0$, which is an improvement of the non-uniformity of $\sup_x (1+|x|)^\gamma \Delta_n(x)\to 0$ in some sense.

3. Bikelis [2] showed that, in i.i.d. case, if

$$\int_{|u|>z} u^2 dF(u)=O(z^{-\delta}) , \qquad 0<\delta<1 , \tag{3.1}$$

then

$$\sup_x (1+|x|)^{2+\delta}\Delta_n(x)=O(n^{-\delta/2}) . \tag{3.2}$$

In connection with this matter, the following result will be shown.

Theorem 7. *Let* $\{X_k\}$ *be a sequence of i.i.d. random variables, and let* $0<\delta<1$. *Under the condition* (3.1),

$$\int_{-\infty}^{\infty} (1+|x|)^{1+\delta-\varepsilon}\Delta_n(x)dx=O(n^{-\delta/2}) \tag{3.3}$$

for any $\varepsilon>0$. *But, in* (3.3), *we cannot put* $\varepsilon=0$.

The first part of the theorem is trivially shown by (3.2). The latter half is derived from the following theorem.

Theorem 8. *If*

$$\int_{-\infty}^{\infty}(1+|x|)^{\beta}\Delta_m(x)dx<\infty \qquad \text{for} \quad m=n,\ n+1\ , \tag{3.4}$$

then $E|X_1|^{\beta+1}<\infty$.

Proof. From (3.4), we have

$$\int_{-\infty}^{0}(1+|x|)^{\beta}\bar{F}_m(x)dx<\infty \qquad \text{and} \qquad \int_{0}^{\infty}(1+|x|)^{\beta}(1-\bar{F}_m(x))dx<\infty\ ,$$

from which $E|S_m|^{\beta+1}<\infty$ for $m=n,\ n+1$. Thus we conclude $E|X_1|^{\beta+1}<\infty$.

However, if we replace the condition (3.1) in Theorem 7 by $E|X_1|^{2+\delta}<\infty$, then (3.3) also holds for $\varepsilon=0$. In fact, the author [10] has proved the following.

Theorem 9 ([10]). *Let* $\{X_k\}$ *be a sequence of independent random variables, and let* $0<\delta<1$. *If* $E|X_k|^{2+\delta}<\infty$, *then*

$$\int_{-\infty}^{\infty}(1+|x|)^{1+\delta}\Delta_n(x)dx\leqq\frac{C}{s_n^{2+\delta}}\sum_{k=1}^{n}E|X_k|^{2+\delta}\ .$$

Combining Theorems 8 and 9, we have the following result for the case of i.i.d. random variables.

Theorem 10. *Let* $\{X_k\}$ *be a sequence of i.i.d. random variables, and let* $0<\delta<1$. *Then*

$$E|X_1|^{2+\delta}<\infty$$

is equivalent to the validity of

$$\int_{-\infty}^{\infty}(1+|x|)^{1+\delta}\Delta_n(x)dx=O(n^{-\delta/2})\ .$$

When $\delta\neq 0$, the non-uniformity of Theorem 5 has been improved as follows.

Theorem 11 ([10]). *Let* $\{X_k\}$ *be a sequence of i.i.d. random variables, and let* $0<\delta<1$. *If* $E|X_1|^{2+\delta}<\infty$, *then*

$$\sum_{n=1}^{\infty}n^{-1+\delta/2}\int_{-\infty}^{\infty}(1+|x|)\Delta_n(x)dx<\infty\ .$$

For the case $\delta=0$, we are going to prove the following.

Theorem 12. *Let* $\{X_k\}$ *be a sequence of i.i.d. random variables. If*

$$EX_1^2 \log(1+|X_1|)<\infty \ , \tag{3.5}$$

then

$$\int_{-\infty}^{\infty}(1+|x|)\Delta_n(x)dx=o(1) \ , \tag{3.6}$$

and if

$$EX_1^2 \log^2(1+|X_1|)<\infty \ , \tag{3.7}$$

then

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_{-\infty}^{\infty}(1+|x|)\Delta_n(x)dx<\infty \ . \tag{3.8}$$

To prove this theorem, we first state two lemmas, which will be also used in the next section.

Lemma 1. *Let* $\{X_k\}$ *be a sequence of independent random variables. We have*

(i)
$$\sup_x (1+|x|)^2\Delta_n(x)\leqq\frac{C}{s_n}\int_0^{s_n}L_n(z)dz$$

and

(ii)
$$\int_{-\infty}^{\infty}(1+|x|)\Delta_n(x)dx\leqq\frac{C}{s_n}\int_0^{s_n}L_n(z)dz+C\int_{s_n}^{\infty}\frac{L_n(z)}{z}dz \ ,$$

where

$$L_n(z)=\frac{1}{s_n^2}\sum_{k=1}^{n}\int_{|u|>z}u^2dF_k(u) \ .$$

Proof of the lemma. Bikelis [1] showed that

$$\Delta_n(x)\leqq\frac{C}{s_n(1+|x|)^3}\int_0^{s_n(1+|x|)}L_n(z)dz \ . \tag{3.9}$$

Noting that $L_n(z)$ is monotone decreasing as z increases, we have

$$\frac{C}{s_n(1+|x|)}\int_0^{s_n(1+|x|)}L_n(z)dz\leqq\frac{C}{s_n}\int_0^{s_n}L_n(z)dz \ ,$$

from which (i) is given. On the other hand, we have, from (3.9),

$$\begin{aligned}\int_{-\infty}^{\infty}(1+|x|)\Delta_n(x)dx&\leqq\frac{C}{s_n}\int_{-\infty}^{\infty}\frac{dx}{(1+|x|)^2}\int_0^{s_n(1+|x|)}L_n(z)dz\\&=\frac{C}{s_n}\int_0^{s_n}L_n(z)dz\int_{-\infty}^{\infty}\frac{dx}{(1+|x|)^2}+\frac{C}{s_n}\int_{s_n}^{\infty}L_n(z)dz\int_{|x|>(z/s_n)-1}\frac{dx}{(1+|x|)^2}\\&=\frac{C}{s_n}\int_0^{s_n}L_n(z)dz+C\int_{s_n}^{\infty}\frac{L_n(z)}{z}dz \ .\end{aligned}$$

The proof is thus completed.

Lemma 2 ([8]). *Let X be a random variable with distribution F(x), and write*

$$L(z)=\int_{|u|>z} u^2 dF(u) .$$

Let $0\leqq\delta<1$. *If*

$$E|X|^{2+\delta}<\infty , \qquad \text{for } 0<\delta<1 ,$$
$$EX^2\log(1+|X|)<\infty , \qquad \text{for } \delta=0 ,$$

then

$$\sum_{n=1}^{\infty} n^{-(3-\delta)/2}\int_0^{\sqrt{n}} L(z)dz<\infty .$$

Proof of Theorem 12. We first show that (3.5) implies (3.6). From Lemma 1 (ii),

$$\begin{aligned}
&\int_{-\infty}^{\infty}(1+|x|)\varDelta_n(x)dx\\
&\leqq\frac{C}{\sqrt{n}}\int_0^{\sqrt{n}}dz\int_{|u|>z}u^2dF(u)+C\int_{\sqrt{n}}^{\infty}\frac{dz}{z}\int_{|u|>z}u^2dF(u)\\
&=\frac{C}{\sqrt{n}}\int_{|u|\leqq\sqrt{n}}u^2dF(u)\int_0^{|u|}dz+\frac{C}{\sqrt{n}}\int_{|u|>\sqrt{n}}u^2dF(u)\int_0^{\sqrt{n}}dz\\
&\quad+C\int_{|u|>\sqrt{n}}u^2dF(u)\int_{\sqrt{n}}^{|u|}\frac{1}{z}dz\\
&\leqq\frac{C}{\sqrt{n}}\int_{|u|\leqq\sqrt{n}}|u|^3dF(u)+C\int_{|u|>\sqrt{n}}u^2dF(u)\\
&\quad+C\int_{|u|>\sqrt{n}}u^2\log(1+|u|)dF(u)\\
&\leqq\frac{C}{\log(1+\sqrt{n})}\int_{|u|\leqq\sqrt{n}}u^2\log(1+|u|)dF(u)+o(1)=o(1) .
\end{aligned}$$

In order to show (3.8), we suppose that (3.7) is satisfied. We have from Lemma 1 again,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n}\int_{-\infty}^{\infty}(1+|x|)\varDelta_n(x)dx\\
&\quad\leqq C\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\int_0^{\sqrt{n}}L(z)dz+C\sum_{n=1}^{\infty}\frac{1}{n}\int_{\sqrt{n}}^{\infty}\frac{L(z)}{z}dz\\
&\quad\equiv\Sigma_1+\Sigma_2 ,
\end{aligned}$$

say. It follows from Lemma 2 with $\delta=0$ that Σ_1 is finite. As to Σ_2, we have

$$\Sigma_2=C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=n}^{\infty}\int_{\sqrt{m}}^{\sqrt{m+1}}\frac{L(z)}{z}dz$$

$$\leqq C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=n}^{\infty}\frac{1}{\sqrt{m}}(\sqrt{m+1}-\sqrt{m})L(\sqrt{m})$$

$$\leqq C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=n}^{\infty}\frac{1}{m}L(\sqrt{m})$$

$$=C\sum_{m=1}^{\infty}\frac{L(\sqrt{m})}{m}\sum_{n=1}^{m}\frac{1}{n}$$

$$\leqq C\sum_{m=1}^{\infty}\frac{\log m}{m}L(\sqrt{m})$$

$$=C\sum_{m=1}^{\infty}\frac{\log m}{m}\sum_{n=m}^{\infty}\int_{\sqrt{n}}^{\sqrt{n+1}}u^2dF(u)$$

$$=C\sum_{n=1}^{\infty}\int_{\sqrt{n}}^{\sqrt{n+1}}u^2dF(u)\sum_{m=1}^{n}\frac{\log m}{m}$$

$$\leqq C\sum_{n=1}^{\infty}\log^2(n+1)\int_{\sqrt{n}}^{\sqrt{n+1}}u^2dF(u)$$

$$\leqq CEX_1^2\log^2(1+|X_1|)<\infty\ ,$$

which completes the proof of the theorem.

4. In this section, we shall extend Theorems 11 and 12 to the case of non-identically distributed random variables. Therefore, throughout this section, we suppose that $\{X_k\}$ is a sequence of independent, but not necessarily identically distributed random variables. We begin with the following lemma.

Lemma 3 ([5]). *Suppose that*

(A) *there exist a random variable X and a positive constant x_0 such that*

$$\frac{1}{n}\sum_{k=1}^{n}P(|X_k|\geqq x)\leqq P(|X|\geqq x)$$

for all n and for all $x\geqq x_0$. Then under the condition $EX^2<\infty$, we have

$$\int_{|u|>x}u^2d\Big(\frac{1}{n}\sum_{k=1}^{n}F_k(u)\Big)\leqq\int_{|u|>x}u^2dF(u)$$

for all n and for all $x\geqq x_0$, where $F(u)=P(X\leqq u)$.

Moreover, we suppose that

(B) there exists ρ such that $s_n^2/n>\rho>0$.

Then, we have the following result which is an extension of Theorem 11 to the case of non-identically distributed random variables.

Theorem 13. *Let $0<\delta<1$. Under the assumptions* (A) *and* (B), *if $E|X|^{2+\delta}<\infty$, then*

(4.1) $$\sum_{n=1}^{\infty} n^{-1+\delta/2}\int_{-\infty}^{\infty}(1+|x|)\Delta_n(x)dx<\infty .$$

Proof. It is readily seen from Lemma 3 and $EX^2<\infty$ that $s_n^2<Cn$. Therefore, in order to prove (4.1), from Lemmas 1 and 3, and the assumption (B), it suffices to show

(4.2) $$\sum_{n=1}^{\infty} n^{-(3-\delta)/2}\int_0^{C\sqrt{n}} L(z)dz<\infty$$

and

(4.3) $$\sum_{n=1}^{\infty} n^{-1+\delta/2}\int_{C\sqrt{n}}^{\infty}\frac{L(z)}{z}<\infty ,$$

where $L(z)$ is the one defined in Lemma 2. (4.2) follows from Lemma 2, and the validity of (4.3) can be shown by the same way as in [10].

Furthermore, the case $\delta=0$ can be handled by a way similar to Theorem 12.

Theorem 14. *Under the assumptions* (A) *and* (B), *if* $EX^2\log(1+|X|)<\infty$, *then*

$$\int_{-\infty}^{\infty}(1+|x|)\Delta_n(x)dx=o(1) ,$$

and if $EX^2\log^2(1+|X|)<\infty$, *then*

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_{-\infty}^{\infty}(1+|x|)\Delta_n(x)dx<\infty .$$

5. In this section, we shall deal with non-uniform estimates in asymptotic expansions in the central limit theorem. As to this problem, Osipov [13], [14] has given some estimates analogous to Bikelis' estimate (3.9). In what follows, using Osipov's estimate, we shall show some results on asymptotic expansions anologous to Theorems 5, 10 and 11.

Throughout this last section, we suppose that $\{X_k\}$ is a sequence of i.i.d. random variables with $EX_1=0$, $EX_1^2=1$ and with distribution $F(x)$, and suppose that $E|X_1|^{p+2}<\infty$ for some integer $p\geqq 1$. Furthermore, we suppose Cramér condition

(C) $\limsup_{|t|\to\infty}|f(t)|<1$

to be satisfied, where $f(t)$ is the characteristic function of X_1. Write

$$R_{np}(x)=|\bar{F}_n(x)-G_{np}(x)| ,$$

where

$$G_{np}(x)=\Phi(x)+\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\sum_{j=1}^{p} n^{-j/2}Q_j(x)$$

is the Chebyshev series corresponding to the X_1. Here $Q_j(x)$ is a polynomial of degree $3j-1$ whose coefficients depend on the first $(j+1)$ moments of X_1. (See Gnedenko-Kolmogorov [7], Section 38.)

We first state Osipov's estimate, which will be used below in having some non-uniform estimates for $R_{np}(x)$.

Theorem 15 ([14]). *If $E|X_1|^{p+2}<\infty$ for some integer $p\geqq 1$, then*

$$R_{np}(x)\leqq C_p\left\{\frac{n}{(\sqrt{n}(1+|x|))^{p+3}}\int_{|u|\leqq\sqrt{n}(1+|x|)}|u|^{p+3}dF(u)\right.$$
$$+\frac{n}{(\sqrt{n}(1+|x|))^{p+2}}\int_{|u|>\sqrt{n}(1+|x|)}|u|^{p+2}dF(u)$$
$$\left.+\left(\sup_{|t|\geqq\gamma}|f(t)|+\frac{1}{2n}\right)^n\frac{(\sqrt{n})^{(p+2)(p+3)}}{(1+|x|)^{p+3}}\right\}, \tag{5.1}$$

where C_p is a positive constant depending on p, and $\gamma=(15E|X_1|^3)^{-1}$.

The first result we are going to prove here is the following.

Theorem 16. *Let $p\geqq 1$ be an integer and $0\leqq\delta<1$, and suppose that Cramér condition* (C) *is satisfied. If*

$$E|X_1|^{p+2+\delta}<\infty\,, \qquad \text{for } 0<\delta<1\,,$$
$$E|X_1|^{p+2}\log(1+|X_1|)<\infty\,, \qquad \text{for } \delta=0\,,$$

then

$$\sum_{n=1}^{\infty}n^{-1+(p+\delta)/2}\sup_x(1+|x|)^{p+2}R_{np}(x)<\infty\,. \tag{5.2}$$

This theorem is a non-uniform extension of results by Galstjan [6] and Heyde-Leslie [9] who proved, under same assumptions,

$$\sum_{n=1}^{\infty}n^{-1+(p+\delta)/2}\sup_x R_{np}(x)<\infty\,.$$

Moreover, this theorem is related to Theorem 5 that Heyde [8] proved (5.2) with $p=0$ without Cramér condition (C).

Proof of Theorem 16. We note that (5.1) is equivalent

$$R_{np}(x)\leqq C_p\left\{\frac{n}{(\sqrt{n}(1+|x|))^{p+3}}\int_0^{\sqrt{n}(1+|x|)}M_p(z)dz\right.$$
$$\left.+\left(\sup_{|t|>\gamma}|f(t)|+\frac{1}{2n}\right)^n\frac{(\sqrt{n})^{(p+2)(p+3)}}{(1+|x|)^{p+3}}\right\}, \tag{5.3}$$

where

$$M_p(z)=\int_{|u|>z}|u|^{p+2}dF(u)\,.$$

Therefore,

$$(1+|x|)^{p+2}R_{np}(x)\leqq\frac{C}{(\sqrt{n})^{p+1}(1+|x|)}\int_0^{\sqrt{n}(1+|x|)}M_p(z)dz$$
$$+C\Big(\sup_{|t|\geqq\gamma}|f(t)|+\frac{1}{2n}\Big)^n\frac{(\sqrt{n})^{(p+2)(p+3)}}{(1+|x|)}\ .$$

Since $M_p(z)$ decreases as z increases, we have

$$\frac{1}{(1+|x|)}\int_0^{\sqrt{n}(1+|x|)}M_p(z)dz\leqq\int_0^{\sqrt{n}}M_p(z)dz\ ,$$

so that

$$\sum_{n=1}^{\infty}n^{-1+(p+\delta)/2}\sup_x(1+|x|)^{p+2}R_{np}(x)$$
$$\leqq C\sum_{n=1}^{\infty}n^{-(3-\delta)/2}\int_0^{\sqrt{n}}M_p(z)dz$$
$$+C\sum_{n=1}^{\infty}n^{-1+(p+\delta)/2}\Big(\sup_{|t|\geqq\gamma}|f(t)|+\frac{1}{2n}\Big)^n(\sqrt{n})^{(p+2)(p+3)}\ ,$$

where the second term is obviously finite, because of Cramér condition (C). As to the first term, we have

$$\sum_{n=1}^{\infty}n^{-(3-\delta)/2}\int_0^{\sqrt{n}}M_p(z)dz$$
$$\leqq CE|X_1|^{p+2+\delta}<\infty\ ,\qquad \text{if}\quad 0<\delta<1\ ,$$
$$\leqq CE|X_1|^{p+2}\log(1+|X_1|)<\infty\ ,\qquad \text{if}\quad \delta=0\ ,$$

by the same way as Heyde [8] showed Lemma 2 in Section 3. This completes the proof of Theorem 16.

The next theorem is an asymptotic expansion analogue of Theorem 10.

Theorem 17. *Let* $p\geqq1$ *be an integer, and let* $0<\delta<1$. *If* $E|X_1|^{p+2+\delta}<\infty$ *and Cramér condition* (C) *is satisfied, then*

$$\int_{-\infty}^{\infty}(1+|x|)^{p+1+\delta}R_{np}(x)dx=O(n^{-(p+\delta)/2})\ .$$

Osipov [14] remarked that, under the same conditions as in Theorem 17, there exists a positive function $\varepsilon(u)$ such that $\lim_{u\to\infty}\varepsilon(u)=0$ and

$$R_{np}(x)\leqq\frac{\varepsilon(\sqrt{n}(1+|x|))}{(\sqrt{n})^{p+\delta}(1+|x|)^{p+2+\delta}}\ .$$

We mention here that Theorem 17 describes that the function $\varepsilon(u)$ also satisfies

$$\int_{-\infty}^{\infty} \frac{\varepsilon(1+|x|)}{(1+|x|)} dx < \infty .$$

Proof of Theorem 17. From (5.1), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} (1+|x|)^{p+1+\delta} R_{np}(x) dx \\
&\leqq C n^{-(p+1)/2} \int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^{2-\delta}} \int_{|u| \leqq \sqrt{n}(1+|x|)} |u|^{p+3} dF(u) \\
&\quad + C n^{-p/2} \int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^{1-\delta}} \int_{|u| > \sqrt{n}(1+|x|)} |u|^{p+2} dF(u) \\
&\quad + C \Big(\sup_{|t| \geqq \gamma} |f(t)| + \frac{1}{2n} \Big)^n (\sqrt{n})^{(p+2)(p+3)} \int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^{2-\delta}} \\
&\equiv I_1 + I_2 + I_3 ,
\end{aligned}$$

say. It is obvious that $I_3 = O(n^{-(p+\delta)/2})$. Furthermore, we have

$$I_1 + I_2 \leqq C n^{-(p+\delta)/2} E|X_1|^{p+2+\delta}$$

by a way similar to the one when the author [10] has proved Theorem 9.

Finally we show the following result which somewhat improves the non-uniformity of Theorem 16 in the case $0<\delta<1$.

Theorem 18. *Let $p \geqq 1$ be an integer, and let $0<\delta<1$. If $E|X_1|^{p+2+\delta}<\infty$, and if Cramér condition* (C) *is satisfied, then*

$$\sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} \int_{-\infty}^{\infty} (1+|x|)^{p+1} R_{np}(x) dx < \infty .$$

Proof. We have from (5.3),

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} \int_{-\infty}^{\infty} (1+|x|)^{p+1} R_{np}(x) dx \\
&\leqq C \sum_{n=1}^{\infty} n^{-(3-\delta)/2} \int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^2} \int_0^{\sqrt{n}(1+|x|)} M_p(z) dz \\
&\quad + C \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} \Big(\sup_{|t| \geqq \gamma} |f(t)| + \frac{1}{2n} \Big)^n (\sqrt{n})^{(p+2)(p+3)} \int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^2} .
\end{aligned}$$

The second series is trivially finite. The first series is equal to

$$\begin{aligned}
&C \sum_{n=1}^{\infty} n^{-(3-\delta)/2} \Big\{ \int_0^{\sqrt{n}} M_p(z) dz \int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^2} + \int_{\sqrt{n}}^{\infty} M_p(z) dz \int_{|x| > (z/\sqrt{n})-1} \frac{dx}{(1+|x|)^2} \Big\} \\
&= C \sum_{n=1}^{\infty} n^{-(3-\delta)/2} \int_0^{\sqrt{n}} M_p(z) dz + C \sum_{n=1}^{\infty} n^{-1+\delta/2} \int_{\sqrt{n}}^{\infty} \frac{M_p(z)}{z} dz \\
&\leqq E|X_1|^{p+2+\delta} < \infty ,
\end{aligned}$$

where the last step is readily seen by the same manner as the author [10] has shown Theorem 11.

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