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A NON-UNIFORM ESTIMATE IN THE LOCAL LIMIT THEOREM FOR DENSITIES, II

By

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1. Introduction and a result.

Let $\{X_k, k=1, 2, \cdots\}$ be a sequence of independent, identically distributed random variables with $EX_1=0$, $EX_1^2=1$ and with distribution $F(x)$. Write $Z_n=n^{-1/2}\sum_{k=1}^{n} X_k$, $F_n(x)=P(Z_n\leqq x)$, and denote the standard normal distribution by $\Phi(x)$. As to the non-uniform convergence of $F_n(x)$ to $\Phi(x)$, two types of convergence have been studied.

Theorem A (Bikelis [2], [3]). *Let* $0<\delta<1$. *In order that*

$$\sup_x (1+|x|)^{2+\delta}|F_n(x)-\Phi(x)|=O(n^{-\delta/2}) , \tag{1}$$

it is necessary and sufficient that

(a) $\int_{|u|>z} u^2dF(u)=O(z^{-\delta})$ *as* $z\to\infty$.

Furthermore, (1) *holds for* $\delta=1$ *if and only if the condition* (a) *with* $\delta=1$ *and*

(b) $\int_{|u|\leqq z} u^3dF(u)=O(1)$ *as* $z\to\infty$

hold together.

Theorem B (Heyde [4]). *Let* $0\leqq\delta<1$. *In order that*

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x (1+|x|)^2|F_n(x)-\Phi(x)|<\infty ,$$

it is necessary and sufficient that

$$E|X_1|^{2+\delta}<\infty , \qquad if \quad 0<\delta<1 ,$$
$$EX_1^2 \log (1+|X_1|)<\infty , \qquad if \quad \delta=0 .$$

As to the local limit theorem, it is natural to ask whether the similar results hold or not. Define the density of Z_n by $p_n(x)$ when it exists and and let $\phi(x)$ denote the standard normal density. The author, in the previous paper [7], has shown the local version of Theorem B. The purpose of this paper is to prove the following local version of Theorem A.

Theorem. *Let* $0<\delta<1$. *In order that*

$$\sup_x (1+|x|)^{2+\delta}|p_n(x)-\phi(x)|=O(n^{-\delta/2}) , \tag{2}$$

it is necessary and sufficient that

(c) *there exists N such that* $\sup_x p_N(x)<\infty$

and the condition (a) *holds. Moreover, in order that*

$$\sup_x (1+|x|)^3|p_n(x)-\phi(x)|=O(n^{-1/2}) , \tag{3}$$

it is necessary and sufficient that (a) *with* $\delta=1$, (b) *and* (c) *hold together.*

We mention here that Petrov [8] (the case $\delta=1$) and Basu [1] (the case $0<\delta<1$) have shown that if (c) is satisfied and if $E|X_1|^{2+\delta}<\infty$, then (3) or (2) is valid, respectively, while our theorem gives the necessary and sufficient conditions, in which it is seen that their moment conditions are relaxed to (a) and (b).

2. Proof.

Ibragimov-Linnik ([6], Theorem 4.5.1) proved that for $0<\delta<1$, $\sup_x |p_n(x)-\phi(x)|=O(n^{-\delta/2})$ if and only if both of the conditions (a) and (c) are satisfied. On the other hand, it can be proved that (a) with $\delta=1$, (b) and (c) hold together if and only if $\sup_x |p_n(x)-\phi(x)|=O(n^{-1/2})$, by the same arguments as Ibragimov [5] did in proving the similar result for distributions. Therefore, for our purpose, it suffices only to prove the sufficiency part and to get

$$\sup_{|x|\geqq 1} |x|^{2+\delta}|p_n(x)-\phi(x)|=O(n^{-\delta/2}) \tag{4}$$

for $0<\delta\leqq 1$.

Write $f(t)=Ee^{itX_1}$ and $\theta_n(t)=Ee^{itZ_n}=\{f(n^{-1/2}t)\}^n$. From the condition (c), we have

$$\theta_n(t)\in L^1 , \qquad n\geqq 2N \tag{5}$$

so that $p_n(x)$, $n\geqq 2N$, exist, and we readily see from the inversion of the Fourier transform that

$$p_n(x)=(2\pi)^{-1}\int_{-\infty}^{\infty} e^{-itx}\theta_n(t)dt .$$

Similarly to Basu [1] and Smith-Basu [9], we write for $n\geqq 2N$,

$$\alpha_n(t, x)=\int_{|u|\leqq(1/2)n^{1/2}|x|} e^{itu}dF(u) ,$$

$$\beta_n(t, x)=f(t)-\alpha_n(t, x) ,$$

$$A_n(t, x)=\{\alpha_n(n^{-1/2}t, x)\}^n ,$$

$$B_n(t, x)=\theta_n(t)-A_n(t, x) \tag{6}$$

$$=\sum_{j=1}^{n}\binom{n}{j}\{\alpha_n(n^{-1/2}t, x)\}^{n-j}\{\beta_n(n^{-1/2}t, x)\}^j . \tag{7}$$

From (5), we have for large n,

$$A_n(t, x) \in L^1 , \qquad B_n(t, x) \in L^1 , \tag{8}$$

(for details, see Smith-Basu [9]), and put

$$a_n(u, x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-itu} A_n(t, x) dt ,$$

$$b_n(u, x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-itu} B_n(t, x) dt ,$$

the existence of which is assured by (8). Hence we have

$$p_n(x) = a_n(x, x) + b_n(x, x) , \qquad x \neq 0 .$$

Basu [1], in proving $\sup_{|x| \geqq 1} |x|^{2+\delta} |b_n(x, x)| \leqq C n^{-\delta/2} E|X_1|^{2+\delta} + O(n^{-\delta/2})$ under the condition $E|X_1|^{2+\delta} < \infty$, used implicitly the estimates

$$\sup_{|x| \geqq 1} |x|^{2+\delta} |b_n(x, x)| \leqq Cn \sup_{|x| \geqq 1} |x|^{2+\delta} P\left(|X_1| > \frac{1}{2} n^{1/2} |x|\right) + O(n^{-\delta/2}) \tag{9}$$

and

$$\sup_{|x| \geqq 1} |x|^{2+\delta} P\left(|X_1| > \frac{1}{2} n^{1/2} |x|\right) \leqq C n^{-(1+\delta/2)} E|X_1|^{2+\delta} \tag{10}$$

([1], Eq. (3.4)). (Here and below, C denotes a positive constant which may differ from one inequality to another.) However, we can see

$$\begin{aligned} \sup_{|x| \geqq 1} |x|^{2+\delta} P\left(|X_1| > \frac{1}{2} n^{1/2} |x|\right) &\leqq 4n^{-1} \sup_{|x| \geqq 1} |x|^{\delta} \int_{|u| > (1/2) n^{1/2} |x|} u^2 dF(u) \\ &\leqq 4n^{-(1+\delta/2)} \sup_{|x| \geqq 1} (n^{1/2} |x|)^{\delta} \int_{|u| > (1/2) n^{1/2} |x|} u^2 dF(u) \\ &= O(n^{-(1+\delta/2)}) , \end{aligned} \tag{11}$$

because of the condition (a). Therefore, using (11) in place of (10), we have from (9)

$$\sup_{|x| \geqq 1} |x|^{2+\delta} |b_n(x, x)| = O(n^{-\delta/2}) \tag{12}$$

under the condition (a). In fact, Basu [1] showed (9) for only the case $0 < \delta < 1$. However we can verify that (9) holds also for $\delta = 1$ by investigating Basu's procedure, so that (12) is valid for $\delta = 1$.

Accordingly, in order to show (4), it is sufficient to prove

$$\sup_{|x| \geqq 1} |x|^{2+\delta} |a_n(x, x) - \phi(x)| = O(n^{-\delta/2}) . \tag{13}$$

In what follows, derivatives of a function denote the derivatives with respect to t.

We first have

(14)
$$A_n'''(t, x)=(n-1)(n-2)n^{-1/2}\{\alpha_n'(n^{-1/2}t, x)\}^3\{\alpha_n(n^{-1/2}t, x)\}^{n-3}$$
$$+3(n-1)n^{-1/2}\alpha_n'(n^{-1/2}t, x)\alpha_n''(n^{-1/2}t, x)\{\alpha_n(n^{-1/2}t, x)\}^{n-2}$$
$$+n^{-1/2}\alpha_n'''(n^{-1/2}t, x)\{\alpha_n(n^{-1/2}t, x)\}^{n-1},$$

so that $A_n'''(t, x)\in L^1$ for large n. Letting $\theta(t)$ denote the characteristic function of $\phi(x)$, we have $\theta'''(t)=(3t-t^3)e^{-t^2/2}\in L^1$ and hence

(15)
$$|x|^3|a_n(x, x)-\phi(x)|=\frac{1}{2\pi}\left|\int_{-\infty}^{\infty} e^{-itx}(A_n'''(t, x)-(3t-t^3)e^{-t^2/2})dt\right|.$$

Here we state some lemmas which are needed for the proof.

Lemma 1 [9]. *For some sufficiently small* $\varepsilon_1>0$ *and for all large* n,

$$|\alpha_n(t, x)|\leqq e^{-t^2/4} \quad \textit{for} \quad |t|<\varepsilon_1 .$$

Lemma 2. *Suppose that*

(a) $M(z)=\int_{|u|>z} u^2dF(u)=O(z^{-\delta})$ *as* $z\to\infty$

is satisfied. If $0<\delta<1$, *then we have*

(16)
$$\int_{|u|>|v|^{-1}}|u|^pdF(u)=O(|v|^{2-p+\delta}) \quad \textit{as} \quad |v|\to 0 \quad \textit{for} \quad p=1, 2$$

and

(17)
$$\int_{|u|\leqq|v|^{-1}}|u|^pdF(u)=O(|v|^{2-p+\delta}) \quad \textit{as} \quad |v|\to 0 \quad \textit{for} \quad p=3, 4, 5 .$$

If $\delta=1$, *then we have* (16) *with* $p=1, 2$ *and* (17) *with* $p=4, 5$.

Proof. Let $p=1, 2$. We have for $0<\delta\leqq 1$,

$$\int_{|u|>|v|^{-1}}|u|^pdF(u)\leqq|v|^{2-p}\int_{|u|>|v|^{-1}}u^2dF(u)=|v|^{2-p}M(|v|^{-1})=O(|v|^{2-p+\delta}) .$$

Let $p=3, 4, 5$. We have

$$\int_{|u|\leqq|v|^{-1}}|u|^pdF(u)=-\int_0^{|v|^{-1}}u^{p-2}dM(u)$$
$$\leqq(p-2)\int_0^{|v|^{-1}}u^{p-3}M(u)du=(p-2)\int_0^{|v|^{-1}}O(u^{p-3-\delta})du ,$$

so that, when $0<\delta<1$, (17) holds for $p=3, 4, 5$, and when $\delta=1$, (17) holds for $p=4, 5$.

Using this lemma, we show the following.

Lemma 3. *Let* $0<\delta\leqq 1$. *If the condition* (a) *is satisfied, and in addition, the further condition* (b) *is satisfied in the case* $\delta=1$, *then we have*

(18)
$$\alpha_n(t, x)=1+\xi_1(n, x)+t\xi_2(n, x)-\frac{1}{2}t^2(1+\xi_3(n, x)+\gamma_{n,1}(t, x)) ,$$

(19) $$\alpha_n'(t, x)=\xi_2(t, x)-t(1+\xi_3(n, x)+\gamma_{n,2}(t, x))\ ,$$

(20) $$\alpha_n''(t, x)=-(1+\xi_3(n, x)+\gamma_{n,3}(t, x))\ ,$$

where $\xi_j(n, x)$, $j=1, 2, 3$, *are independent of* t *and*

$$\xi_1(n, x)=O(n^{-(2+\delta)/2}) \quad \text{as} \quad n\to\infty\ ,$$
$$\xi_2(n, x)=O(n^{-(1+\delta)/2}) \quad \text{as} \quad n\to\infty\ ,$$
$$\xi_3(n, x)=O(n^{-\delta/2}) \quad \text{as} \quad n\to\infty$$

hold independently of x, *and*
$\gamma_{n,j}(t, x)=O(|t|^\delta)$, $j=1, 2, 3$, *as* $|t|\to 0$, *independently of* n *and* x.

Proof. By Taylor's expansion, we have

$$\begin{aligned}\alpha_n(t, x)&=\int_{|u|\leqq(1/2)n^{1/2}|x|}dF(u)+it\int_{|u|\leqq(1/2)n^{1/2}|x|}udF(u)\\&\quad-\frac{1}{2}t^2\Big(\int_{|u|\leqq(1/2)n^{1/2}|x|}u^2dF(u)+\gamma_{n,1}(t, x)\Big)\\&=1-\int_{|u|>(1/2)n^{1/2}|x|}dF(u)-it\int_{|u|>(1/2)n^{1/2}|x|}udF(u)\\&\quad-\frac{1}{2}t^2\Big(1-\int_{|u|>(1/2)n^{1/2}|x|}u^2dF(u)+\gamma_{n,1}(t, x)\Big)\ .\end{aligned}$$

Here noting that $|x|\geqq 1$ and using the condition (a), we have

$$\int_{|u|>(1/2)n^{1/2}|x|}dF(u)\leqq\int_{|u|>(1/2)n^{1/2}}dF(u)=O(n^{-(2+\delta)/2})\ ,$$
$$\Big|\int_{|u|>(1/2)n^{1/2}|x|}udF(u)\Big|\leqq\int_{|u|>(1/2)n^{1/2}}|u|dF(u)=O(n^{-(1+\delta)/2})\ ,$$
$$\int_{|u|>(1/2)n^{1/2}|x|}u^2dF(u)\leqq\int_{|u|>(1/2)n^{1/2}}u^2dF(u)=O(n^{-\delta/2})\ ,$$

and, by Lemma 2,

(21) $$\begin{aligned}|\gamma_{n,1}(t, x)|&\leqq\mathrm{Re}|\gamma_{n,1}(t, x)|+\mathrm{Im}|\gamma_{n,1}(t, x)|\\&=2t^{-2}\Big|\int_{|u|\leqq(1/2)n^{1/2}|x|}\Big(\cos tu-1+\frac{1}{2}t^2u^2\Big)dF(u)\Big|\\&\quad+2t^{-2}\Big|\int_{|u|\leqq(1/2)n^{1/2}|x|}(\sin tu-tu)dF(u)\Big|\\&\leqq Ct^{-2}\int_{|u|\leqq|t|^{-1}}t^4u^4dF(u)+Ct^{-2}\int_{|u|>|t|^{-1}}u^2dF(u)\\&\quad+2t^{-2}\Big|\int_{|u|\leqq\min(|t|^{-1},(1/2)n^{1/2}|x|)}(\sin tu-tu)dF(u)\Big|\end{aligned}$$

$$+Ct^{-2}\int_{|u|>|t|^{-1}}|tu|dF(u)$$
$$=O(|t|^{\delta})+2t^{-2}\left|\int_{|u|\leqq\min(|t|^{-1},(1/2)n^{1/2}|x|)}(\sin tu-tu)dF(u)\right| .$$

If $0<\delta<1$, then

$$(22)\qquad t^{-2}\left|\int_{|u|\leqq\min(|t|^{-1},(1/2)n^{1/2}|x|)}(\sin tu-tu)dF(u)\right|\leqq t^{-2}\int_{|u|\leqq|t|^{-1}}|tu|^{3}dF(u)=O(|t|^{\delta}) .$$

If $\delta=1$, then

$$(23)\qquad t^{-2}\left|\int_{|u|\leqq\min(|t|^{-1},(1/2)n^{1/2}|x|)}(\sin tu-tu)dF(u)\right|$$
$$\leqq t^{-2}\left|\int_{|u|\leqq\min(|t|^{-1},(1/2)n^{1/2}|x|)}t^{3}u^{3}dF(u)\right|$$
$$+t^{-2}\left|\int_{|u|\leqq\min(|t|^{-1},(1/2)n^{1/2}|x|)}\left(\sin tu-tu-\frac{1}{3!}t^{3}u^{3}\right)dF(u)\right|$$
$$=|t|\left|\int_{|u|\leqq\min(|t|^{-1},(1/2)n^{1/2}|x|)}u^{3}dF(u)\right|+C|t|^{3}\int_{|u|\leqq|t|^{-1}}|u|^{5}dF(u)$$
$$=O(|t|) ,$$

which follows from the condition (b) and Lemma 2. Thus we have $|\gamma_{n,1}(t, x)|=O(|t|^{\delta})$ as $|t|\to 0$ for $0<\delta\leqq 1$ from (21)-(23), and so (18) is proved. As to (19) and (20), we have

$$\alpha_n'(t, x)=i\int_{|u|\leqq(1/2)n^{1/2}|x|}udF(u)-t\left(\int_{|u|\leqq(1/2)n^{1/2}|x|}u^{2}dF(u)+\gamma_{n,2}(t, x)\right)$$
$$=\xi_2(n, x)-t(1+\xi_3(n, x)+\gamma_{n,2}(t, x)) ,$$
$$\alpha_n''(t, x)=-\int_{|u|\leqq(1/2)n^{1/2}|x|}u^{2}dF(u)-\gamma_{n,3}(t, x)$$
$$=-(1+\xi_3(n, x)+\gamma_{n,3}(t, x)) ,$$

and further, by a way similar to the one when we have shown $|\gamma_{n1}(t, x)|=O(|t|^{\delta})$ above, we have, from Lemma 2 and the condition (b),

$$|\gamma_{n,2}(t, x)|\leqq\mathrm{Re}|\gamma_{n,2}(t, x)|+\mathrm{Im}|\gamma_{n,2}(t, x)|$$
$$=|t|^{-1}\left|\int_{|u|\leqq(1/2)n^{1/2}|x|}u(-\sin tu+tu)dF(u)\right|$$
$$+|t|^{-1}\left|\int_{|u|\leqq(1/2)n^{1/2}|x|}u(\cos tu-1)dF(u)\right|$$
$$=O(|t|^{\delta}) ,$$

$$\begin{aligned}|\gamma_{n,3}(t, x)| &\leqq \mathrm{Re}|\gamma_{n,3}(t, x)|+\mathrm{Im}|\gamma_{n,3}(t, x)| \\ &= \left|\int_{|u|\leqq(1/2)n^{1/2}|x|} u^2(\cos tu-1)dF(u)\right| \\ &\quad +\left|\int_{|u|\leqq(1/2)n^{1/2}|x|} u^2 \sin tu dF(u)\right| \\ &= O(|t|^{\delta})\end{aligned}$$

for $0<\delta\leqq 1$. Therefore, we see the validity of (19) and (20), and the proof of the lemma is completed.

Now let us return to the proof of the theorem. From Lemma 3, we can choose ε_2 so small that

$$|\gamma_{n,j}(t, x)|\leqq C|t|^{\delta} \quad \text{for} \quad |t|\leqq\varepsilon_2\,, \quad j=1, 2, 3\,. \tag{24}$$

From (15), we have

$$\begin{aligned}|x|^3|a_n(x, x)-\phi(x)| &\leqq \left|\int_{|t|\leqq n^{1/2}\varepsilon} e^{-itx}(A_n'''(t, x)-(3t-t^3)e^{-t^2/2})dt\right| \\ &\quad +\int_{|t|>n^{1/2}\varepsilon}|A_n'''(t, x)|dt+\int_{|t|>n^{1/2}\varepsilon}|3t-t^3|e^{-t^2/2}dt \\ &\equiv I_1+I_2+I_3\,,\end{aligned} \tag{25}$$

say, where $\varepsilon=\min(\varepsilon_1, \varepsilon_2)$.

$$I_3=O(n^{-\delta/2}) \tag{26}$$

is obvious, and

$$I_2=O(n^{-\delta/2}) \tag{27}$$

has also been shown implicitly by Basu [1] under our assumptions. In fact, the validity of (27) can be verified in both cases $0<\delta<1$ and $\delta=1$. Hence we need only prove

$$I_1=O(n^{-\delta/2})$$

for $0<\delta\leqq 1$. Using (14) and Lemma 3, we have

$$\begin{aligned}A_n'''(t, x) &= \Big\{(n-1)(n-2)n^{-1/2}[\xi_2(n, x)-n^{-1/2}t(1+\xi_3(n, x)+\gamma_{n,2}(n^{-1/2}t, x))]^3 \\ &\quad +3(n-1)n^{-1/2}[\xi_2(n, x)-n^{-1/2}t(1+\xi_3(n, x)+\gamma_{n,2}(n^{-1/2}t, x))] \\ &\quad \times[-1-\xi_3(n, x)-\gamma_{n,3}(n^{-1/2}t, x)] \\ &\quad \times\left[1+\xi_1(n, x)+n^{-1/2}t\xi_2(n, x)-\frac{1}{2}n^{-1}t(1+\xi_3(n, x)+\gamma_{n,1}(n^{-1/2}t, x))\right]\Big\} \\ &\quad \times\{\alpha_n(n^{-1/2}t, x)\}^{n-3}+n^{-1/2}\alpha_n'''(n^{-1/2}t, x)\{\alpha_n(n^{-1/2}t, x)\}^{n-1}\,,\end{aligned}$$

so that

$$A_n'''(t, x)=(3t-t^3)\{\alpha_n(n^{-1/2}t, x)\}^{n-3}+R_n(t, x)+Q_n(t, x) ,$$

where

(28)
$$\begin{aligned}R_n(t, x)=\Big\{&(-3n+2)n^{-1/2}[\xi_2(n, x)-n^{-1/2}t(1+\xi_3(n, x)+\gamma_{n,2}(n^{-1/2}t, x))]^3\\
&+n^{3/2}[\{\xi_2(n, x)\}^3-3\{\xi_2(n, x)\}^2 n^{-1/2}t(1+\xi_3(n, x)+\gamma_{n,2}(n^{-1/2}t, x))\\
&\quad+3\xi_2(n, x)n^{-1}t^2(1+\xi_3(n, x)+\gamma_{n,2}(n^{-1/2}t, x))^2]\\
&-n^{3/2}(n^{-1/2}t)^3[3(\xi_3(n, x)+\gamma_{n,2}(n^{-1/2}t, x))\\
&\quad+3(\xi_3(n, x)+\gamma_{n,2}(n^{-1/2}t, x))^2+(\xi_3(n, x)+\gamma_{n,2}(n^{-1/2}t,x))^3]\\
&-3n^{-1/2}[\xi_2(n, x)-n^{-1/2}t(1+\xi_3(n, x)+\gamma_{n,2}(n^{-1/2}t, x))]\\
&\quad\times[-1-\xi_3(n, x)-\gamma_{n,3}(n^{-1/2}t, x)]\\
&\quad\times\Big[1+\xi_1(n, x)+n^{-1/2}t\xi_2(n, x)-\frac{1}{2}n^{-1}t^2(1+\xi_3(n, x)+\gamma_{n,1}(n^{-1/2}t, x))\Big]\\
&+3n^{1/2}[\xi_2(n, x)-n^{-1/2}t(\xi_3(n, x)+\gamma_{n,2}(n^{-1/2}t, x))]\\
&\quad\times[-1-\xi_3(n, x)-\gamma_{n,3}(n^{-1/2}t, x)]\\
&\quad\times\Big[1+\xi_1(n, x)+n^{-1/2}t\xi_2(n, x)-\frac{1}{2}n^{-1}t^2(1+\xi_3(n, x)+\gamma_{n,1}(n^{-1/2}t, x))\Big]\\
&+3n^{1/2}(-n^{-1/2}t)[-\xi_3(n, x)-\gamma_{n,3}(n^{-1/2}t, x)]\\
&\quad\times\Big[1+\xi_1(n, x)+n^{-1/2}t\xi_2(n, x)-\frac{1}{2}n^{-1}t^2(1+\xi_3(n, x)+\gamma_{n,1}(n^{-1/2}t, x))\Big]\\
&+3n^{1/2}(-n^{-1/2}t)(-1)\Big[\xi_1(n, x)+n^{-1/2}t\xi_2(n, x)\\
&\quad-\frac{1}{2}n^{-1}t^2(1+\xi_3(n, x)+\gamma_{n,1}(n^{-1/2}t, x))\Big]\Big\}\{\alpha_n(n^{-1/2}t, x)\}^{n-3}\end{aligned}$$

and

$$Q_n(t, x)=n^{-1/2}\alpha_n'''(n^{-1/2}t, x)\{\alpha_n(n^{-1/2}t, x)\}^{n-1} .$$

Therefore, we have

(29)
$$\begin{aligned}I_1&=\left|\int_{|t|\leqq n^{1/2}\varepsilon}e^{-itx}[(3t-t^3)(\{\alpha_n(n^{-1/2}t, x)\}^{n-3}-e^{-t^2/2})+R_n(t, x)+Q_n(t, x)]dt\right|\\
&\leqq\int_{|t|\leqq n^{1/2}\varepsilon}|3t-t^3||\{\alpha_n(n^{-1/2}t, x)\}^{n-3}-e^{-t^2/2}|dt\\
&\quad+\int_{|t|\leqq n^{1/2}\varepsilon}|R_n(t, x)|dt+\left|\int_{|t|\leqq n^{1/2}\varepsilon}e^{-itx}Q_n(t, x)dt\right| .\end{aligned}$$

We first prove

(30)
$$J_1\equiv\int_{|t|\leqq n^{1/2}\varepsilon}|3t-t^3||\{\alpha_n(n^{-1/2}t, x)\}^{n-3}-e^{-t^2/2}|dt=O(n^{-\delta/2}) .$$

In fact,

$$\begin{aligned} J_1 &\leqq \int_{|t|\leqq n^{1/2}\varepsilon} |3t-t^3||\{\alpha_n(n^{-1/2}t, x)\}^{n-3}-\{\alpha_n(n^{-1/2}t, x)\}^n|dt \\ &+\int_{|t|\leqq n^{1/2}\varepsilon} |3t-t^3||\{\alpha_n(n^{-1/2}t, x)\}^{n}-\{f(n^{-1/2}t)\}^n|dt \\ &+\int_{|t|\leqq n^{1/2}\varepsilon} |3t-t^3||\{f(n^{-1/2}t)\}^{n}-e^{-t^2/2}|dt \\ &\equiv J_{11}+J_{12}+J_{13}\ , \end{aligned}$$

say. $J_{13}=O(n^{-\delta/2})$ can be shown by a way similar to the one when $\sup_x|p_n(x)-\phi(x)|=O(n^{-\delta/2})$ was proved. As to J_{11}, we have, from Lemmas 1 and 2,

$$\begin{aligned} J_{11} &\leqq C\int_{|t|\leqq n^{1/2}\varepsilon} |3t-t^3||1-\{\alpha_n(n^{-1/2}t, x)\}^3|e^{-t^2/4}dt \\ &\leqq 3C\int_{|t|\leqq n^{1/2}\varepsilon} |3t-t^3||1-\alpha_n(n^{-1/2}t, x)|e^{-t^2/4}dt \\ &\leqq C\int_{|t|\leqq n^{1/2}\varepsilon} |3t-t^3|\Big(|\xi_1(n, x)|+n^{-1/2}t|\xi_2(n, x)|+\frac{1}{2}n^{-1}t^2 \\ &+\frac{1}{2}n^{-1}t^2|\xi_3(n, x)|+\frac{1}{2}n^{-1}t^2|\gamma_{n,1}(n^{-1/2}t, x)|\Big)e^{-t^2/4}dt \\ &=O(n^{-1})=O(n^{-\delta/2})\ . \end{aligned}$$

It remains to prove $J_{12}=O(n^{-\delta/2})$. Keeping (6) in mind, we need only show

$$J_{12}^*\equiv\int_{|t|\leqq n^{1/2}\varepsilon} |t|^3|B_n(t, x)|dt=O(n^{-\delta/2})\ . \tag{31}$$

From (7),

$$J_{12}^*\leqq \sum_{j=1}^{n}\binom{n}{j}\int_{|t|\leqq n^{1/2}\varepsilon} |t|^3|\alpha_n(n^{-1/2}t, x)|^{n-j}|\beta_n(n^{-1/2}t, x)|^j dt\ .$$

Making use of Lemma 1 and the fact that $|\beta_n(\cdot, x)|\leqq P(|X_1|>(1/2)n^{1/2}|x|)=O(n^{-(1+\delta/2)})$ which is assured by (11), we have

$$\begin{aligned} J_{12}^* &\leqq C\sum_{j=1}^{n}\binom{n}{j}\int_{|t|\leqq n^{1/2}\varepsilon} |t|^3 e^{-(n-j)t^2/(4n)}n^{-j(1+\delta/2)}dt \\ &=\sum_{j=1}^{[10/\delta]}+\sum_{j=[10/\delta]+1}^{n}\equiv\Sigma_1+\Sigma_2\ , \end{aligned}$$

say. We find that

$$\Sigma_1\leqq C\sum_{j=1}^{[10/\delta]}n^j\int_{|t|\leqq n^{1/2}\varepsilon} |t|^3 e^{-Ct^2}n^{-j(1+\delta/2)}dt=O(n^{-\delta/2})$$

and

$$\Sigma_2 \leqq C \sum_{j=[10/\delta]+1}^{n} n^j n^2 (Kn^{-(1+\delta/2)})^j \leqq Cn^2 \sum_{j=[10/\delta]+1}^{\infty} (Kn^{-\delta/2})^j ,$$

where K is a positive constant such that $|\beta_n(\cdot, x)| \leqq Kn^{-(1+\delta/2)}$. Since we can make $0 < Kn^{-\delta/2} < 1$ for large n, we conclude $\Sigma_2 = O(n^{-\delta/2})$. Therefore we get (31), and hence (30) is shown.

We next prove

$$\int_{|t| \leqq n^{1/2}\varepsilon} |R_n(t, x)| dt = O(n^{-\delta/2}) . \tag{32}$$

As to the first term on the right hand side of (28), (J_2, say,) we have from Lemmas 1, 3 and (24),

$$\begin{aligned} &\int_{|t| \leqq n^{1/2}\varepsilon} |J_2| dt \\ &\quad \leqq Cn^{1/2} \int_{|t| \leqq n^{1/2}\varepsilon} |\xi_2(n, x) - n^{-1/2} t (1 + \xi_3(n, x) + \gamma_{n,2}(n^{-1/2}t, x))|^3 |\alpha_n(n^{-1/2}t, x)|^{n-3} dt \\ &\quad \leqq Cn^{1/2} \int_{|t| \leqq n^{1/2}\varepsilon} |O(n^{-(1+\delta)/2}) + n^{-1/2} t (1 + O(n^{-\delta/2}) + C|n^{-1/2}t|^{\delta})|^3 e^{-t^2/4} dt \\ &\quad = O(n^{-\delta/2}) \end{aligned}$$

for $0 < \delta \leqq 1$. By an argument similar to the above, using Lemmas 1 and 3, we see that the other terms on the right hand side of (28) have the order of $O(n^{-\delta/2})$, so that (32) is proved for $0 < \delta \leqq 1$.

Finally, we show that for $0 < \delta < 1$,

$$\left| \int_{|t| \leqq n^{1/2}\varepsilon} e^{-itx} Q_n(t, x) dt \right| \leqq C|x|^{1-\delta} n^{-\delta/2} , \tag{33}$$

and

$$\left| \int_{|t| \leqq n^{1/2}\varepsilon} e^{-itx} Q_n(t, x) dt \right| = O(n^{-1/2}) \tag{34}$$

in order to prove the case $\delta = 1$. If we could show (33) and (34), then we can conclude from (25), (26), (27), (29), (30) and (32) that for $|x| \geqq 1$,

$$|x|^3 |a_n(x, x) - \phi(x)| = O(n^{-\delta/2}) + |x|^{1-\delta} O(n^{-\delta/2}) ,$$

which implies (13). However, the case $0 < \delta < 1$ is easily handled. In fact, by Lemma 2 with $p = 3$,

$$|\alpha_n'''(n^{-1/2}t, x)| \leqq \int_{|u| \leqq (1/2) n^{1/2}|x|} |u|^3 dF(u) = |x|^{1-\delta} O(n^{(1-\delta)/2})$$

and so, by Lemma 1, we have

$$\int_{|t|\leq n^{1/2}\varepsilon}|Q_n(t, x)|dt\leq Cn^{-1/2}|x|^{1-\delta}n^{(1-\delta)/2}\int_{|t|\leq n^{1/2}\varepsilon}e^{-t^2/4}dt$$
$$\leq C|x|^{1-\delta}n^{-\delta/2} ,$$

which proves (33).

To prove (34) under the conditions (a) with $\delta=1$ and (b), we need some preliminaries.

Lemma 4. *If the conditions* (a) *with* $\delta=1$ *and* (b) *are satisfied, then we have*

(35) $$f(t)=\exp\left\{-\frac{1}{2}t^2(1+\zeta_1(t))\right\}$$

and

(36) $$f'(t)=-t(1+\zeta_2(t)) ,$$

where $\zeta_j(t)=O(|t|)$, $j=1, 2$, *as* $|t|\to 0$.

For the proof of (35), see Ibragimov [5]. Further, we can show (36) by a way similar to one when we have shown (19) for $\alpha_n'(t, x)$.

The following simple known lemma on the characteristic function of a non-degenerate distribution will be also used.

Lemma 5. *There exist positive constants* ε_3 *and* c *such that* $|f(t)|\leq e^{-ct^2}$ *for* $|t|\leq\varepsilon_3$.

Now we retern to prove (34). We choose ε not larger than ε_3 in Lemma 5, with

(37) $$\max_{0<|t|\leq\varepsilon}|\zeta_1(t)|\leq\frac{1}{10} .$$

We have

$$\left|\int_{|t|\leq n^{1/2}\varepsilon}e^{-itx}Q_n(t, x)dt\right|=\left|\int_{|t|\leq n^{1/2}\varepsilon}e^{-itx}n^{-1/2}\alpha_n'''(n^{-1/2}t, x)\{\alpha_n(n^{-1/2}t, x)\}^{n-1}dt\right|$$
$$\leq n^{-1/2}\left|\int_{|t|\leq n^{1/2}\varepsilon}e^{-itx}\alpha_n'''(n^{-1/2}t, x)(\{\alpha_n(n^{-1/2}t, x)\}^{n-1}-e^{-t^2/2})dt\right|$$
$$+n^{-1/2}\left|\int_{|t|\leq n^{1/2}\varepsilon}e^{-itx}\alpha_n'''(n^{-1/2}t, x)e^{-t^2/2}dt\right|$$
$$\equiv L_1+L_2 ,$$

say. We first estimate L_2. We have

$$L_2=n^{-1/2}\left|\int_{|t|\leq n^{1/2}\varepsilon}e^{-itx}e^{-t^2/2}dt\int_{|u|\leq(1/2)n^{1/2}|x|}u^3e^{in^{-1/2}tu}dF(u)\right|$$
$$=n^{-1/2}\left|\int_{|u|\leq(1/2)n^{1/2}|x|}u^3dF(u)\int_{|t|\leq n^{1/2}\varepsilon}\exp\left\{-\frac{1}{2}t^2+i(n^{-1/2}u-x)t\right\}dt\right|$$

$$=n^{-1/2}\left|\int_{|u|\leqq(1/2)n^{1/2}|x|} u^3 \exp\left\{-\frac{1}{2}(n^{-1/2}u-x)^2\right\}dF(u)\right.$$
$$\left.\times\int_{|t|\leqq n^{1/2}\varepsilon} \exp\left\{-\frac{1}{2}(t-i(n^{-1/2}u-x))^2\right\}dt\right|$$

and, by Cauchy's integral theorem, we have

$$L_2\leqq n^{-1/2}\left|\int_{|u|\leqq(1/2)n^{1/2}|x|} u^3 \exp\left\{-\frac{1}{2}(n^{-1/2}u-x)^2\right\}dF(u)\int_{|t|\leqq n^{1/2}\varepsilon} e^{-t^2/2}dt\right|$$
$$+2n^{-1/2}\left|\int_{|u|\leqq(1/2)n^{1/2}|x|} u^3 \exp\left\{-\frac{1}{2}(n^{-1/2}u-x)^2\right\}dF(u)\right.$$
$$\left.\times\int_0^{|n^{-1/2}u-x|} \exp\left\{-\frac{1}{2}(n^{1/2}\varepsilon+iw)^2\right\}dw\right|$$
$$\equiv L_{21}+L_{22}\ ,$$

say. As to L_{21}, letting $H(v)=\int_{|u|\leqq v} u^3dF(u)$, we have

$$L_{21}\leqq Cn^{-1/2}\left|\int_{|u|\leqq(1/2)n^{1/2}|x|} u^3 \exp\left\{-\frac{1}{2}(n^{-1/2}u-x)^2\right\}dF(u)\right|$$
$$\leqq Cn^{-1/2}\left|\int_0^{(1/2)n^{1/2}|x|} \exp\left\{-\frac{1}{2}(n^{-1/2}u-|x|)^2\right\}dH(u)\right|\ ,$$

which is, by the integration by parts,

$$\leqq Cn^{-1/2}\left|H\left(\frac{1}{2}n^{1/2}|x|\right)\right|$$
$$+Cn^{-1/2}\left|\int_0^{(1/2)n^{1/2}|x|} n^{-1/2}(n^{-1/2}u-|x|)\exp\left\{-\frac{1}{2}(n^{-1/2}u-|x|)^2\right\}H(u)du\right|$$
$$\leqq O(n^{-1/2})+Cn^{-1/2}\int_0^\infty ye^{-y^2/2}dy\ ,$$

where we have used $H(u)=O(1)$ by the condition (b), so that we conclude $L_{21}=O(n^{-1/2})$. Next we have

$$L_{22}\leqq 2n^{-1/2}\int_{|u|\leqq(1/2)n^{1/2}|x|} |u|^3 \exp\left\{-\frac{1}{2}(n^{-1/2}u-x)^2\right\}dF(u)\int_0^{|n^{-1/2}u-x|} e^{-n\varepsilon^2/2}e^{w^2/2}dw$$
$$\leqq 2n^{-1/2}e^{-n\varepsilon^2/2}\int_{|u|\leqq(1/2)n^{1/2}|x|} |u|^3 \exp\left\{-\frac{1}{2}(n^{-1/2}u-x)^2\right\}dF(u)$$
$$\times\int_0^1 \exp\left\{\frac{1}{2}(n^{-1/2}u-x)^2v^2\right\}|n^{-1/2}u-x|dv\ ,$$

which is, since $|n^{-1/2}u-x|\leqq(3/2)|x|$ for $|u|\leqq(1/2)n^{1/2}|x|$,

$$\leqq 3n^{-1/2}e^{-n\varepsilon^2/2}|x|\int_{|u|\leqq(1/2)n^{1/2}|x|}|u|^3dF(u)\int_0^1\exp\left\{-\frac{1}{2}(n^{-1/2}u-x)^2(1-v^2)\right\}dv$$

$$\leqq 3n^{-1/2}e^{-n\varepsilon^2/2}|x|\int_{|u|\leqq(1/2)n^{1/2}|x|}|u|^3dF(u)\int_0^1\exp\left\{-\frac{1}{8}x^2(1-v^2)\right\}dv\ ,$$

where we have used $(1/2)|x|\leqq|n^{-1/2}u-x|$ for $|u|\leqq(1/2)n^{1/2}|x|$. Since

$$\int_{|u|\leqq(1/2)n^{1/2}|x|}|u|^3dF(u)=O\Big(\int_{|u|\leqq(1/2)n^{1/2}|x|}u^4dF(u)\Big)=O(n^{1/2}|x|)$$

by Lemma 2, we have

$$\begin{aligned}L_{22}&\leqq Ce^{-n\varepsilon^2/2}x^2e^{-x^2/8}\int_0^1e^{x^2v/8}dv\\&\leqq Ce^{-n\varepsilon^2/2}x^2e^{-x^2/8}x^{-2}e^{x^2/8}=O(n^{-1/2})\ .\end{aligned}$$

We are now going to prove $L_1=O(n^{-1/2})$. We have

$$\begin{aligned}L_1\leqq\ & n^{-1/2}\left|\int_{|t|\leqq n^{1/2}\varepsilon}e^{-itx}\alpha_n'''(n^{-1/2}t,x)[\{\alpha_n(n^{-1/2}t,x)\}^{n-1}-\{\alpha_n(n^{-1/2}t,x)\}^n]dt\right|\\&+n^{-1/2}\left|\int_{|t|\leqq n^{1/2}\varepsilon}e^{-itx}\alpha_n'''(n^{-1/2}t,x)[\{\alpha_n(n^{-1/2}t,x)\}^n-\{f(n^{-1/2}t)\}^n]dt\right|\\&+n^{-1/2}\left|\int_{|t|\leqq n^{1/2}\varepsilon}e^{-itx}\alpha_n'''(n^{-1/2}t,x)[\{f(n^{-1/2}t)\}^n-e^{-t^2/2}]dt\right|\\&\equiv L_{11}+L_{12}+L_{13},\end{aligned}$$

say.

$$\begin{aligned}L_{11}=\ & n^{-1/2}\left|\int_{|t|\leqq n^{1/2}\varepsilon}e^{-itx}\left[\int_{|u|\leqq(1/2)n^{1/2}|x|}u^3e^{in^{-1/2}tu}dF(u)\right]\right.\\&\left.\times\{\alpha_n(n^{-1/2}t,x)\}^{n-1}[\alpha_n(n^{-1/2}t,x)-1]dt\right|\\=\ & n^{-1/2}\left|\int_{|u|\leqq(1/2)n^{1/2}|x|}u^3dF(u)\int_{|t|\leqq n^{1/2}\varepsilon}\exp\{i(n^{-1/2}u-x)t\}\right.\\&\left.\times\{\alpha_n(n^{-1/2}t,x)\}^{n-1}[\alpha_n(n^{-1/2}t,x)-1]dt\right|\\=\ & n^{-1/2}\left|\int_{|u|\leqq(1/2)n^{1/2}|x|}u^3dF(u)\{2i^{-1}(n^{-1/2}u-x)^{-1}\theta\{\alpha_n(\varepsilon,x)\}^{n-1}[\alpha_n(\varepsilon,x)-1]\right.\\&-i^{-1}(n^{-1/2}u-x)^{-1}\int_{|t|\leqq n^{1/2}\varepsilon}\exp\{i(n^{-1/2}u-x)t\}[n^{1/2}\{\alpha_n(n^{-1/2}t,x)\}^{n-1}\alpha_n'(n^{-1/2}t,x)\\&\left.-(n^{1/2}-n^{-1/2})\{\alpha_n(n^{-1/2}t,x)\}^{n-2}\alpha_n'(n^{-1/2}t,x)]dt\}\right|\ ,\end{aligned}$$

the integration by parts being used in the inner integral, where $|\theta|=1$. We note here that $|n^{-1/2}u-x|^{-1}\leqq n^{1/2}|u|^{-1}$ for $|u|\leqq(1/2)n^{1/2}|x|$. Hence

$$
\begin{aligned}
L_{11} \leqq & n^{-1/2} \int_{|u| \leqq (1/2) n^{1/2} |x|} u^2 dF(u) \Big\{ C n^{1/2} |\alpha_n(\varepsilon, x)|^{n-1} \\
& + n \int_{|t| \leqq n^{1/2} \varepsilon} |\alpha_n(n^{-1/2}t, x)|^{n-2} |\alpha_n'(n^{-1/2}t, x)| |\alpha_n(n^{-1/2}t, x) - 1| dt \\
& + \int_{|t| \leqq n^{1/2} \varepsilon} |\alpha_n(n^{-1/2}t, x)|^{n-2} |\alpha_n'(n^{-1/2}t, x)| dt \Big\} \\
\equiv & L_{111} + L_{112} + L_{113} ,
\end{aligned}
$$

say. Since $|\alpha_n(\varepsilon, x)| < 1$ by Lemma 1, we have $L_{111} = O(n^{-1/2})$, and $L_{113} = O(n^{-1/2})$ also follows from Lemma 1. Further, we have, using Lemma 3,

$$
\begin{aligned}
L_{112} \leqq & n^{1/2} \int_{-\infty}^{\infty} u^2 dF(u) \int_{|t| \leqq n^{1/2} \varepsilon} |\alpha_n(n^{-1/2}t, x)|^{n-2} |\alpha_n'(n^{-1/2}t, x)| |\alpha_n(n^{-1/2}t, x) - 1| dt \\
\leqq & C n^{1/2} \int_{|t| \leqq n^{1/2} \varepsilon} e^{-t^2/4} |\xi_2(n, x) + n^{-1/2}(1 + \xi_3(n, x) + \gamma_{n,2}(n^{-1/2}t, x))| \\
& \times \left| \xi_1(n, x) + n^{-1/2} t \xi_2(n, x) + \frac{1}{2} n^{-1} t^2 (1 + \xi_3(n, x) + \gamma_{n,1}(n^{-1/2}t, x)) \right| dt \\
\equiv & O(n^{-1/2}) .
\end{aligned}
$$

As to L_{12}, using the integration by parts again, we have

$$
\begin{aligned}
L_{12} = & n^{-1/2} \Bigg| \int_{|t| \leqq n^{1/2} \varepsilon} \left[\int_{|u| \leqq (1/2) n^{1/2} |x|} u^3 \exp\{i(n^{-1/2}u - x)t\} dF(u) \right] \\
& \times [\{\alpha_n(n^{-1/2}t, x)\}^n - \{f(n^{-1/2}t)\}^n] dt \Bigg| \\
= & n^{-1/2} \Bigg| \int_{|u| \leqq (1/2) n^{1/2} |x|} u^3 dF(u) \int_{|t| \leqq n^{1/2} \varepsilon} \exp\{i(n^{-1/2}u - x)t\} \\
& \times [\{\alpha_n(n^{-1/2}t, x)\}^n - \{f(n^{-1/2}t)\}^n] dt \Bigg| \\
= & n^{-1/2} \Bigg| \int_{|u| \leqq (1/2) n^{1/2} |x|} u^3 dF(u) \Big\{ 2 i^{-1} (n^{-1/2}u - x)^{-1} \theta [\{\alpha_n(\varepsilon, x)\}^n - \{f(\varepsilon)\}^n] \\
& - i^{-1} (n^{-1/2}u - x)^{-1} \int_{|t| \leqq n^{1/2} \varepsilon} \exp\{i(n^{-1/2}u - x)t\} [n^{1/2} \{\alpha_n(n^{-1/2}t, x)\}^{n-1} \alpha_n'(n^{-1/2}t, x) \\
& - n^{1/2} \{f(n^{-1/2}t)\}^{n-1} f'(n^{-1/2}t)] dt \Big\} \Bigg| \\
\leqq & n^{-1/2} \int_{|u| \leqq (1/2) n^{1/2} |x|} u^2 dF(u) \Big\{ C n^{1/2} (|\alpha_n(\varepsilon, x)|^n + |f(\varepsilon)|^n) \\
& + n^{1/2} \int_{|t| \leqq n^{1/2} \varepsilon} |n^{1/2} \{\alpha_n(n^{-1/2}t, x)\}^{n-1} \alpha_n'(n^{-1/2}t, x) - n^{1/2} \{f(n^{-1/2}t)\}^{n-1} f'(n^{-1/2}t)| dt \Big\} \\
\equiv & L_{121} + L_{122} ,
\end{aligned}
$$

say. Since $|\alpha_n(\varepsilon, x)| < 1$ and $|f(\varepsilon)| < 1$, it is seen that $L_{121} = O(n^{-1/2})$. As to L_{122}, making use of Lemmas 3 and 4, we have

$$L_{122}\leqq\int_{-\infty}^{\infty}u^2dF(u)\int_{|t|\leqq n^{1/2}\varepsilon}|n^{1/2}\{\alpha_n(n^{-1/2}t,\ x)\}^{n-1}$$
$$\times[\xi_2(n,\ x)-n^{-1/2}t(1+\xi_3(n,\ x)+\gamma_{n,2}(n^{-1/2}t,\ x))]$$
$$-n^{1/2}\{f(n^{-1/2}t)\}^{n-1}[-n^{-1/2}t(1+\zeta_2(n^{-1/2}t))]|dt$$
$$\leqq C\int_{|t|\leqq n^{1/2}\varepsilon}|-t\{\alpha_n(n^{-1/2}t,x)\}^{n-1}+t\{f(n^{-1/2}t)\}^{n-1}|dt$$
$$+C\int_{|t|\leqq n^{1/2}\varepsilon}|n^{1/2}\{\alpha_n(n^{-1/2}t,\ x)\}^{n-1}\xi_2(n,\ x)$$
$$-t\{\alpha_n(n^{-1/2}t,\ x)\}^{n-1}[\xi_3(n,\ x)+\gamma_{n,2}(n^{-1/2}t,\ x)]+t\{f(n^{-1/2}t)\}^{n-1}\zeta_2(n^{-1/2}t)|dt$$
$$\equiv L_{1221}+L_{1222}\ ,$$

say. $L_{1222}=O(n^{-1/2})$ is easily shown by Lemmas 1, 3, 4 and 5. Moreover we have

$$L_{1221}=C\int_{|t|\leqq n^{1/2}\varepsilon}|t||\{\alpha_n(n^{-1/2}t,\ x)\}^{n-1}-\{f(n^{-1/2}t)\}^{n-1}|dt$$
$$=C\int_{|t|\leqq n^{1/2}\varepsilon}|t||B_{n-1}((n-1)^{1/2}n^{-1/2}t)|dt\ ,$$

so that $J_{1221}=O(n^{-1/2})$ can be shown by a way similar to the one by which we have got (31). Hence $L_{12}=O(n^{-1/2})$ is concluded.

Finally we prove $L_{13}=O(n^{-1/2})$. We have

$$L_{13}=n^{-1/2}\left|\int_{|t|\leqq n^{1/2}\varepsilon}\left[\int_{|u|\leqq(1/2)n^{1/2}|x|}u^3\exp\{i(n^{-1/2}u-x)t\}dF(u)\right][\{f(n^{-1/2}t)\}^n-e^{-t^2/2}]dt\right|$$
$$=n^{-1/2}\left|\int_{|u|\leqq(1/2)n^{1/2}|x|}u^3dF(u)\int_{|t|\leqq n^{1/2}\varepsilon}\exp\{i(n^{-1/2}u-x)t\}[\{f(n^{-1/2}t)\}^n-e^{-t^2/2}]dt\right|$$
$$=n^{-1/2}\left|\int_{|u|\leqq(1/2)n^{1/2}|x|}u^3dF(u)\left\{\{2i^{-1}(n^{-1/2}u-x)^{-1}\theta(\{f(\varepsilon)\}^n-e^{\varepsilon^2n/2})\right.\right.$$
$$-i^{-1}(n^{-1/2}u-x)^{-1}\int_{|t|\leqq n^{1/2}\varepsilon}\exp\{i(n^{-1/2}u-x)t\}$$
$$\left.\left.\times[n^{1/2}\{f(n^{-1/2}t)\}^{n-1}f'(n^{-1/2}t)+te^{-t^2/2}]dt\right\}\right|$$
$$\leqq n^{-1/2}\int_{|u|\leqq(1/2)n^{1/2}|x|}u^2dF(u)\left\{Cn^{1/2}(|f(\varepsilon)|^n+e^{-\varepsilon^2n/2})\right.$$
$$\left.+n^{1/2}\int_{|t|\leqq n^{1/2}\varepsilon}|n^{1/2}\{f(n^{-1/2}t)\}^{n-1}f'(n^{-1/2}t)+te^{-t^2/2}|dt\right\}$$
$$\equiv L_{131}+L_{132}\ ,$$

say. L_{131} is trivially of the order $O(n^{-1/2})$. As to L_{132}, we have from Lemma 4,

$$L_{132}\leqq C\int_{|t|\leqq n^{1/2}\varepsilon}|n^{1/2}\exp\{-(n-1)(2n)^{-1}t^2(1+\zeta_1(n^{-1/2}t))\}$$
$$\times[-n^{-1/2}t(1+\zeta_2(n^{-1/2}t))]+te^{-t^2/2}|dt$$

$$\leqq C\int_{|t|\leqq n^{1/2}\varepsilon}|t|e^{-t^2/2}\left|1-\exp\left\{\frac{1}{2}n^{-1}t^2(1+\zeta_1(n^{-1/2}t))\right\}\exp\left\{-\frac{1}{2}t^2\zeta_1(n^{-1/2}t)\right\}\right|dt$$

$$+C\int_{|t|\leqq n^{1/2}\varepsilon}|t||\zeta_2(n^{-1/2}t)||\exp\{-(n-1)(2n)^{-1}t^2(1+\zeta_1(n^{-1/2}t))\}|dt$$

$$\leqq C\int_{|t|\leqq n^{1/2}\varepsilon}|t|e^{-t^2/2}\left|1-\exp\left\{-\frac{1}{2}t^2\zeta_1(n^{-1/2}t)\right\}\right|dt$$

$$+C\int_{|t|\leqq n^{1/2}\varepsilon}|t|e^{-t^2/2}\left|1-\exp\left\{\frac{1}{2}n^{-1}t^2(1+\zeta_1(n^{-1/2}t))\right\}\right|\left|\exp\left\{-\frac{1}{2}t^2\zeta_1(n^{-1/2}t)\right\}\right|dt$$

$$+C\int_{|t|\leqq n^{1/2}\varepsilon}|t||\zeta_2(n^{-1/2}t)||\exp\{-(n-1)(2n)^{-1}t^2(1+\zeta_1(n^{-1/2}t))\}|dt$$

$$\equiv L_{1321}+L_{1322}+L_{1323}\ ,$$

say. Using the inequality $|e^z-1|\leqq|z|e^{|z|}$, we have

$$L_{1321}\leqq C\int_{|t|\leqq n^{1/2}\varepsilon}|t|\frac{1}{2}t^2|\zeta_1(n^{-1/2}t)|\exp\left\{-\frac{1}{2}t^2(1-|\zeta_1(n^{-1/2}t)|)\right\}dt$$

$$\leqq C\int_{|t|\leqq n^{1/2}\varepsilon}|t|^3|\zeta_1(n^{-1/2}t)|e^{-9t^2/20}dt\ ,$$

because of (37). Noting that $|\zeta_1(t)|\leqq C|t|$ for $|t|\leqq\varepsilon$ by Lemma 4, we conclude that that $L_{1321}=O(n^{-1/2})$. Similarly we have,

$$L_{1322}\leqq C\int_{|t|\leqq n^{1/2}\varepsilon}|t|e^{-9t^2/20}\left|1-\exp\left\{\frac{1}{2}n^{-1}t^2(1+\zeta_1(n^{-1/2}t))\right\}\right|dt$$

$$\leqq C\int_{|t|\leqq n^{1/2}\varepsilon}|t|^3e^{-9t^2/20}\frac{1}{2}n^{-1}(1+|\zeta_1(n^{-1/2}t)|)\exp\left\{\frac{1}{2}n^{-1}t^2(1+|\zeta_1(n^{-1/2}t)|)\right\}dt$$

$$=O(n^{-1/2})\ .$$

Finally we have

$$L_{1323}\leqq C\int_{|t|\leqq n^{1/2}\varepsilon}|t||\zeta_2(n^{-1/2}t)|e^{-9t^2/20}dt=O(n^{-1/2})\ ,$$

since $\zeta_2(t)=O(|t|)$ as $|t|\to 0$. This gives us $L_{13}=O(n^{-1/2})$, and hence (34) is proved. The proof of the theorem is thus completed.

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