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ON GENERALIZED PARANORMAL OPERATORS I

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The aim of this note is to obtain some structure theorems for a class of generalized paranormal operators which contain the class of M-hyponormal operators and is not included in normaloid operators in general.

Furuta [5] has defined a bounded linear operator T on a Hilbert space H as paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector x in H. He has also remarked that a paranormal operator includes a hyponormal operator and is included in normaloid operators. Wadhawa [3] introduced a new class of operators—M-hyponormal operators, which includes hyponormal operators but is not included in normaloid operators in general, because on the basis of example [3] one can see easily that spectral radius inequality is not true in general. Moreover, he also proved [4] a result for every unit vector x,

$$\|T^{n+1}x\| \geq \frac{1}{M^n}\|T^n x\|\,\|Tx\|$$

or

$$\|T^{n+1}x\| \geq \frac{1}{M^{n(n+1)/2}}\|Tx\|^{n+1}$$

where T is a bounded linear operator on H and $M>0$.

On the basis of the above result, we define the generalized paranormal operator as follows:

A bounded linear operator T on H is called *generalized paranormal* if for every unit vector $x \in H$ and $M>0$, T satisfies

$$\|T^2x\| \geq \frac{1}{M}\|Tx\|^2 .$$

The following example (due to Wadhawa [3]) shows that operator T can be generalized paranormal without being normal or hyponormal.

Example. [3] Let $\{e_i\}_{i=1}$ be an orthonormal basis of a Hilbert space H. Let T be a weighted shift defined as $Te_1=e_2$, $Te_2=2e_3$ and $Te_i=e_{i+1}$ for $i\geq 3$. Then $T^*e_1=0$, $T^*e_2=e_1$, $T^*e_3=2e_2$ and $T^*e_i=e_{i-1}$ for $i\geq 4$.

It includes M-hyponormal but is not included in normaloid, in general. The results which are true for generalized paranormal will be true for paranormal if $M=1$. This means that paranormality implies generalized paranormality, but the converse is not true.

In section 1, we have proved the main result in the form of Theorem 1.4 which gives the characterization of generalized paranormal operators. We have also formulated a fundamental inequality, for generalized paranormal operator, which holds for every vector x, not necessarily for every unit vector x, by the homogeneity of operator, we have also formulated the inequality for inverse operator.

In section 2, we have taken an idea from Shah and Sheth [8] and drawn some similar results for generalized paranormal operators.

1. We prove the following theorems.

Theorem 1.1. *Let T be a generalized paranormal operator, then*

$$\|T^3x\| \geq \frac{1}{M^2}\|T^2x\|\|Tx\|, \qquad \text{for every unit vector } x \in H.$$

Proof. For a unit vector x in H, we may assume $Tx \neq 0$. We have

$$\begin{aligned}
\|T^3x\| &= \|Tx\|\left\|T^2\frac{Tx}{\|Tx\|}\right\| \geq \frac{1}{M}\|Tx\|\left\|T\frac{Tx}{\|Tx\|}\right\|^2 \\
&\geq \frac{1}{M}\|Tx\|\left\|\frac{T^2x}{\|Tx\|^2}\right\|^2 \geq \frac{1}{M}\|Tx\|\frac{1}{M}\frac{\|T^2x\|^2}{\|Tx\|^2} \\
&\geq \frac{1}{M}\frac{\|T^2x\|^2}{\|Tx\|} \geq \frac{1}{M^2}\|T^2x\|\frac{\|Tx\|^2}{\|Tx\|}
\end{aligned}$$

$$\|T^3x\| \geq \frac{1}{M^2}\|T^2x\|\|Tx\|.$$

Hence the theorem.

Theorem 1.2. *Let T be a generalised paranormal operator, then*

$$\|T^{k+1}x\|^2 \geq \frac{1}{M^{2k-1}}\|T^kx\|^2\|T^2x\|$$

for a positive integer $k \geq 1$ and every unit vector $x \in H$.

Proof. For case $k=1$

$$\|T^2x\|^2 = \|T^2x\|\|T^2x\| \geq \frac{1}{M}\|Tx\|^2\frac{1}{M}\|Tx\|^2$$

$$\|T^2x\|^2 \geq \frac{1}{M^2}\|Tx\|^4$$

and GP_1 (generalized paranormal operator for $k=1$) is true. Now suppose that (GP_k) is valid for any k and assume that $\|Tx\|\neq 0$, then

$$\begin{aligned}\|T^{k+2}x\|^2 &= \|Tx\|^2\left\|T^{k+1}\frac{Tx}{\|Tx\|}\right\|^2\\ &\geq \|Tx\|^2\frac{1}{M^{2k-1}}\left\|T^k\frac{Tx}{\|Tx\|}\right\|^2\left\|T^2\frac{Tx}{\|Tx\|}\right\|\\ &\geq \|Tx\|^2\frac{1}{M^{2k-1}}\left\|\frac{T^{k-1}x}{\|Tx\|}\right\|^2\frac{\|T^3x\|}{\|Tx\|}\\ &\geq \frac{1}{M^{2k-1}}\|T^{k+1}x\|^2\frac{1}{M^2}\frac{\|T^2x\|\|Tx\|}{\|Tx\|}\\ &\geq \frac{1}{M^{2k-1}}\|T^{k+1}x\|^2\frac{1}{M^2}\|T^2x\|\\ &\geq \frac{1}{M^{2k+1}}\|T^{k+1}x\|^2\|T^2x\|\end{aligned}$$

i.e.

$$\|T^{k+2}x\|^2\geq\frac{1}{M^{2k+1}}\|T^{k+1}x\|^2\|T^2x\| .$$

Hence the Theorem.

Remark. There exists a generalised paranormal operator which is not M-hyponormal. T^2 is not M-hyponormal but by Theorem 1.2 T^2 is a generalized paranormal operator.

Let T be a generalized paranormal, then the following inequality holds for every vector x, not necessarily for unit vector x, by the homogeneity of T.

$$\frac{M^n\|T^{n+1}x\|}{\|T^nx\|}\geq\cdots\geq M^4\frac{\|T^5x\|}{\|T^4x\|}\geq M^3\frac{\|T^4x\|}{\|T^3x\|}\geq M^2\frac{\|T^3x\|}{\|T^2x\|}\geq M\frac{\|T^2x\|}{\|Tx\|}\geq\frac{\|Tx\|}{\|x\|} .$$

From this inequality we can easily obtain the properties of generalized paranormal operators. For any unit vector x,

$$\frac{M\|T^{2n}x\|}{\|T^nx\|}\geq\frac{\|T^nx\|}{\|x\|} ,$$

so T^n is a generalized paranormal operator.

Theorem 1.2 of this paper follows from

$$\left(\frac{M^k\|T^{k+1}x\|}{\|T^kx\|}\right)^2\geq\frac{M\|T^2x\|}{\|Tx\|}\,\frac{\|Tx\|}{\|x\|}=M\|T^2x\|$$

or

$$\frac{M^{2k}\|T^{k+1}\|^2}{\|T^kx\|^2}\geq M\|T^2x\|$$

or

$$\|T^{k+1}x\|^2 \geq \frac{1}{M^{2k-1}}\|T^k x\|^2\|T^2 x\| .$$

Hence the theorem.

If T is invertible, then the following inequality holds for every vector x,

$$\frac{\|x\|}{\|T^{-1}x\|} \geq \frac{\|T^{-1}x\|}{M\|T^{-2}x\|} \geq \frac{\|T^{-2}x\|}{M^2\|T^{-3}x\|} \geq \cdots \geq \frac{\|T^{-n}x\|}{M^n\|T^{-(n+1)}x\|} \geq \cdots .$$

So

$$\|T^{-3}x\| \geq \frac{1}{M^2}\|T^{-2}x\|\|T^{-1}x\| ,$$

thus showing that T^{-1} is generalized paranormal.

Theorem 1.4. *If a generalized paranormal operator T has compact power T^k, then T is compact.*

Proof. We shall introduce a new notation: An operator T is generalized paranormal, symbolically $T \in GP_n$ $(n>0)$, if T satisfies

$$\|T^{n+1}x\| \geq \frac{1}{M^{n(n+1)/2}}\|Tx\|^{n+1} , \tag{1.4.1}$$

for any $x \in H$ with $\|x\|=1$. Obviously, GP concides with GP_1 (GP means generalized paranormal operator). Moreover, we have

$$T \in GP \Rightarrow T \in GP_n , \quad n>0 . \tag{1.4.2}$$

For $n=1$ (1.4.2) is trivial. If (1.4.2) is true for $(n-1)$, then we have

$$\begin{aligned}
\|T^{n+1}x\| &= \|Tx\|\left\|\frac{Tx}{\|Tx\|}\right\| \\
&\geq \frac{1}{M^{(n-1)n/2}}\|Tx\|\left\|\frac{T^2x}{\|Tx\|}\right\|^n \\
&\geq \frac{1}{M^{(n-1)n/2}}\|Tx\|\frac{1}{M^n}\frac{\|T^2x\|^{2n}}{\|Tx\|^n} \\
&\geq \frac{1}{M^{n(n+1)/2}}\frac{\|Tx\|^{2n}}{\|Tx\|^{n-1}} \\
&\geq \frac{1}{M^{n(n+1)/2}}\|Tx\|^{n+1} .
\end{aligned}$$

By (1.4.2) we shall prove that

$$T \in GP_{n+1} , \quad T^n \in \mathfrak{C} \Rightarrow T \in \mathfrak{C} . \tag{1.4.3}$$

where $\mathfrak{C}$ is an algebra of all compact operators. Let us suppose that

$$x_\alpha \to 0 \text{ (weakly)} , \qquad \|x_\alpha\|=1 .$$

Since $T \in GP_{n-1}$, (1.4.1) implies

$$\|T^n x_\alpha\| \geq \frac{1}{M^{n(n-1)/2}} \frac{\|Tx_\alpha\|^n}{\|x_\alpha\|^n} \geq \frac{1}{M^{n(n-1)/2}} \|Tx_\alpha\|^n ,$$

which tells us that Tx_α converges strongly to 0, since

$$\|T^n x_\alpha\| \to 0 \quad \text{by compactness of} \quad T^n .$$

Therefore T is compact.

2. T. Ando [1] has characterized paranormal operators as follows.

Theorem A. *An operator T is paranormal if and only if $T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0$ for all $\lambda > 0$.*

Shah and Sheth [8] gave an essentially similar characterization for paranormal operators with a simple proof. Their result includes two theorems of [1] and [6]. They have used an elementary property of real quadratic forms. If $a > 0$, b and c are real numbers, then $at^2 + bt + c \geq 0$ for every real t if and only if $b^2 - 4ac \leq 0$. In an analogous manner, we have proved some results for generalized paranormal operators.

Theorem 2.1. *An operator T is generalized paranormal if and only if*

$$T^{*2}T^2 + \frac{2\lambda}{M} T^*T + \lambda^2 I \geq 0 ,$$

for all real λ, $M > 0$.

Proof.

$$T^{*2}T^2 + \frac{2\lambda}{M} T^*T + \lambda^2 I \geq 0$$

implies that

$$\left(\left(T^{*2}T^2 + \frac{2\lambda}{M} T^*T + \lambda^2 I\right)x, x\right) \geq 0$$

or

$$(T^{*2}T^2 x, x) + \frac{2\lambda}{M}(T^*Tx, x) + \lambda^2 (x, x) \geq 0$$

or

$$(T^2 x, T^2 x) + \frac{2\lambda}{M}(Tx, Tx) + \lambda^2 (x, x) \geq 0$$

or

$$\|T^2 x\|^2 + \frac{2\lambda}{M} \|Tx\| + \lambda^2 \|x\| \geq 0 .$$

By the above argument, this will happen only if

$$\frac{4}{M^2}\|Tx\|^2-4\|T^2x\|^2\leq 0$$

or

$$\|Tx\|^2-M^2\|T^2x\|^2\leq 0$$

or

$$\|Tx\|^2\leq M\|T^2x\|$$

or

$$\|T^2x\|\geq\frac{1}{M}\|Tx\|^2\ .$$

Hence the theorem.

Theorem 2.2. *If T is a generalized paranormal operator and* (2.1.1) *is satisfied then T^2 also is a generalized paranormal operator.*

Proof. Since T satisfies (2.1.1) then

$$T^{*2}T^2+\frac{2\lambda}{M}T^*T+\lambda^2I\geq 0$$

implies

$$T^*\left(T^{*2}T^2+\frac{2\lambda}{M}T^*T+\lambda^2I\right)T\geq 0$$

or

$$\left(\left(T^{*3}T^3+\frac{2\lambda}{M}T^{*2}T^2+\lambda^2T^*T\right)x,\ x\right)\geq 0$$

or

$$(T^3x,\ T^3x)+\frac{2\lambda}{M}(T^2x,\ T^2x)+\lambda^2(Tx,\ Tx)\geq 0$$

$$\|T^3x\|^2+\frac{2\lambda}{M}\|T^2x\|^2+\lambda^2\|Tx\|^2\geq 0\ .$$

This will be true only if

$$\frac{4}{M^2}\|T^2x\|^4-4\|Tx\|^2\|T^3x\|^2\leq 0$$

or

$$\|T^2x\|^4-M^2\|Tx\|^2\|T^3x\|^2\leq 0$$

$$\|T^3x\|^2\|Tx\|^2\geq\frac{1}{M^2}\|T^2x\|^4$$

$$\geq\frac{1}{M^4}\|T^2x\|^2\|Tx\|^4$$

$$\geq\frac{1}{M^2}\frac{1}{M^2}\|T^2x\|^2\|Tx\|^4$$

$$\|T^3x\|\|Tx\|\geq\frac{1}{M^2}\|T^2x\|\|Tx\|^2$$

or

$$\|T^3x\| \geq \frac{1}{M^2}\|T^2x\|\,\|Tx\| .$$

Hence the theorem.

Theorem 2.3. *If T is a generalized paranormal operator, then T^n is generalized paranormal for every integer $n \geq 1$.*

Proof. It is sufficient to show that if T and T^k are generalized paranormal, then T^{k+1} is generalized paranormal too. We may assume $\|T^2x\| \neq 0$, then

$$(2.3.1) \qquad \begin{aligned} \|T^{2(k+1)}x\| &= \left\| T^{2k}\frac{T^2x}{\|T^2x\|} \right\| \|T^2x\| \\ &\geq \left\| T^k\frac{T^2x}{\|T^2x\|} \right\|^2 \frac{1}{M}\|T^2x\| \\ &= \frac{1}{M}\frac{\|T^{k+2}x\|^2}{\|T^2x\|^2}\|T^2x\| . \end{aligned}$$

We know from Theorem 2.2 that

$$(2.3.2) \qquad \|T^{k+2}x\| \leq \frac{1}{M^{2k+1}}\|T^{k+1}x\|^2\|T^2x\| .$$

Therefore, from (2.3.1) and (2.3.2) we get

$$\|T^{2(k+1)}x\| \geq \frac{1}{M^{2k+1}}\,\frac{1}{M}\,\frac{\|T^{k+1}x\|^2}{\|T^2x\|^2}\|T^2x\|\,\|T^2x\|$$

$$\|T^{2(k+1)}x\| \geq \frac{1}{M^{2(k+1)}}\|T^{k+1}x\|^2 .$$

So T^k is a generalized paranormal operator. Now we assume that T^k is generalized paranormal and we show that T^{k+1} is also generalized paranormal. By the Theorem 2.1, we have for any real number $\lambda > 0$.

$$T^{*(2k+1)}T^{(2k+1)} + \frac{2\lambda}{M}T^{*(k+1)}T^{(k+1)} + \lambda^2T^*T = T^*(T^{*2k}T^{2k} + 2\lambda T^{*k}T^k + \lambda^2 I)T \geq 0 .$$

This implies that

$$\|T^{2k+1}x\|^2 + \frac{2\lambda}{M}\|T^{k+1}x\|^2 + 2\lambda^2\|Tx\|^2 \geq 0 .$$

Hence

$$\frac{4}{M^2}\|T^{k+1}x\|^4 - 4\|T^{2k+1}x\|^2\|Tx\|^2 \leq 0$$

or

$$\frac{1}{M^2}\|T^{k+1}x\|^4 \leq \|T^{2k+1}x\|^2\|Tx\|^2 .$$

From Theorem 1.2 we know that T will be generalized paranormal if

$$\|T^{n+1}x\|^2 \geq \frac{1}{M^{2n-1}}\|T^n x\|^2 \|T^2 x\| ,$$

for every positive integer n. Thus

$$\|T^{2(k+1)}x\|^2 \geq \frac{1}{M^{2(2k+1)}}\|T^{k+1}x\|^4$$

or

$$\|T^{2(k+1)}x\| \geq \frac{1}{M^{2k+1}}\|T^{k+1}x\|^2 .$$

Therefore T^{k+1} is a generalized paranormal operator. An operator T is said to be a *partial isometry* if $T=TT^*T$. An operator T is called *generalized quasi-hyponormal* if $\|T^2x\| \geq \frac{1}{M}\|T^*Tx\|$ for all $x \in H$, or equivalently, if

$$M^2(T^*T)^2 - T^{*2}T^2 \geq 0 .$$

Theorem 2.4. *If T is a generalized paranormal partial isometry then T is a generalized quasi-hyponormal operator.*

Proof. By Theorem 2.1, $A=T^{*2}T^2+\frac{2\lambda}{M}T^*T+\lambda^2 I\geq 0$ for any real λ and $M>0$. Now if $B=T^*T$, then $B=B^2\geq 0$. Since $AB=BA$, we have $AB\geq 0$ (if [2, ex. 7, p. 149]).

Then this implies that

$$T^{*2}T^2+\frac{2\lambda}{M}(T^*Tx)^2+\lambda^2(T^*T)^2\geq 0$$

or

$$\|T^2x\|^2+\frac{2\lambda}{M}\|T^*Tx\|^2+\lambda^2\|T^*Tx\|^2\geq 0 .$$

For any real λ and $M>0$

$$\frac{4}{M^2}\|T^*Tx\|^4 \leq 4\|T^2x\|^2\|T^*Tx\|^2$$

or

$$\frac{1}{M}\|T^*Tx\|^2 \leq \|T^2x\|\,\|T^*Tx\|$$

or

$$\|T^2x\| \geq \frac{1}{M}\|T^*Tx\|^2 .$$

Hence the theorem.

Theorem 2.5. *Every generalized quasi-hyponormal is generalized paranormal.*

Proof. Since T is generalized quasi-hyponormal, then

$$\|T^2x\| \geq \frac{1}{M}\|T^*Tx\|^2 . \tag{2.5.1}$$

To prove T is a generalized paranormal, we have to show that

$$\|T^2x\| \geq \frac{1}{M}\|Tx\|^2 .$$

We know that for any bounded linear operator on H

(2.5.2) $$\|Tx\|^2 \leq \|T^*Tx\| .$$

Therefore, from (2.5.1) and (2.5.2), we get

$$\|T^2x\| \geq \frac{1}{M}\|Tx\|^2 .$$

Hence T is a generalized paranormal operator.

T. Saito [7] has proved the following theorem using the spectral integral:

Theorem B. *If a paranormal operator T is doubly commutative with a hyponormal operator S (i.e. $TS=ST$ and $TS^*=S^*T$) then TS is paranormal.*

Shah and Sheth [8] proved a similar theorem using the technique of quadratic equations.

Theorem C. *If a paranormal operator T commutes with an isometric operator S, then TS is paranormal.*

We shall prove the above theorem for generalized paranormal operators.

Theorem 2.6. *If a generalized paranormal operator T commutes with an isometric operator S, then TS is generalized paranormal.*

Proof. Let $A=TS$, we have for any real number λ and $M>0$.

$$A^{*2}A^2+\frac{2\lambda}{M}A^*A+\lambda^2 I=S^*T^*S^*T^*TSTS+\frac{2\lambda}{M}S^*T^*TS+\lambda^2 I .$$

Using $TS=ST$, $T^*S^*=S^*T^*$ and $S^*S=I$, we have

$$A^{*2}A^2+\frac{2\lambda}{M}A^*A+\lambda^2 I=T^{*2}T^2+\frac{2\lambda}{M}T^*T+\lambda^2 I \geq 0 ,$$

so that A is generalized paranormal by Theorem 2.1.

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