YOKOHAMA MATHEMATICAL
JOURNAL VOL. 26, 1978

NORMAL APPROXIMATE SPECTRUM OF OPERATORS

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(Received October 3, 1977)

ABSTRACT. Putnam [5] considered λ and T as accessible point of spectrum of T and semi-normal operator $((TT^*-T^*T)=D\geq 0$ or $D\leq 0)$ respectively and proved some inequalities, which I have proved again in more general form for λ as a normal approximate spectrum and T as any bounded linear operator on a Hilbert space.

1. Only bounded linear operators on a Hilbert space are to be considered. A bounded linear operator T is called semi-normal if $TT^*-T^*T=D$, $D\geq 0$ or $D\leq 0$. Clearly any normal operator is also semi-normal. It is easy to see that the converse is also true in case the space is finite dimensional. For if, say, $D\geq 0$, its eigenvalues are non-negative while their sum is the trace of D, which is zero. Hence all eigenvalues are 0 and so $D=0$: In the infinite dimensional case however it is possible that an operator be semi-normal without being normal. In fact, any isometric but not unitary operator V has this property; for $VV^*-V^*V\leq 0$, $\neq 0$. On l^2 the operator T given by the matrix $T=(a_{i,j})$ with $a_{i+1,i}=1$ and $a_{i,j}=0$ otherwise $(i,j=1,2\cdots)$ is such an operator.

The above statement shows that an operator be semi-normal without being normal, it can also be proved in the case of T being unbounded. First we defined the notions of hyponormality and semi-normality for not necessarily bounded operators.

An operator T is called *hyponormal* if it is closed, densely defined and satisfies the conditions:

$$D_T=D_{T^*},$$
$$\|T^*x\|\leq\|Tx\| \text{ for } x\in D_T, \text{ where } D_T \text{ is domain of } T.$$

An operator T is called *semi-normal* if it is closed, densely defined and if T or T^* is hyponormal. It is clear that every normal operator is hyponormal and therefore semi-normal. Moreover the above definitions are extensions of the definitions of hyponormality and semi-normality for bounded operators.

Let $H=l^2$ and let T be the infinite matrix

$$(a_{ij})_{i,j=1}^{\infty},$$

where $a_{i,i+1}=i$ for $i=1, 2, \cdots$, and $a_{ij}=0$ for $j\neq i+1$. We will show that T^* is hyponormal, it is clear that T is closed and densely defined. It remains therefore to show that $D_T=D_{T^*}$, $\|Tx\|\leq\|T^*x\|$ for each x in D_T and that there exists vector in D_T for which $\|Tx\|\leq\|T^*x\|$.

$$D_T=\{x=(\xi_1, \xi_2, \cdots);\ \sum_{i=1}^{\infty}(i-1)^2|\xi_i|^2<\infty\}\ .$$

$$D_{T^*}=\{x=(\xi_1, \xi_2, \cdots);\ \sum_{i=1}^{\infty}i^2|\xi_i|^2<\infty\}\ .$$

For each $x\in D_{T^*}$, $x\neq 0$ we have

$$\|Tx\|^2=\sum_{i=2}^{\infty}(i-1)^2|\xi_i|^2<\sum_{i=1}^{\infty}i^2|\xi_i|^2\ .$$

Moreover, since $i^2<2(i-1)^2$ for $i\geq 2$, the inequality $\sum_{i=2}^{\infty}(i-1)^2|\xi_i|^2<\infty$ implies $\sum_{i=1}^{\infty}i^2|\xi_i|^2<\infty$. The semi-normality of T is proved.

If θ is real it is clear that $e^{i\theta}T$ is also semi-normal whenever T is, if $T=H+iJ$ is replaced by $e^{i\theta}T=H_\theta+iJ_\theta$. Then

$$H_\theta=\frac{1}{2}(e^{i\theta}T+e^{-i\theta}T^*)$$

and

$$J_\theta=\left(\frac{1}{2i}\right)(e^{i\theta}T-e^{-i\theta}T^*)\ .$$

Let

$$T_\theta=e^{i\theta}T \qquad \text{for real } \theta\ .$$

Then

$$H_\theta=\frac{1}{2}(T_\theta+T_\theta^*)\ .$$

It is seen that H_θ is the real or imaginary part of T according as $\theta=0$ or $\theta=-\pi/2$.

If $\lambda\in sp(T)$ then λ is called accessible if there exists a sequence $\{\lambda_n\}$, $\lambda_n\notin sp(T)$, satisfying $\lambda_n\to\lambda$ as $n\to\infty$. A complex number λ is an approximate proper value of T provided that λ and T satisfy

$$\|Tx_n-\lambda x_n\|\to 0 \quad (n\to\infty) \tag{1.1}$$

for a sequence $\{x_n\}$ of unit vectors. Furthermore, if λ and T satisfy (1.1) and

$$\|T^*x_n-\bar{\lambda}x_n\|\to 0 \quad (n\to\infty)\ , \tag{1.2}$$

then λ is called a normal approximate proper value of T. The normal approximate spectrum denoted by $\Pi_n(T)$ is defined to be the set of all normal approximate proper values of T.

For a (continuous linear) operator T on a Hilbert space, we use the following notation and terminology: spectrum $sp(T)$, continuous spectrum $C(T)$, approximate point spectrum $\Pi(T)$, point spectrum $p(T)$, spectral radius $r(T)=\sup\{|\lambda|: \lambda\in sp(T)\}$, boundary ∂, numerical range $W(T)$ is convex, and $\Sigma T\subset Cl\ W(T)$ (Σ=convex hull of the spectrum, Cl=closure), if $\Sigma T=Cl\ W(T)$ then T is called convexoid operator and we say that an operator T is restriction convexoid if the restriction of T to every invariant subspace has property convexoid.

2. Putnam [5] proved the following theorem:

Theorem A. *Let T be bounded and satisfy $TT^*-T^*T=D\geq 0$ and let $\lambda=re^{-i\theta}$ ($r\geq 0$) be an accessible point of $sp(T)$. Then*

$$(\max H_\theta)^2\geq \min TT^*$$

and

$$|r-\max H_\theta|\leq[(\max H_\theta)^2-\min TT^*]^{1/2}\ .$$

We prove the Putnam inequalities for bounded linear operator taking λ in the normal approximate spectrum. Actually, we will prove the following.

Theorem 2.1. *If T is any operator and*

$$\lambda\in\Pi_n(T)\ ,\qquad \lambda=re^{-i\theta}\quad (r\geq 0)\ ,$$

then

$$(\max H_\theta)\geq r\geq(\min T^*T)^{1/2} \tag{2.1}$$

and

$$|r-\max H_\theta|\leq[(\max H_\theta)^2-\min T^*T]\ . \tag{2.2}$$

Proof. Since $\lambda\in\Pi_n(T)$, there exists a sequence $\{x_n\}$ of unit vectors such that

$$\|(T-\lambda I)x_n\|\to 0 \quad\text{and}\quad \|(T^*-\bar\lambda I)x_n\|\to 0\ .$$

We get

$$(T-\lambda I)x_n\to 0 \quad\text{and}\quad (T^*-\bar\lambda I)x_n\to 0\ .$$

Therefore

$$((T-\lambda I)x_n, x_n)\to 0\ .$$

Similarly

$$((T^*-\bar\lambda I)x_n, x_n)\to 0\ .$$

It follows that

$$(e^{i\theta}Tx_n, x_n)+(e^{-i\theta}T^*x_n, x_n)\to\lambda e^{i\theta}+\bar\lambda e^{-i\theta}$$

or

$$\frac{1}{2}((e^{i\theta}T+e^{-i\theta}T^*)x_n, x_n)\to\frac{1}{2}(r+r)$$

or

$$(H_\theta x_n, x_n)\to r\ ,$$

and therefore

$$\max H_\theta \geq r\ .$$

Since

$$0=\lim\,((T-\lambda I)x_n, Tx_n)$$
$$0=\lim\,(T^*Tx_n, x_n)-\lambda(x_n, Tx_n)\ ,$$

we get

$$\lim\,(T^*Tx_n, x_n)=\lambda\bar{\lambda}=r^2$$

and

$$\min T^*T\leq r^2\ .$$

Thus

$$\max H_\theta\geq r\geq(\min T^*T)^{1/2}\ .$$

Since

$$\begin{aligned}(T-\lambda I)^*(T-\lambda I)&=(T^*-\bar{\lambda}I)(T-\lambda I)\\ &=(T^*T-\bar{\lambda}T-\lambda T^*+\lambda\bar{\lambda})\\ &=T^*T-2rH_\theta+r^2\ ,\end{aligned}$$

one has

$$\|(T-\lambda I)x_n\|=(T^*Tx_n, x_n)-2r(H_\theta x_n, x_n)+r^2\ .$$

Hence

$$\min T^*T\leq(T^*Tx_n, x_n)=\|(T-\lambda I)x_n\|+2r(H_\theta x_n, x_n)-r^2\ .$$

Letting $n\to\infty$

$$\begin{aligned}\min T^*T&\leq 2r\max H_\theta-r^2\\ &\leq(\max H_\theta)^2-(\max H_\theta-r)^2\end{aligned}$$

or

$$|r-\max H_\theta|\leq[(\max H_\theta)^2-\min T^*T]^{1/2}\ .$$

Theorem 2.2. *Let T be a restriction convexoid operator and $\lambda=re^{-i\theta}$, $(r\geq 0)\in sp(T)$ is finite, then* (2.1) *and* (2.2) *hold.*

Before giving the proof of theorem 2.2 we need the following lemmas:

Lemma A [3, Theorem 1]: *If λ belongs to $\partial W(T)$ and $\Pi(T)$ then $\lambda\in\Pi_n(T)$, where $\partial W(T)$ is the frontier of numerical range of T.*

Lemma B [4, Theorem 3]: *If λ is a normal approximate proper values of A, then there exists a character ϕ on the C^*-algebra U generated by A and I which satisfies*

$$\phi(A)=\lambda\ .$$

Lemma C [1, Lemma 2]: *If T is restriction-convexoid and λ is an isolated point of $sp(T)$, then λ is an eigenvalue.*

Proof of Theorem 2.2. By lemma C $sp(T)=p(T)\subset W(T)$ since T is convexoid, $Cl\ W(T)=\Sigma(T)$, $\Sigma(T)\subset W(T)$, thus $W(T)$ is closed, $W(T)=\Sigma(T)$ and the extreme points of $W(T)$ belong to $sp(T)$. Therefore, if λ be the extreme point of $W(T)$ then $\lambda \in sp(T)\cap\partial W(T)\subset\Pi_n(T)$ by Hildebrandt's theorem [3] Satz 2. Hence we have

$$\lambda \in \Pi_n(T) .$$

Now, from Theorem 2.1 we get the result.

Remark. If T is hyponormal and $\lambda=re^{-i\theta}$ $(r\geq 0)$ and $sp(T)=C(T)$, then (2.1) and (2.2) hold for T.

Proof. Since $sp(T)=C(T)$ implies that there exists a sequence of unit vectors, say $\{x_n\}$, such that

$$\|Tx_n-\lambda x_n\|\to 0 \quad (n\to\infty) .$$

From hyponormality, we get

$$\|T^*x_n-\bar{\lambda}x_n\|\to 0 \quad (n\to\infty) .$$

Now from Theorem 2.1 we get the result.

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